INTRODUCTION TO A THEORY OF CLOSURE SPACES

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INTRODUCTION TO A THEORY OF CLOSURE SPACES

by

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THESIS

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Preface

This text is a self-contained introduction to a theory of closure spaces. "Closure" is a simple concept that appears very frequently in mathematics and theoretical computer science. Examples are the "topological closure" of a set of points of a topological space, the "convex closure" of a set of geometrical points, the "linear closure" of a set of vectors (i.e. the vector space spanned by these vectors), the "deductive closure" of a set of logical formulae, or the "reflexive and transitive closure" of a binary relation. If relations are – for that purpose – regarded as sets of pairs, these examples have in common that the intersection of any number of "closed sets" (that is, of sets that equal their own closure) is again closed. A "closure space" is in this text essentially understood as an "intersection-stable" set of sets. At the outset, a theory about such structures might not appear very rewarding. However, we will demonstrate that it can provide a number of nice results. The material is also relevant to computer science because it provides a thorough introduction to structures defined in terms of a complete partial order (also called a 'complete lattice'). We think a familiarity with these structures should make more accessible the subject of continuous lattices, which are used for instance in theories of computation (cf. *Scott* [Sc76], [Sc80], [CL]).

Our investigations started out with a study of logical systems based on properties of the "derivation operator". Regarding well-formed formulae as points in a given collection, this operator maps points to sets of points. (This level of description is for instance to be found in *Tarski* [Ta], where a number of our results are presented in the setting of logic.) The result of applying the derivation operator to a set X of points (i.e., formulae), called the "deductive closure of X", is commonly understood to be the set of the formulae that can eventually be derived from those given in X. They include the elements of X (by a derivation in "zero steps"), which defines the derivation operator as 'increasing'. The second distinctive property of derivation operators as 'closure operations' is that the deductive closure of a set is "deductively closed". We call a set X 'closed' under a set mapping f if $f(X) \subseteq X$, and

'deductively closed' if f is the derivation operator. A somewhat more basic, and also necessary, property is *monotonicity*; that is, the derivation operator has to preserve set inclusion. This property alone suffices to prove that the set of deductively closed sets is a closure space. As concerns the study of logical systems, it is possible to demonstrate many notions of model theory in terms of the set of deductively closed sets alone, as for instance done in *Chang/Keisler* [CK]. Related notions in logic and topology, e.g. inconsistency and denseness, can also be observed (cf. *Martin* [Ma]). However, the results presented here are interesting in a more general framework, which belongs to the area of universal algebra rather than logic or topology.

First, the notion of a complete partial order, introduced in chapter one, can replace set inclusion (with "infimum" instead of intersection). Second, a closure space can be defined by a monotonic function, not necessarily a closure operation. In the example above, the suitable monotonic function, call it m, is indeed more basic than the described derivation operator. It maps any set of formulae to the set of those formulae that can derived in *one* step using a rule of the logical system. This function yields the same deductively closed sets as the derivation operator before. The derivation operator is definable in terms of the closure space of closed sets of m: the deductive closure of a set X is the intersection of the deductively closed sets that include X. Along these lines, it can be proved that there is a one-to-one correspondence between closure spaces and closure operations. Furthermore, mapping a monotonic function m to the closure operation corresponding to the closure space of closed sets of m is itself a closure operation (cf. theorem (2.24)). Another "meta-closure" is given by the fact that the set of all closure spaces is a closure space (theorem (2.25)). These are the essential results of chapter two, the central chapter of the text; they have also been discovered by others (for example, in [Sc76, pp. 549-553], [Sc80, p.239], [CL, p.21]).

The most original results of this text are contained in chapter three. However, the overall presentation of the paper also deserves comment. The subject matter is treated in this text in very detailed form. Every proof is reduced to steps justified by selected axioms of predicate calculus. This formal treatment helped us

substantially in seeing repetitions and similarities, yielding – as we think – a more structured exposition than in common mathematical texts. The proofs are in our opinion a rewarding exercise in formality.

The basic axioms are presented in chapter zero, and are interesting insofar as negation in not used, and disjunction only in the form of existential quantification. The proof system is part of intuitionistic and classical logic. However, the text is not directed to the logician. The proofs are only semi-formal in that words are used besides symbols, and derivations are presented as chains of implications or equivalences, not as successive applications of a derivation operator. In short, we do not subject to formal treatment the way predicates are given and derived. The syntactical process of substitution is also left informal.

As another remark on how this text differs from usual mathematics, we mention that we use types as a basis for the discussion instead of sets. The main motivation for types is to reduce the specification of "ranges" in quantified predicates, which would be necessary if a universal domain were assumed (e.g., the class of all sets). It would have been possible to identify many more constructs, by reducing sets, relations, and possibly predicates, to functions and cartesian products, but we considered the level of discourse abstract enough

In chapter one, complete partial orders are introduced as the underlying structure in subsequent chapters. The infimum of a set is defined by an equation ([1.13]), instead of by the usual twofold description as "lower bound" and "greatest lower bound" (cf. (1.15)). This definition permits convenient proofs of equivalences (e.g. (1.20)). In chapter two, the condition of "sub-idempotency" replaces the common "idempotency" in the definition of a closure operation (cf. [2.13]). In chapter three, we can then express the result that reflexive and transitive relations can be regarded as increasing and sub-idempotent disjunctive set mappings, respectively. Also, a threefold characterization of the reflexive and transitive closure of a relation is given. One is the first "meta-closure" mentioned above, because a reflexive and

transitive relation corresponds to a disjunctive closure operation. Another uses relational composition as a monotonic operator to define a suitable closure space. The third is the usual "iterative" definition. The proof (cf. (3.39)) can be regarded as representative for the known iteration of continuous functions to obtain a closure operation, which is not proved here. We refer again to the above example of the derivation operator. The formulae that can eventually be derived from a given set X of formulae are obtained by applying successively, but only finitely often, rules of the logical system. This corresponds to iterating the simpler "one-step" function m above. The reason that finitely many applications of rules are enough is that every rule has only finitely many "antecedents". In other words, the derivation operator is "finitely representable" as will be discussed in chapter four (cf. also *Tarski* [Ta, p.64, Axiom 4]). This chapter treats continuous functions, in particular the equivalence of "distribution over unions of directed sets" and "finite representability". This equivalence is cited, for example, in *Scott* [Sc80, p.229].

For reasons of space, some considerations had to be omitted that would have fit the level of presentation. One concerns the subtype of a type P that has a complete partial order, consisting of the *fixed points* of a monotonic function on P. The restriction of the partial order on this subtype is complete, but the "infimum" and "supremum" functions are different from the original functions, although definable in terms of these. This theorem can be proved using (1.17). The concept of fixed points occurs frequently in semantics, in connection with recursively defined programs (for instance in *Dijkstra* [Di] or *Scott* [Sc76]). Other applications of the theory are mentioned in the text, but are not explained in detail.

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B.V.S.

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Chapter 0

Notation and Proof System

The first part of this chapter explains the notation used in the text. It deviates from standard notation mostly in the usage of sets; we will use *types* instead of sets for the ranges of quantifiers and as domains for functions, and are interested in sets only as elements of a collection usually called "power set" (here: power type). The rest of the notation is standard; the casual reader can start at chapter one and will understand most of our conventions. In the second part of this chapter, we will list the rules (of predicate calculus) actually used in the text. We do this for the sake of completeness, because the effort is relatively small. Our results can be expressed with fewer symbols and proved with less axioms than usually used in mathematics. This justifies to some extent presenting a complete "calculus". (Some familiarity with common predicate calculus is useful, however.) Finally, our logical system might be of interest, because it is a subsystem of both classical and intuitionistic logic.

Statements and Proofs.

The statements (axioms or theorems) of the text are usually written down in the following way. Some predicates are listed ("Let ..; ..; ..") as *assumptions*. The conclusion is then usually of the form $a \Rightarrow b$ or $a \Leftrightarrow b$ (disregarding a possible universal quantification). A statement used later is labeled by a number which is surrounded by brackets [] if it is an axiom or a definition (requiring no proof), by parentheses () if it is a proposition or theorem (to be proved using previously made statements). A statement is labeled with a number followed by a prime ('), if it differs so little from the corresponding "un-primed" statement, that the latter is of nearly equal help as a reference.

In the proof of a statement, the assumptions are assumed to be given. We have chosen to write down derivations as sequences of lines of the form

...comment: inference-symbol predicate,

where the inference-symbol is either $\Rightarrow$ or $\Leftrightarrow$. The predicate is derived from the previous line, and the comment explains which statement is used as a transformation rule, as specified further below. The first line contains no comment or inference-symbol, only a predicate. If the predicate of the first line is a, and that of the last line is b, then the whole sequence of lines is considered as a proof for $a \Rightarrow b$; if in every line the inference-symbol is $\Leftrightarrow$, as a proof for $a \Leftrightarrow b$. An assertion of the form $a \wedge b$ can be proved by separately proving a and b. Universally quantified predicates of the form $(\forall x \,.\, c(x))$, e.g. $(\forall x \,.\, a(x) \Rightarrow b(x))$, are proved by proving $c(x)$ for an arbitrary choice of x.

This proof format was chosen because alternating comment lines and predicates would reduce the legibility. Furthermore, a skilled reader can frequently skip the comments. The comments appear on the left hand side, because in most cases the transformation deserves a comment rather than the predicate obtained. If we want to refer to this predicate later within the proof, we label it on the right by a lower-case letter in parentheses.

Some more comments on using statements in derivation steps are appropriate. The derivation step

a

... (statement X): $\Rightarrow$ b

is possible if, after proper (uniform) substitutions, the assumptions of statement X are met and its conclusion is of the form $a \Rightarrow b$. If it is a universally quantified predicate, it is allowed to *instantiate* the variable bound by the quantifier, say x, to a particular term, say t. We preferred not to indicate this in a comment, for example by "x:=t", because the particular name of a bound variable is irrelevant, so that the symbol x should not even appear outside the scope of the universal quantification. An instantiation in the form "x:=t" seems to be only appropriate for "free" identifiers x. We therefore do not explicitly mention an instantiation which is mostly obvious, or refer only to axiom [0.30] (s.b.).

The preceding comments hold correspondingly for a proof step of the form a ⇔ b. The inference-symbol ⇔ can also be used by replacing any predicate by an equivalent predicate, or by replacing any term by an equal term; the respective statements used are equivalences or equations. Several derivation steps can be contracted into one, in particular if very simple statements are used, or if this sequence of steps has occurred before.

Predicates.

We use ∧, ⇒ and ⇔ as the only logical connectives. To avoid extensive use of parentheses, we give precedence rules for binary operators. We will assume ∧ has higher binding power than ⇒ (that is, a∧b⇒c and a⇒b∧c are to be read as (a∧b)⇒c and a⇒(b∧c), respectively). The use of parentheses can be further restricted if connectives are associative, which holds in particular for ∧, i.e. (a∧b)∧c and a∧(b∧c) are equivalent and will therefore be written as a∧b∧c. For symbols with equal binding power, and unless associativity is explicitly stated, the use of parentheses is required. ⇒ and ⇔ have equal binding power.

We write quantified expressions in the form (Qx . a(x)) , where Q stands for the existential quantifier ∃ or for the universal quantifier ∀. The variable bound by the quantifier (i.e. the variable following the quantifier) has its meaning only within the outer parentheses, which thus denote the (syntactical) scope of the quantification. By a(x) we mean a predicate with a free occurrence of x, that is, a(x) depends on the variable x if interpreted as a truth-valued expression. It is allowed that x does not appear at all in a(x), but x must not be bound by a quantifier within the predicate. The result of substituting y for x in a(x) is written as a(y). In using the notation a(x) for general predicates, we only require that within the scope of a quantification, all free occurrences of the bound variable are mentioned. For example, in the "axiom schema" (∃x . a ∧ b(x)) ⇔ a ∧ (∃x . b(x)), x must not occur free in a; there can be other free variables in a, however. We will automatically rename bound variables, if the scopes of the respective quantifications would overlap otherwise. For instance, we will use (∀y . (∀x . a(x)) ⇒ b(y)) instead of (∀x . (∀x . a(x)) ⇒ b(x)).

The interpretation of $(Qx\,.\,a(x))$ is that the bound variable x "ranges" over (i.e. denotes an object of) some *type*. This type can in general be *inferred* from syntactical parts of the expression a(x) which are different from x. $(\exists x\,.\,a(x))$ means "there is an x within the type such that a(x) holds", $(\forall x\,.\,a(x))$ means "for all x of the type a(x) holds". Equivalent predicates are $(\exists x\,.\,x{:}T \wedge a(x))$ and $(\forall x\,.\,x{:}T \Rightarrow a(x))$, respectively, where $x{:}T$ indicates explicitly that "x is of type T"; $x{:}T \wedge a(x)$, for instance, is therefore a predicate where one can infer that x must be of type T. The type system is described in the next section.

Types.

We distinguish predicates (as purely syntactic entities) form *objects* and *types* on the semantic side. Each object belongs to a type. Objects are denoted by terms, which are either variables, constants or functional expressions. Types are in general mutually exclusive collections of objects and will never be regarded as objects themselves. We assume some types are given (e.g., the type of natural numbers), and will describe how new types can be obtained from given ones. Capital letters P, S, T, U will denote types.

The notation $x{:}T$ expresses that object x belongs to type T. Objects of the same type can be enumerated using commas; for example, $x,y{:}T$ is short for $x{:}T \wedge y{:}T$. (Remark: As a rule, when commas are used, the order of the enumerated items (here variables) is not important; in this case, because $\wedge$ is symmetric and associative).

The expression $x{:}T$ is a predicate. Alternatively, an (atomic) predicate can be formed by putting a symbol for a (binary) relation between two terms. Predicates can also be of the form $x \in A$ where x *and* A denote objects. In this case, if x is of type T, then the type of A is the so-called *power type* of T; in other words, A is a *set*. Symbols for relations, colon (:) and $\in$ have higher binding power than any logical connective.

We regard functions, relations and sets as (typed) objects. If S and T are types, $S \to T$ is the type of all functions from S to T. Thus $f{:}S \to T$ asserts that f is an object of this type. The result of applying the function f to the argument x is written as f(x); this term always denotes an object of the range T of f, and the type of x must be the domain S of f. A function *on* S is a function with domain and range S.

As concerns relations, we will only deal with binary relations between objects of the same type. We introduce **RT** as the type of all binary relations on T. Hence, R:**RT** expresses that R is such a relation, and sRt is then a predicate, where s and t must be terms denoting objects of type T.

The *power type* of a type T is written as **PT**. An object A of this type is a set of objects of type T. The predicate $t \in A$ expresses that a particular object of type T (only this case is allowed) is an element of A. The empty set $\varnothing$ is a constant of type **PT**. The "type-creating" symbols **R** and **P** have higher binding power than $\rightarrow$.

It would be possible to introduce more general schemes (e.g. cartesian products; relations can then be defined as "subsets" of cartesian squares). Our particular choice of only regarding functions, binary relations and sets is motivated by the kinds of structures we want to study.

Disjointness of types: we assume that each object belongs to exactly one type, with the exception of subtypes. In particular, for different types S, T, any two of the types $S, T, S \rightarrow T, T \rightarrow S, S \rightarrow S, T \rightarrow T$, **RS**, **RT**, **PS**, **PT** are assumed to be disjoint. However, in particular cases we may allow one symbol to be used for several types. It is thereby understood that this symbol denotes a different object for each type. We will use this "type-overloading" of a symbol if it denotes objects with similar properties: an example is $\varnothing$, and another is the equality symbol introduced below. An object can belong to two types only if one is a subtype of the other. The concept of subtypes will be introduced in chapter two, where it is needed.

Basic Axioms and Propositions.

In this last section, the "proof system" will be explained, i.e. general statements that are used in the proofs of the text. These axioms and propositions are listed at the end of the chapter for easy reference. The axioms are not claimed to be independent, only those statements are indicated as propositions which follow more or less obviously from others; the proofs will be given. We comment here mainly on how to interpret the statements; part of their use can be demonstrated in the proofs.

The assertions [0.0] to [0.16] describe our "propositional calculus", which does not involve the type system. They have to be read as "axiom schemata", i.e. the identifiers a, b, c, d are to be replaced by (arbitrary) actual predicates. The only such predicate not involving types is the logical constant 'true', which by [0.0] always holds. [0.1], [0.2], [0.3] state that 'true ⇒', 'true ⇔' and 'true ∧' can be prefixed to any predicate without changing its meaning. In particular, [0.1] allows us to prove predicate a by proving true ⇒ a, as we will frequently do. [0.2'] states that any known predicate a can be used to replace true in a derivation; inparticular, a can be added as a conjunct to any predicate because of [0.3]. An example is given in the following proof.

Proof of (0.6):

	true	
... [0.3], [0.3]:	⇔ true ∧ true ∧ true	
... [0.2'] with [0.5], [0.8], [0.8], resp.:	⇔ (a∧a ⇒ a) ∧ (a⇒a) ∧ (a⇒a)	
... [0.11]:	⇔ (a∧a ⇒ a) ∧ (a ⇒ a∧a)	
... [0.13]:	⇔ (a∧a ⇔ a).	*(End of Proof.)*

Note that we adopt the associativity of ∧ (regardless of whether needed or not) by our writing conventions instead of explicitly referring to [0.7]. [0.4] asserts the symmetry of ∧. It allows us to derive the symmetry of ⇔:

Proof of (0.14):

	(a ⇔ b)	
... [0.13]:	⇔ (a⇒b) ∧ (b⇒a)	
... [0.4]:	⇔ (b⇒a) ∧ (a⇒b)	
... [0.13]:	⇔ (b ⇔ a).	*(End of Proof.)*

A simple proof like this will be abbreviated as "By [0.13], [0.4], [0.13].". For brevity's sake, we will leave the use of [0.14] implicit, and sometimes also of other simple statements like [0.2'], [0.3] or [0.4].

Proof of (0.5'): By [0.4], [0.5]. *(End of Proof.)*

Proof of (0.13'): By [0.13], [0.5]. *(End of Proof.)*

Proof of (0.16):

	true	
... [0.8]:	⇔ ((a⇒b) ⇒ (a⇒b))	
... [0.10]:	⇔ ((a⇒b) ∧ a ⇒ b)	
... [0.4]:	⇔ (a ∧ (a⇒b) ⇒ b).	*(End of Proof.)*

Many of the axioms of "propositional calculus" justify the inference-rules given in the first section of this chapter. Therefore, statements like [0.15] are not introduced as theorems. One might regard [0.13] as a definition of the symbol ⇔; we do not adopt this view, because we want to use ⇔ from the beginning without restriction.

The statements [0.17] to [0.33] are axiom schemata for quantified expressions. It is assumed that variables occurring in predicates, like x in a(x), are consistently typed. An example is given (further below) by [0.38], where the variable s bound by the quantifier must range over S, because f and g are functions with domain S; this "type-inferencing" is used throughout the text and should require no further explanation.

[0.17] to (0.20) describe the possible *interchange* of quantifiers, which will be a crucial step in many proofs. [0.17] states that consecutive quantifiers of the same kind (Q is to be uniformly substituted by ∀ or ∃) can be replaced by a single one followed by a list of the bound variables (read (∀x,y . c(x,y)) as "for all x and y c(x,y)"). The order of these variables is unimportant, as stated in [0.18]. For any axiom schema with quantifiers, bound variables can also be vectors, where a vector is either a variable or of the form u,v with vectors u and v, as e.g. x,y,z in [0.36]. The reader might convince himself that these syntactic conventions allow free rearrangements of the variables following a quantifier. (0.19) and (0.20) are easy to memorize by thinking of the predicates a(x) and b(y) as "subranges" of the type of the variables x and y bound by the first and second quantifier of the left hand side, respectively. By assuming 'true' for these subranges, (0.18') is because of [0.1], [0.3] a special case of (0.19) or (0.20), respectively.

Proof of (0.18'): By [0.17], [0.18], [0.17]. *(End of Proof.)*

Proof of (0.19): $(\forall x\,.\,a(x) \Rightarrow (\forall y\,.\,b(y) \Rightarrow c(x,y)))$

... [0.21]: $\Leftrightarrow\ (\forall x\,.\,(\forall y\,.\,a(x) \Rightarrow (b(y) \Rightarrow c(x,y))))$

... [0.10]: $\Leftrightarrow\ (\forall x\,.\,(\forall y\,.\,a(x) \wedge b(y) \Rightarrow c(x,y)))$

... (0.18'), [0.4]: $\Leftrightarrow\ (\forall y\,.\,(\forall x\,.\,b(y) \wedge a(x) \Rightarrow c(x,y)))$

... [0.10]: $\Leftrightarrow\ (\forall y\,.\,(\forall x\,.\,b(y) \Rightarrow (a(x) \Rightarrow c(x,y))))$

... [0.21]: $\Leftrightarrow\ (\forall y\,.\,b(y) \Rightarrow (\forall x\,.\,a(x) \Rightarrow c(x,y)))$. *(End of Proof.)*

Proof of (0.20): $(\exists x\,.\,a(x) \wedge (\exists y\,.\,b(y) \wedge c(x,y)))$

... [0.22]: $\Leftrightarrow\ (\exists x\,.\,(\exists y\,.\,a(x) \wedge b(y) \wedge c(x,y)))$

... (0.18'), [0.4]: $\Leftrightarrow (\exists y \,.\, (\exists x \,.\, b(y) \wedge a(x) \wedge c(x,y)))$

... [0.22]: $\Leftrightarrow (\exists y \,.\, b(y) \wedge (\exists x \,.\, a(x) \wedge c(x,y)))$. *(End of Proof.)*

[0.21] to [0.24] describe how logical connectives can be "distributed" over quantified expressions. [0.24] can be read as 'a "disjunction" of predicates implies a predicate b if each of these predicates implies b'. The assertions [0.25] and [0.26] can be regarded as "one-point rules". For, if the "subrange" for the bound variable x, as above, consists only of the single object (or "point") denoted by the term t, then the quantification can be omitted by substituting this term for x.

Whereas [0.17] to [0.26] are equivalences, [0.27] to [0.33] are (in general proper) implications. [0.27] is used to prove universally quantified predicates, as already stated in the first section. (0.28) and [0.29] assert that $(Qx \,.\, b(x))$ can be inferred from $(Qx \,.\, a(x))$ if $a(x) \Rightarrow b(x)$ is known for all x. If Q stands for the existential quantifier, $a(x) \Rightarrow b(x)$ is in many applications of the form of a "propositional" axiom. [0.29] is therefore stated with x as a free variable. An example is given in the proof of (0.31) below.

Proof of (0.28): Let $(\forall x \,.\, a(x) \Rightarrow b(x))$ hold.

Then for y:S, $(\forall x \,.\, a(x))$

... [0.30]: $\Rightarrow a(y)$

... assumption (with [0.30]): $\Rightarrow b(y)$.

This proves $(\forall y \,.\, (\forall x \,.\, a(x)) \Rightarrow b(y))$ by [0.27], which is equivalent to

$(\forall x \,.\, a(x)) \Rightarrow (\forall y \,.\, b(y))$ by [0.21]. *(End of Proof.)*

Assertion [0.30] is the "instantiation rule"; in most cases, we will not mention explicitly when it is applied.

Proof of (0.31): $a(t)$

... [0.26]: $\Leftrightarrow (\exists x \,.\, x = t \wedge a(x))$

... (0.5'), [0.29]: $\Rightarrow (\exists x \,.\, a(x))$. *(End of Proof.)*

(0.32) and [0.33] are assertions about predicates of the form $(Qx \,.\, a)$. In such a predicate, the type of the variable x bound by the quantifier can not be inferred, since a has no reference to x. It is understood that this type is the same as in a predicate derived from $(\forall x \,.\, a)$ or from which $(\exists x \,.\, a)$ is derived (e.g. from $(\exists x \,.\, a \wedge b(x))$ by [0.5], [0.29]).

This type can be left unspecified because it never matters when (0.32) or [0.33] is applied. Note that it can in particular be empty; otherwise, we could assert $(Qx\,.\,a) \Leftrightarrow a$.

Proof of (0.32): (0.32) is a special case of [0.30]. *(End of Proof.)*

The predicate calculus presented can be further demonstrated in the remaining paragraphs which treat *equality* and definitions of sets. For a type T, equality is a particular relation on T and is denoted by the symbol $=$. The predicate $s=t$ asserts that the terms s and t (which must be of type T) denote the same object. If the equation $s=t$ holds, any occurrence of s in a predicate can be replaced by t yielding an equivalent predicate (and vice versa, according to [0.35]). Using the equality symbol, one can also *define* the meaning of a new identifier. Definitions are treated like axioms.

The symbol $=$ denotes a separate relation for each type. [0.34] to [0.36] must therefore be read as shorthands for predicates that are separately given for each type. The use of the same symbol is justified because the type of the equality relation in a predicate can cause no confusion. (Some readers might also prefer to imagine a subscript, e.g. in the form $=_T$, for each equality relation of type **R**T.)

[0.37] asserts that the equality of two *relations* can be reduced to the respective equivalence of all the predicates that can be formed with the corresponding relation symbols. Two *functions* are equal if they give the same results for all arguments, as stated in [0.38]. By [0.39], two *sets* are equal if they contain the same elements. [0.38] and [0.39] state the so-called the *extensionality* of functions and sets, respectively.

[0.40] and [0.41] state that functions are total (i.e. everywhere defined) and single-valued. In particular, [0.41] asserts that terms can be replaced by equal terms if they appear as arguments for functions.

For defining sets, we also allow a *comprehension* scheme. That is, for a predicate a(s), s of type T, it is possible to define a set X of type **P**T by $(\forall s\,.\,s \in X \Leftrightarrow a(s))$. This set is expressed in the form $\{s \mid a(s)\}$, if the type of the variable s can be inferred, or as $\{s \mid s{:}T \wedge a(s)\}$ otherwise. Because of [0.39], a(s) can thereby be replaced by any equivalent predicate, in accordance with the proof rules. The variable left of the bar is to be treated like a variable bound by a quantifier; that is, it can be freely renamed. [0.42] is a more general form of comprehension (note that in [0.42] $\{f(s) \mid a(s)\}$ is of type **P**T). We will sometimes omit references to the very simple proposition (0.42').

Proof of (0.42'): For any t of the given type,

$$t \in \{s \mid a(s)\}$$

... [0.42]: $\Leftrightarrow (\exists s \,.\, a(s) \wedge t = s)$

... [0.4], [0.35]: $\Leftrightarrow (\exists s \,.\, s = t \wedge a(s))$

... [0.26]: $\Leftrightarrow a(t)$. *(End of Proof.)*

(0.43) and (0.44) will allow us to rewrite predicates that assert a particular predicate b(t) for all or some elements t (of type T) of a set that is defined by comprehension.

Proof of (0.43): $(\forall t \,.\, t \in \{f(s) \mid a(s)\} \Rightarrow b(t))$

... [0.42]: $\Leftrightarrow (\forall t \,.\, (\exists s \,.\, a(s) \wedge t = f(s)) \Rightarrow b(t))$

... [0.24]: $\Leftrightarrow (\forall t \,.\, (\forall s \,.\, a(s) \wedge t = f(s) \Rightarrow b(t)))$

... (0.18'), [0.10]: $\Leftrightarrow (\forall s \,.\, (\forall t \,.\, a(s) \Rightarrow (t = f(s) \Rightarrow b(t))))$

... [0.21]: $\Leftrightarrow (\forall s \,.\, a(s) \Rightarrow (\forall t \,.\, t = f(s) \Rightarrow b(t)))$

... [0.25]: $\Leftrightarrow (\forall s \,.\, a(s) \Rightarrow b(f(s)))$. *(End of Proof.)*

Proof of (0.44): $(\exists t \,.\, t \in \{f(s) \mid a(s)\} \wedge b(t))$

... [0.42]: $\Leftrightarrow (\exists t \,.\, (\exists s \,.\, a(s) \wedge t = f(s)) \wedge b(t))$

... [0.4], [0.22], [0.4]: $\Leftrightarrow (\exists t \,.\, (\exists s \,.\, a(s) \wedge t = f(s) \wedge b(t)))$

... (018'), [0.22]: $\Leftrightarrow (\exists s \,.\, a(s) \wedge (\exists t \,.\, t = f(s) \wedge b(t)))$

... [0.26]: $\Leftrightarrow (\exists s \,.\, a(s) \wedge b(f(s)))$. *(End of Proof.)*

It is common practice to define special sets, like sets with zero, one, or two elements, by comprehension. We do this only for the case of the *singleton* by [0.36]. We can not define the empty set this way, because we do not want to use negation. [0.37] asserts that the expression $t \in \varnothing$ is always "false". If the variable t bound by the quantifier ranges over T, $\varnothing$ is of type **PT**. The use of a constant for the empty set is justified by the following "uniqueness" theorem:

Let $\varnothing, \varnothing' : \mathbf{PT}$ and for any predicates c(t), d(t) $(\forall t \,.\, t \in \varnothing \Rightarrow c(t))$ and $(\forall t \,.\, t \in \varnothing' \Rightarrow d(t))$. Then $\varnothing = \varnothing'$.

Proof: true

... [0.3]: $\Leftrightarrow$ true $\wedge$ true

... assumption: $\Rightarrow (\forall t \,.\, t \in \varnothing \Rightarrow t \in \varnothing') \wedge (\forall t \,.\, t \in \varnothing' \Rightarrow t \in \varnothing)$

... [0.24]: $\Leftrightarrow (\forall t \,.\, (t \in \varnothing \Rightarrow t \in \varnothing') \wedge (t \in \varnothing' \Rightarrow t \in \varnothing))$

... [0.13]: $\Leftrightarrow\ (\forall t\,.\ t\in\varnothing \Leftrightarrow t\in\varnothing')$

... [0.34]: $\Leftrightarrow\ \varnothing=\varnothing'$. *(End of Proof.)*

In a similar way, [0.47] defines the set of two elements without using an "or"-connective. (From the viewpoint of conventional set theory, [0.47] indicates that conjunction is a special case of universal quantification, namely one over a two-element range.) In [0.47], {x,y} should for the moment be read as an indivisible syntactic entity, denoting an object of PT. (0.48) states one expected property of this set.

Proof of (0.48): true

... (0.32): $\Rightarrow\ (\forall t\,.\ \text{true})$

... [0.8]: $\Leftrightarrow\ (\forall t\,.\ t\in\{x,y\} \Rightarrow t\in\{x,y\})$

... [0.47]: $\Leftrightarrow\ x\in\{x,y\} \wedge y\in\{x,y\}$. *(End of Proof.)*

The following proposition asserts that in {x,y} x and y can indeed be replaced by equal terms, so that the notation for the two-element set can be used in the usual way.

Let x,y,u,v:T. Then $x=u \wedge y=v \ \Rightarrow\ \{x,y\}=\{u,v\}$.

Proof.

$x=u \wedge y=v$

$\Leftrightarrow\ x=u \wedge y=v \wedge \text{true} \wedge \text{true}$

... (0.48), (0.48): $\Leftrightarrow\ x=u \wedge y=v \wedge\ u\in\{u,v\} \wedge v\in\{u,v\}\ \wedge\ x\in\{x,y\} \wedge y\in\{x,y\}$

$\Rightarrow\ x\in\{u,v\} \wedge y\in\{u,v\}\ \wedge\ u\in\{x,y\} \wedge v\in\{x,y\}$

... [0.47], [0.47]: $\Leftrightarrow\ (\forall t\,.\ t\in\{x,y\} \Rightarrow t\in\{u,v\}) \wedge (\forall t\,.\ t\in\{u,v\} \Rightarrow t\in\{x,y\})$

... [0.23], [0.13]: $\Leftrightarrow\ (\forall t\,.\ t\in\{x,y\} \Leftrightarrow t\in\{u,v\})$

... [0.39]: $\Leftrightarrow\ \{x,y\}=\{u,v\}$. *(End of Proof.)*

We conclude with the list of axioms and propositions.

True.

[0.0] true.

[0.1] $(\text{true} \Rightarrow a) \Leftrightarrow a$.

[0.2] $(\text{true} \Leftrightarrow a) \Leftrightarrow a$.

[0.2'] $a \Rightarrow (\text{true} \Leftrightarrow a)$.

[0.3] $\text{true} \wedge a \Leftrightarrow a$.

Conjunction, Implication and Equivalence.

[0.4] $a \wedge b \Leftrightarrow b \wedge a.$

[0.5] $a \wedge b \Rightarrow a.$

(0.5') $a \wedge b \Rightarrow b.$

(0.6) $a \wedge a \Leftrightarrow a.$

[0.7] $a \wedge (b \wedge c) \Leftrightarrow (a \wedge b) \wedge c.$

[0.8] $a \Rightarrow a.$

[0.9] $(a \Rightarrow b) \wedge (b \Rightarrow c) \Rightarrow (a \Rightarrow c).$

[0.10] $(a \Rightarrow (b \Rightarrow c)) \Leftrightarrow (a \wedge b \Rightarrow c).$

[0.11] $(a \Rightarrow b \wedge c) \Leftrightarrow (a \Rightarrow b) \wedge (a \Rightarrow c).$

[0.12] $(a \Rightarrow c) \wedge (b \Rightarrow d) \Rightarrow (a \wedge b \Rightarrow c \wedge d).$

[0.13] $(a \Leftrightarrow b) \Leftrightarrow (a \Rightarrow b) \wedge (b \Rightarrow a).$

(0.13') $(a \Leftrightarrow b) \Rightarrow (a \Rightarrow b).$

(0.14) $(a \Leftrightarrow b) \Leftrightarrow (b \Leftrightarrow a).$

[0.15] $(a \Leftrightarrow b) \wedge (b \Leftrightarrow c) \Rightarrow (a \Leftrightarrow c).$

(0.16) $a \wedge (a \Rightarrow b) \Rightarrow b.$

Quantification.

[0.17] $(Qx,y \,.\, c(x,y)) \Leftrightarrow (Qx \,.\, (Qy \,.\, c(x,y))).$

[0.18] $(Qx,y \,.\, c(x,y)) \Leftrightarrow (Qy,x \,.\, c(x,y)).$

(0.18') $(Qx \,.\, (Qy \,.\, c(x,y))) \Leftrightarrow (Qy \,.\, (Qx \,.\, c(x,y))).$

(0.19) $(\forall x \,.\, a(x) \Rightarrow (\forall y \,.\, b(y) \Rightarrow c(x,y))) \Leftrightarrow (\forall y \,.\, b(y) \Rightarrow (\forall x \,.\, a(x) \Rightarrow c(x,y))).$

(0.20) $(\exists x \,.\, a(x) \wedge (\exists y \,.\, b(y) \wedge c(x,y))) \Leftrightarrow (\exists y \,.\, b(y) \wedge (\exists x \,.\, a(x) \wedge c(x,y))).$

[0.21] $(a \Rightarrow (\forall x \,.\, b(x))) \Leftrightarrow (\forall x \,.\, a \Rightarrow b(x)).$

[0.22] $a \wedge (\exists x \,.\, b(x)) \Leftrightarrow (\exists x \,.\, a \wedge b(x)).$

[0.23] $(\forall x \,.\, a(x) \wedge b(x)) \Leftrightarrow (\forall x \,.\, a(x)) \wedge (\forall x \,.\, b(x)).$

[0.24] $((\exists x \,.\, a(x)) \Rightarrow b) \Leftrightarrow (\forall x \,.\, a(x) \Rightarrow b).$

[0.25] For any term t, $(\forall x \,.\, x = t \Rightarrow a(x)) \Leftrightarrow a(t).$

[0.26] For any term t, $(\exists x \,.\, x = t \wedge a(x)) \Leftrightarrow a(t).$

[0.27] For free x, $a(x) \Rightarrow (\forall x\,.\,a(x))$.

(0.28) $(\forall x\,.\,a(x) \Rightarrow b(x)) \Rightarrow ((\forall x\,.\,a(x)) \Rightarrow (\forall x\,.\,b(x)))$.

[0.29] For free x, $(a(x) \Rightarrow b(x)) \Rightarrow ((\exists x\,.\,a(x)) \Rightarrow (\exists x\,.\,b(x)))$.

[0.30] For any term t, $(\forall x\,.\,a(x)) \Rightarrow a(t)$.

(0.31) For any term t, $a(t) \Rightarrow (\exists x\,.\,a(x))$.

(0.32) $a \Rightarrow (\forall x\,.\,a)$.

[0.33] $(\exists x\,.\,a) \Rightarrow a$.

Equality.

[0.34] $(\forall x\,.\,x = x)$.

[0.35] $(\forall x,y\,.\,x = y \Leftrightarrow y = x)$.

[0.36] $(\forall x,y,z\,.\,x = y \wedge y = z \Rightarrow x = z)$.

[0.37] Let $R,Q: \mathbf{RT}$. Then $R = Q \Leftrightarrow (\forall s,t\,.\,sRt \Leftrightarrow sQt)$.

[0.38] Let $f,g: S \to T$. Then $f = g \Leftrightarrow (\forall s\,.\,f(s) = g(s))$.

[0.39] Let $X,Y: \mathbf{PT}$. Then $X = Y \Leftrightarrow (\forall s\,.\,s \in X \Leftrightarrow s \in Y)$.

[0.40] Let $f: S \to T$. Then $(\forall s\,.\,(\exists t\,.\,t = f(s))$.

[0.41] Let $f: S \to T$. Then $(\forall s,t\,.\,s = t \Rightarrow f(s) = f(t))$.

Defining Sets.

[0.42] Let $f: S \to T$. Then $(\forall t\,.\,t \in \{f(s) \mid a(s)\} \Leftrightarrow (\exists s\,.\,a(s) \wedge t = f(s)))$.

(0.42') $(\forall t\,.\,t \in \{s \mid a(s)\} \Leftrightarrow a(t))$.

[0.43] Let $f: S \to T$. Then $(\forall t\,.\,t \in \{f(s) \mid a(s)\} \Rightarrow b(t)) \Leftrightarrow (\forall s\,.\,a(s) \Rightarrow b(f(s)))$.

[0.44] Let $f: S \to T$. Then $(\exists t\,.\,t \in \{f(s) \mid a(s)\} \wedge b(t)) \Leftrightarrow (\exists s\,.\,a(s) \wedge b(f(s)))$.

[0.45] Let $t:T$. Then $\{t\} = \{s \mid s = t\}$.

[0.46] $(\forall t\,.\,t \in \varnothing \Rightarrow a(t))$.

[0.47] Let $x,y:T$. Then $(\forall t\,.\,t \in \{x,y\} \Rightarrow a(t)) \Leftrightarrow a(x) \wedge a(y)$.

(0.48) Let $x,y:T$. Then $x \in \{x,y\} \wedge y \in \{x,y\}$.

Chapter 1

Complete Partial Orders

A One-to-One Correspondence between RS and S → PS.

It is well known that a relation R on a type S can be viewed as a function mapping each object t of type S to a set whose elements are exactly the objects of S that are "related to t by R". Conversely, to any function R of type S → PS corresponds a unique relation on S that appears as R if regarded in the described way. We therefore chose to use one and and the same letter for corresponding objects of type RS and S → PS, respectively. Symbols for relations and functions are used in different notation, so that it is clear which object is meant. The described correspondence is stated in the following proposition.

(1.0) Let R:RS and R:S → PS. Then the following are equivalent:

(i) $(\forall t\,.\, R(t) = \{s \mid sRt\})$,

(ii) $(\forall s,t\,.\, sRt \Leftrightarrow s \in R(t))$.

Proof.

$$(\forall t\,.\, R(t) = \{u \mid uRt\})$$

... [0.39]: $$\Leftrightarrow (\forall t\,.\, (\forall s\,.\, s \in R(t) \Leftrightarrow s \in \{u \mid uRt\}))$$

... (0.18'), [0.45], (0.42'): $$\Leftrightarrow (\forall t,s\,.\, s \in R(t) \Leftrightarrow sRt)$$

... (0.18), (0.14): $$\Leftrightarrow (\forall s,t\,.\, sRt \Leftrightarrow s \in R(t)).$$ *(End of Proof.)*

Given a relation R on S, (1.0)(i) is a definition of the function R from S to PS (cf.[0.38]). Conversely, for R:S → PS, (1.0)(ii) specifies completely the relation R (cf.[0.37]). If either object (relation or "set-valued" function) is given, we will use the other, if necessary, referring to the appropriate definition in (1.0).

Partial Orders.

[1.1] ≤ is called a *partial order* on S if ≤:RS and

[ref] ≤ is *reflexive*: $(\forall x\,.\, x \leq x)$,

[tra] $\leq$ is *transitive*: $(\forall x,y,z \,.\, x \leq y \wedge y \leq z \Rightarrow x \leq z)$,

[asy] $\leq$ is *antisymmetric*: $(\forall x,y \,.\, x \leq y \wedge y \leq x \Rightarrow x = y)$.

Reflexivity, transitivity and antisymmetry of a given partial order $\leq$ will be so frequently used in proofs that we will omit the reference to [1.1] and simply refer to these properties by [ref $\leq$], [tra $\leq$], [asy $\leq$], respectively (the symbol for the partial order is mentioned because it may vary).

At this point, it is worth mentioning that the property [asy] can essentially be dropped in the study of partial orders, thereby regarding only reflexive and transitive relations (frequently called 'preorders'). This is briefly explained as follows: Given a reflexive and transitive relation $\leq$, the relation E defined by $(\forall x,y \,.\, xEy \Leftrightarrow x \leq y \wedge y \leq x)$ is reflexive, transitive and symmetric. Two objects x,x' with xEx' "behave essentially alike" with respect to the original relation $\leq$, because of $(\forall x,x',y,y' \,.\, xEx' \wedge yEy' \Rightarrow (x \leq y \Leftrightarrow x' \leq y'))$. That is, for the purpose of studying properties of $\leq$, E can be regarded as a substitute for equality (if $\leq$ is antisymmetric, E equals $=$); the formal construction involves "taking the quotient with respect to the congruence relation E". Details and proofs are left to the interested reader. Whether condition [ref] can also be dropped without substantially weakening the theory, is a question of possible interest, but beyond the scope of the current text.

When using the symbol $\leq$ for a partial order on S, we will assume that the relation $\geq$ on S is given by $(\forall x,y \,.\, x \geq y \Leftrightarrow y \leq x)$. We will call the relation $\geq$ the *dual* of $\leq$.

(1.2) Let $\leq$ be a partial order on S, $\geq$:RS and $(\forall x,y \,.\, x \geq y \Leftrightarrow y \leq x)$.
Then $\geq$ is a partial order on S.

Proof: By the assumptions and [0.4]. *(End of Proof.)*

Set Inclusion.

In this section, we define $\subseteq$, known as *inclusion*, as a relation on **PS** for any type S. For A,B:**PS**, $A \subseteq B$ is to be read as "A is a subset of B".We will prove that $\subseteq$ is a partial order on **PS**; after that, some familiar propositions will be given.

[1.3] Define $\subseteq$ by $\subseteq$:**PS** and $(\forall X,Y\,.\,X\subseteq Y \Leftrightarrow (\forall s\,.\,s\in X \Rightarrow s\in Y))$.

(1.4) $\subseteq$ is a partial order on **PS**.

Proof: Prove [ref $\subseteq$]: true

... (0.32): $\Rightarrow$ $(\forall X,s\,.\,\text{true})$

... [0.17], [0.8]: $\Leftrightarrow$ $(\forall X\,.\,(\forall s\,.\,s\in X \Rightarrow s\in X))$

... [1.3]: $\Leftrightarrow$ $(\forall X\,.\,X\subseteq X)$;

to prove [tra $\subseteq$]:, let X,Y,Z:**PS**. Then

true

... (0.32): $\Rightarrow$ $(\forall s\,.\,\text{true})$

... [0.9]: $\Leftrightarrow$ $(\forall s\,.\,(s\in X \Rightarrow s\in Y)\wedge(s\in Y \Rightarrow s\in Z) \Rightarrow (s\in X \Rightarrow s\in Z))$

... (0.28): $\Rightarrow$ $(\forall s\,.\,(s\in X \Rightarrow s\in Y)\wedge(s\in Y \Rightarrow s\in Z)) \Rightarrow (\forall s\,.\,s\in X \Rightarrow s\in Z)$

... [0.23]: $\Leftrightarrow$ $(\forall s\,.\,s\in X \Rightarrow s\in Y)\wedge(\forall s\,.\,s\in Y \Rightarrow s\in Z) \Rightarrow (\forall s\,.\,s\in X \Rightarrow s\in Z)$

... [1.3]: $\Leftrightarrow$ $X\subseteq Y \wedge Y\subseteq Z \Rightarrow X\subseteq Z$;

to prove [asy $\subseteq$]:, let X,Y:**PS**. Then

true

... (0.32): $\Rightarrow$ $(\forall s\,.\,\text{true})$

... [0.13], (0.13'): $\Leftrightarrow$ $(\forall s\,.\,(s\in X \Rightarrow s\in Y)\wedge(s\in Y \Rightarrow s\in X) \Rightarrow (s\in X \Leftrightarrow s\in Y))$

... (0.28): $\Rightarrow$ $(\forall s\,.\,(s\in X \Rightarrow s\in Y)\wedge(s\in Y \Rightarrow s\in X)) \Rightarrow (\forall s\,.\,s\in X \Leftrightarrow s\in Y)$

... [1.3]: $\Leftrightarrow$ $X\subseteq Y \wedge Y\subseteq X \Rightarrow X=Z$. *(End of Proof.)*

(1.5) Let s:S; X,Y:**PS**. Then $t\in X \wedge X\subseteq Y \Rightarrow t\in Y$.

Proof: $t\in X \wedge X\subseteq Y$

... [1.3]: $\Leftrightarrow$ $t\in X \wedge (\forall s\,.\,s\in X \Rightarrow s\in Y)$

... [0.30], [0.12]: $\Rightarrow$ $t\in X \wedge (t\in X \Rightarrow t\in Y)$

... (0.16): $\Rightarrow$ $t\in Y$. *(End of Proof.)*

(1.6) For $\varnothing$:**PT**, $(\forall X\,.\,\varnothing\subseteq X)$.

Proof: true

... (0.32): $\Rightarrow$ $(\forall X\,.\,\text{true})$

... [0.46]: $\Leftrightarrow$ $(\forall X\,.\,(\forall t\,.\,t\in\varnothing \Rightarrow t\in X))$

... [1.3]: $\Rightarrow$ $(\forall X\,.\,\varnothing\subseteq X)$. *(End of Proof.)*

(1.7) Let t:S; X:**PS**. Then $\{t\}\subseteq X \Leftrightarrow t\in X$.

Proof:

$\{t\}\subseteq X$

... [1.3]: $\Leftrightarrow (\forall s\,.\, s\in\{t\} \Rightarrow s\in X)$

... [0.45], (0.42'): $\Leftrightarrow (\forall s\,.\, s=t \Rightarrow s\in X)$

... [0.25]: $\Leftrightarrow t\in X$. *(End of Proof.)*

(1.8) Let x,y:S; A:**PS**. Then $\{x,y\}\subseteq A \Leftrightarrow x\in A \wedge y\in A$.

Proof: By [1.3], [0.47]. *(End of Proof.)*

The next three propositions apply the concept introduced in the first section of this chapter. They will be useful in applications.

(1.9) Let R:**RS**. Then R is reflexive $\Leftrightarrow (\forall x,y\,.\, R(x)\subseteq R(y) \Rightarrow xRy)$.

Proof: Let R be reflexive and x,y:S. Then

$R(x)\subseteq R(y)$

... [1.3]: $\Leftrightarrow (\forall z\,.\, z\in R(x) \Rightarrow z\in R(y))$

... (1.0)(ii): $\Leftrightarrow (\forall z\,.\, zRx \Rightarrow zRy)$

... [0.30]: $\Rightarrow (xRx \Rightarrow xRy)$

... assumption, [0.1]: $\Leftrightarrow xRy$.

Let $(\forall x,y\,.\, R(x)\subseteq R(y) \Rightarrow xRy)$. Then for x:S,

true

... [ref $\subseteq$]: $\Leftrightarrow R(x)\subseteq R(x)$

... assumption: $\Rightarrow xRx$;

this proves R is reflexive. *(End of Proof.)*

(1.10) Let R:**RS**. Then R is transitive $\Leftrightarrow (\forall x,y\,.\, xRy \Rightarrow R(x)\subseteq R(y))$.

Proof:

R is transitive

... [1.1]: $\Leftrightarrow (\forall z,x,y\,.\, zRx \wedge xRy \Rightarrow zRy)$

... [0.18], [0.4]: $\Leftrightarrow (\forall x,y,z\,.\, xRy \wedge zRx \Rightarrow zRy)$

... [0.17], [0.10]: $\Leftrightarrow (\forall x,y\,.\, (\forall z\,.\, xRy \Rightarrow (zRx \Rightarrow zRy)))$

... [0.21], (1.0)(ii): $\Leftrightarrow (\forall x,y\,.\, xRy \Rightarrow (\forall z\,.\, z\in R(x) \Rightarrow z\in R(y)))$

... [1.3]: $\Leftrightarrow (\forall x,y\,.\, xRy \Rightarrow R(x)\subseteq R(y))$. *(End of Proof.)*

(1.11) Let R:**RS**. Then R is reflexive and transitive $\Leftrightarrow (\forall x,y\,.\, xRy \Leftrightarrow R(x)\subseteq R(y))$.

Proof: By (1.9), (1.10); [0.23]; [0.13]. *(End of Proof.)*

(1.11) is interesting because it shows that any partial order R on S can be "represented" by the inclusion relation on an appropriate power type, namely PS. The function from S to PS, with the property that the relation $\subseteq$ between two function values "represents" the given relation between the respective arguments, as in (1.11), is simply given by R of type $S \to PS$ as defined in (1.0)(i). In this sense, we could confine ourselves to inclusion when studying partial orders; for applications of this kind, in particular with respect to results of the next chapters, see *Burris/Sankappanavar* [BS, Thm. 5.3].

Completeness.

When dealing with general partial orders, we will use the symbol $\leq$. We will thereby understand that its dual is denoted by $\geq$ (cf. (1.2)), and also employ the following notational conventions:

[1.12] Let $\leq$ be a partial order on S. Then define lb,ub as follows: $lb,ub:PS \to PS$, $(\forall A \,.\, lb(A) = \{y \mid (\forall x \,.\, x \in A \Rightarrow y \leq x)\})$, $(\forall A \,.\, ub(A) = \{y \mid (\forall x \,.\, x \in A \Rightarrow y \geq x)\})$; for $A:PS$, lb(A) is called the set of *lower bounds* of A and ub(A) the set of *upper bounds* of A.

The definition of ub is identical to the definition of lb, except that for any occurrence of $\leq$, $\geq$ is used instead; in other words, for a given set A, the set of its upper bounds is the set of its lower bounds in terms of the dual of the original partial order, which is a partial order by (1.2). In this sense, lb and ub are *dual notions* (an obviously symmetric condition, because the dual of $\geq$ is again $\leq$).

In the following definition, inf A and sup A are to be read as "infimum of A" and "supremum of A", respectively. The parentheses around the argument, here A, will be omitted for the functions inf and sup; if A is a term, the function symbols used in A are given higher binding power than inf and sup.

[1.13] $\leq$,inf,sup is called a *complete partial order* on P if $\leq$ is a partial order on P, $inf,sup:PP \to P$ and $(\forall A \,.\, lb(A) = \leq(\inf A))$, $(\forall A \,.\, ub(A) = \geq(\sup A))$.

(1.13') see below.

(The letter P will be reserved for types with a complete partial order.) inf and sup are dual notions. The advantages of the concept of dual notions are the following: It is immediately seen that if $\leq$,inf,sup is a complete partial order, then so is its dual notion $\geq$,sup,inf (with the meanings of lb and ub exchanged, too). Any proof remains valid if $\leq$ is replaced by $\geq$ within the proof and in the definitions of derived terms like inf. Therefore, properties of one "notion", e.g. the object inf, hold without further proof for the dual notion, "*by duality*", as we will say. An example is given in the proof of the next proposition, which asserts that in a complete partial order $\leq$,inf,sup, the functions inf and sup are unique.

(1.14) Let $\leq$,inf,sup be a complete partial order on P and inf',sup':$\mathbf{PP} \to \mathrm{P}$ such that $(\forall A \,.\, \mathrm{lb}(A) = \leq(\mathrm{inf}'\, A))$, $(\forall A \,.\, \mathrm{ub}(A) = \geq(\mathrm{sup}'\, A))$. Then inf' = inf, sup' = sup.

Proof.

		true
... [1.13]:	$\Leftrightarrow$	$(\forall A \,.\, \mathrm{lb}(A) = \leq(\mathrm{inf}'\, A)) \wedge (\forall A \,.\, \mathrm{lb}(A) = \leq(\mathrm{inf}\, A))$
... [0.23],[0.35],[0.36]:	$\Rightarrow$	$(\forall A \,.\, \leq(\mathrm{inf}'\, A) = \leq(\mathrm{inf}\, A))$
... [ref $\subseteq$], (0.28):	$\Rightarrow$	$(\forall A \,.\, \leq(\mathrm{inf}'\, A) \subseteq \leq(\mathrm{inf}\, A) \wedge \leq(\mathrm{inf}\, A) \subseteq \leq(\mathrm{inf}'\, A))$
... (1.11):	$\Leftrightarrow$	$(\forall A \,.\, \mathrm{inf}'\, A \leq \mathrm{inf}\, A \wedge \mathrm{inf}\, A \leq \mathrm{inf}'\, A)$
... [asy $\leq$], (0.28):	$\Rightarrow$	$(\forall A \,.\, \mathrm{inf}'\, A = \mathrm{inf}\, A)$
... [0.38]:	$\Leftrightarrow$	inf' = inf.

sup' = sup holds by duality. *(End of Proof.)*

By (1.14), there is at most one infimum and one supremum function for a given partial order: if these functions exist, the partial order is complete. We identify a complete partial order by the triple $\leq$,inf,sup in order to be able to substitute other symbols for the components. lb and ub will only be used if the partial order is denoted by $\leq$. Because [1.13] will be so widely used in what follows, we rephrase it again entirely in terms of the given partial order $\leq$:

(1.13') $\leq$,inf,sup is a complete partial order on P $\Leftrightarrow$ inf,sup:$\mathbf{PP} \to \mathrm{P}$ and
$(\forall A, y \,.\, (\forall x \,.\, x \in A \Rightarrow y \leq x) \Leftrightarrow y \leq \mathrm{inf}\, A)$,
$(\forall A, y \,.\, (\forall x \,.\, x \in A \Rightarrow x \leq y) \Leftrightarrow \mathrm{sup}\, A \leq y)$.

Proof: It is sufficient to show $(\forall A \,.\, \mathrm{lb}(A) = \leq(\mathrm{inf}\, A))$

... [0.39]:	$\Leftrightarrow$	$(\forall A \,.\, (\forall y \,.\, y \in \mathrm{lb}(A) \Leftrightarrow y \in \leq(\mathrm{inf}\, A))$
... [0.17], [1.12], (0.42'):	$\Leftrightarrow$	$(\forall A, y \,.\, (\forall x \,.\, x \in A \Rightarrow y \leq x) \Leftrightarrow y \leq \mathrm{inf}\, A)$;

the corresponding equivalence for the supremum holds by duality and by the definition of the dual of $\leq$. *(End of Proof.)*

The following proposition finally states the familiar property that the infimum of a set A is the *"greatest" lower bound* of A, and that the supremum of A is the *"least" upper bound*. The labels in square brackets, e.g. [glb], will be used in comments without naming (1.15). The interested reader may wish to prove that (1.15) can be used as a definition for inf and sup instead of [1.13].

(1.15) Let $\leq$,inf,sup be a complete partial order on P and A:**PP**. Then

[lb] $\inf A \in lb(A)$, [glb] $lb(A) \subseteq \leq(\inf A)$,

[ub] $\sup A \in ub(A)$, [lub] $ub(A) \subseteq \geq(\sup A)$.

Proof:

		true
... [ref $\leq$], [1.13]:	$\Leftrightarrow$	$\inf A \leq \inf A \wedge lb(A) = \leq(\inf A)$
... (1.0)(ii):	$\Leftrightarrow$	$\inf A \in \leq(\inf A) \wedge lb(A) = \leq(\inf A)$
... [ref $\subseteq$], [0.12]:	$\Rightarrow$	$\inf A \in lb(A) \wedge lb(A) \subseteq \leq(\inf A)$;

this proves [lb] and [glb]; [ub], [lub] hold by duality. *(End of Proof.)*

In the following it will be shown that a partial order is already complete if the infimum function alone exists (or, by duality, the supremum function). We prove an auxiliary lemma first.

(1.16) Let $\leq$ be a partial order on S and A:**PS**. Then $A \subseteq lb(ub(A))$.

Proof:

		$A \subseteq lb(ub(A))$
... [1.3]:	$\Leftrightarrow$	$(\forall s \,.\, s \in A \Rightarrow s \in lb(ub(A)))$
... [1.12], (0.42'):	$\Leftrightarrow$	$(\forall s \,.\, s \in A \Rightarrow (\forall y \,.\, y \in ub(A) \Rightarrow s \leq y))$
... [1.12], (0.42'):	$\Leftrightarrow$	$(\forall s \,.\, s \in A \Rightarrow (\forall y \,.\, (\forall x \,.\, x \in A \Rightarrow x \leq y) \Rightarrow s \leq y))$
... [0.21]:	$\Leftrightarrow$	$(\forall s \,.\, (\forall y \,.\, s \in A \Rightarrow ((\forall x \,.\, x \in A \Rightarrow x \leq y) \Rightarrow s \leq y)))$
... [0.17], [0.10]:	$\Leftrightarrow$	$(\forall s,y \,.\, s \in A \wedge (\forall x \,.\, x \in A \Rightarrow x \leq y) \Rightarrow s \leq y)$ (a).

It suffices to prove (a): Let s,y:S. Then

		$s \in A \wedge (\forall x \,.\, x \in A \Rightarrow x \leq y)$
... [0.30], [0.12]:	$\Rightarrow$	$s \in A \wedge (s \in A \Rightarrow s \leq y)$
... (0.16):	$\Rightarrow$	$s \leq y$. *(End of Proof.)*

(1.17) Let $\leq$ be a partial order on S; inf: $\mathbf{PS} \to S$; $(\forall A \,.\, \mathrm{lb}(A) = \leq(\inf A))$.
Then $\leq$,inf,sup is a complete partial order on S, where sup: $\mathbf{PS} \to S$ and $(\forall A \,.\, \sup A = \inf \mathrm{ub}(A))$.

Proof: By (1.13') and (1.14) it is sufficient to prove $(\forall A, y \,.\, y \in \mathrm{ub}(A) \Leftrightarrow \inf \mathrm{ub}(A) \leq y)$.
Let A:PS; y:S. Then

		true
... [lb, A:= ub(A)]:	$\Leftrightarrow$	$\inf \mathrm{ub}(A) \in \mathrm{lb}(\mathrm{ub}(A))$
... [1.12], (0.42'):	$\Leftrightarrow$	$(\forall x \,.\, x \in \mathrm{ub}(A) \Rightarrow \inf \mathrm{ub}(A) \leq x)$
... [0.30]:	$\Rightarrow$	$(y \in \mathrm{ub}(A) \Rightarrow \inf \mathrm{ub}(A) \leq y)$;
furthermore,		$\inf \mathrm{ub}(A) \leq y$
... (1.16), [glb, A:= ub(A)], (1.11):	$\Leftrightarrow$	$A \subseteq \mathrm{lb}(\mathrm{ub}(A)) \wedge \mathrm{lb}(\mathrm{ub}(A)) \subseteq \leq(\inf \mathrm{ub}(A)) \wedge \leq(\inf \mathrm{ub}(A)) \subseteq \leq(y)$
... [tra $\subseteq$], [tra $\subseteq$]:	$\Rightarrow$	$A \subseteq \leq(y)$
... [1.3], (1.0)(ii), [1.12], (0.42'):	$\Leftrightarrow$	$y \in \mathrm{ub}(A)$.

By [0.13], this proves $y \in \mathrm{ub}(A) \Leftrightarrow \inf \mathrm{ub}(A) \leq y$. *(End of Proof.)*

We conclude this section with the observation that for a complete partial order $\leq$,inf,sup on P, inf $\emptyset$ is the "greatest" and sup $\emptyset$ the "least" object of P with respect to $\leq$.

(1.18) Let $\leq$,inf,sup be a complete partial order on P.
Then $(\forall y \,.\, y \leq \inf \emptyset)$, $(\forall y \,.\, y \geq \sup \emptyset)$.

Proof:

		true
... [glb]:	$\Leftrightarrow$	$\mathrm{lb}(\emptyset) \subseteq \leq(\inf \emptyset)$
... [1.3]:	$\Leftrightarrow$	$(\forall y \,.\, y \in \mathrm{lb}(\emptyset) \Rightarrow y \in \leq(\inf \emptyset))$
... [1.12], (0.42'), (1.0)(ii):	$\Leftrightarrow$	$(\forall y \,.\, (\forall x \,.\, x \in \emptyset \Rightarrow y \leq x) \Rightarrow y \leq \inf \emptyset)$
... [0.46]:	$\Leftrightarrow$	$(\forall y \,.\, \mathrm{true} \Rightarrow y \leq \inf \emptyset)$
... [0.1]:	$\Leftrightarrow$	$(\forall y \,.\, y \leq \inf \emptyset)$;

$(\forall y \,.\, y \geq \sup \emptyset)$ holds by duality. *(End of Proof.)*

Intersection and Union.

We will show that the *intersection* over a set A of sets, denoted by $\bigcap A$, is the "infimum" of A with respect to the partial order $\subseteq$ on **PS** (A:**P**(**PS**)), and that the *union* over A, denoted by $\bigcup A$, is the "supremum" of A with respect to $\subseteq$. In other words, $\subseteq, \bigcap, \bigcup$ will prove to be a complete partial order on **PS**.

[1.19] For any given type S, define $\bigcap, \bigcup$ as follows: $\bigcap, \bigcup$:**P**(**PS**) $\rightarrow$ **PS**,

$(\forall A \,.\, \bigcap A = \{s \mid (\forall X \,.\, X \in A \Rightarrow s \in X)\})$, $(\forall A \,.\, \bigcup A = \{s \mid (\exists X \,.\, X \in A \wedge s \in X)\})$.

(1.20) $\subseteq, \bigcap, \bigcup$ is a complete partial order on **PS**, i.e. (cf. (1.13'))

$(\forall A,Y \,.\, (\forall X \,.\, X \in A \Rightarrow Y \subseteq X) \Leftrightarrow Y \subseteq \bigcap A)$,

$(\forall A,Y \,.\, (\forall X \,.\, X \in A \Rightarrow X \subseteq Y) \Leftrightarrow \bigcup A \subseteq Y)$.

Proof: Let A:**P**(**PS**); Y:**PS**. Then

		$(\forall X \,.\, X \in A \Rightarrow Y \subseteq X)$
... [1.3]:	$\Leftrightarrow$	$(\forall X \,.\, X \in A \Rightarrow (\forall s \,.\, s \in Y \Rightarrow s \in X))$
... (0.19):	$\Leftrightarrow$	$(\forall s \,.\, s \in Y \Rightarrow (\forall X \,.\, X \in A \Rightarrow s \in X))$
... [1.19], (0.42'):	$\Leftrightarrow$	$(\forall s \,.\, s \in Y \Rightarrow s \in \bigcap A)$
... [1.3]:	$\Leftrightarrow$	$Y \subseteq \bigcap A$;
		$(\forall X \,.\, X \in A \Rightarrow X \subseteq Y)$
... [1.3]:	$\Leftrightarrow$	$(\forall X \,.\, X \in A \Rightarrow (\forall s \,.\, s \in X \Rightarrow s \in Y))$
... [0.21]:	$\Leftrightarrow$	$(\forall X \,.\, (\forall s \,.\, X \in A \Rightarrow (s \in X \Rightarrow s \in Y)))$
... (0.18'), [0.10]:	$\Leftrightarrow$	$(\forall s \,.\, (\forall X \,.\, X \in A \wedge s \in X \Rightarrow s \in Y))$
... [0.24]:	$\Leftrightarrow$	$(\forall s \,.\, (\exists X \,.\, X \in A \wedge s \in X) \Rightarrow s \in Y)$
... [1.19], (0.42'):	$\Leftrightarrow$	$(\forall s \,.\, s \in \bigcup A \Rightarrow s \in Y)$
... [1.3]:	$\Leftrightarrow$	$\bigcup A \subseteq Y$. *(End of Proof.)*

We next define the symbols $\cap$ and $\cup$. The binding power of $\cap$ and $\cup$ shall be equal, lower than that of functions with domain **PS**, and higher than that of $\in$ and relations on **PS**.

[1.21] Let X,Y:**PS**. Then let $X \cap Y = \bigcap\{X,Y\}$ and $X \cup Y = \bigcup\{X,Y\}$.

(1.22) Let X,Y:**PS**. Then $X \cap Y = \{s \mid s \in X \wedge s \in Y\}$.

Proof:

		true
... [0.34]:	$\Leftrightarrow$	$\bigcap\{X,Y\} = \bigcap\{X,Y\}$

... [1.21], [1.19]: $\Leftrightarrow\ X \cap Y = \{\forall\, s \mid (Z\,.\, Z \in \{X,Y\} \Rightarrow s \in Z)\}$

... [0.47]: $\Leftrightarrow\ X \cap Y = \{\, s \mid s \in X \wedge s \in Y\}.$ *(End of Proof.)*

(1.23) Let A,B:**PS**. Then $A \cap B \subseteq A$.

Proof: true

... (0.32), [0.12]: $\Rightarrow\ (\forall s\,.\, s \in A \wedge s \in B \Rightarrow s \in A)$

... (1.22), (0.42'): $\Leftrightarrow\ (\forall s\,.\, s \in A \cap B \Rightarrow s \in A)$

... [1.3]: $\Leftrightarrow\ A \cap B \subseteq A.$ *(End of Proof.)*

(1.24) Let A,B,C,D:**PS**. Then $A \subseteq C \wedge B \subseteq D \Rightarrow A \cap B \subseteq C \cap D$.

Proof: true

... [0.12]: $\Leftrightarrow\ (\forall s\,.\, (s \in A \Rightarrow s \in C) \wedge (s \in B \Rightarrow s \in D) \Rightarrow (s \in A \wedge s \in B \Rightarrow s \in C \wedge s \in D))$

... (0.28): $\Rightarrow\ (\forall s\,.\, (s \in A \Rightarrow s \in C) \wedge (s \in B \Rightarrow s \in D)) \Rightarrow (\forall s\,.\, s \in A \wedge s \in B \Rightarrow s \in C \wedge s \in D)$

... [0.23], (1.22):

$\Leftrightarrow\ (\forall s\,.\, s \in A \Rightarrow s \in C) \wedge (\forall s\,.\, s \in B \Rightarrow s \in D) \Rightarrow (\forall s\,.\, s \in A \cap B \Rightarrow s \in C \cap D)$

... [1.3]: $\Leftrightarrow\ A \subseteq C \wedge B \subseteq D \Rightarrow A \cap B \subseteq C \cap D.$ *(End of Proof.)*

In the next proposition, it should be noted that the symbol $\varnothing$ denotes different objects, namely for the three occurrences of $\varnothing$, the type of $\varnothing$ is **P(PS)**, **P(PS)**, **PS**, respectively.

(1.25) Let $\cap, \cup$:**P(PS)** $\rightarrow$ **PS**. Then $(\forall s\,.\, s \in \cap\varnothing),\ \cup\varnothing = \varnothing$.

Proof: Let s:S. Then true

... (1.20), (1.18): $\Leftrightarrow\ (\forall X\,.\, X \subseteq \cap\varnothing)$

... [0.30]: $\Rightarrow\ \{s\} \subseteq \cap\varnothing$

... (1.7): $\Leftrightarrow\ s \in \cap\varnothing.$

Furthermore, true

... (1.20), (1.18), (1.6): $\Leftrightarrow\ (\forall X\,.\, \cup\varnothing \subseteq X) \wedge \varnothing \subseteq \cup\varnothing$

... [0.30], [0.12]: $\Rightarrow\ \cup\varnothing \subseteq \varnothing \wedge \varnothing \subseteq \cup\varnothing$

... [asy $\subseteq$]: $\Rightarrow\ \cup\varnothing = \varnothing.$ *(End of Proof.)*

Partial Orders on Function Types.

Let R be a relation on T and consider $S \to T$ (the objects of $S \to T$ shall be simply called "functions"). Then a relation R' on $S \to T$ can be defined by $(\forall f,g \,.\, f\,R'g \Leftrightarrow (\forall s \,.\, f(s)Rg(s)))$. In other words, R' is obtained by "comparing" any two functions "argumentwise" by R. We will show that R' is similar to R in that it is a partial order if R is. (As an aside, we remark here that if R,inf,sup is a complete partial order on T, then INF and SUP can be defined in terms of inf and sup such that R',INF,SUP is a ocmplete partial order on PT.) One could use the same symbol R instead of R', as is in fact the case for the equality symbol = (cf. [0.38]); it would be "natural" to interpret fRg as $(\forall s \,.\, f(s)Rg(s))$. We will not do this, however, but will use instead the typographically different symbol $\leqq$ for R' if R is a partial order $\leq$ on T.

[1.26] Let $\leq$ be a partial order on T. Then define $\leqq$ by $\leqq : \mathbf{R}(S \to T)$ and $(\forall f,g \,.\, f \leqq g \Leftrightarrow (\forall s \,.\, f(s) \leq g(s)))$.

(1.27) $\leqq$ as defined in [1.25] is a partial order on $S \to T$.

Proof: Prove [ref $\leqq$]: true

... (0.32): $\Rightarrow$ $(\forall f,s \,.\, \text{true})$

... [0.17], [ref $\leq$]: $\Leftrightarrow$ $(\forall f \,.\, (\forall s \,.\, f(s) \leq f(s)))$

... [1.25]: $\Leftrightarrow$ $(\forall f \,.\, f \leqq f)$;

to prove [tra $\leqq$], let f,g,h: $S \to T$. Then

$f \leqq g \wedge g \leqq h$

... [1.25], [0.23]: $\Leftrightarrow$ $(\forall s \,.\, f(s) \leq g(s) \wedge g(s) \leq h(s))$

... [tra $\leq$], (0.28): $\Rightarrow$ $(\forall s \,.\, f(s) \leq h(s))$

... [1.25]: $\Leftrightarrow$ $f \leqq h$;

to prove [asy $\leqq$], let f,g: $S \to T$. Then

$f \leqq g \wedge g \leqq f$

... [1.25], [0.23]: $\Leftrightarrow$ $(\forall s \,.\, f(s) \leq g(s) \wedge g(s) \leq f(s))$

... [asy $\leq$], (0.28): $\Rightarrow$ $(\forall s \,.\, f(s) = g(s))$

... [0.38]: $\Leftrightarrow$ $f = g$. *(End of Proof.)*

This concludes our introduction of complete partial orders. In the next chapters, we will investigate functions on P, where a complete partial order on P is given.

Chapter 2

Closure Spaces

In the informal discussions of this chapter, it is understood that $\leq$ is a partial order on S and $\leq$,inf,sup is a complete partial order on P. Objects of type S or P are called *points*.

Closed Points of Monotonic Functions.

[2.0] Let $\leq$ be a partial order on S. f is called a *monotonic* function on S if $f:S \to S$ and $(\forall x,y\,.\, x \leq y \Rightarrow f(x) \leq f(y))$.

Monotonic functions on S are in a sense the "homomorphisms" with respect to the given partial order $\leq$, that is, functions "preserving" the mathematical structure given by $\leq$.

[2.1] For a monotonic function f on P, the set $\{x \mid f(x) \leq x\}$ is called the set of *closed points* of f.

[2.2] Let $\leq$,inf,sup be a complete partial order on P. C is called a *closure space* in P if $C:PP$ and $(\forall A\,.\, A \subseteq C \Rightarrow \inf A \in C)$.

Closure spaces in P can be viewed as "infimum-stable" sets of points: the infimum of any subset of C belongs to C.

(2.3) Let $\leq$,inf,sup be a complete partial order on P, f be a monotonic function on P and C be the set of closed points of f. Then C is a closure space in P.

Proof: Let $A:PP$. Then $A \subseteq C$

... [0.3], [Ib], [2.1]: $\Leftrightarrow \inf A \in lb(A) \wedge A \subseteq \{x \mid f(x) \leq x\}$

... [1.12], [1.3]: $\Leftrightarrow (\forall x\,.\, x \in A \Rightarrow \inf A \leq x) \wedge (\forall x\,.\, x \in A \Rightarrow f(x) \leq x)$

... [2.0], [0.9], (0.20): $\Rightarrow (\forall x\,.\, x \in A \Rightarrow f(\inf A) \leq f(x)) \wedge (\forall x\,.\, x \in A \Rightarrow f(x) \leq x)$

... [0.23], [0.11]: $\Leftrightarrow (\forall x\,.\, x \in A \Rightarrow f(\inf A) \leq f(x) \wedge f(x) \leq x)$

... [tra $\leq$], [0.9], (0.20): $\Rightarrow (\forall x\,.\, x \in A \Rightarrow f(\inf A) \leq x)$

... (1.13'): $\Leftrightarrow$ $f(\inf A) \leq \inf A$

... (0.42'): $\Leftrightarrow$ $\inf A \in C$.

Hence C is a closure space by [0.27], [2.2]. *(End of Proof.)*

The auxiliary statements given next will be used to prove that every closure space in P is equal to the set of closed points of some monotonic function on P.

[2.4] For A:PP and x:P, define $A/x = A \cap \geq(x)$.

The symbol / shall have higher binding power that inf. For A:PP and x:P, A/x can be read as the set of elements of A "extending" x. A/x is a subset of A by (1.23).

(2.5) Let A,B:PP; x,y:P. Then

(i) $A \subseteq B \Rightarrow A/x \subseteq B/x$,

(ii) $x \leq y \Rightarrow A/y \subseteq A/x$,

(iii) $A \subseteq B \Rightarrow lb(B) \subseteq lb(A)$,

(iv) $A \subseteq B \Rightarrow \inf B \leq \inf A$,

(v) $x \leq y \Rightarrow \inf A/x \leq \inf A/y$.

Proof: Prove (i): $A \subseteq B$

... [ref $\subseteq$], (1.24): $\Rightarrow$ $A \cap \geq(x) \subseteq B \cap \geq(x)$

... [2.4]: $\Leftrightarrow$ $A/x \subseteq B/x$;

prove (ii): $x \leq y$

$\Leftrightarrow$ $y \geq x$

... (1.2), (1.11): $\Leftrightarrow$ $\geq(y) \subseteq \geq(x)$

... [ref $\subseteq$], (1.24): $\Rightarrow$ $A/y \subseteq A/x$.

To prove (iii), let $A \subseteq B$ and s:P. Then

$s \in lb(B)$

... [1.12]: $\Leftrightarrow$ $(\forall t \,.\, t \in B \Rightarrow s \leq t)$

... assumption $A \subseteq B$, [0.23]: $\Leftrightarrow$ $(\forall t \,.\, (t \in A \Rightarrow t \in B) \wedge (t \in B \Rightarrow s \leq t))$

... [0.9], (0.28): $\Rightarrow$ $(\forall t \,.\, t \in A \Rightarrow s \leq t)$

... [1.12]: $\Leftrightarrow$ $s \in lb(A)$;

this proves $lb(B) \subseteq lb(A)$ by [0.27], [1.3]. Hence (iii) holds.

Prove (iv): $A \subseteq B$

... (iii): $\Rightarrow$ $lb(B) \subseteq lb(A)$

... [1.13]: $\Leftrightarrow$ $\leq(\inf B) \subseteq \leq(\inf A)$

... (1.11): $\Leftrightarrow$ $\inf B \leq \inf A$;

prove (v) by (ii), (iv). *(End of Proof.)*

(2.6) Let $\leq$,inf,sup be a complete partial order on P; C:PP; c:P $\rightarrow$ P;
$(\forall x \,.\, c(x) = \inf C/x)$. Then c is a monotonic function on P.

Proof: By (2.5, A:=C)(v), [0.27], [2.0]. *(End of Proof.)*

(2.7) Let $\leq$,inf,sup be a complete partial order on P and C:PP.
Then $(\forall y \,.\, y \leq \inf C/y)$.

Proof: Let y:P. Then true

... (0.32), [0.5]: $\Rightarrow$ $(\forall x \,.\, x \in C \wedge x \in \geq(y) \Rightarrow x \in \geq(y))$

... (1.22), (1.0)(ii): $\Leftrightarrow$ $(\forall x \,.\, x \in C \cap \geq(y) \Rightarrow x \geq y)$

... [2.4]: $\Leftrightarrow$ $(\forall x \,.\, x \in C/y \Rightarrow y \leq x)$

... (1.13'): $\Leftrightarrow$ $y \leq \inf C/y$. *(End of Proof.)*

(2.8) Let $\leq$,inf,sup be a complete partial order on P; C be a closure space in P.
Then $(\forall x \,.\, x \in C \Rightarrow \inf C/x \leq x)$.

Proof: Let x:P. Then $x \in C$

... [ref $\geq$], [lb]: $\Leftrightarrow$ $x \in C \wedge x \in \geq(x) \wedge \inf C/x \in lb(C/x)$

... (1.22), [1.12]: $\Leftrightarrow$ $x \in C \cap \geq(x) \wedge (\forall z \,.\, z \in C/x \Rightarrow \inf C/x \leq z)$

... [2.4], [0.30], [0.12]: $\Rightarrow$ $x \in C/x \wedge (x \in C/x \Rightarrow \inf C/x \leq x)$

... (0.16): $\Rightarrow$ $\inf C/x \leq x$. *(End of Proof.)*

For C:PP, the function c on P defined by $(\forall x \,.\, c(x) = \inf C/x)$ is monotonic by (2.6). The following theorem gives two characterizations of closure spaces C in terms of c, namely (2.9)(ii) and (2.9)(iii). (2.9)(ii) asserts that the result of applying c to an argument is always an element of C. (2.9)(iii) states that C equals the set of closed points of c.

(2.9) Let $\leq$,inf,sup be a complete partial order on P and C:PP.
Then the following are equivalent:

(i) C is a closure space in P,

(ii) $(\forall x \,.\, \inf C/x \in C)$,

(iii) $C = \{x \mid \inf C/x \leq x\}$.

Proof: Let C be a closure space in P. Then for x:P,

true

... [2.2]: $\Leftrightarrow$ $(\forall A \,.\, A \subseteq C \Rightarrow \inf A \in C)$

... [0.30]: $\Rightarrow$ $(C/x \subseteq C \Rightarrow \inf C/x \in C)$

... [2.4], (1.23): $\Leftrightarrow$ $\inf C/x \in C$;

this proves $(\forall x \,.\, \inf C/x \in C)$. Hence (i) $\Rightarrow$ (ii).

Let $(\forall x \,.\, \inf C/x \in C)$; y:P. Then $y \in C$

... (2.8): $\Rightarrow$ $\inf C/y \leq y$

... (0.42'): $\Leftrightarrow$ $y \in \{ x \mid \inf C/x \leq x \}$;

furthermore, $y \in \{ x \mid \inf C/x \leq x \}$

... (2.7), (0.42'): $\Leftrightarrow$ $y \leq \inf C/y \ \wedge \ \inf C/y \leq y$

... [asy $\leq$]: $\Rightarrow$ $y = \inf C/y$

... instantiate assumption: $\Rightarrow$ $y \in C$;

thus $y \in C \Leftrightarrow y \in \{ x \mid \inf C/x \leq x \}$. By [0.27], this proves $C = \{ x \mid \inf C/x \leq x \}$. Hence (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) holds by (2.6), (2.3).

(i), (ii) and (ii) are equivalent by [0.13] and [0.9]. *(End of Proof.)*

We will investigate the connection between a given monotonic function f on P and the the function c on P defined by $(\forall x \,.\, c(x) = \inf C/x)$, where $C = \{ x \mid f(x) \leq x \}$. Mapping f to c will be the purpose of a function *clo*. The domain of *clo* shall be the type of all monotonic functions on P; this is a *subtype* of $P \to P$, as introduced in the next section.

Intermezzo on Subtyping.

It will be convenient to have the collection of all monotonic functions on P given as a type. We shall denote this type by **MO**$(P, \leq)$. The boldface indicates the construction of a type (as with **P** and **R**), and using two (or more) letters for the identifier **MO** shall indicate that it is a subtype of a given type, here of $P \to P$.

[2.10] For a given partial order $\leq$ on S,

$f: \mathbf{MO}(P, \leq) \ \Leftrightarrow \ f: S \to S \ \wedge \ (\forall x,y \,.\, x \leq y \Rightarrow f(x) \leq f(y))$.

In the previous definiton, $\leq$, f, S are free identifiers for which any other symbols can be substituted. In particular, the given equivalence has to hold for *any* function f on S. The next definition gives a general schema of subtyping, which will be explained below.

[2.11] Let $t{:}\mathbf{sub}(T) \Leftrightarrow t{:}T \wedge a(t)$. Then by [0.5],

(i) $t{:}\mathbf{sub}(T) \Rightarrow t{:}T$.

sub(T) is called a *subtype* of T. Further conventions are

[ii] $f{:}\, T \to S \Rightarrow f{:}\, \mathbf{sub}(T) \to S$,

[iii] $f{:}\, S \to \mathbf{sub}(T) \Rightarrow f{:}\, S \to T$,

[iv] $f{:}\, S \to T \wedge (\forall s\,.\, a(f(s))) \Rightarrow f{:}\, S \to \mathbf{sub}(T)$,

[v] $R{:}\, \mathbf{R}T \Rightarrow R{:}\, \mathbf{R}(\mathbf{sub}(T))$,

[vi] $A{:}\, \mathbf{P}(\mathbf{sub}(T)) \Rightarrow A{:}\, \mathbf{P}T$,

[vii] $A{:}\, \mathbf{P}T \wedge (\forall s\,.\, s \in A \Rightarrow a(s)) \Rightarrow A{:}\, \mathbf{P}(\mathbf{sub}(T))$.

In [2.11], a(t) is a predicate in t, t ranging over T. The type **sub**(T) consists of exactly those objects of T for which this predicate holds. Consequently, any object of type **sub**(T) belongs also to T, as stated in (i). The assertions [ii] to [vii] express ways in which objects of type **sub**(T) can also be used with functions, relations and sets involving objects of type T. [ii] describes the *restriction* of a function to a subtype of its domain. [iii] accounts for the fact that if an object of type **sub**(T) is denoted by the term f(s), then this object is also of type T, for all arguments s of f. By [iv], the range of a function f with range T can also be assumed as **sub**(T), provided any value f(s) of the function fulfills the predicate defining the subtype **sub**(T), i.e. provided a(f(s)) holds. [v] allows one to restrict any relation R on T to a relation on **sub**(T); in other words, given R:**R**T, sRt is an admissible predicate if s,t:**sub**(T). According to [vi], any set A of objects of type **sub**(T) is also a set of objects of type T. Conversely, by [vii], a set A of objects of type T all belonging to **sub**(T) can also be regarded as of type **P**(**sub**(T)). (Note the analogy between [vi]-[vii] and [iii]-[iv].)

The assertions given in [2.11] are to guarantee that the usage of functions, relations and sets is consistent with the involved objects of "lower type" like arguments of functions or elements of sets, as described in chapter zero. In a particular predicate, there should be only one way to consistently choose the types for the objects (if necessary,

.with the aid of explicit type assertions like t:**sub**(T)). We will explain the types of bound variables, which have to be inferred, in non-obvious cases.

Some of the assertions can be interpreted as ways to "derive" subtypes of "higher" types if a subtype is given. For example, under the assumptions of [2.11], the subtype **fsub**(S,T) of $S \to T$ given by $f: \mathbf{fsub}(S,T) \Leftrightarrow f: S \to T \wedge (\forall s\,.\, a(f(s)))$ consists of the same objects as $S \to \mathbf{sub}(T)$ by [iii], [iv]. This is not the case for [ii]; $\mathbf{sub}(T) \to S$ can not be regarded as a subtype of $T \to S$.

The numerous conceivable complications of a type system allowing subtyping and multiple kinds of "inheritances" will not be discussed here. We hope the use of subtypes will be clear in the comparatively simple cases presented in this text.

Closure Operations.

For a monotonic function f on P, the function c on P defined by $(\forall y\,.\, c(y) = \inf\{x \mid f(x) \leq x\}/y)$ is monotonic by (2.6). Consequently, a function *clo* mapping f to c can be assumed as a function with domain and range $\mathbf{MO}(P,\leq)$. As the function symbol inf, *clo* will be simply prefixed to its argument to denote functional application.

[2.12] Let $\leq$,inf,sup be a complete partial order on P. Then define *clo* by

$clo: \mathbf{MO}(P,\leq) \to \mathbf{MO}(P,\leq)$ and $(\forall f\,.\,(\forall y\,.\,(clo\ f)(y) = \inf\{x \mid f(x) \leq x\}/y))$.

We demonstrate briefly how the function *clo* is obtained by a stepwise application of the conventions in [2.11]. In [2.11] let $T,\mathbf{sub}(T) := P \to P, \mathbf{MO}(P,\leq)$; $a(f) := (\forall x,y\,.\, x \leq y \Rightarrow f(x) \leq f(y))$ (cf. [2.10]). Define *clo* temporarily by asserting $clo: \mathbf{MO}(P,\leq) \to (P \to P)$ and the equation given in [2.12]. The variables f and y bound by the first and second quantifier in this equation are of type $\mathbf{MO}(P,\leq)$ and P, because they are arguments of *clo* and (*clo* f), respectively. f is of type $P \to P$ by [2.11](i); hence $\leq$ is properly used in $\{x \mid f(x) \leq x\}$. This set, and thus $\{x \mid f(x) \leq x\}/y$, is of type **P**P, where the latter serves as argument for inf. Having thus correctly established that *clo* f is of type $P \to P$, a(*clo* f) holds by (2.6), for any f. By [0.27] and [2.11][iv], this allows us to assert $clo: \mathbf{MO}(P,\leq) \to \mathbf{MO}(P,\leq)$.

We postpone here the study of the function *clo* for a more basic consideration. This concerns properties of *clo* f; because of (2.7) and (2.9)(ii), *clo* f is not only monotonic, but a so-called *closure operation* on P.

[2.13] Let $\leq$ be a partial order on S. c is called a *closure operation* on S if $c : S \to S$ and

[mon] c is *monotonic*: $(\forall x,y \,.\, x \leq y \Rightarrow c(x) \leq c(y))$,

[inc] c is *increasing*: $(\forall x \,.\, x \leq c(x))$,

[subip] c is *sub-idempotent*: $(\forall x \,.\, c(c(x)) \leq c(x))$.

(As before in [1.1] and [1.], the labels in square brackets will be referred to without mentioning [2.13].) Closure operations are usually defined as monotonic, increasing and *idempotent* functions c, i.e. [subip] is replaced by $(\forall x \,.\, c(c(x)) = c(x))$. The reader may verify that for increasing functions idempotency is equivalent to sub-idempotency: we chose the latter as a condition that is weaker, hence more easily established, and independent of [inc]. Obviously, a function c on P is sub-idempotent if every result of applying c to an argument is a closed point of c:

(2.14) Let $\leq$ be a partial order on S and $c : S \to S$. Then

c is sub-idempotent $\Leftrightarrow$ $(\forall y \,.\, c(y) \in \{x \mid c(x) \leq x\})$.

Proof: By (0.42'). *(End of Proof.)*

(2.15) Let $\leq$,inf,sup be a complete partial order on P; C be a closure space in P; $c : P \to P$; $(\forall x \,.\, c(x) = \inf C/x)$. Then c is a closure operation on P.

Proof: By (2.6) and (2.7), c is monotonic and increasing. By (2.9)(ii), (2.9)(iii) and (2.14), c is sub-idempotent. *(End of Proof.)*

We remark here without proof that the condition 'C is a closure space in P' is not necessary for c being a closure operation on P, given $C : PP$ and $(\forall x \,.\, c(x) = \inf C/x)$. We can not state this remark within our formal system, but is is not difficult to find a suitable counterexample.

A closure space C in P is equal to the set of closed points of the closure operation c on P given by $(\forall x \,.\, c(x) = \inf C/x)$. We will prove conversely that any closure operation c on P can be defined in terms of a closure space C by $(\forall x \,.\, c(x) = \inf C/x)$. It will then be easy to show that closure spaces in P and closure operations on P are in a one-to-one correspondence, as for instance C and c before.

At this point, it is appropriate to consider the partial order $\leqq$ on "function types" as introduced in the last section of chapter 1. If $\leqq$ is defined as in [1.26, S,T:=P,P], we can assume $\leqq:\mathbf{R}(\mathbf{MO}(P,\leq))$ by [2.11][iv]. However, it has to be shown that $\leqq$ as a relation on $\mathbf{MO}(P,\leq)$ is a partial order.

(2.16) Let $\leq$ be a partial order on S, and **sub**(S) be a subtype of S.
Then $\leq$ is a partial order on **sub**(S).

Proof: $\leq:\mathbf{R}(\mathbf{sub}(S))$ holds by [2.11][iv]. If $\leq:\mathbf{R}(\mathbf{sub}(S))$ is given, the variables bound by the quantifiers in [ref $\leq$], [tra $\leq$], [asy $\leq$] range over **sub**(S).
To prove, for example, $(\forall x\,.\,x\leq x)$, let x:**sub**(S). Then x:S by [2.11](i), hence $x\leq x$ by instantiating [ref $\leq$, $\leq:\mathbf{RS}$]. This proves [ref $\leq$] for $\leq:\mathbf{R}(\mathbf{sub}(S))$.
[tra $\leq$] and [asy $\leq$] are proved analogously. *(End of Proof.)*

(2.17) Let $\leq$ be a partial order on P, $\leqq:\mathbf{R}(\mathbf{MO}(P,\leq))$ and $(\forall f,g\,.\,f\leqq g \Leftrightarrow (\forall x\,.\,f(x)\leq g(x)))$. Then $\leqq$ is a partial order on $\mathbf{MO}(P,\leq)$.

Proof: By (1.27) and (2.16). *(End of Proof.)*

Whenever the symbol $\leqq$ is used in the remaining statements of this chapter, $\leqq$ is assumed to be given as in the assumptions of (2.17). After the following two propositions, we will state the theorem asserting the correspondence mentioned above.

(2.18) Let $\leq$,inf,sup be a complete partial order on P. Then $(\forall f\,.\,f\leqq clo\,f)$.

Proof: Let f:$\mathbf{MO}(P,\leq)$; y:P. Then

		true
... [2.12], (2.7), (2.3),(2.9) (ii)	$\Leftrightarrow$	$y\leq(clo\,f)(y) \wedge \inf\{x \mid f(x)\leq x\}/y \in \{x \mid f(x)\leq x\}$
... [2.12]:	$\Leftrightarrow$	$y\leq(clo\,f)(y) \wedge (clo\,f)(y) \in \{x \mid f(x)\leq x\}$
... [2.10], [0.12], (0.42'):	$\Rightarrow$	$f(y)\leq f((clo\,f)(y)) \wedge f((clo\,f)(y))\leq(clo\,f)(y)$
... [tra $\leq$]:	$\Rightarrow$	$f(y)\leq(clo\,f)(y)$.

This proves $(\forall y\,.\,f(y)\leq(clo\,f)(y))$, i.e. $f\leqq clo\,f$. *(End of Proof.)*

(2.19) Let $\leq$,inf,sup be a complete partial order on P.
Then $(\forall f\,.\,f$ is a closure operation on P $\Rightarrow$ $clo\,f\leqq f)$.

Proof: Let f be a closure operation on P; y:P. Then

		true
... [subip],(2.14): [inc], (1.0)(ii):	$\Leftrightarrow$	$f(y) \in \{x \mid f(x)\leq x\} \wedge f(y) \in \geq(y)$
... (1.22), [2.4]:	$\Leftrightarrow$	$f(y) \in \{x \mid f(x)\leq x\}/y$

... [lb], [1.12], [0.30]: $\Rightarrow \inf\{x \mid f(x) \leq x\}/y \leq f(y)$

... [2.12]: $\Leftrightarrow (clo\, f)(y) \leq f(y)$. *(End of Proof.)*

(2.20) Let $\leq$,inf,sup be a complete partial order on P; C:PP; c:P $\rightarrow$ P.

Then the following are equivalent:

(i) C is a closure space in P $\wedge$ $(\forall y \,.\, c(y) = \inf C/y)$,

(ii) c is a closure operation on P $\wedge$ $C = \{x \mid c(x) \leq x\}$.

Proof: (i) $\Rightarrow$ (ii) by (2.15), (2.9)(iii) and [0.11].

Conversely, let (ii) hold. Then C is a closure space in P by [mon], (2.3).

Furthermore, true

... (2.18), (2.19): $\Leftrightarrow c \leqq clo\, c \wedge clo\, c \leqq c$

... (2.17), [asy $\leqq$]: $\Rightarrow c = clo\, c$

... [0.38],[2.12]: $\Leftrightarrow (\forall y \,.\, c(y) = \inf\{x \mid c(x) \leq x\}/y)$

... (ii): $\Leftrightarrow (\forall y \,.\, c(y) = \inf C/y)$.

Thus (ii) $\Rightarrow$ (i) holds by [0.11]. By [0.13], (i) $\Leftrightarrow$ (ii). *(End of Proof.)*

Theorem (2.20) can be rephrased in the following way: Let the closure space C in P correspond to the closure operation c on P given by $(\forall y \,.\, c(y) = \inf C/y)$, and vice versa by $C = \{x \mid c(x) \leq x\}$. The equivalence of (2.20)(i) and (2.20)(ii) asserts that this correspondence, regarded as a mapping in either direction between closure spaces in P and closure operations on P, has an inverse, viz. the mapping in the other direction. Thus it is a one-to-one correspondence.

The Set of Closure Operations on P and the Set of Closure Spaces in P.

We next investigate the correspondence stated in (2.20) with respect to the partial order $\leqq$ on $\mathbf{MO}(P, \leq)$ and the partial order $\subseteq$ on $\mathbf{PP}$.

(2.21) Let $\leq$,inf,sup be a complete partial order on P; f,g:$\mathbf{MO}(P, \leq)$.

Then $f \leqq g \Rightarrow \{x \mid g(x) \leq x\} \subseteq \{x \mid f(x) \leq x\}$.

Proof: Let $f \leqq g$, i.e. $(\forall x \,.\, f(x) \leq g(x))$; y:P. Then

$y \in \{x \mid g(x) \leq x\}$

... (0.42'): $\Leftrightarrow g(y) \leq y$

... instantiate assumption: $\Leftrightarrow f(y) \leq g(y) \wedge g(y) \leq y$

... [tra $\leq$]: $\Rightarrow$ $f(y) \leq y$

... (0.42'): $\Leftrightarrow$ $y \in \{x \mid f(x) \leq x\}$.

This proves $\{x \mid g(x) \leq x\} \subseteq \{x \mid f(x) \leq x\}$ by [0.27], [1.3]. *(End of Proof.)*

(2.22) Let $\leq$,inf,sup be a complete partial order on P; F,G:**PP**; f,g:$P \to P$; $(\forall x \,.\, f(x) = \inf F/x)$; $(\forall x \,.\, g(x) = \inf G/x)$. Then $G \subseteq F \Rightarrow f \leqq g$.

Proof: Let x:P. Then $G \subseteq F$

... (2.5)(i): $\Rightarrow$ $G/x \subseteq F/x$

... (2.5)(iv): $\Rightarrow$ $\inf F/x \leq \inf G/x$

$\Leftrightarrow$ $f(x) \leq g(x)$.

This proves $(\forall x \,.\, G \subseteq F \Rightarrow f(x) \leq g(x))$, i.e. $G \subseteq F \Rightarrow (\forall x \,.\, f(x) \leq g(x))$ by [0.21]. *(End of Proof.)*

Note that in (2.22), the assertion f,g:$P \to P$ can be replaced by f,g:**MO**$(P, \leq)$ because of (2.6). We will shortly discuss the connection between **MO**$(P, \leq)$ and **PP** given by (2.21) and (2.22). Before that however we regard the case that f,g in (2.21) are closure operations, $F = \{x \mid f(x) \leq x\}$, $G = \{x \mid g(x) \leq x\}$, and F,G in (2.22) are closure spaces. Under these conditions, f,g and F,G are in respective correspondence by (2.20). This correspondence can then be viewed as "order-reversing", as expressed by the following statement.

(2.23) Let $\leq$,inf,sup be a complete partial order on P and the closure operations c,d on P correspond to the closure spaces C,D in P as in (2.20). Then $c \leqq d \Leftrightarrow D \subseteq C$.

Proof: By (2.21), (2.22) with f,g,F,G:=c,d,C,D, respectively, and [0.13]. *(End of Proof.)*

We briefly comment on the two functions that map f to $\{x \mid f(x) \leq x\}$ and G to g given by $(\forall x \,.\, g(x) = \inf G/x)$, respectively, where f,g:**MO**$(P, \leq)$ and G:**PP**. With the "order-reversing" properties stated in (2.21), (2.22), this pair of functions is a so-called *"Galois connection"* between **MO**$(P, \leq)$ and **PP** with respect to the partial orders $\leqq$ and $\subseteq$ on these types. The reader is referred to *Cohn* [Co, p.44] for a general definition. Many closure operations used in mathematics arise in the context of Galois connections. For instance, it is possible to derive the following theorem in this framework, cf. [Co, p.44].

(2.24) Let $\leq$,inf,sup be a complete partial order on P.
Then *clo* is a closure operation on $\mathbf{MO}(P,\leq)$.

Proof: Let $f,g:\mathbf{MO}(P,\leq)$. Then $f \leqq g$

... (2.21): $\Rightarrow \{x \mid g(x) \leq x\} \subseteq \{x \mid f(x) \leq x\}$

... (2.22): $\Rightarrow clo\, f \leqq clo\, g$;

this proves *clo* is a monotonic function on $\mathbf{MO}(P,\leq)$. *clo* is increasing by (2.18). By (2.3) and (2.15), *clo* f is a closure operation on P. Thus by (2.19), *clo* (*clo* f) $\leqq$ *clo* f, i.e. *clo* is sub-idempotent. *(End of Proof.)*

Last, we will show that the set of closure spaces in P is a closure space in PP. $\subseteq,\cap,\cup$ is thereby the complete partial order on PP. Closure spaces in a power type PS appear throughout mathematics. We restate [2.2] with $\leq$,inf,sup,$P:=\subseteq,\cap,\cup,\mathbf{PS}$: For $C:\mathbf{P}(\mathbf{PS})$, C is a closure space in PS $\Leftrightarrow$ $(\forall A \,.\, A\subseteq C \Rightarrow \cap A\in C)$. An important example of a closure space in PS is a so-called *topology* on S, that is a closure space C in PS such that $(\forall X,Y \,.\, X\in C \wedge Y\in C \Rightarrow X\cup Y\in C) \wedge \varnothing\in C$; the elements of C are commonly called 'closed sets'. The reader may be familiar with another example, informally described as the set of *subgroups of a given group* S, which is also an "intersection-stable" set of sets. In these examples, the result of applying the corresponding closure operation on PS to a set X is usually called "topological closure of X" and "the subgroup of S generated by X", respectively.

(2.25) Let $\leq$,inf,sup be a complete partial order on P.
Then the set *CS* of closure spaces in P is a closure space in PP.

Proof: Let $B:\mathbf{P}(\mathbf{PP})$; $B\subseteq CS$. Then for $A:\mathbf{PP}$

$A \subseteq \cap B$

... [1.3], [1.19]: $\Leftrightarrow (\forall x \,.\, x\in A \Rightarrow (\forall C \,.\, C\in B \Rightarrow x\in C))$

... (0.19): $\Leftrightarrow (\forall C \,.\, C\in B \Rightarrow (\forall x \,.\, x\in A \Rightarrow x\in C))$

... [1.3]: $\Leftrightarrow (\forall C \,.\, C\in B \Rightarrow A\subseteq C)$

... $B\subseteq CS$, (1.5) (i.e. $C\in CS$), [2.2]: $\Leftrightarrow (\forall C \,.\, C\in B \Rightarrow A\subseteq C \wedge (A\subseteq C \Rightarrow \inf A\in C)$

... (1.16), [0.9], (0.28): $\Rightarrow (\forall C \,.\, C\in B \Rightarrow \inf A\in C)$

... [1.19], (0.42'): $\Leftrightarrow \inf A \in \cap B$.

Thus by [0.27], $\cap B$ is a closure space in P according to [2.2]. *(End of Proof.)*

The interested reader may verify that the set of topologies on S is a closure space in P(PS); the proof is similar to that for (2.25). The result of applying the corresponding closure operation on P(PS) to an argument A is the so-called topology "generated from A".

Chapter 3

Disjunctive Functions

In the informal discussions of this chapter, it is understood that $\leq$,inf,sup is a complete partial order on P. Objects of type P are called *points*. For some fixed type S, objects of type **RS**, **PS** and **PS**$\rightarrow$**PS** are for simplicity called *relations*, *sets* and *set mappings*, respectively.

Disjunctive Functions and Closure Operations.

In this chapter, we investigate functions on P that "distribute over suprema". The study of these functions is an interesting counterpart to closure spaces in P, which are "infimum-stable" sets of points. "Conjunctive" functions on P, functions that "distribute over infima", do not have the properties in connection with closure spaces that will be demonstrated in this section.

[3.0] Let $\leq$,inf,sup be a complete partial order on P. f is called a *disjunctive* function on P if $f:P \rightarrow P$ and $(\forall A \,.\, f(\sup A) = \sup \{f(x) \mid x \in A\})$.

The equation in [3.0] can be rewritten in the form $f(\sup \{x \mid x \in A\}) = \sup \{f(x) \mid x \in A\}$, because the set A of points is equal to $\{x \mid x \in A\}$.

We also define the type of all disjunctive functions on P.

[3.1] Let $\leq$,inf,sup be a complete partial order on P. Then define **DIS**(P,sup) as follows: $f:\mathbf{DIS}(P,\sup) \Leftrightarrow f:P \rightarrow P \wedge (\forall A \,.\, f(\sup A) = \sup \{f(x) \mid x \in A\})$.

We will show that the so-called "functional composition" of disjunctive functions on P is disjunctive. This will be proved using the following proposition about "iterated comprehension".

(3.2) Let $g:S \rightarrow T$, $f:T \rightarrow U$, and a(s) be a predicate in s, s ranging over S. Then $\{f(t) \mid t \in \{g(s) \mid a(s)\}\} = \{f(g(s)) \mid a(s)\}$.

Proof: Let u:U. Then $u \in \{ f(t) \mid t \in \{g(s) \mid a(s)\}\}$

... [0.42]: $\Leftrightarrow (\exists t \,.\, t \in \{g(s) \mid a(s)\} \wedge u = f(t))$

... (0.44): $\Leftrightarrow (\exists s \,.\, a(s) \wedge u = f(g(s)))$

... [0.42]: $\Leftrightarrow u \in \{ f(g(s)) \mid a(s) \}$.

This proves the assertion by [0.27] ,[1.3]. *(End of Proof.)*

(3.3) Let $\leq$,inf,sup be a complete partial order on P; $f,g,h:P \to P$; $(\forall x \,.\, h(x) = f(g(x)))$; let f,g be disjunctive functions on P. Then h is a disjunctive function on P.

Proof: Let A:PP. Then true

... [3.0]: $\Leftrightarrow f(\sup \{g(x) \mid x \in A\}) = \sup \{f(y) \mid y \in \{g(x) \mid x \in A\}\}$

... [3.0, f:=g], (3.2): $\Leftrightarrow f(g(\sup A)) = \sup \{f(g(x)) \mid x \in A\}$

$\Leftrightarrow h(\sup A) = \sup \{h(x) \mid x \in A\}$. *(End of Proof.)*

The previous proposition is very intuitive in viewing disjunctivity as a kind of "distributive" property. Similarly, monotonic functions on P can be regarded as "distributing over the partial order $\leq$"; it is easily seen that the "functional composition" of monotonic functions is monotonic. We next investigate the connection between monotonic and disjunctive functions on P.

(3.4) Let $\leq$,inf,sup be a complete partial order on P and f be a monotonic function on P. Then $(\forall A \,.\, \sup \{f(x) \mid x \in A\} \leq f(\sup A))$.

Proof: Let A:PP. Then true

... [ub]: $\Leftrightarrow \sup A \in ub(A)$

... [1.12]: $\Leftrightarrow (\forall x \,.\, x \in A \Rightarrow x \leq \sup A)$

... [2.0], [0.9], (0.28): $\Rightarrow (\forall x \,.\, x \in A \Rightarrow f(x) \leq f(\sup A))$

... (0.43): $\Leftrightarrow (\forall y \,.\, y \in \{f(x) \mid x \in A\} \Rightarrow y \leq f(\sup A))$

... (1.13'): $\Leftrightarrow \sup \{f(x) \mid x \in A\} \leq f(\sup A)$. *(End of Proof.)*

Because of (3.4) and [asy $\leq$], a monotonic function f on P is disjunctive if $(\forall A \,.\, f(\sup A) \leq \sup \{f(x) \mid x \in A\})$. We remark here without proof that this predicate does not hold for all monotonic functions. Therefore, not every monotonic function on P is disjunctive. The converse is true, however; this will be proved using the following propositions.

(3.5) Let $\leq$,inf,sup be a complete partial order on P.

Then $(\forall x,y\,.\,x\leq y \Leftrightarrow \sup\{x,y\}=y)$.

Proof: Let x,y:P. Then

		true
... [ub]:	$\Leftrightarrow$	$\sup\{x,y\} \in ub(\{x,y\})$
... [1.12]:	$\Leftrightarrow$	$(\forall t\,.\,t\in\{x,y\} \Rightarrow t\leq\sup\{x,y\})$
... [0.47]:	$\Leftrightarrow$	$x\leq\sup\{x,y\} \wedge y\leq\sup\{x,y\}$
... (0.5'):	$\Rightarrow$	$y\leq\sup\{x,y\}$. (a)

Therefore,		$x\leq y$
... [ref $\leq$]:	$\Leftrightarrow$	$x\leq y \wedge y\leq y$
... [0.47]:	$\Leftrightarrow$	$(\forall t\,.\,t\in\{x,y\} \Rightarrow t\leq y)$
... (1.13'):	$\Leftrightarrow$	$\sup\{x,y\}\leq y$
... (a):	$\Leftrightarrow$	$\sup\{x,y\}\leq y \wedge y\leq\sup\{x,y\}$
... [asy $\leq$], [ref $\leq$]:	$\Leftrightarrow$	$\sup\{x,y\}=y$. *(End of Proof.)*

(3.6) Let $f:S\to T$ and $x,y:S$. Then $\{f(x),f(y)\} = \{f(s)|\,s\in\{x,y\}\}$.

Proof:

		true
... (0.48):	$\Leftrightarrow$	$x\in\{x,y\} \wedge y\in\{x,y\}$
... [0.26]:	$\Leftrightarrow$	$(\exists s\,.\,s=x\wedge s\in\{x,y\}) \wedge (\exists s\,.\,s=y\wedge s\in\{x,y\})$
... [0.41], [0.12], [0.29]:	$\Rightarrow$	$(\exists s\,.\,f(s)=f(x)\wedge s\in\{x,y\}) \wedge (\exists s\,.\,f(s)=f(y)\wedge s\in\{x,y\})$
... [0.4], [0.35]:	$\Leftrightarrow$	$(\exists s\,.\,s\in\{x,y\}\wedge f(x)=f(s)) \wedge (\exists s\,.\,s\in\{x,y\}\wedge f(y)=f(s))$
... [0.42]:	$\Leftrightarrow$	$f(x)\in\{f(s)|\,s\in\{x,y\}\} \wedge f(y)\in\{f(s)|\,s\in\{x,y\}\}$
... [0.47]:	$\Leftrightarrow$	$(\forall t\,.\,t\in\{f(x),f(y)\} \Rightarrow t\in\{f(s)|\,s\in\{x,y\}\})$
... [1.3]:	$\Leftrightarrow$	$\{f(x),f(y)\}\subseteq\{f(s)|\,s\in\{x,y\}\}$.

Conversely,		true
... (0.48):	$\Leftrightarrow$	$f(x)\in\{f(x),f(y)\} \wedge f(y)\in\{f(x),f(y)\}$
... [0.47]:	$\Leftrightarrow$	$(\forall s\,.\,s\in\{x,y\} \Rightarrow f(s)\in\{f(x),f(y)\})$
... (0.43):	$\Leftrightarrow$	$(\forall t\,.\,t\in\{f(s)|\,s\in\{x,y\}\} \Rightarrow t\in\{f(x),f(y)\})$
... [1.3]:	$\Leftrightarrow$	$\{f(s)|\,s\in\{x,y\}\}\subseteq\{f(x),f(y)\}$.

The result follows by [asy $\subseteq$]. *(End of Proof.)*

(3.7) Let $\leq$,inf,sup be a complete partial order on P and f be a disjunctive function on P. Then f is a monotonic function on P.

Proof: Let x,y:P. Then $x \leq y$

... (3.5): $\Leftrightarrow \sup\{x,y\} = y$

... [0.41]: $\Rightarrow f(\sup\{x,y\}) = f(y)$

... f is disjunctive: $\Leftrightarrow \sup\{f(s) \mid s \in \{x,y\}\} = f(y)$

... (3.6): $\Leftrightarrow \sup\{f(x), f(y)\} = f(y)$

... (3.5): $\Leftrightarrow f(x) \leq f(y)$. *(End of Proof.)*

We next regard the set of closed points of a disjunctive function f on P. This set is by (3.7), (2.3) a closure space in P. We will show that it is not only "infimum-stable", but "supremum-stable": it contains the supremum of any of its subsets. We give an auxiliary proposition first.

(3.8) Let $\leq$,inf,sup be a complete partial order on P; A:PP; f,g:P $\rightarrow$ P; $(\forall x \,.\, x \in A \Rightarrow f(x) \leq g(x))$. Then $\sup\{f(x) \mid x \in A\} \leq \sup\{g(x) \mid x \in A\}$.

Proof: Let $G = \sup\{g(x) \mid x \in A\}$. Then true

... [ub]: $\Leftrightarrow G \in ub(\{g(x) \mid x \in A\})$

... [1.12]: $\Leftrightarrow (\forall y \,.\, y \in \{g(x) \mid x \in A\} \Rightarrow y \leq G)$

... (0.43): $\Leftrightarrow (\forall x \,.\, x \in A \Rightarrow g(x) \leq G)$

... assumption, [0.23], [0.11]: $\Leftrightarrow (\forall x \,.\, x \in A \Rightarrow f(x) \leq g(x) \wedge g(x) \leq G)$

... [tra $\leq$], [0.9], (0.28): $\Rightarrow (\forall x \,.\, x \in A \Rightarrow f(x) \leq G)$

... (0.43): $\Leftrightarrow (\forall y \,.\, y \in \{f(x) \mid x \in A\} \Rightarrow y \leq G)$

... (1.13'): $\Leftrightarrow \sup\{f(x) \mid x \in A\} \leq G$. *(End of Proof.)*

(3.9) Let $\leq$,inf,sup be a complete partial order on P; f be a disjunctive function on P; $C = \{x \mid f(x) \leq x\}$. Then $(\forall A \,.\, A \subseteq C \Rightarrow \sup A \in C)$.

Proof: Let A:PP; $A \subseteq C$.

Then for x:P, $x \in A$

... (1.5): $\Rightarrow x \in C$

$\Leftrightarrow f(x) \leq x$.

Hence $(\forall x \,.\, x \in A \Rightarrow f(x) \leq x)$. (a)

Note that in the following application of (3.8), g in (3.8) is given by $(\forall x \,.\, g(x) = x)$.

true

... (a), (3.8): $\Leftrightarrow \sup\{f(x) \mid x \in A\} \leq \sup\{x \mid x \in A\}$

... f is disjunctive: $\Leftrightarrow f(\sup A) = \sup\{f(x) \mid x \in A\} \wedge \sup\{f(x) \mid x \in A\} \leq \sup A$

... [tra ≤]: $\Rightarrow$ $f(\sup A) \leq \sup A$

$\Leftrightarrow$ $\sup A \in C$. *(End of Proof.)*

As mentioned before, the set C of closed points of a disjunctive function f on P has the property $(\forall A \,.\, A \subseteq C \Rightarrow \inf A \in C \wedge \sup A \in C)$, because of (3.7), (2.3), [0.23] and [0.11]. This condition does not change if we regard the dual of the given partial order. Consequently, the set $\{x \mid f(x) \geq x\}$ for a "conjunctive" function f on P has the same property; "conjunctivity" of f is thereby understood as $(\forall A \,.\, f(\inf A) = \inf\{f(x) \mid x \in A\})$. We will not pursue this observation further, however.

The next theorem shows that any closure space C in P with the property $(\forall A \,.\, A \subseteq C \Rightarrow \sup A \in C)$ is equal to the set of closed points of a suitable disjunctive function c on P. This follows from (2.20), because c will be the familiar closure operation corresponding to C.

(3.10) Let ≤,inf,sup be a complete partial order on P; C be a closure space in P; $c: P \to P$; $(\forall x \,.\, c(x) = \inf C/x)$; $(\forall A \,.\, A \subseteq C \Rightarrow \sup A \in C)$. Then c is a disjunctive function on P.

Proof: Let B:PP. Then true

... (2.6): $\Leftrightarrow$ c is monotonic

... (3.4): $\Rightarrow$ $\sup\{c(x) \mid x \in B\} \leq c(\sup B)$. (a)

Furthermore, true

... (2.15), [inc], (3.8): $\Rightarrow$ $\sup\{x \mid x \in B\} \leq \sup\{c(x) \mid x \in B\}$

... [mon]: $\Rightarrow$ $c(\sup B) \leq c(\sup\{c(x) \mid x \in B\})$. (b)

For x:P, $x \in B$

... (2.8)(ii): $\Rightarrow$ $c(x) \in C$.

Hence true

... [0.27]: $\Leftrightarrow$ $(\forall x \,.\, x \in B \Rightarrow c(x) \in C)$

... (0.43): $\Leftrightarrow$ $(\forall y \,.\, y \in \{c(x) \mid x \in B\} \Rightarrow y \in C)$

... [1.3]: $\Leftrightarrow$ $\{c(x) \mid x \in B\} \subseteq C$

... assumption: $\Rightarrow$ $\sup\{c(x) \mid x \in B\} \in C$

... (2.9)(ii): $\Leftrightarrow$ $c(\sup\{c(x) \mid x \in B\}) \leq \sup\{c(x) \mid x \in B\}$. (c)

Thus true

... (b), (c), [tra $\leq$]: $\Rightarrow$ $c(\sup B) \leq \sup\{c(x) \mid x \in B\}$

... (a), [asy $\leq$]: $\Rightarrow$ $c(\sup B) = \sup\{c(x) \mid x \in B\}$.

By [0.27] and [3.0], this proves c is disjunctive. *(End of Proof.)*

We can state a main result of this section as a corollary of (3.9) and (3.10):

(3.11) Let $\leq$,inf,sup be a complete partial order on P. Then
f is a disjunctive function on P $\Rightarrow$ *clo* f is a disjunctive function on P.

Proof: By (3.9), (3.10), [2.12]. *(End of Proof.)*

Relations as Disjunctive Set Mappings.

For the rest of this chapter, we will be concerned with disjunctive set mappings, that is disjunctive functions on **PS**. The complete partial order on **PS** is thereby understood as $\subseteq, \cap, \cup$. Since f is a disjunctive function on PS if for all A, A:**P**(PS), $f(\cup A) = \cup\{f(X) \mid X \in A\}$, a disjunctive set mapping can be regarded as "distributing over unions". The following proposition states a useful and very intuitive assertion about membership in a set like $\cup\{f(X) \mid X \in A\}$.

(3.12) Let $f:T \rightarrow \mathbf{PS}$ and a(t) a predicate in t, t ranging over T.
Then $(\forall s \,.\, s \in \cup\{f(t) \mid a(t)\} \Leftrightarrow (\exists t \,.\, a(t) \wedge s \in f(t)))$.

Proof: Let s:S. Then $s \in \cup\{f(t) \mid a(t)\}$

... [1.19]: $\Leftrightarrow$ $(\exists X \,.\, X \in \{f(t) \mid a(t)\} \wedge s \in X)$

... (0.44): $\Leftrightarrow$ $(\exists t \,.\, a(t) \wedge s \in f(t))$. *(End of Proof.)*

We will show that a one-to-one correspondence between **DIS**(PS,$\cup$) and **RS** exists. Informally speaking, this holds because a disjunctive set mapping is uniquely specified by its images of singletons, and thus equivalent to a function of type $S \rightarrow \mathbf{PS}$, or a relation on S by (1.0). The following lemma will be used to prove this result; it states that every set can be written as the "union of its singletons".

(3.13) Let X:**PS**. Then $X = \cup\{\{t\} \mid t \in X\}$.

Proof: In the following application of (3.12), note that "{t} is a function of t", that is, t is mapped to {t} by a suitable function of type $S \rightarrow \mathbf{PS}$.

For s:S, $s \in \bigcup\{\{t\} \mid t \in X\}$

... [3.12]: $\Leftrightarrow (\exists t \,.\, t \in X \wedge s \in \{t\})$

... [0.45],[0.35],[0.4]: $\Leftrightarrow (\exists t \,.\, t = s \wedge t \in X)$

... [0.26]: $\Leftrightarrow s \in X.$

Thus $X = \bigcup\{\{t\} \mid t \in X\}$ by [0.27], [0.39]. *(End of Proof.)*

(3.14) Let $r: \mathbf{PS} \to \mathbf{PS}$; $R: \mathbf{RS}$; $R: S \to \mathbf{PS}$ according to (1.0)(i).

Then the following are equivalent:

(i) r is disjunctive $\wedge$ $(\forall s,t \,.\, sRt \Leftrightarrow s \in r(\{t\}))$,

(ii) $(\forall X \,.\, r(X) = \bigcup\{R(t) \mid t \in X\})$.

Proof: Let (i) hold. Then true

... (0.5'), [0.18]: $\Leftrightarrow (\forall t,s \,.\, sRt \Leftrightarrow s \in r(\{t\}))$

... [0.17], (1.0)(ii): $\Leftrightarrow (\forall t \,.\, (\forall s \,.\, s \in R(t) \Leftrightarrow s \in r(\{t\})))$

... [0.39]: $\Leftrightarrow (\forall t \,.\, R(t) = r(\{t\}))$. (a)

Let furthermore X:**PS**. Then

true

... r is disjunctive, (3.2): $\Leftrightarrow r(\bigcup\{\{t\} \mid t \in X\}) = \bigcup\{r(\{t\}) \mid t \in X\}$

... (3.13), (a): $\Leftrightarrow r(X) = \bigcup\{R(t) \mid t \in X\}$.

This proves (i) $\Rightarrow$ (ii).

Let (ii) hold. Then for s,t:S, $s \in r(\{t\})$

... assumption: $\Leftrightarrow s \in \bigcup\{R(u) \mid u \in \{t\}\}$

... (3.12): $\Leftrightarrow (\exists u \,.\, u \in \{t\} \wedge s \in R(u))$

... [0.45], (1.0)(ii): $\Leftrightarrow (\exists u \,.\, u = t \wedge sRu)$

... [0.26]: $\Leftrightarrow sRt;$

thus $(\forall s,t \,.\, sRt \Leftrightarrow s \in r(\{t\}))$.

Furthermore, for A:**P**(**PS**) and s:S,

$s \in r(\bigcup A)$

... assumption (ii): $\Leftrightarrow s \in \bigcup\{R(t) \mid t \in \bigcup A\}$

... (3.12): $\Leftrightarrow (\exists t \,.\, t \in \bigcup A \wedge s \in R(t))$

... [1.19]: $\Leftrightarrow (\exists t \,.\, (\exists X \,.\, X \in A \wedge t \in X) \wedge s \in R(t))$

... [0.4], (0.20), [0.4]: $\Leftrightarrow (\exists X \,.\, X \in A \wedge (\exists t \,.\, t \in X \wedge s \in R(t)))$

... (3.12): $\Leftrightarrow (\exists X \,.\, X \in A \wedge s \in \bigcup\{R(t) \mid t \in X\})$

... (ii): $\Leftrightarrow (\exists X \,.\, X \in A \wedge s \in r(X))$

... (3.12): $\Leftrightarrow$ $s \in \bigcup\{r(X) \mid X \in A\}$.

By [0.27], [0.39] and again [0.27], this proves $(\forall A \,.\, r(\bigcup A) = \bigcup\{r(X) \mid X \in A\})$, i.e. r is disjunctive. Hence (ii) $\Rightarrow$ (i).

(i) $\Leftrightarrow$ (ii) by [0.13]. *(End of Proof.)*

We will also show that the established correspondence between disjunctive set mappings and relations "preserves" the partial orders $\leqq$ on **DIS**(**PS**,$\cup$) and $\leq$ on **RS**, which will be defined in what follows.

[3.15] For f,g: **DIS**(**PS**,$\cup$), let $f \leqq g \Leftrightarrow (\forall X \,.\, f(X) \subseteq g(X))$.

If T is a type with a partial order, we use uniformly $\leqq$ for "comparing" functions on T "argument-wise". Note that by (1.27) and (2.16), $\leqq$ is a partial order on **DIS**(**PS**,$\cup$).

The partial order on **RS** will be defined in analogy to the inclusion on **PS**. It will be denoted by the symbol $\leq$. In more conventional terms, if the relations R,Q on S are regarded as sets of pairs of objects of S, then $R \leq Q$ if R is a subset of Q. We choose the symbol $\leq$ here instead of $\subseteq$ to avoid confusion with the inclusion on **PS**; the symbol $\leq$ is currently not used otherwise. Furthermore, it will be easier to follow applications of previous results, because fewer "translations", i.e. substitutions of symbols, are necessary.

[3.16] Define $\leq$ as follows: $\leq$:**R**(**RS**) and $(\forall R,Q \,.\, R \leq Q \Leftrightarrow (\forall s,t \,.\, sRt \Rightarrow sQt))$.

(3.17) $\leq$ is a partial order on **RS**.

Proof: The proof is entirely analogous to that of (1.4), where the definition [3.16] is used instead of [1.3], $\leq$ instead of $\subseteq$, and the equality of sets, as defined in [0.39], is replaced by the equality of relations, [0.37]. To demonstrate this, we prove

[asy $\leq$]: true

... (0.32): $\Rightarrow$ $(\forall s,t \,.\, \text{true})$

... [0.13], (0.13'): $\Leftrightarrow$ $(\forall s,t \,.\, (sRt \Rightarrow sQt) \wedge (sQt \Rightarrow sRt) \Rightarrow (sRt \Leftrightarrow sQt))$

... (0.28): $\Rightarrow$ $(\forall s,t \,.\, (sRt \Rightarrow sQt) \wedge (sQt \Rightarrow sRt)) \Rightarrow (\forall s,t \,.\, sRt \Leftrightarrow sQt)$

... [3.16], [0.37]: $\Leftrightarrow$ $R \leq Q \wedge Q \leq R \Rightarrow R = Q$. *(End of Proof.)*

(3.18) Let $R,Q:\mathbf{RS}$ and r,s be the disjunctive functions on **PS** corresponding to R,Q according to (3.14)(ii). Then $R \leq Q \Leftrightarrow r \leqq q$.

Proof: Let $r \leqq q$; t:S. Then

	true
... [3.15]:	$\Leftrightarrow (\forall X \,.\, r(X) \subseteq q(X))$
... [0.30]:	$\Rightarrow r(\{t\}) \subseteq q(\{t\})$
... [1.3]:	$\Leftrightarrow (\forall s \,.\, s \in r(\{t\}) \Rightarrow s \in q(\{t\}))$
... (1.0)(ii):	$\Leftrightarrow (\forall s \,.\, sRt \Rightarrow sQt)$.

This proves $(\forall t \,.\, (\forall s \,.\, sRt \Rightarrow sQt))$. Thus $R \leq Q$ holds by [0.17], [0.18], [3.16].

Let $R \leq Q$; X:**PS**; s:S. Then	$s \in r(X)$
... (3.14)(ii):	$\Leftrightarrow s \in \bigcup\{R(t) \mid t \in X\}$
... (3.12), (1.0)(ii):	$\Leftrightarrow (\exists t \,.\, t \in X \wedge sRt)$
... [3.16], [0.12], [0.29]:	$\Rightarrow (\exists t \,.\, t \in X \wedge sQt)$
... (1.0)(ii), (3.12):	$\Leftrightarrow s \in \bigcup\{Q(t) \mid t \in X\}$
... (3.14)(ii):	$\Leftrightarrow s \in q(X)$.

This proves $r(X) \subseteq q(X)$ by [0.27], [1.3], and thus $r \leqq q$ by [0.27], [3.16].

By [0.13], $R \leq Q \Leftrightarrow r \leqq q$. *(End of Proof.)*

In the remainder of this section, we will investigate the properties of relations that correspond to disjunctive closure operations on **PS**. The next statement defines the so-called "relational composition" $R \circ Q$ of two relations R,Q. The symbol $\circ$ shall have higher binding power than relation symbols.

[3.19] Let $R,Q:\mathbf{RS}$. Then define $R \circ Q$ by $R \circ Q:\mathbf{RS}$ and
$(\forall s,t \,.\, sR \circ Qt \Leftrightarrow (\exists u \,.\, sRu \wedge uRt))$.

The "relational composition" of two relations R,Q corresponds to the "functional composition" of the corresponding disjunctive set mappings r,s, as the following proposition shows.

(3.20) Let $R,Q:\mathbf{RS}$ and r,s be the disjunctive functions on **PS** corresponding to R,Q. Then $(\forall X \,.\, r(q(X)) = \bigcup\{R \circ Q(t) \mid t \in X\})$.

Proof: Let X:**PS**; s:S. Then

	$s \in r(q(X))$
... (3.14)(ii):	$\Leftrightarrow s \in \bigcup\{R(u) \mid u \in q(X)\}$
... (3.12):	$\Leftrightarrow (\exists u \,.\, u \in q(X) \wedge s \in R(u))$

... (3.14)(ii), (1.0)(ii): $\Leftrightarrow (\exists u\,.\, u\in \bigcup\{Q(t)\mid t\in X\} \wedge sRu)$

... (3.12): $\Leftrightarrow (\exists u\,.\, (\exists t\,.\, t\in X \wedge u\in Q(t)) \wedge sRu)$

... [0.4], (0.20), (1.0)(ii): $\Leftrightarrow (\exists t\,.\, t\in X \wedge (\exists u\,.\, sRu \wedge uQt))$

... [3.19]: $\Leftrightarrow (\exists t\,.\, t\in X \wedge sR\circ Qt)$

... (3.14)(ii): $\Leftrightarrow s\in \bigcup\{R\circ Q(t) \mid t\in X\})$.

This proves the assertion by [0.27], [0.39], [0.27]. *(End of Proof.)*

(3.21) Let R:**RS**. Then R is transitive $\Leftrightarrow$ $R\circ R\leq R$.

Proof:

R is transitive

... [1.1]: $\Leftrightarrow (\forall s,u,t\,.\, sRu \wedge uRt \Rightarrow sRt)$

... [0.18], (0.18'): $\Leftrightarrow (\forall s,t\,.\, (\forall u\,.\, sRu \wedge uRt \Rightarrow sRt))$

... [0.24]: $\Leftrightarrow (\forall s,t\,.\, (\exists u\,.\, sRu \wedge uRt) \Rightarrow sRt)$

... [3.19]: $\Leftrightarrow (\forall s,t\,.\, sR\circ Rt \Rightarrow sRt)$

... [3.16]: $\Leftrightarrow R\circ R\leq R$. *(End of Proof.)*

The preceding propositions allow us to show that sub-idempotent disjunctive set mappings correspond exactly to transitive relations.

(3.22) Let R:**RS** and r be the disjunctive function on **PS** corresponding to R according to (3.14)(ii). Then r is sub-idempotent $\Leftrightarrow$ R is transitive.

Proof:

r is sub-idempotent

... [2.13]: $\Leftrightarrow (\forall X\,.\, r(r(X)) \subseteq r(X))$

... [3.15], (3.18): $\Leftrightarrow R\circ R\leq R$

... (3.21): $\Leftrightarrow$ R is transitive. *(End of Proof.)*

We will furthermore show that increasing disjunctive set mappings correspond to reflexive relations. We give another characterization of reflexive relations first, as we did in (3.21) for transitive relations.

[3.23] Let ID be defined by ID:**RS** $\wedge$ $(\forall s,t\,.\, sIDt \Leftrightarrow s=t)$.

The relation ID defined in [3.23] is the equality relation on S. We chose the new symbol ID in order to use it according to (1.0) as a function of type $S\rightarrow$ **PS**, too; this would be confusing for $=$.

(3.24) Let R:**RS**. Then R is reflexive $\Leftrightarrow$ $ID\leq R$.

Proof: R is reflexive

... [1.1]: $\Leftrightarrow (\forall t \,.\, tRt)$

... [0.25]: $\Leftrightarrow (\forall t \,.\, (\forall s \,.\, s=t \Rightarrow sRt))$

... [0.17], [0.18], [3.23]: $\Leftrightarrow (\forall s,t \,.\, sIDt \Rightarrow sRt)$

... [3.16]: $\Leftrightarrow ID \leq R.$ *(End of Proof.)*

(3.25) Let id be the disjunctive function on **PS** corresponding to ID according to [3.23] and (3.14)(ii). Then $(\forall X \,.\, id(X) = X)$.

Proof: Let X:**PS**. Then true

... [3.23]: $\Leftrightarrow \cup\{\{s|sIDt\} \mid t \in X\} = \cup\{\{s|s=t\} \mid t \in X\}$

... (1.0)(i), [0.45]: $\Leftrightarrow \cup\{ID(t) \mid t \in X\} = \cup\{\{s\} \mid t \in X\}$

... (3.14)(ii). $\Leftrightarrow id(X) = X.$ *(End of Proof.)*

(3.26) Let R:$\dot{RS}$ and r be the disjunctive function on **PS** corresponding to R.
Then r is increasing $\Leftrightarrow$ R is reflexive.

Proof: r is increasing

... [2.13]: $\Leftrightarrow (\forall X \,.\, X \subseteq r(X))$

... (3.25): $\Leftrightarrow (\forall X \,.\, id(X) \subseteq r(X))$

... [3.15]: $\Leftrightarrow id \leqq r$

... (3.18): $\Leftrightarrow ID \leq R$

... (3.24): $\Leftrightarrow$ R is reflexive. *(End of Proof.)*

The following statement summarizes the previous results in characterizing the disjunctive closure operations on **PS**.

(3.27) Let R:**RS** and r be the disjunctive function on **PS** corresponding to R.
Then r is a closure operation on **PS** $\Leftrightarrow$ R is reflexive and transitive.
If r is a closure operation on **PS**, then $(\forall s,t \,.\, sRt \Leftrightarrow r(\{s\}) \subseteq r(\{t\}))$.

Proof: Any disjunctive function on **PS** is monotonic. Hence the first equivalence follows from (3.26) and (3.22). To prove the second assertion, let r be a closure operation on **PS** and s,t:S. Then sRt

... (3.26), (3.22), (1.11): $\Leftrightarrow R(s) \subseteq R(t)$

... (3.14)(i), (1.0)(ii), [0.39]: $\Leftrightarrow r(\{s\}) \subseteq r(\{t\}).$ *(End of Proof.)*

The second assertion in (3.27) describes which reflexive and transitive relation corresponds to a given disjunctive closure operation r on **PS**: It is "represented" by the inclusion of the "closures" of the corresponding singletons; the "closure" of a set X is thereby understood as r(X).

The Reflexive and Transitive Closure of a Relation.

In this last section of this chapter, we will investigate the so-called reflexive and transitive closure R* of a relation R on S. This is the "least" reflexive and transitive relation "extending" R, with respect to the partial order $\leq$ on **RS** as defined in [3.16]. With inf and sup defined in analogy to the intersection and union of sets, this partial order is in fact complete, as the following statements assert.

[3.28] Define inf,sup as follows: Let inf,sup: $\mathbf{P}(\mathbf{RS}) \to \mathbf{RS}$ and for A:$\mathbf{P}(\mathbf{RS})$,

$(\forall s,t\,.\, s(\inf A)t \;\Leftrightarrow\; (\forall R\,.\, R\in A \Rightarrow sRt))$,

$(\forall s,t\,.\, s(\sup A)t \;\Leftrightarrow\; (\exists R\,.\, R\in A \wedge sRt))$.

(3.29) $\leq$,inf,sup is a complete partial order on **RS**.

Proof: As for (3.17), the proof is analogous to that of (1.20), with inf,sup instead of $\cap,\cup$.

(End of Proof.)

We will show that the set RT of reflexive and transitive relations on S is a closure space in **RS**. The result of applying the closure operation on **RS** corresponding to RT to a relation R will be denoted by R*. The following is a formal definition of RT and "*", according to (3.24), (3.21) and (2.20).

[3.30] Define RT:$\mathbf{P}(\mathbf{RS})$, $RT = \{R \mid ID \leq R \wedge R\circ R \leq R\}$ and $(\forall R\,.\, R^* = \inf RT/R)$.

(3.31) Let R,R',Q,Q':**RS**. Then $\quad R\leq Q \wedge R'\leq Q' \;\Rightarrow\; R\circ R'\leq Q\circ Q'$.

Proof: Let $R\leq Q \wedge R'\leq Q'$.

Then for s,t:S,		$sR\circ R't$
... [3.19]:	$\Leftrightarrow$	$(\exists u\,.\, sRu \wedge uR't)$
... [3.16], [0.29]:	$\Rightarrow$	$(\exists u\,.\, sQu \wedge uQ't)$
... [3.19]:	$\Leftrightarrow$	$sQ\circ Q't$,

i.e. $R\circ R'\leq Q\circ Q'$ by [0.27],[3.16]. *(End of Proof.)*

(3.32) RT is a closure space in **RS**.

Proof: For R,Q:**RS**, $R \leq Q \Rightarrow ID \leq ID$ by [ref $\leq$], and

$$\begin{aligned} & & R \leq Q \\ \ldots (0.6): & \Leftrightarrow & R \leq Q \wedge R \leq Q \\ \ldots (3.31): & \Rightarrow & R \circ R \leq Q \circ Q. \end{aligned}$$

In other words, the functions mapping R to ID and R to R∘R, respectively, are monotonic functions on **RS**. By (2.3), the sets C,D defined by C,D:**P**(**RS**) and $C=\{R \mid ID \leq R\}$, $D=\{R \mid R \circ R \leq R\}$ are closure spaces in **RS**. Thus $C \cap D$ is the intersection over a set of closure spaces in **RS** by [1.21], and therefore a closure space in **RS** by (2.25). But $C \cap D=\{R \mid ID \leq R \wedge R \circ R \leq R\}$ by (1.22).

(End of Proof.)

In the previous section it was shown that relations on S correspond to disjunctive functions on **PS**. The following theorem asserts that the reflexive and transitive closure R* of a relation R corresponds to the "closure" *clo* r of the corresponding disjunctive set mapping r.

(3.33) Let R:**RS** and r be the corresponding disjunctive function on **PS** according to (3.14)(ii). Then R* corresponds to *clo* r.

Proof: *clo* r is a disjunctive function on **PS** by (3.11). Let Q be the relation on S that corresponds to *clo* r according to (3.14)(i). Let r* be the disjunctive function on **PS** corresponding to R*. *clo* r is a closure operation on **PS** by [2.12], (2.3), (2.15). Hence Q is reflexive and transitive by (3.27), i.e. $Q \in RT$. Therefore

$$\begin{aligned} & & \text{true} \\ \ldots (2.18): & \Leftrightarrow & r \leqq clo\, r \\ \ldots (3.18): & \Leftrightarrow & R \leq Q \\ \ldots (3.27), [3.30], [2.4]: & \Rightarrow & Q \in RT/R \\ \ldots [1b], [1.12], [0.30]: & \Rightarrow & \inf RT/R \leq Q \\ \ldots [3.30]: & \Leftrightarrow & R^* \leq Q \\ \ldots (3.18): & \Leftrightarrow & r^* \leqq clo\, r. \end{aligned} \qquad \text{(a)}$$

Conversely, by (3.27), r* is a closure operation on **PS**, (b)

and thus

$$\begin{aligned} & & \text{true} \\ \ldots (2.7): & \Leftrightarrow & R \leq \inf RT/R \end{aligned}$$

... [3.30]: $\Leftrightarrow R \leq R^*$

... (3.18): $\Leftrightarrow r \leqq r^*$

... (2.24), [mon]: $\Rightarrow clo\, r \leqq clo\, r^*$

... (b), (2.19): $\Leftrightarrow clo\, r \leqq clo\, r^* \wedge clo\, r^* \leqq r^*$

... (2.17), [tra $\leqq$]: $\Rightarrow clo\, r \leqq r^*$

... (a), [asy $\leqq$]: $\Rightarrow clo\, r = r^*$. *(End of Proof.)*

On the remaining pages of this chapter, we show that R* can also be defined by "iterating" the relation R, or, more precisely, that $(\forall s,t\,.\, sR^*t \Leftrightarrow (\exists n\,.\, sR^n t))$. **N** is thereby the type of natural numbers, and for a natural number n, R^n is the relation obtained by "composing" R with itself n times. This is the common definition of R*. It requires some effort to prove that with this definition, R* is the "least" reflexive and transitive relation "extending" R. On the other hand, this is directly expressed by [3.30], where the fact that R* exists is based on theorem (2.3); propositions (3.31), (3.21) and (3.24), which were also used in proving (3.32), will be used in the following "conventional" derivation of R* as well. This suggests regarding R* in the general framework of closure spaces as presented in this text.

With little extra effort, we can formally introduce the properties of the type **N** of natural numbers we need. We assume only the existence of a constant 0 and a "successor" function *s* on **N**, and the *induction* scheme. That is, let $0{:}\mathbf{N}$ and $s{:}\mathbf{N} \to \mathbf{N}$; *s* will be simply prefixed to its arguments (*s*n can be read as "n + 1"). A predicate a(n) can be proved "by induction" to hold for all natural numbers n, using the following axiom.

[3.34] Let a(n) be a predicate in n, n ranging over **N**. Then

$a(0) \wedge (\forall n\,.\, a(n) \Rightarrow a(sn)) \Rightarrow (\forall n\,.\, a(n))$.

In particular, for functions f,g, $f,g{:}\mathbf{N} \to T$, $f(0)=g(0) \wedge (\forall n\,.\, f(n)=g(n) \Rightarrow f(sn)=g(sn))$ implies f = g by [3.34] and [0.38]. Therefore an "inductive definition" of a function f with domain **N** is unique, that is, a definition of f(0) in combination with a definition of f(*s*n) in terms of f(n) for all n. An example is the following definition. In the "exponential" notation given there, superscribing natural numbers shall have higher binding power than the symbol $\circ$.

[3.35] Let $R{:}\mathbf{RS}$. Then define R^n for $n{:}\mathbf{N}$ as follows: $R^0 = ID$ and $(\forall n\,.\, R^{sn} = R \circ R^n)$.

The following proposition asserts that the operation $\circ$ is associative, and that ID is a "unit" with respect to this operation.

(3.36) Let R,Q,M:**RS**. Then $R\circ(Q\circ M)=(R\circ Q)\circ M$, $ID\circ R=R$ and $R\circ ID=R$.

Proof: Let s,t:S. Then

		$sR\circ(Q\circ M)t$
... [3.19]:	$\Leftrightarrow$	$(\exists u\,.\,sRu \wedge uQ\circ Mt)$
... [3.19]:	$\Leftrightarrow$	$(\exists u\,.\,sRu \wedge (\exists v\,.\,uQv \wedge vMt))$
... [0.22]:	$\Leftrightarrow$	$(\exists u\,.\,(\exists v\,.\,sRu \wedge uQv \wedge vMt))$
... (0.18'), [0.4], [0.22], [0.4]:	$\Leftrightarrow$	$(\exists v\,.\,(\exists u\,.\,sRu \wedge uQv) \wedge vMt)$
... [3.19]:	$\Leftrightarrow$	$(\exists v\,.\,sR\circ Qv \wedge vMt)$
... [3.19]:	$\Leftrightarrow$	$s(R\circ Q)\circ Mt$;
furthermore,		sRt
... [0.26]:	$\Leftrightarrow$	$(\exists u\,.\,u=s \wedge uRt)$
... [3.23]:	$\Leftrightarrow$	$(\exists u\,.\,sIDu \wedge uRt)$
... [3.19]:	$\Leftrightarrow$	$sID\circ Rt$,
and		sRt
... [0.26]:	$\Leftrightarrow$	$(\exists u\,.\,u=t \wedge sRu)$
... [0.4], [0.30], [3.23]:	$\Leftrightarrow$	$(\exists u\,.\,sRu \wedge uIDt)$
... [3.19]:	$\Leftrightarrow$	$sR\circ IDt$.

The assertions follow from [0.27] and [0.37], respectively. *(End of Proof.)*

The addition of natural numbers, denoted by $+$, is also defined inductively.

[3.37] Let m:**N**. Then define $0+m=m$ and $(\forall n\,.\,(sn)+m = s(n+m))$.

(3.38) Let R:**RS**. Then $(\forall n,m\,.\,R^{n+m} = R^n\circ R^m)$.

Proof: Let n:**N**.

Then		true
... (3.36):	$\Leftrightarrow$	$R^n = ID\circ R^n$
... [3.35]:	$\Leftrightarrow$	$R^{0+n} = R^0\circ R^n$.
(a)		
For m:**N**,		$R^{n+m} = R^n\circ R^m$
	$\Rightarrow$	$R\circ R^{n+m} = R\circ(R^n\circ R^m)$
... [3.35], (3.36):	$\Leftrightarrow$	$R^{s(n+m)} = (R\circ R^n)\circ R^m$

... [3.37], [3.35]: $\Leftrightarrow R^{(sn)+m} = R^{sn} \circ R^m$.

By [0.27], this proves $(\forall m \,.\, R^{n+m} = R^n \circ R^m \Rightarrow R^{(sn)+m} = R^{sn} \circ R^m)$. (b)

By (a), (b) and [3.34], $(\forall m \,.\, R^{n+m} = R^n \circ R^m)$.

Hence the assertion holds by [0.27], [0.17]. *(End of Proof.)*

The following theorem asserts the equation for R* given above. The reader may wish to verify that it is equivalent to $R^* = \sup\{R^n \mid n{:}\mathbf{N}\}$, according to [3.28].

(3.39) Let R:**RS**. Then $(\forall s,t \,.\, sR^*t \Leftrightarrow (\exists n \,.\, sR^n t))$.

Proof: Let Q:**RS** and $(\forall s,t \,.\, sQt \Leftrightarrow (\exists n \,.\, sR^n t))$.

We first prove $R^* \leq Q$.

Let s,t:S. Then sRt

... (3.36): $\Leftrightarrow sR \circ IDt$

... [3.35]: $\Leftrightarrow sR \circ R^0 t$

... [3.35]: $\Leftrightarrow sR^{s0} t$

... (0.31): $\Rightarrow (\exists n \,.\, sR^n t)$

$\Leftrightarrow sQt$.

Thus $R \leq Q$ by [0.27], .

Similarly, let s,t:S. Then $sIDt$

... [3.35]: $\Leftrightarrow sR^0 t$

... (0.31): $\Rightarrow (\exists n \,.\, sR^n t)$

$\Leftrightarrow sQt$.

Hence $ID \leq Q$ (i.e. Q is reflexive).

Finally, for s,t:S, $sQ \circ Qt$

... [3.19]: $\Leftrightarrow (\exists u \,.\, sQu \wedge uQt)$

$\Leftrightarrow (\exists u \,.\, (\exists n \,.\, sR^n u) \wedge (\exists m \,.\, uR^m t))$

... [0.4], [0.22], [0.4]: $\Leftrightarrow (\exists u \,.\, (\exists n \,.\, sR^n u \wedge (\exists m \,.\, uR^m t)))$

... [0.22], [0.17]: $\Leftrightarrow (\exists u \,.\, (\exists n,m \,.\, sR^n u \wedge uR^m t))$

... (0.18'): $\Leftrightarrow (\exists n,m \,.\, (\exists u \,.\, sR^n u \wedge uR^m t))$

... [3.19]: $\Leftrightarrow (\exists n,m \,.\, sR^n \circ R^m t)$

... (3.38): $\Leftrightarrow (\exists n,m \,.\, sR^{n+m} t)$

... [0.26]: $\Leftrightarrow (\exists n,m \,.\, (\exists k \,.\, k = n+m \wedge sR^k t))$

... (0.5'), [0.29], [0.29]: $\Rightarrow (\exists n,m \,.\, (\exists k \,.\, sR^k t))$

... [0.29]: $\Rightarrow (\exists k \,.\, sR^k t)$

$\Leftrightarrow$ sQt.

This proves $Q \circ Q \leq Q$ (i.e. Q is transitive).

Therefore, true

... [3.30], [2.4]: $\Leftrightarrow$ $Q \in RT/R$

... [1b], [1.12], [0.20]: $\Rightarrow$ $\inf RT/R \leq Q$

... [3.30]: $\Leftrightarrow$ $R^* \leq Q$.

By [asy $\leq$], it suffices to prove $Q \leq R^*$.

$Q \leq R^*$

... [3.16]: $\Leftrightarrow$ $(\forall s,t \,.\, (\exists n \,.\, sR^n t) \Rightarrow sR^*t)$

... [0.24]: $\Leftrightarrow$ $(\forall s,t \,.\, (\forall n \,.\, sR^n t \Rightarrow sR^*t))$

... (0.18'): $\Leftrightarrow$ $(\forall n \,.\, (\forall s,t \,.\, sR^n t \Rightarrow sR^*t))$

... [3.16]: $\Leftrightarrow$ $(\forall n \,.\, R^n \leq R^*)$.

(a)

We prove (a) by induction over n. true

... (2.9)(ii) (i.e. $R^* \in RS$), [3.30]: $\Leftrightarrow$ $ID \leq R^*$

... [3.35]: $\Leftrightarrow$ $R^0 \leq R^*$.

Let n:**N**. Then $R^n \leq R^*$

... (0.6): $\Leftrightarrow$ $R \leq R^* \wedge R^n \leq R^*$

... (3.31): $\Rightarrow$ $R \circ R^n \leq R^* \circ R^*$

... [3.35], $R^* \in RS$: $\Leftrightarrow$ $R^{sn} \leq R^* \circ R^* \wedge R^* \circ R^* \leq R^*$

... (3.17), [tra $\leq$]: $\Rightarrow$ $R^{sn} \leq R^*$.

This proves $(\forall n \,.\, R^n \leq R^* \Rightarrow R^{sn} \leq R^*)$.

Hence (a) holds by [3.34]. *(End of Proof).*

Chapter 4

Continuous Functions

In this chapter, P will be a type with a given partial order or complete partial order. The objects of P will be called *points*. S will denote some fixed type. The objects of **PS** and **PS → PS** will be called sets and set mappings, respectively.

Continuous Functions and Closure Operations.

In this chapter, we will investigate functions on P that "distribute over suprema of directed sets".

[4.0] Let ≤ be a partial order on P; D:PP. D is called *directed* if
$(\exists u \,.\, u \in D) \;\wedge\; (\forall u,v \,.\, u \in D \wedge v \in D \Rightarrow (\exists z \,.\, z \in D \wedge u \le z \wedge v \le z))$.

The predicate $(\exists u \,.\, u \in D)$ should be read as "D is not empty". [4.0] asserts that a directed set D of points is not empty and that any two-element subset {u,v} of D has an upper bound z in D. This condition can be expressed if only a partial order on P is given, not necessarily a complete one. The following definition involves directed sets of points, assuming a complete partial order on P. We will later comment on how one could weaken the assumption of a given complete partial order.

[4.1] Let ≤,inf,sup be a complete partial order on P. Then f is called a *continuous* function on P if $f:P \to P$ and $(\forall D \,.\, D \text{ is directed} \Rightarrow f(\sup D) = \sup \{f(x) \mid x \in D\})$.

Obviously, any disjunctive function on P (cf. [3.0]) is continuous. The converse is not true. We can informally describe a simple counterexample: any constant function f on P that results into a value different from sup ∅ is continuous, but not disjunctive. For, $\{f(x) \mid x \in \emptyset\} = \emptyset$ and thus $f(\sup \emptyset) = \sup \emptyset$ for a disjunctive function f; but this equation does not have to hold for a continuous function f, since ∅ is not directed.

The following investigations on continuous functions are similar to those on disjunctive functions in the first section of chapter three.

(4.2) Let $\leq$ be a partial order on P and x,y:P. Then $x \leq y \Rightarrow \{x,y\}$ is directed.

Proof:

		true
... (0.47), [0.4]:	$\Rightarrow$	$x \in \{x,y\}$
... (0.31):	$\Rightarrow$	$(\exists u \,.\, u \in \{x,y\})$.
Furthermore,		$x \leq y$
... [ref $\leq$]:	$\Leftrightarrow$	$x \leq y \wedge y \leq y$
... (0.6):	$\Leftrightarrow$	$x \leq y \wedge y \leq y \wedge x \leq y \wedge y \leq y$
... [0.46]:	$\Leftrightarrow$	$(\forall u \,.\, u \in \{x,y\} \Rightarrow u \leq y) \wedge (\forall v \,.\, v \in \{x,y\} \Rightarrow v \leq y)$
... (0.32), [0.17], [0.18]:	$\Rightarrow$	$(\forall u,v \,.\, u \in \{x,y\} \Rightarrow u \leq y) \wedge (\forall u,v \,.\, v \in \{x,y\} \Rightarrow v \leq y)$
... [0.23]:	$\Leftrightarrow$	$(\forall u,v \,.\, (u \in \{x,y\} \Rightarrow u \leq y) \wedge (v \in \{x,y\} \Rightarrow v \leq y))$
... [0.12], (0.28):	$\Rightarrow$	$(\forall u,v \,.\, u \in \{x,y\} \wedge v \in \{x,y\} \Rightarrow u \leq y \wedge v \leq y)$
... (0.47), (0.5'):	$\Leftrightarrow$	$(\forall u,v \,.\, u \in \{x,y\} \wedge v \in \{x,y\} \Rightarrow y \in \{x,y\} \wedge u \leq y \wedge v \leq y)$
... (0.31), [0.9], (0.28):	$\Rightarrow$	$(\forall u,v \,.\, u \in \{x,y\} \wedge v \in \{x,y\} \Rightarrow$
		$(\exists z \,.\, z \in \{x,y\} \wedge u \leq z \wedge v \leq z))$.

Thus by [0.12] and [4.0], true $\wedge$ $x \leq y \Rightarrow \{x,y\}$ is directed. *(End of Proof.)*

(4.3) Let $\leq$,inf,sup be a complete partial order on P and f be a continuous function on P. Then f is monotonic.

Proof: Let x,y:P. Then

		$x \leq y$
... (0.6):	$\Leftrightarrow$	$x \leq y \wedge x \leq y$
... (3.5):	$\Leftrightarrow$	$\sup\{x,y\} = y \wedge x \leq y$
... [0.41], [0.8], [0.12]:	$\Rightarrow$	$f(\sup\{x,y\}) = f(y) \wedge x \leq y$
... (4.2), [4.1], [0.5]:	$\Rightarrow$	$\sup\{f(s) \mid s \in \{x,y\}\} = f(y)$
... (3.6):	$\Leftrightarrow$	$\sup\{f(x), f(y)\} = f(y)$
... (3.5):	$\Leftrightarrow$	$f(x) \leq f(y)$. *(End of Proof.)*

Proposition (4.3) can replace (3.7), because every disjunctive function is continuous.

(4.4) Let $\leq$ be a partial order on P; let f be a monotonic function on P, D:PP and D be directed. Then $\{f(x) \mid x \in D\}$ is directed.

Proof:

		true
... [4.0], [0.5]:	$\Rightarrow$	$(\exists x \,.\, x \in D)$
... [0.40]:	$\Leftrightarrow$	$(\exists x \,.\, x \in D \wedge (\exists u \,.\, u = f(x)))$

... [0.2], (0.18'): $\Leftrightarrow\ (\exists u\,.\,(\exists x\,.\,x\in D\ \wedge\ u=f(x)))$

... [0.42]: $\Leftrightarrow\ (\exists u\,.\,u\in\{f(x)|\ x\in D\})$.

Let u,v:P. Then $u\in\{f(x)|\ x\in D\}\ \wedge\ v\in\{f(x)|\ x\in D\}$

... [0.42]: $\Leftrightarrow\ (\exists x\,.\,x\in D\wedge u=f(x))\ \wedge\ (\exists y\,.\,y\in D\wedge v=f(y))$

... [0.4], [0.22], [0.4]: $\Leftrightarrow\ (\exists x\,.\,x\in D\ \wedge\ u=f(x)\ \wedge\ (\exists y\,.\,y\in D\wedge v=f(y)))$

... [0.22], [0.4], [0.18]: $\Leftrightarrow\ (\exists x,y\,.\,x\in D\ \wedge\ y\in D\ \wedge\ u=f(x)\wedge v=f(y))$

... [4.0], [0.29]: $\Rightarrow\ (\exists x,y\,.\,(\exists z\,.\,z\in D\wedge x\leq z\wedge y\leq z)\ \wedge\ u=f(x)\wedge v=f(y))$

... f monotonic, [0.29]: $\Rightarrow\ (\exists x,y\,.\,(\exists z\,.\,z\in D\wedge f(x)\leq f(z)\ \wedge\ f(y)\leq f(z))$
$\wedge\ u=f(x)\wedge v=f(y))$

... [0.29], [0.33]: $\Rightarrow\ (\exists z\,.\,z\in D\ \wedge\ u\leq f(z)\ \wedge\ v\leq f(z))$

... [0.44]: $\Leftrightarrow\ (\exists w\,.\,w\in\{f(x)|\ x\in D\}\ \wedge\ u\leq w\ \wedge\ v\leq w)$.

This proves $\{f(x)|\ x\in D\}$ is directed. *(End of Proof.)*

In view of the previous proposition, continuity can be defined even if the underlying partial order is not complete. It is sufficient to assume that every directed set of points "has a supremum". That is to say, for every directed D, D:PP, there is an object sup D of P such that $ub(D)=\geq(\sup D)$. This condition is indeed weaker than the existence of a complete partial order $\leq$,inf,sup on P, by an argument similar to that following [4.1]: sup $\varnothing$ does not have to exist (cf. also (1.18) and preceding remarks). Assuming that suprema for directed sets of points exist, a function f on P can be defined as monotonic and fulfilling $(\forall D\,.\,D \text{ is directed} \Rightarrow f(\sup D) = \sup\{f(x)|\ x\in D\})$. Alternatively, one could only require the last predicate, with the understanding that $\sup\{f(x)|\ x\in D\}$ exists for directed D; then f is monotonic by (4.2), (4.3).

To continue the previous consideration, we also remark that partial orders with suprema for directed sets of points are exactly those partial orders such that every *chain* has a supremum. C is a chain if C:PP and $(\exists u\,.\,u\in C)\ \wedge\ (\forall u,v\,.\,u\in C\wedge v\in C\ \Rightarrow\ (\exists z\,.\,z\in\{u,v\}\wedge u\leq z\wedge v\leq z))$. In other words, a chain C is a non-empty set of points such that for any two elements u,v of C, either $u\leq v$ or $v\leq u$ holds. Every chain is directed, but not vice versa. For a proof of the claim above cf. *Cohn* [Co, p.33]. It has some intuitive appeal to call a function that "distributes over suprema of chains" 'continuous'. The supremum of a chain can be regarded as its "limit", being "approximated" by the elements of the chain

We will regard continuous functions only if a complete partial order is given, however. With the aid of propositions (4.3) and (4.4), the set C of closed points of a continuous function f on P will prove to contain the supremum of any of its directed subsets. Also, the closure operation on P corresponding to such a closure space will be shown to be continuous. The proofs are indeed identical to those of (3.9) and (3.10), apart from the propositions used. We abstained from an ad-hoc generalization of the respective statements in chapters three and four, which would have avoided this duplication.

(4.5) Let $\leq$,inf,sup be a complete partial order on P; f be a continuous function on P; $C=\{x \mid f(x)\leq x\}$. Then $(\forall D \,.\, D \text{ is directed} \wedge D\subseteq C \Rightarrow \sup D\in C)$.

Proof: Let $D:\mathbf{P}P$, D be directed and $D\subseteq C$.

Then for $x:P$, $x\in D$

... (1.5): $\Rightarrow x\in C$

$\Leftrightarrow f(x)\leq x$.

Hence by [0.27], $(\forall x \,.\, x\in D \Rightarrow f(x)\leq x)$. (a)

Therefore true

... (a), (3.8): $\Leftrightarrow \sup\{f(x)\mid x\in D\}\leq \sup\{x\mid x\in D\}$

... f is continuous: $\Leftrightarrow f(\sup D)=\sup\{f(x)\mid x\in D\} \wedge \sup\{f(x)\mid x\in D\}\leq \sup D$

... [tra $\leq$]: $\Rightarrow f(\sup D)\leq \sup D$

$\Leftrightarrow \sup D\in C$. *(End of Proof.)*

(4.6) Let $\leq$,inf,sup be a complete partial order on P; C be a closure space in P; $c:P\rightarrow P$; $(\forall x \,.\, c(x)=\inf C/x)$; $(\forall D \,.\, D \text{ is directed} \wedge D\subseteq C \Rightarrow \sup D\in C)$.
Then c is a continuous function on P.

Proof: Let $B:\mathbf{P}P$ and B be directed.

Then true

... (2.6): $\Leftrightarrow$ c is monotonic

... (3.4): $\Rightarrow \sup\{c(x)\mid x\in B\}\leq c(\sup B)$. (a)

Furthermore, true

... (2.15), [inc], (3.8): $\Rightarrow \sup\{x\mid x\in B\}\leq \sup\{c(x)\mid x\in B\}$

... [mon]: $\Rightarrow c(\sup B)\leq c(\sup\{c(x)\mid x\in B\})$. (b)

For x:P,	$x \in B$	
... (2.8)(ii), [0.30]:	$\Rightarrow$	$c(x) \in C$.
Hence		true
... [0.27]:	$\Leftrightarrow$	$(\forall x \,.\, x \in B \Rightarrow c(x) \in C)$
... (0.43):	$\Leftrightarrow$	$(\forall y \,.\, y \in \{c(x) \mid x \in B\} \Rightarrow y \in C)$
... [1.3]:	$\Leftrightarrow$	$\{c(x) \mid x \in B\} \subseteq C$
... (4.4), assumption:	$\Rightarrow$	$\sup\{c(x) \mid x \in B\} \in C$
... (2.8)(ii):	$\Leftrightarrow$	$c(\sup\{c(x) \mid x \in B\}) \leq \sup\{c(x) \mid x \in B\}$. (c)
Thus		true
... (b), (c), [tra $\leq$]:	$\Rightarrow$	$c(\sup B) \leq \sup\{c(x) \mid x \in B\}$
... (a), [asy $\leq$]:	$\Rightarrow$	$c(\sup B) = \sup\{c(x) \mid x \in B\}$.

By [0.27] and [4.1], this proves c is continuous. *(End of Proof.)*

(4.7) Let $\leq$,inf,sup be a complete partial order on P. Then

f is a continuous function on P $\Rightarrow$ *clo* f is a continuous function on P.

Proof: By (4.6), (4.7), [2.12]. *(End of Proof.)*

"Finitely Representable" Set Mappings.

In the second section of chapter three, we saw that disjunctive set mappings are uniquely determined by their images of singletons. In this section, we will investigate set mappings that are uniquely determined by their images of finite sets, therefore called "finitely representable" set mappings. They will prove to be exactly the continuous set mappings.

We introduce the type **Fin**S of finite sets of objects of S as a subtype of **PS**.

[4.8] Let E:**Fin**S $\Leftrightarrow$ E:**PS** $\wedge$ E is finite.

This of course leaves open what "finite" means; so far 'is finite' is only a predicate that can hold for objects of **PS**. A finite set is either empty or it is obtained recursively as the union of a finite set with a singleton. The following axiom asserts that some finite sets can be obtained this way.

[4.9] $\varnothing$ is finite $\wedge$ $(\forall E,t \,.\, E$ is finite $\Rightarrow E \cup \{t\}$ is finite).

We will also assert that in some sense, [4.9] characterizes all finite sets. We do this by postulating the existence of a "cardinality" function *card* that maps finite sets to natural numbers.

[4.10] Let $card$:**FinS** $\rightarrow$ **N**, $(\forall E \,.\, card(E)=0 \Leftrightarrow E=\varnothing)$ and
$(\forall n \,.\, (\forall F \,.\, card(F)=sn \Rightarrow (\exists E,t \,.\, F=E\cup\{t\} \wedge card(E)=n))$.

Intuitively, $card(E)$ is the number of elements contained in E. We however do not require that there is only one function with the properties of *card* expressed in [4.10]. We use *card* only to show that a predicate a(E) can be proved "inductively" to hold for all finite sets E, by proving $a(\varnothing)$ and $a(E) \Rightarrow a(E\cup\{t\})$ for all E,t, where E:**FinS** and t:S. We will prove this assertion using the induction scheme [3.34] for natural numbers.

(4.11) Let a(E) be a predicate where E is of type **FinS**. Let $a(\varnothing)$ hold and
$(\forall E,t \,.\, a(E) \Rightarrow a(E\cup\{t\}))$. Then $(\forall E \,.\, a(E))$.

Proof:

$(\forall E \,.\, a(E))$

... [0.1], [0.40]: $\Leftrightarrow (\forall E \,.\, (\exists n \,.\, n=card(E)) \Rightarrow a(E))$

... [0.24], (0.18'), [0.35]: $\Leftrightarrow (\forall n \,.\, (\forall E \,.\, card(E)=n \Rightarrow a(E)))$. (a)

It suffices to prove (a) by induction on n.

true

... assumption: $\Leftrightarrow a(\varnothing)$

... [0.26]: $\Leftrightarrow (\forall E \,.\, E=\varnothing \Rightarrow a(E))$

... [4.10]: $\Leftrightarrow (\forall E \,.\, card(E)=0 \Rightarrow a(E))$.

Let n:**N** and $(\forall E \,.\, card(E)=n \Rightarrow a(E))$.

Then for F:**FinS**, $card(F)=sn$

... [4.10]: $\Rightarrow (\exists E,t \,.\, F=E\cup\{t\} \wedge card(E)=n)$

... [0.12], (0.29): $\Rightarrow (\exists E,t \,.\, F=E\cup\{t\} \wedge a(E))$

... assumption, [0.12], (0.29): $\Rightarrow (\exists E,t \,.\, F=E\cup\{t\} \wedge a(F))$

... (0.29), [0.33]: $\Rightarrow a(F)$.

Thus $(\forall F \,.\, card(F)=sn \Rightarrow a(F))$ by [0.27].

This proves $(\forall n \,.\, (\forall E \,.\, card(E)=n \Rightarrow a(E)) \Rightarrow (\forall F \,.\, card(F)=sn \Rightarrow a(F)))$.

Hence (a) holds by [3.34]. *(End of Proof.)*

We next give some auxialiary propositions about the union $X \cup Y$ of two sets X,Y, in particular for the case that X and Y are finite.

(4.12) Let X,Y,Z:**PS**. Then $X \cup Y \subseteq Z \Leftrightarrow X \subseteq Z \wedge Y \subseteq Z$.

Proof:

$X \cup Y \subseteq Z$

... [1.3], [1.21]: $\Leftrightarrow (\forall s \,.\, s \in \cup\{X,Y\} \Rightarrow s \in Z)$

... [1.19]: $\Leftrightarrow (\forall s \,.\, (\exists V \,.\, V \in \{X,Y\} \wedge s \in V) \Rightarrow s \in Z)$

... [0.24], [0.10]: $\Leftrightarrow (\forall s \,.\, (\forall V \,.\, V \in \{X,Y\} \Rightarrow (s \in V \Rightarrow s \in Z)))$

... (0.18'), (0.21): $\Leftrightarrow (\forall V \,.\, V \in \{X,Y\} \Rightarrow (\forall s \,.\, s \in V \Rightarrow s \in Z))$

... [1.3]: $\Leftrightarrow (\forall V \,.\, V \in \{X,Y\} \Rightarrow V \subseteq Z)$

... [0.47]: $\Leftrightarrow X \subseteq Z \wedge Y \subseteq Z$. *(End of Proof.)*

(4.13) Let X:**PS**. Then $X \cup \varnothing = X$.

Proof:

true

... [ref $\subseteq$], (1.6), [ref $\subseteq$]: $\Leftrightarrow X \subseteq X \wedge \varnothing \subseteq X \wedge X \cup \varnothing \subseteq X \cup \varnothing$

... (4.12), (4.12): $\Leftrightarrow X \cup \varnothing \subseteq X \wedge X \subseteq X \cup \varnothing \wedge \varnothing \subseteq X \cup \varnothing$

... [0.5], [asy $\subseteq$]: $\Rightarrow X \cup \varnothing = X$. *(End of Proof.)*

The previous lemma asserts that $\varnothing$ is "neutral" with respect to $\cup$. The next proposition shows $\cup$ is associative.

(4.14) Let X,Y,Z:**PS**. Then $X \cup (Y \cup Z) = (X \cup Y) \cup Z$.

Proof:

true

... [ref $\subseteq$]: $\Leftrightarrow X \cup Y \subseteq X \cup Y \wedge (X \cup Y) \cup Z \subseteq (X \cup Y) \cup Z$

... (4.12): $\Leftrightarrow X \subseteq X \cup Y \wedge Y \subseteq X \cup Y \wedge (X \cup Y) \subseteq (X \cup Y) \cup Z \wedge Z \subseteq (X \cup Y) \cup Z$

... (0.6), [0.4]: $\Leftrightarrow X \subseteq X \cup Y \wedge (X \cup Y) \subseteq (X \cup Y) \cup Z \wedge Y \subseteq X \cup Y$
$\wedge (X \cup Y) \subseteq (X \cup Y) \cup Z \wedge Z \subseteq (X \cup Y) \cup Z$

... [tra$\subseteq$],[0.12]: $\Rightarrow X \subseteq (X \cup Y) \cup Z \wedge Y \subseteq (X \cup Y) \cup Z \wedge Z \subseteq (X \cup Y) \cup Z$

... (4.12): $\Leftrightarrow X \subseteq (X \cup Y) \cup Z \wedge Y \cup Z \subseteq (X \cup Y) \cup Z$

... (4.12): $\Leftrightarrow X \cup (Y \cup Z) \subseteq (X \cup Y) \cup Z$.

In a similar way, one can prove

true

$\Rightarrow X \subseteq X \cup (Y \cup Z) \wedge Y \subseteq X \cup (Y \cup Z) \wedge Z \subseteq X \cup (Y \cup Z)$

$\Leftrightarrow$ $(X \cup Y) \cup Z \subseteq X \cup (Y \cup Z)$,

which yields the assertion by [asy $\subseteq$]. *(End of Proof.)*

(4.15) Let E,F:**FinS**. Then E∪F is finite.

Proof:

		true
... (4.13):	$\Leftrightarrow$	$E \cup \emptyset = E$
... E is finite:	$\Rightarrow$	$E \cup \emptyset$ is finite.
Let G:**FinS**; t:S. Then		E∪G is finite
... [4.9]:	$\Rightarrow$	$(E \cup G) \cup \{t\}$ is finite
... (4.14):	$\Leftrightarrow$	$E \cup (G \cup \{t\})$ is finite.

Thus by [0.27], (4.11), $(\forall G \,.\, G{:}\mathbf{FinS} \Rightarrow E \cup G \text{ is finite})$.

Hence E∪F is finite by [0.30]. *(End of Proof.)*

We are now in the position to prove properties of "finitely representable" set mappings. As such, we regard functions f on **PS** with the property $(\forall X \,.\, f(X) = \bigcup\{f(E) \mid E \subseteq X \wedge E \text{ is finite}\})$. This is analogous to the assertion $(\forall X \,.\, r(X) = \bigcup\{ r(\{t\}) \mid \{t\} \subseteq X \})$ for disjunctive set mappings r, which holds by (3.14)(ii), and (a) in the proof of (3.14). On the right hand side of both equations only specific objects of **PS** appear as arguments of the set mapping, namely finite sets and singletons, respectively. The equations assert that these arguments suffice to specify the set mapping. The result of applying the set mapping to any set X is given by the union of the values of the "specified" arguments contained in X. The union is used (and not the intersection, for instance), because the set mapping should be monotonic.

(4.16) Let $f{:}\mathbf{PS} \to \mathbf{PS}$ and $(\forall X \,.\, f(X) = \bigcup\{f(E) \mid E \subseteq X \wedge E \text{ is finite}\})$.

Then f is a monotonic function on **PS**.

Proof: Let X,Y:**PS**; X⊆Y. Then

		$s \in f(X)$
	$\Leftrightarrow$	$s \in \bigcup\{f(E) \mid E \subseteq X \wedge E \text{ is finite}\}$
... (3.12):	$\Leftrightarrow$	$(\exists E \,.\, E \subseteq X \wedge E \text{ is finite} \wedge s \in f(E))$
... assumption X⊆Y:	$\Leftrightarrow$	$(\exists E \,.\, E \subseteq X \wedge X \subseteq Y \wedge E \text{ is finite} \wedge s \in f(E))$
... [tra ⊆], [0.29]:	$\Rightarrow$	$(\exists E \,.\, E \subseteq Y \wedge E \text{ is finite} \wedge s \in f(E))$
... (3.12):	$\Leftrightarrow$	$s \in \bigcup\{f(E) \mid E \subseteq Y \wedge E \text{ is finite}\}$

$\Leftrightarrow$ $s \in f(Y)$.

Thus $f(X) \subseteq f(Y)$ by [0.27], [1.3]. *(End of Proof.)*

The following two propositions will allow us to prove that the "finitely representable" set mappings are the continuous functions on **PS**.

(4.17) Let X:**PS**. Then $\{E \mid E \subseteq X \wedge E \text{ is finite}\}$ is directed.

Proof:

	true
	$\Leftrightarrow \emptyset \subseteq X \wedge \emptyset \text{ is finite}$
	$\Leftrightarrow \emptyset \in \{E \mid E \subseteq X \wedge E \text{ is finite}\}$
... (0.31):	$\Rightarrow (\exists F \,.\, F \in \{E \mid E \subseteq X \wedge E \text{ is finite}\})$.
Let F,G: **PS**. Then	$F \in \{E \mid E \subseteq X \wedge E \text{ is finite}\} \wedge G \in \{E \mid E \subseteq X \wedge E \text{ is finite}\}$
... (0.42'):	$\Leftrightarrow F \subseteq X \wedge F \text{ is finite} \wedge G \subseteq X \wedge G \text{ is finite}$
... [0.4], [ref $\subseteq$]:	$\Leftrightarrow F \subseteq X \wedge G \subseteq X \wedge F \text{ is finite} \wedge G \text{ is finite} \wedge F \cup G \subseteq F \cup G$
... (4.12), (4.15):	$\Rightarrow F \cup G \subseteq X \wedge F \cup G \text{ is finite} \wedge F \cup G \subseteq F \cup G$
... (0.42'), (4.12):	$\Leftrightarrow F \cup G \in \{E \mid E \subseteq X \wedge E \text{ is finite}\} \wedge F \subseteq F \cup G \wedge G \subseteq F \cup G$
... (0.31):	$\Rightarrow (\exists Z \,.\, Z \in \{E \mid E \subseteq X \wedge E \text{ is finite}\} \wedge F \subseteq Z \wedge G \subseteq Z)$.

Thus $\{E \mid E \subseteq X \wedge E \text{ is finite}\}$ is directed by [4.0]. *(End of Proof.)*

(4.18) Let D:**P**(**PS**) and D be directed. Then

$(\forall E \,.\, E{:}\mathbf{FinS} \Rightarrow (E \subseteq \cup D \Rightarrow (\exists X \,.\, X \in D \wedge E \subseteq X)))$.

Proof:

	$\emptyset \subseteq \cup D$
... (1.6):	$\Leftrightarrow$ true
... [4.0]:	$\Rightarrow (\exists X \,.\, X \in D)$
... (1.6):	$\Leftrightarrow (\exists X \,.\, X \in D \wedge \emptyset \subseteq X)$.

Let E:**FinS**; t:S. Let $E \subseteq \cup D \Rightarrow (\exists X \,.\, X \in D \wedge E \subseteq X)$.

Then	$E \cup \{t\} \subseteq \cup D$
... (4.12):	$\Leftrightarrow E \subseteq \cup D \wedge \{t\} \subseteq \cup D$
... (1.7):	$\Leftrightarrow E \subseteq \cup D \wedge t \in \cup D$
... assumption, [0.26]:	$\Rightarrow (\exists X \,.\, X \in D \wedge E \subseteq X) \wedge (\exists Y \,.\, Y \in D \wedge t \in Y)$
... [0.4], [0.22], etc. :	$\Leftrightarrow (\exists X, Y \,.\, X \in D \wedge E \subseteq X \wedge Y \in D \wedge t \in Y)$
... [0.4], (1.7):	$\Leftrightarrow (\exists X, Y \,.\, X \in D \wedge Y \in D \wedge E \subseteq X \wedge \{t\} \subseteq Y)$
... [4.0], [0.29]:	$\Rightarrow (\exists X, Y \,.\, (\exists Z \,.\, Z \in D \wedge X \subseteq Z \wedge Y \subseteq Z) \wedge E \subseteq X \wedge \{t\} \subseteq Y)$

... [0.4], (0.20), [0.4]: $\Leftrightarrow (\exists Z\,.\, Z\in D \wedge (\exists X,Y\,.\, E\subseteq X \wedge X\subseteq Z \wedge \{t\}\subseteq Y \wedge Y\subseteq Z))$

... [tra $\subseteq$], (0.29): $\Rightarrow (\exists Z\,.\, Z\in D \wedge (\exists X,Y\,.\, E\subseteq Z \wedge \{t\}\subseteq Z))$

... [0.33], (0.29): $\Rightarrow (\exists Z\,.\, Z\in D \wedge E\subseteq Z \wedge \{t\}\subseteq Z)$

... (4.12): $\Leftrightarrow (\exists Z\,.\, Z\in D \wedge E\cup\{t\}\subseteq Z)$.

Thus the assertion holds by [0.27], (4.11). *(End of Proof.)*

(4.19) Let $f:\mathbf{PS}\rightarrow\mathbf{PS}$. Then the following are equivalent:

(i) f is a continuous function on **PS**,

(ii) $(\forall X\,.\, f(X) = \cup\{f(E)\mid E\subseteq X \wedge E \text{ is finite}\})$.

Proof: Let (i) hold.

Let $X:\mathbf{PS}$. Then for $t:S$, $t\in X$

... (1.6), (1.7), [ref $\subseteq$]: $\Leftrightarrow \varnothing\subseteq X \wedge \{t\}\subseteq X \wedge \varnothing\cup\{t\}\subseteq\varnothing\cup\{t\}$

... (4.12): $\Leftrightarrow \varnothing\cup\{t\}\subseteq X \wedge \varnothing\subseteq\varnothing\cup\{t\} \wedge \{t\}\subseteq\varnothing\cup\{t\}$

... (0.5'), [4.9]: $\Rightarrow \varnothing\cup\{t\}\subseteq X \wedge \varnothing\cup\{t\}$ is finite $\wedge \{t\}\subseteq\varnothing\cup\{t\}$

... (1.7): $\Leftrightarrow \varnothing\cup\{t\}\in\{E\mid E\subseteq X \wedge E \text{ is finite}\} \wedge t\in\varnothing\cup\{t\}$

... (0.31): $\Rightarrow (\exists F\,.\, F\in\{E\mid E\subseteq X \wedge E \text{ is finite}\} \wedge t\in F)$

... [1.19]: $\Leftrightarrow t\in\cup\{E\mid E\subseteq X \wedge E \text{ is finite}\}$.

Thus $X\subseteq\cup\{E\mid E\subseteq X \wedge E \text{ is finite}\}$ by [0.27], [1.99].

Furthermore, true

... (0.32), [0.5]: $\Rightarrow (\forall E\,.\, E\subseteq X \wedge E \text{ is finite} \Rightarrow E\subseteq X)$

... (0.43): $\Leftrightarrow (\forall F\,.\, f\in\{E\mid E\subseteq X \wedge E \text{ is finite}\} \Rightarrow F\subseteq X)$

... (1.20): $\Leftrightarrow \cup\{E\mid E\subseteq X \wedge E \text{ is finite}\}\subseteq X$.

Hence true

... [asy $\subseteq$]: $\Leftrightarrow X = \cup\{E\mid E\subseteq X \wedge E \text{ is finite}\}$

... [0.41]: $\Rightarrow f(X) = f(\cup\{E\mid E\subseteq X \wedge E \text{ is finite}\})$

... (4.17), (i): $\Leftrightarrow f(X) = \cup\{f(F)\mid F\in\{E\mid E\subseteq X \wedge E \text{ is finite}\}\}$

... (3.2): $\Leftrightarrow f(X) = \cup\{f(E)\mid E\subseteq X \wedge E \text{ is finite}\}$.

Thus (ii) holds by [0.27].

Since (i) $\Rightarrow$ (ii), it suffices by [0.13] to prove (ii) $\Rightarrow$ (i).

Let (ii) hold. Let $D:\mathbf{P}(\mathbf{PS})$ and D be directed.

Then by (4.16) and (3.4), $\cup\{f(X)\mid X\in D\}\subseteq f(\cup D)$. (a)

Let s:S. Then $s \in f(\bigcup D)$

... (ii): $\Leftrightarrow\ s \in \bigcup\{E \mid E \subseteq \bigcup D \wedge E \text{ is finite}\}$

... [1.19], [0.4]: $\Leftrightarrow\ (\exists E\,.\, E \text{ is finite} \wedge E \subseteq \bigcup D \wedge s \in f(E))$

... (4.18), (0.29): $\Rightarrow\ (\exists E\,.\,(\exists X\,.\, X \in D \wedge E \subseteq X) \wedge s \in f(E))$

... [0.4], (0.20): $\Leftrightarrow\ (\exists X\,.\, X \in D \wedge (\exists E\,.\, s \in f(E) \wedge E \subseteq X))$

... (4.16), [0.12], (0.29): $\Rightarrow\ (\exists X\,.\, X \in D \wedge (\exists E\,.\, s \in f(E) \wedge f(E) \subseteq f(X)))$

... [tra $\subseteq$], [0.33], (0.29): $\Rightarrow\ (\exists X\,.\, X \in D \wedge s \in f(X))$

... (3.12): $\Leftrightarrow\ s \in \bigcup \{f(X) \mid X \in D\}$.

Thus by [0.27], [1.3], (a) and [asy $\subseteq$], $f(\bigcup D) = \bigcup \{f(X) \mid X \in D\}$.

This proves f is continuous according to [4.1]. *(End of Proof.)*

This concludes our presentation of continuous functions. We briefly point to some literature for further reading. Theorems (2.20), (4.5) and (4.6) assert that closure spaces in **PS** that are "stable under unions of directed sets" correspond exactly to continuous closure operations on **PS**. These in turn are "finitely representable" by (4.19), and sometimes also called "algebraic" closure operations (cf. *Cohn* [Co, p.45]). The properties of the corresponding closure space can be expressed in terms of general partial orders, namely as those of a so-called "algebraic lattice" (cf. *Burris/Sankappanavar* [BS, p.17,19]). The set of *subalgebras* of a given algebra is a very general example of a closure space corresponding to an "algebraic" closure operation. This example includes subgroups of a group, subspaces of a vector space, or convex sets of geometric points. Subalgebras are in the intuitive sense "closed" under the operations of the algebra. These operations are for instance the group operations; in the case of convex sets, there is one binary operation for each real number α between 0 and 1 for the "convex combination" of two geometrical points x and y, usually denoted by $\alpha x + (1-\alpha)y$. Refer to [BS, §§ II.1-3] for details on algebras. The so-called "multi-relations", introduced in *Scott* [Sc80] as part of a set-theoretic model of lambda-calculus, are also examples of continuous set mappings.

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