Game Theory Explorer – Software for the Applied Game Theorist

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Abstract

This paper presents the "Game Theory Explorer" software tool to create and analyze games as models of strategic interaction. A game in extensive or strategic form is created and nicely displayed with a graphical user interface in a web browser. State-of-the-art algorithms then compute all Nash equilibria of the game after a mouseclick. In tutorial fashion, we present how the program is used, and the ideas behind its main algorithms. We report on experiences with the architecture of the software and its development as an open-source project.

Keywords Game theory, Nash equilibrium, scientific software

1 Introduction

Game theory provides mathematical concepts and tools for modeling and analyzing interactive scenarios. In *noncooperative* game theory, the possible actions of the players are represented explicitly, together with payoffs that the players want to maximize for themselves. Basic models are the *extensive* form represented by a game tree with possible imperfect information represented by information sets, and the *strategic* (or "normal") form that lists the players' strategies, which they choose independently, together with a table of the players' payoffs for each strategy profile.

The central concept for noncooperative games is the *Nash equilibrium* which prescribes a strategy for each player that is optimal when the other players keep their prescribed strategies fixed. Every finite game has an equilibrium when players are allowed to *mix* (randomize) their actions (Nash, 1951). A game may have more than one Nash equilibrium. Finding one or all Nash equilibria of a game is often a laborious task.

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In this paper, we describe the *Game Theory Explorer* (GTE) software that allows to create games in extensive or strategic form, and to compute their Nash equilibria. GTE is primarily intended for the applied game theorist who is not an expert in equilibrium computation, for example an experimental economist who designs experiments to test if subjects play equilibrium strategies. The user can easily vary the parameters of the game-theoretic model, and study more complex games, because their equilibrium analysis is quickly provided by the algorithm. The analysis of a game with general mathematical parameters is also aided by knowing its equilibria for specific numerical values of those parameters. As our exposition will demonstrate, GTE can also be used for more theoretical research in game theory, for example on strategic stability (see Section 4). The ease of creating, displaying, and analyzing games suggests GTE also as an *educational tool* for game theory.

Computing equilibria is a main research topic of the authors, and in later sections we will explain some of our results on finding equilibria of two-player games in strategic form (Avis et al., 2010) and extensive form based on the "sequence form" (von Stengel, 1996). Scientific algorithms are often implemented as prototopes to show that they work, as done, for example, by Audet et al. (2001), or von Stengel, van den Elzen, and Talman (2002). However, providing a robust user interface to create games is much more involved, and necessary to make such algorithms useful for a wider research community. In particular, the drawing of game trees should be done with a friendly *graphical user interface* (GUI) where the game tree can be created and seen on the screen, and be stored, retrieved, and changed in an intuitive manner. This is one of the purposes of the GTE software presented in this article.

An existing suite of software for game-theoretic analysis is *Gambit* (McKelvey, McLennan, and Turocy, 2010). Gambit has been developed over the course of nearly 25 years and presents a library of solution algorithms, formats for storing games, ways to program the creation of games with the help of the Python programming language, and a GUI for creating game trees. It is open-source software that is free to use and that can be extended by anyone. Given the mature state of Gambit and the joint research interests and close contacts with its developers, it is clear that any improvements offered by GTE should eventually be integrated into Gambit.

Other existing game solvers are GamePlan (Langlois, 2006) and XGame (Belhaiza, Mve, and Audet, 2010).

The main difference of GTE to Gambit is the provided access to the software and the user interface. In terms of access, Gambit needs to be downloaded and installed; it is offered on the main personal computing platforms Windows, Linux or Mac. Getting the program to run may require some patience and technical experience with software installions, which may present a "barrier to entry" for its use. In contrast, GTE is started in a *web browser* via the web address `http://www.gametheoryexplorer.org`. All interaction with the software is via the browser interface. The created games and their output can be saved as files by the user on their local computer. This avoids the technical hurdles of installing software on the user side, and simplifies updating the software.

The graphical display of game trees in GTE is user-friendly and can be customized, such as growing the tree in the vertical or horizontal direction. GTE can even be used just as a drawing tool for games, which can be exported as pictures to file formats for use in papers or presentations.

Providing application software via the web has the following disadvantages compared to installed software. First, a higher complexity of the program for the required communication over the internet; however, manifold standard solutions are freely available. Second, limited control over the user's local computing resources for security reasons. This is an issue because equilibrium computation for larger games is computationally intensive. For that reason, this computation takes place on the central public server rather than the user's client computer; we will explain the technical issues in Section 7. Third, in our case, currently very limited use of the existing Gambit software. We envisage a loosely coupled integration, in particular with Gambit's game solvers. GTE is very much under development in this respect.

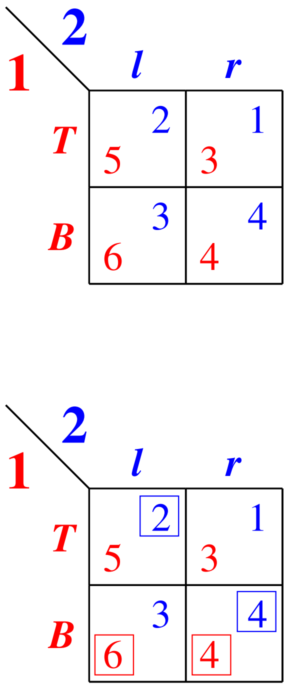
We describe GTE, first from the perspective of the user, with an example in Section 2, and the general creation of extensive and strategic-form games in Sections 3 and 4. We explain the main algorithms for finding all equilibria for games in strategic form in Section 5, including issues for handling larger games where one has to restrict oneself to finding sample equilibria (which, in particular, does not allow to decide if the game has a unique equilibrium). For extensive games, the computation of behavior strategies, which are exponentially less complex than mixed strategies, is outlined in Section 6. The software architecture and the communication between server and client computers is discussed, with as little technical jargon as possible, in Section 7. In conclusion, we mention the difficulties of incentives and funding for implementing user-friendly scientific software, and call for contributions of volunteers.

2 Example of using GTE

In this section we describe a simple 2×2 game, due to Bagwell (1995), together with its "commitment" or "Stackelberg" variant which has more strategies. This game is created and analyzed very simply with GTE, where we demonstrate the computed equilibria. Our detailed analysis may also serve as a tutorial introduction to the basics of noncooperative game theory. For a textbook on game theory see Osborne (2004).

A basic model of a noncooperative game is a game in *strategic form*. Each player has a finite list of strategies. Players choose simultaneously a strategy each, which defines a strategy *profile*, with a given *payoff* to each player. The strategic form is the table of payoffs for all strategy profiles. For two players, the strategies of player 1 are the m rows and those of player 2 the n columns of a table, which in each cell has a payoff pair. The two $m \times n$ matrices of payoffs to player 1 and 2 define such a two-player game in strategic form, which is also called a bimatrix game.

Fig. 1 shows a 2×2 game with the payoff to player 1 shown in the bottom left corner and the payoff to player 2 in the top right corner of each cell. In GTE, such a game is entered by giving the two payoff matrices (Fig. 13 shows the input screen for a larger example), with a graphical display as shown at the top left of Fig. 1. At the bottom left the same table is shown with a box around each payoff that is maximal against the respective strategy of the other player (these boxes are currently not part of GTE output). This shows that the bottom strategy B of player 1 is the only *best*




```
Strategic form:
2 x 2 Payoff player 1

  l r
T 5 3
B 6 4

2 x 2 Payoff player 2

  l r
T 2 1
B 3 4

EE = Extreme Equilibrium, EP = Expected Payoffs

Rational:
EE 1 P1: (1) 0 1 EP= 4 P2: (1) 0 1 EP= 4

Decimal:
EE 1 P1: (1) 0 1.0 EP= 4.0 P2: (1) 0 1.0 EP= 4.0

Connected component 1:
{1}  x  {1}
```

Figure 1 Top left: Graphical output of a 2×2 bimatrix game. Bottom left: Best response payoffs surrounded by boxes (currently not part of GTE output). Right: Output of the computed equilibria of this game.

response (payoff-maximizing strategy) against both columns l and r of player 2. For player 2, the best responses are l against T and r against B.

In this game, strategy T is strictly dominated by B and can therefore be disregarded because it will never be played by player 1. The best response r against B then defines the strategy profile (B, r) as the unique Nash equilibrium of the game, that is, a pair of mutual best responses. The text output at the right of Fig. 1, which pops up in a window after clicking a button that starts the equilibrium computation, shows the two payoff matrices and the Nash equilibria. The equilibrium strategies are shown as vectors of probabilities, both in rational output as exact fractions of integers and in decimal. In this case there is only one equilibrium, which is the pair of vectors $(0,1)$ and $(0,1)$ that describe the probabilities that player 1 plays his pure strategies T and B and player 2 her strategies l and r, respectively. This pair of strategies also defines the unique equilibrium component shown as `{1} x {1}`, an output which is of more interest in the example of Fig. 3 below.

The game in Fig. 1 can also be represented in *extensive form*, as shown in Fig. 2 which is automatically generated by GTE. An extensive game is a game tree with nodes as game states and outgoing edges as moves chosen by the player to move at that node. *Information sets*, due to Kuhn (1953), describe a player's lack of information about the current game state and have the same outgoing moves at each node. Here, player 2 is not informed about the move of player 1, in accordance with the players' simultaneous choice of moves in the strategic form.

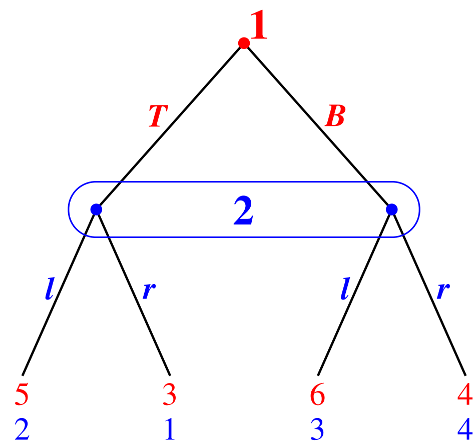


Figure 2 The game in Fig. 1 as an extensive game with an information set for player 2 who is uninformed about the move of player 1.

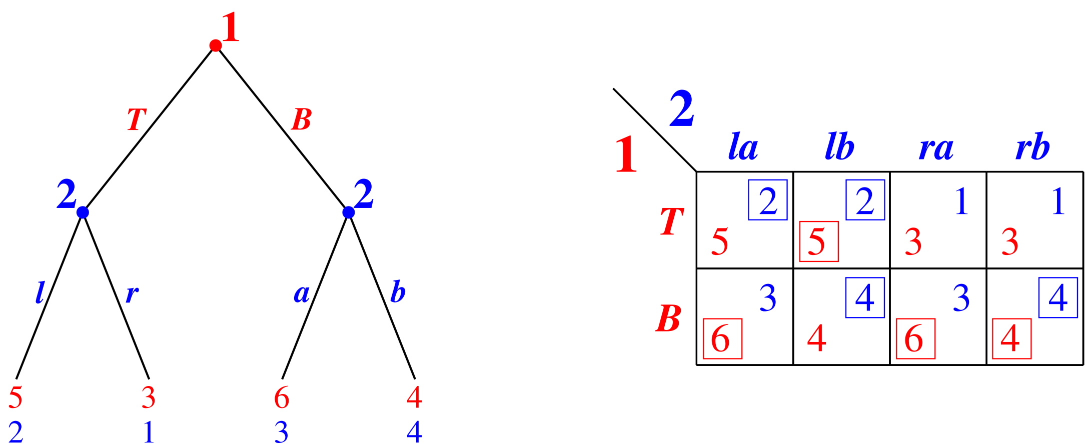


Figure 3 Left: Commitment version game of the game in Fig.s 1 and 2 where player 2 is informed about the first move of player 1. Right: strategic form of this game as generated by GTE (except for the boxes around the best-response payoffs).

The game in Fig. 1 is an example due to Bagwell (1995). It is a simple version of a "Cournot" game. In the corresponding "Stackelberg" or *commitment* game, player 1 is a leader who commits to his strategy, about which player 2, as a follower, is informed. The game tree, and thus the game, is changed by becoming a game of perfect information where each information set is a singleton.[1] To change the game in this way the information set is dissolved, which is a simple operation in GTE. The new game tree is shown on the left in Fig. 3. Then the moves of player 2, who can react to the choice of player 1, get new names, here at the right node *a* and *b* instead of the original moves *l* and *r* at the original information set that remain the moves at the left node of player 2.

A game tree with perfect information can be solved by "backward induction" which defines a *subgame perfect equilibrium* or SPE, which is indeed a Nash equilibrium. Here, player 2's optimal moves are *l* and *b*, which defines her strategy *lb*.

[1]GTE does not show singleton information sets as ovals that contain a single node, only information sets with two or more nodes.

Given these choices of player 2, the optimal move of player 1 is T. This defines the SPE (T, lb). In general, a strategy in an extensive game specifies a move at each information set of the player, so player 2 has the four strategies la, lb, ra, rb listed as columns in the strategic form on the right in Fig. 3, which is generated by GTE. Because player 1 has only one information set given by the singleton that contains the root node of the tree, his strategies are just the moves T and L. The SPE (T, lb) is one of the cells in the strategic form with the two best-response payoffs 5 and 2 for the two players.

In a game tree with perfect information and different payoffs at each terminal node, backward induction defines a unique SPE. However, the game has in general additional Nash equilibria that are not subgame perfect. In this example, the strategy pair (B, rb) is also an equilibrium, which can be seen from the strategic form. Here player 2 chooses the right move r and b at both information sets, and the best response of player 1 is then B, with payoffs 4 and 4 to the two players. This is an equilibrium because neither player can unilaterally improve his or her payoff, under the crucial assumption of equilibrium that the strategy of the other player stays fixed: When player 1 chooses T instead of B and player 2 plays rb, then player 1 receives a payoff of 3 rather than 4 and therefore prefers to stay with B. In turn, b is an optimal move when player 1 chooses B. Player 2 cannot improve her payoff by changing from r to l because that part of the game tree is not reached due to the move B by player 1. This equilibrium is in effect the equilibrium (B, r) in the original simultaneous game in Fig. 1 translated to the commitment game where player 2, even though she can now react to the move of player 1, always chooses the equivalent of the original move r. However, this equilibrium is not subgame perfect because it prescribes the suboptimal move r in the subgame that starts with the left node of player 2.

In addition to these two pure-strategy equilibria, the game in Fig. 3 has additional equilibria where player 2 makes a random choice at the node that is unreached due to the move of player 1. Because the node is unreached, any choice of player 2 is optimal because it has no effect on her payoffs, but that random choice must not change the preference of player 1 for his move in order to keep the equilibrium property. Move B of player 1, followed necessarily by move b of player 2, is optimal for player 1 as long as his expected payoff for T is at most 4, which is the case whenever player 2 makes move r with probability at least $1/2$. The two extreme cases are the pure strategy equilibrium (B, rb) already discussed and the mixed strategy equilibrium where player 1 chooses B and player 2 mixes between lb and rb with probability $1/2$ each. This is represented by the probability vector $(0, 1/2, 0, 1/2)$ for the four strategies of player 2. Similarly, an equilibrium that has the same outcome with payoffs 5 and 2 as the SPE but a suboptimal random choice between a and b of player 2 is $(T, (1/2, 1/2, 0, 0))$ where player mixes between la and lb with probability $1/2$ each; $1/2$ is the largest probability that player 2 can assign to a so that T stays a best response of player 1.

The preceding analysis is straightforward and simple, but a complete list of all equilibria is nevertheless of interest. This is provided by GTE with the following output:

```
Strategic form:
```

```
2 x 4 Payoff player 1

  la lb ra rb
T  5  5  3  3
B  6  4  6  4

2 x 4 Payoff player 2

  la lb ra rb
T  2  2  1  1
B  3  4  3  4

EE = Extreme Equilibrium, EP = Expected Payoffs

Rational:
EE 1 P1: (1) 0 1 EP= 4 P2: (1)   0 1/2 0 1/2 EP= 4
EE 2 P1: (1) 0 1 EP= 4 P2: (2)   0   0 0   1 EP= 4
EE 3 P1: (2) 1 0 EP= 5 P2: (3)   0   1 0   0 EP= 2
EE 4 P1: (2) 1 0 EP= 5 P2: (4) 1/2 1/2 0   0 EP= 2

Decimal:
EE 1 P1: (1)   0 1.0 EP= 4.0 P2: (1)   0 0.5 0 0.5 EP= 4.0
EE 2 P1: (1)   0 1.0 EP= 4.0 P2: (2)   0   0 0 1.0 EP= 4.0
EE 3 P1: (2) 1.0   0 EP= 5.0 P2: (3)   0 1.0 0   0 EP= 2.0
EE 4 P1: (2) 1.0   0 EP= 5.0 P2: (4) 0.5 0.5 0   0 EP= 2.0

Connected component 1:
{1}  x  {1, 2}

Connected component 2:
{2}  x  {3, 4}
```

This output gives the four "extreme" equilibria described above, in both rational and decimal description. Each equilibrium strategy of a player is preceded by an identifying number in parentheses such as `(1)` and `(2)` for the two strategies of player 1, each of which appears in two equilibria. All four equilibrium strategies of player 2 are distinct, marked with `(1)` to `(4)`. The connected components listed at the end of the output show how these extreme equilibria can be arbitrarily combined: The first connected component `{1} x {1, 2}` says that strategy `(1)` of player 1, which is $(0,1)$ (the pure strategy B), together with *any convex combination* of strategies `(1)` and `(2)` of player 2, which are the strategies $(0,1/2,0,1/2)$ and $(0,0,0,1)$ (the latter being the pure strategy rb), defines an equilibrium. This and the other connected component `{2} x {3, 4}` describe the full set of Nash equilibria of the game.

In principle, GTE provides such a complete description of all equilibria for any two-player game, except that the computation time, which is in general exponential in the size of the game, may be prohibitively long for larger games.

3 Creating and analyzing extensive form games

In this section, we describe the construction in GTE of an extensive game which corresponds to the game of the previous section but where the first player's commitment is imperfectly observed, as studied by Bagwell (2005). The game is created in five stages: Drawing the raw tree, assigning players, combining nodes into information sets, defining moves and chance probabilities, and setting payoffs. Graphical and other more permanent settings, such as the orientation of the game tree, can also be changed. After explaining the GUI operations, we consider the interesting equilibrium structure of the constructed game.

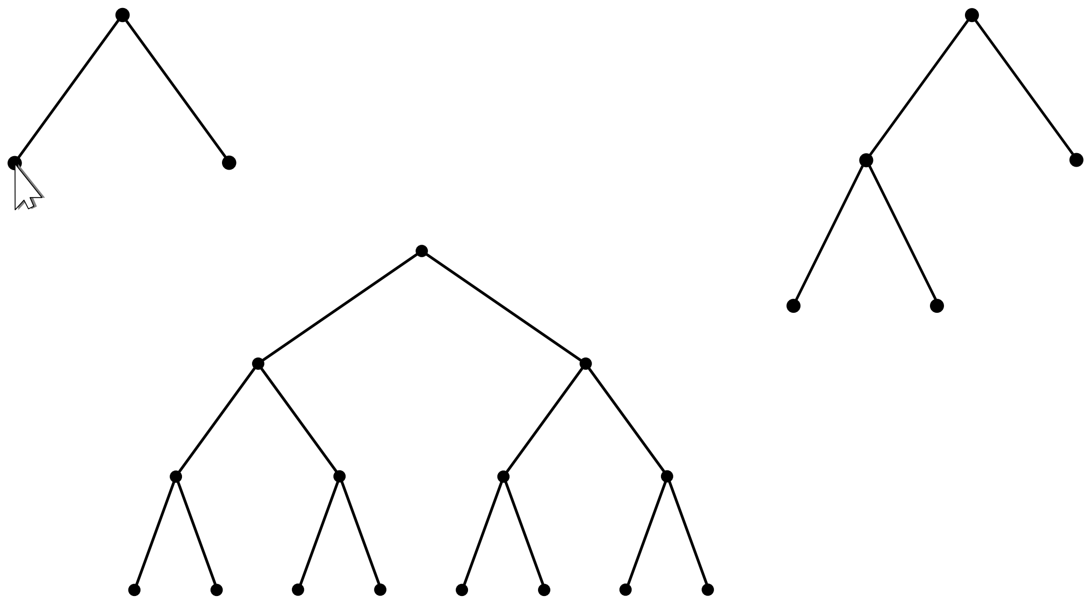


Figure 4 Constructing a tree in stages by clicking on nodes, as shown with the starting tree at the top left, second stage at the top right, and final stage at the bottom. In this tree, every nonterminal node has two children, but in general it may have any number of children.

The first stage of creating a new extensive game defines the tree structure, beginning from a simple tree that has a root node with two children, as shown in Fig. 4. Clicking on a leaf (terminal node) creates two children for that node, and clicking on a nonterminal node creates an additional child.

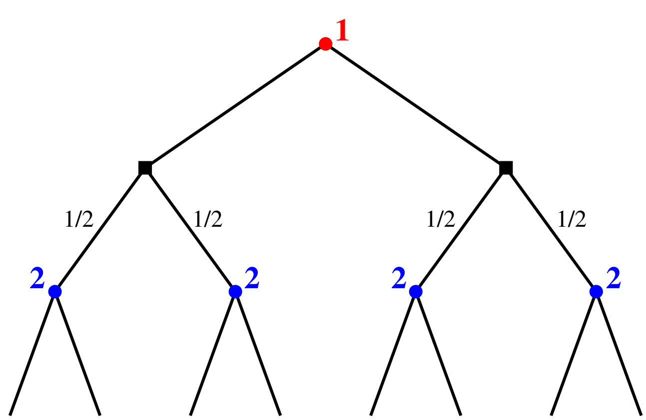


Figure 5 Choosing the player to move for each node. The square nodes are chance nodes, with initially uniform probabilities for the outgoing edges.

The next stage is to select players, by selecting a "player assignment" button for each player, where the name of the player can also be changed, for example to "Alice" and "Bob" instead of the defaults "1" and "2" (which we have not done). Clicking on a nonterminal node then asigns the player (originally unassigned with a black node), for example player 2 for the four nonterminal nodes closest to the leaves in Fig. 5. Assigning a chance node, represented by a black square, defines per default uniform probabilities on the outgoing edges, which can be changed. All nonterminal nodes have to be assigned a player before the next stage can be reached. It is possible to go back to a previous stage at any time, here to alter the tree structure by adding or deleting nodes.

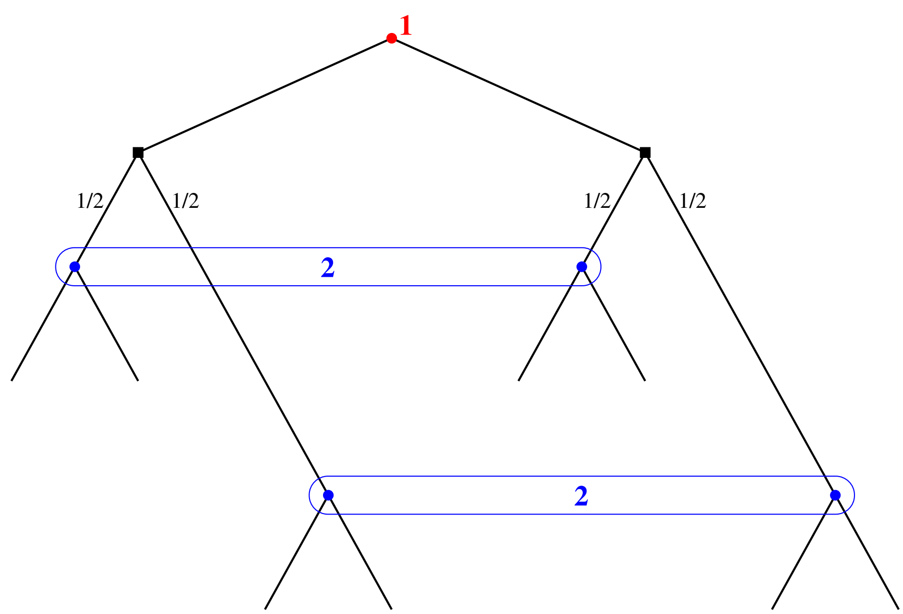


Figure 6 Creating information sets, which are automatically drawn so as to minimize the number of crossing edges.

The next stage is to create information sets by clicking on two nodes (or information sets) of the same player, which are then merged into a single information set. It is necessary that the respective nodes have the same number of children because these will be the moves at the newly created information set. When an information set is created, the program adjusts, as far as possible, the levels of nodes in the tree so that all nodes in the information set are at the same level and the information set appears horizontally (if the game tree grows in the vertical direction, otherwise vertically as in Fig. 11). In addition, crossings between edges and information sets are minimized. Fig. 6 shows the resulting game tree, which at this stage has its final shape, except for the definition of moves and payoffs.

Figure 7 Browser headline bar that guides through the stages of creating a game tree, here indicating the "information set" creation stage.

Fig. 7 shows a “headline bar” that indicates the current stage of the creation of the game tree, which defines either the tree, the players, the information sets, the moves, or the payoffs. The location of the headline bar relative to the entire browser window can be seen in Fig. 11. In Fig. 7, the current stage is that of creating information sets. Underneath the stage indicator are buttons that define the *mode* of operation for the computer mouse. The default mode at the “information set” stage is that of merging two nodes (more generally the information set that they are currently contained in) into a larger information set. The two other mode buttons at this stage are to dissolve an information set into singletons, or (indicated by the scissors) to cut it into two smaller sets.

The default mode for the earlier “tree” stage is to add children to a node (indicated by the ⊕ sign). The alternative mode button (with the ⊖ sign) is to delete a node and all its descendants. At the “players” stage, the mode buttons correspond to the player (including chance) to be assigned to the node when clicking on the node.

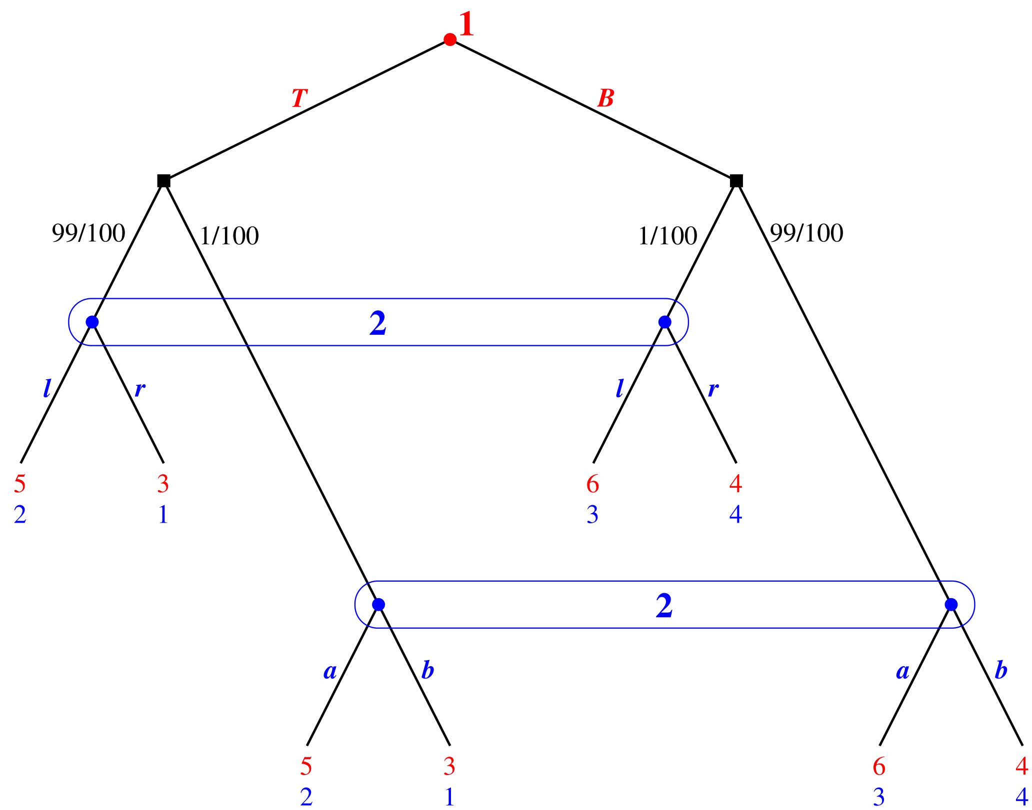


Figure 8 The final game tree after adding moves, modified chance probabilities, and payoffs.

Fig. 8 shows the game tree after the completed stages of assigning moves (and chance probabilities) and payoffs, which work as follows. When the “moves” stage is chosen in the headline bar, all outgoing edges at an information set get unique pre-assigned names from a list by traversing the tree in “breadth-first” manner. Per default these are the upper-case letters in alphabetical order for player 1 and the lower-case letters for player 2. There are two ways to change these default move names: First, by clicking on a move label, which allows to enter an alternative name via the keyboard;

this is not restricted to a single letter and need not be unique. On the left in Fig. 9, this is shown for the chance probability of the right chance move which is changed from $1/2$ to 0.99 (which could also be entered as $99/100$). The remaining probability for the other chance move is automatically set to 0.01 so that the probabilities sum to one. A second, quick way to change all move names is the mode button below the "players" stage in the headline bar where all move names for a player are entered at once, or altered from the displayed current list; this is shown in the right picture of Fig. 9 for player 2 whose default moves a, b, c, d are changed to l, r, a, b.

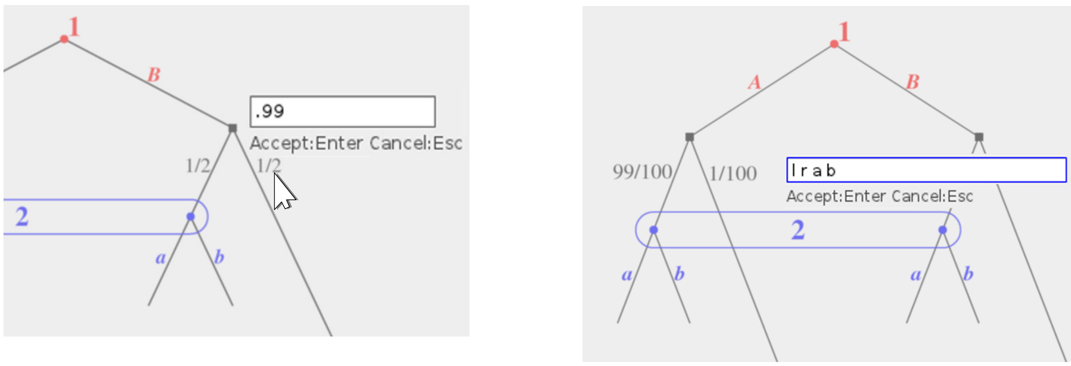


Figure 9 Changing move names, either by clicking on a single move name or chance probability (left), or by changing all moves of a player at once (right).

At the final "payoffs" stage, the leaves of the tree get payoffs to the two players, which are at first consecutively given as $0, 1, 2, \ldots$ in order to have a unique payoff to identify each leaf. As shown in Fig. 10, a player's payoffs can then be changed at once by replacing them with the intended payoffs for the game, where the numbering helps to identify the leaves. Payoffs can also be changed individually. In addition, payoffs can be generated randomly. A zero-sum option can be chosen which automatically sets the other player's payoff to the negative of the payoff to the current player.

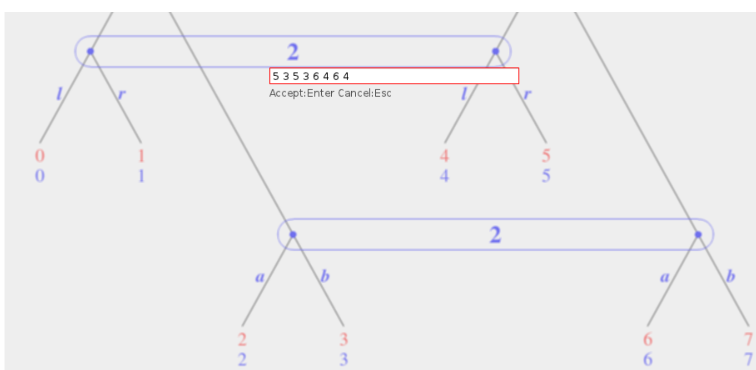


Figure 10 Changing all payoffs of player 1 from the pre-set values 0 1 2 3 4 5 6 7 (for easy identification) to their intended values.

The game tree is stored as its logical structure. Its graphical layout is generated automatically. Its parameters can be changed, such as the orientation of the game tree, which can grow vertically or horizontally in either direction, the default being top-down. Fig. 11 shows a change of settings so that the game tree grows from left to right. Other parameters such as the colors used for the players, line thickness, fonts, and other dimensions can also be changed.

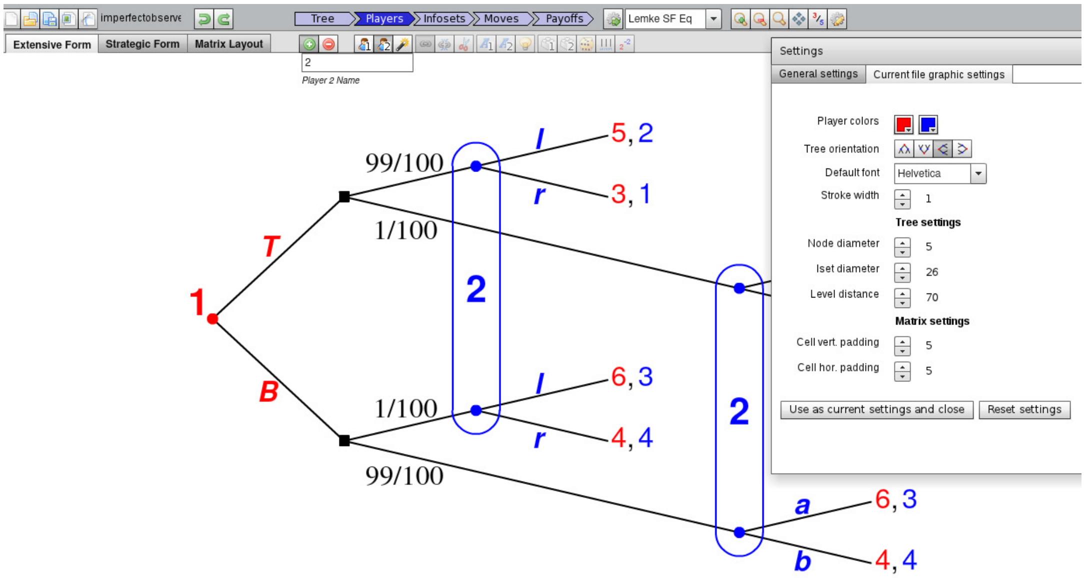


Figure 11 Changing the graphics settings, in this case to a left-to-right tree orientation and a Helvetica font.

Fig. 11 also shows the general layout of the graphical interface in the web browser. The top left offers file manipulation functions such as starting a new game, storing and loading the current game (together with its settings), and exporting it to a picture format (.png) and a scalable graphics format (.fig) that can be further manipulated with the `xfig` drawing program and converted to .pdf or .eps files for inclusion in documents. On the top right various solution algorithms can be selected and started, and at the very right zoom operations and change of settings can be selected.

1 \ 2	*la*	*lb*	*ra*	*rb*
T	2 5	1.99 4.98	1.01 3.02	1 3
B	3 6	3.99 4.02	3.01 5.98	4 4

Figure 12 Strategic form of the game in Fig. 8.

The game in Fig. 8 has an interesting equilibrium structure. It is due to Bagwell (1995) and represents the commitment game shown earlier in Fig. 3, but where the commitment of player 1 is imperfectly observed due to some transmission noise: Namely, with a small probability (here 0.01), the second player observes the commitment incorrectly as the opposite move, which is represented by the chance moves and the two information sets of player 2. The resulting strategic form is shown in Fig. 12. Finding all equilibria with GTE gives the following output, with three equilibria that are isolated, each in a separate component.

```
Strategic form:
2 x 4 Payoff player 1

  la     lb     ra rb
T  5 249/50 151/50  3
B  6 201/50 299/50  4

2 x 4 Payoff player 2

  la      lb      ra rb
T  2 199/100 101/100  1
B  3 399/100 301/100  4

EE = Extreme Equilibrium, EP = Expected Payoffs

Rational:
EE 1 P1: (1)  1/100 99/100 EP= 393/98 P2: (1)     0 25/49 0 24/49 EP= 397/100
EE 2 P1: (2)      0      1 EP=      4 P2: (2)     0     0 0     1 EP=       4
EE 3 P1: (3) 99/100  1/100 EP= 489/98 P2: (3) 24/49 25/49 0     0 EP= 201/100

Decimal:
EE 1 P1: (1) 0.01 0.99 EP= 4.0102 P2: (1)      0 0.5102 0 0.4898 EP= 3.97
EE 2 P1: (2)    0  1.0 EP=    4.0 P2: (2)      0      0 0    1.0 EP=  4.0
EE 3 P1: (3) 0.99 0.01 EP= 4.9898 P2: (3) 0.4898 0.5102 0      0 EP= 2.01

Connected component 1:
{1}  x  {1}

Connected component 2:
{2}  x  {2}

Connected component 3:
{3}  x  {3}
```

In particular, the original SPE (T, lb) of the commitment game with perfect observation and payoffs 5 and 2 has disappeared. Because the game does not have any nontrivial subgames (which are subtrees where each player knows that they are in the subgame), the concepts of subgame perfection and backward induction no longer apply in the game with imperfectly observable commitment in Fig. 8. The reason why (T, lb) is no longer an equilibrium is the following. Because player 1 commits to move T with certainty, player 2 should choose move l when seeing T, because this gives her a higher payoff than r. However, when player 2 sees move B of player 1, it must be due to an error in the observation and player 2 would therefore also choose a rather than b; in short, la and not lb is a best response to T. However, (T, la) is not an equilibrium either because against lb player 1 would choose B; in a sense, player 1 would exploit the fact that player 2 interprets B as an erroneous signal. This, however, seems to imply that player 1 has lost his commitment power due to the noise in the observation.

A Bagwell (1995) pointed out, and as is shown in the above list of equilibria, this loss of commitment power only applies when the players are restricted to use pure strategies. There is in fact a mixed equilibrium, listed as the component `{3} x {3}`, which has payoffs $489/98$ and $201/100$ to the two players that are close to the "Stackelberg" payoffs 5 and 2 when no noise is present. Here player 1 himself adds a small amount of noise to the commitment and plays T and B with probabilities 0.99 and 0.01. In turn, player 2 mixes between la and lb with probabilities $24/49$ and $25/49$. That is, player 2 chooses l with certainty and is indifferent between a and b because when she sees move B, this signal may equally likely be received due to the chance move or due to player 1's randomization. The mixture between a and b is such that player 1 in turn is indifferent between T and B. The game also has the pure strategy equilibrium (B, rb) as before, listed as `{2} x {2}` with payoffs 4 and 4, and another mixed equilibrium `{1} x {1}` with similar payoffs.

With GTE, the game in Fig. 8 is created in a few minutes, and the equilibria are computed instantly. The game does not allow abstract parameters, such as ε for the error probability as in the analysis by Bagwell (1995), which is here chosen as 0.01. However, as a quick way to test a typical case of this game-theoretic model, GTE is a valuable tool.

4 Examples of analyzing games in strategic form

In this section we give examples of strategic-form games analyzed with GTE. These also demonstrate the use of GTE as a research tool for game theory, for questions on the possible number of equilibria, or the description of equilibrium components in the context of strategic stability.

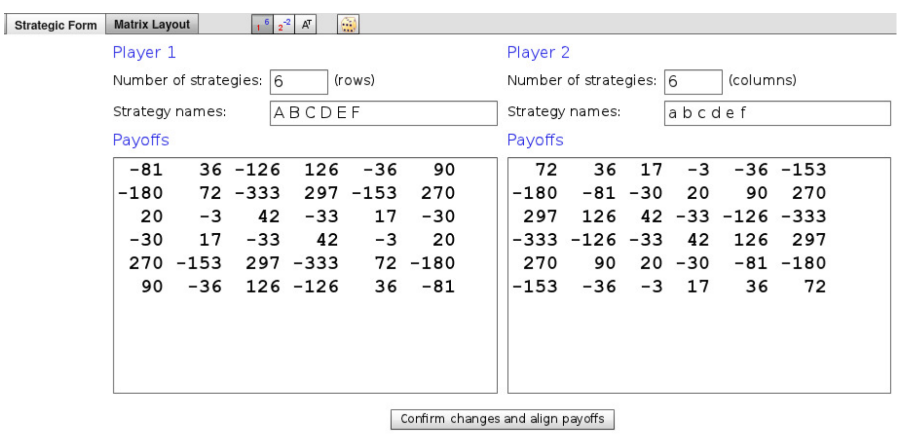


Figure 13 GTE input of a 6×6 game in strategic form.

Fig. 13 shows the input of a game in strategic form, currently implemented for two players only. For each player one needs to specify the number of strategies, which are the rows for player 1 and columns for player 2, and a payoff matrix. Names for the strategies are generated automatically as upper case letters for player 1 and lower case letters for player 2, and can be changed. If the strategic form has been generated from the extensive form, then the strategies are shown as tuples of moves, one for each information set.

One can choose a *zero-sum* input mode where the payoffs to player 2 are automatically the negative of the payoffs to player 1. Similarly, one can input a *symmetric* game where the square payoff matrices to player 1 and 2 are a matrix A and its transpose $A^\top$; the game is symmetric because it does not change when the players are exchanged. In both cases, only the payoff matrix of player 1 is entered.

Payoffs can be entered as integers, fractions, or with a decimal point where the display can be switched between fractions and decimals; internally they are all stored as fractions. The "Align and Update" button aligns the entries in columns and updates the second player's payoffs for zero-sum and symmetric games.

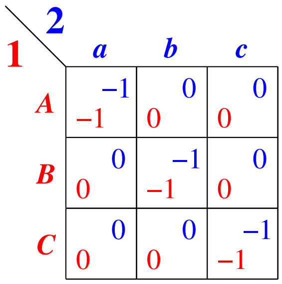
The input of bimatrix games and the computation of their equilibria has the functionality of the popular webpage of Savani (2005) which has been used tens of thousands of times. The extra feature in GTE is the graphical output as in Fig. 12, for example, which is accessed by the "Matrix Layout" tab shown at the top of Fig. 13.

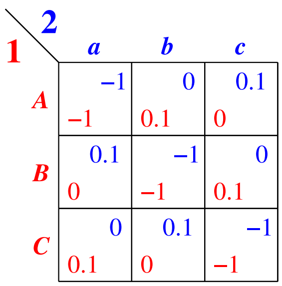
The game in Fig. 13 has 75 equilibria, listed as follows; the output displays each equilibrium in a single line, which we have here broken into two lines, one per player, to fit the page.

```
EE  1 P1:  (1)   1/30    1/6    3/10    3/10    1/6    1/30 EP=     3/2
      P2:  (1)    1/6   1/30    3/10    3/10   1/30     1/6 EP=     3/2
EE  2 P1:  (2)      0      0    1/33    5/33   4/11    5/11 EP=   24/11
      P2:  (2)      0      0    5/33    1/33   5/11    4/11 EP=   24/11
EE  3 P1:  (3)  1/128      0       0    7/64 47/128   33/64 EP=    12/7
      P2:  (3)      0      0    5/21  13/189 59/189    8/21 EP= 297/128
...
EE 74 P1: (74)      0      0   13/32       0      0   19/32 EP= 450/103
      P2: (74) 33/103 70/103       0       0      0       0 EP=  477/16
EE 75 P1: (75)      0      0       0       0      1       0 EP=     270
      P2: (75)      1      0       0       0      0       0 EP=     270

Connected component 1:
{1}  x  {1}
...
Connected component 75:
{75}  x  {75}
```

This game is the smallest known example that refutes a conjecture by Quint and Shubik (1995) that an $n \times n$ game has at most $2^n - 1$ equilibrium components, here for $n = 6$. It has been constructed by von Stengel (1999) using methods from polytope theory, with the specific small integers in Fig. 13 described by Savani and von Stengel (2004, p. 25). Studying games with large numbers of equilibria is obviously greatly aided by computational tools; for another example see von Stengel (2012).

Fig. 14 shows at the top left a symmetric "anti-coordination" game where the only nonzero payoffs are -1 to both players on the diagonal. Because no payoff is positive, any cell with payoff zero to both players is an equilibrium, and these equilibria are connected by line segments to form a "ring" that defines the topologically connected component 2 as in the output shown on the right in Fig. 14. The game also has an isolated completely mixed equilibrium shown as component 1. Interestingly, only

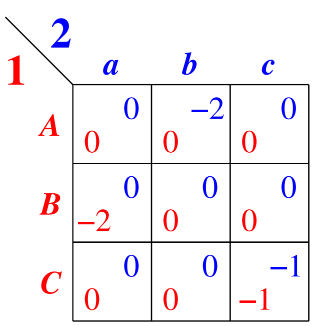





```
EE 1 P1: (1) 1/3 1/3 1/3 EP= -1/3 P2: (1) 1/3 1/3 1/3 EP= -1/3
EE 2 P1: (2)   1   0   0 EP=    0 P2: (2)   0   1   0 EP=    0
EE 3 P1: (2)   1   0   0 EP=    0 P2: (3)   0   0   1 EP=    0
EE 4 P1: (3)   0   1   0 EP=    0 P2: (3)   0   0   1 EP=    0
EE 5 P1: (3)   0   1   0 EP=    0 P2: (4)   1   0   0 EP=    0
EE 6 P1: (4)   0   0   1 EP=    0 P2: (4)   1   0   0 EP=    0
EE 7 P1: (4)   0   0   1 EP=    0 P2: (2)   0   1   0 EP=    0

Connected component 1:
{1}  x  {1}

Connected component 2:
{2, 4}  x  {2}
{3, 4}  x  {4}
{4}  x  {2, 4}
{2, 3}  x  {3}
{3}  x  {3, 4}
{2}  x  {2, 3}

----------------------------------------------------------------

EE 1 P1: (1) 1/3 1/3 1/3 EP= -3/10 P2: (1) 1/3 1/3 1/3 EP= -3/10

Connected component 1:
{1}  x  {1}
```

Figure 14 Top left: 3×3 game which has two equilibrium components, shown on the right. Bottom left: The same game but with *perturbed* payoffs so that only component 1 remains (see bottom right).

component 1 is *strategically stable* in the sense that there is always an equilibrium nearby when the payoffs are slightly perturbed. In the game shown at the bottom left, the payoffs are perturbed so that the only equilibrium that remains is the completely mixed equilibrium in component 1; the perturbations from zero to 0.1 can be changed to independent arbitrarily small positive reals with the same effect.




```
EE 1 P1: (1) 0 1 0 EP= 0 P2: (1) 0 1 0 EP= 0
EE 2 P1: (1) 0 1 0 EP= 0 P2: (2) 0 0 1 EP= 0
EE 3 P1: (2) 1 0 0 EP= 0 P2: (3) 1 0 0 EP= 0
EE 4 P1: (2) 1 0 0 EP= 0 P2: (2) 0 0 1 EP= 0
EE 5 P1: (3) 0 0 1 EP= 0 P2: (3) 1 0 0 EP= 0
EE 6 P1: (3) 0 0 1 EP= 0 P2: (1) 0 1 0 EP= 0

Connected component 1:
{1, 3}  x  {1}
{2, 3}  x  {3}
{3}  x  {1, 3}
{1, 2}  x  {2}
{2}  x  {2, 3}
{1}  x  {1, 2}
```

Figure 15 3×3 game which has only a "ring" of equilibria as its sole stable component, shown on the right.

The concept of strategic stability is due to Kohlberg and Mertens (1986). Fig. 15 shows a game that is also symmetric like that at the top left of Fig. 14 (the symmetry is easy to see due to the staggered payoffs in the lower left and upper right of each cell). It is strategically equivalent (by subtracting constants from columns of the row player's payoffs and from rows of the column player's payoffs) to the game of Kohlberg and Mertens (1986, p. 1034) and has a "ring" consisting of line segments as its only equilibrium component. As Kohlberg and Mertens have shown, any point on that ring can be chosen so that a suitably perturbed game has its only equilibrium near that point, which can be verified by experimenting with numeric perturbations of the game. Correspondingly, the entire ring defines the minimal stable component of the game.

1 \ 2	*a*	*b*	*c*
A	5, 4	−15, 0	−11, 2
B	−23, 0	−1, 8	3, 1
C	−1, 2	−21, −2	1, 3
D	0, 0	0, 0	0, 0

```
EE 1 P1: (1) 0 0 0 1 EP= 0 P2: (1) 31/282 5/141 241/282 EP= 0
EE 2 P1: (1) 0 0 0 1 EP= 0 P2: (2)      0   3/4     1/4 EP= 0
EE 3 P1: (1) 0 0 0 1 EP= 0 P2: (3)      0     1       0 EP= 0
EE 4 P1: (1) 0 0 0 1 EP= 0 P2: (4)    3/4   1/4       0 EP= 0
EE 5 P1: (1) 0 0 0 1 EP= 0 P2: (5)    1/2     0     1/2 EP= 0
EE 6 P1: (1) 0 0 0 1 EP= 0 P2: (6)  11/16     0    5/16 EP= 0
EE 7 P1: (2) 1 0 0 0 EP= 5 P2: (7)      1     0       0 EP= 4

Connected component 1:
{1}  x  {1, 2, 3, 4, 5, 6}

Connected component 2:
{2}  x  {7}
```

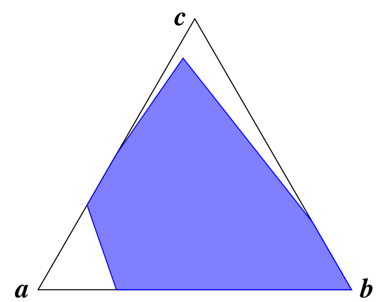


Figure 16 Left: 4×3 game with two equilibrium components, shown on the right. In component 1, player 1 plays D and player 2 plays so that player 1 gets at most payoff 0, shown by the hexagon at the bottom right (see also Hauk and Hurkens, 2002, Fig. 4). Component 2 is the pure-strategy equilibrium (A,a).

Fig. 16 shows another game related to strategic stability, due to Hauk and Hurkens (2002, p. 74). This game corresponds to an extensive game where player 1 can first choose between an "outside option" D with constant payoff zero to both players or else play in a 3×3 simultaneous game. The game has two equilibrium components, the pure equilibrium (A,a) and the outside option component where player 2 randomizes in such a way that player 1 gets at most payoff zero and therefore will not deviate from D.

Interestingly, the larger component in this game has the property that it has *index* zero but is nevertheless strategically stable, that is, any perturbed game has equilibria near that component, as shown by Hauk and Hurkens (2002). The larger component in the game in Fig. 14 has also index zero and is not stable, which is what one would normally expect. The index is a certain integer defined for an equilibrium component, with the important property that the sum over all equilibrium indices is one, and that for generic perturbations each equilibrium has index 1 or -1, which is always 1 for a pure-strategy equilibrium (see Shapley, 1974). A symbolic computation of the index of an equilibrium component for a bimatrix game is described by Balthasar (2009, Chapter 2). Its implementation is planned as a future additional feature of GTE.

All currently implemented algorithms of GTE apply only to two-player games. A simple next stage is to allow three or more players at least for the purpose of drawing games and generating their strategic form. Algorithms for finding all equilibria of a game with any number of players using polynomial algebra are described by Datta (2010); some of these require computer algebra packages, so far not part of GTE.

5 Equilibrium computation for games in strategic form

In this section, we describe the algorithm used in GTE that finds all equilibria of a two-player game in strategic form. This algorithm searches the vertices of certain polyhedra defined by the payoff matrices, which is practical for up to about 20 strategies per player. For larger games, one can normally still find in reasonable time one equilibrium, or possibly several equilibria with varying starting points, using the path-following algorithms by Lemke and Howson (1964) and Lemke (1965) mentioned at the end of this section. We assume some familiarity with *pivoting* as used in the simplex algorithm for linear programming, which in the present context is explained further in Avis et al. (2010).

The *best response condition* describes, in a general finite game in strategic form, when a profile of mixed strategies is a Nash equilibrium. If player 1, say, has m pure strategies, then his set X of mixed strategies is the set of probability vectors $x = (x_1, \ldots, x_m)$ so that $x_i \geq 0$ and $\sum_{i=1}^{m} x_i = 1$. Geometrically, X is a *simplex* which is the convex hull of the m unit vectors in $\mathbb{R}^m$. If x is part of a Nash equilibrium, then x has to be a best response against the strategies of the other players. The best response condition states that this is the case if and only if every *pure strategy* i so that $x_i > 0$ is such a best response (the condition is easy to see and was used by Nash, 1951). This is a finite condition, which requires to compute player 1's expected payoff for each pure strategy i, and to check if the strategies i in the *support* $\{i \mid x_i > 0\}$ of x give indeed maximal payoff. These maximal payoffs have to be *equal*, and the resulting equations typically determine the probabilities of the mixed strategies of the other players.

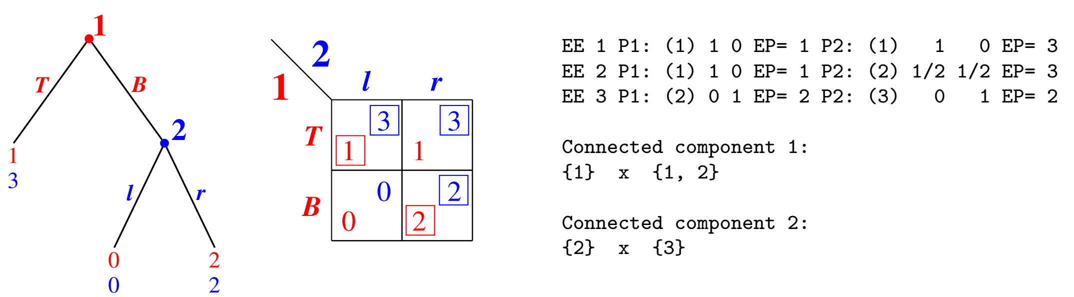


Figure 17 A "threat game" in extensive form, its 2×2 strategic form, and its equilibrium components.

For two players, these constraints define "best response polyhedra" that simplify the search for equilibria. We illustrate this geometric apprach with an example; for a general exposition and detailed references see von Stengel (2002; 2007). Fig. 17 shows a simple game tree which has the SPE (B, r), and another equilibrium component where player 2 "threatens" to choose l with probability at least $1/2$ and player 1 chooses T.

Fig. 18 shows on the left the mixed strategy y of player 2, which is defined by the probability that she chooses move r, say, together with the expected payoff to player 1 for his two pure strategies T and B. These expected payoffs are linear functions of y (because the game has only two players; for more than two players, the expected

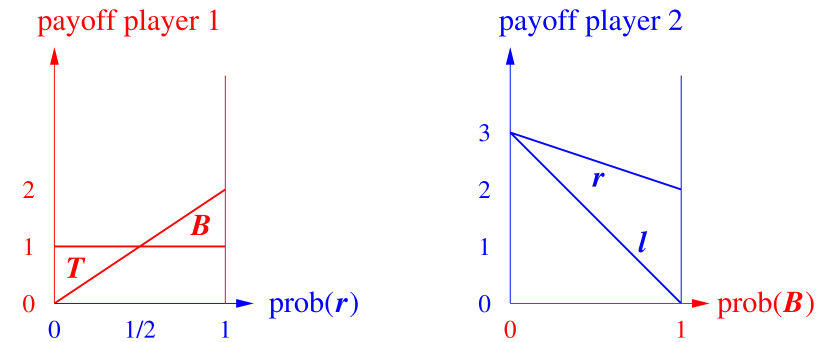


Figure 18 Expected payoffs to the two players in the threat game in Fig. 17 as a function of the mixed strategy of the other player.

payoff for a pure strategy of player 1 is a product of mixed strategy probabilities of the other players, which is no longer linear). The right picture shows the expected payoffs for the pure strategies l and r of player 2 as a function of the mixed strategy of player 1.

These pictures show when a pure strategy is a *best response* against a mixed strategy of the other player: T is a best response of player 1 when $\text{prob}(r) \leq 1/2$, and B is a best response when $\text{prob}(r) \geq 1/2$. Strategy r of player 2 is always a best response, and l is a best response when $\text{prob}(B) = 0$.

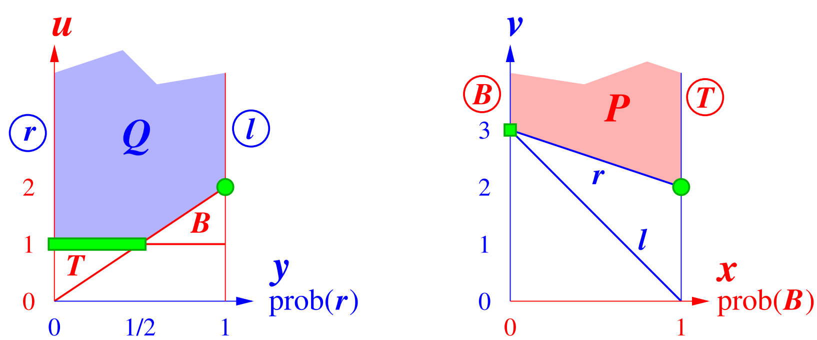


Figure 19 The polyhedra Q and P in (1) that define the "upper envelope" of the payoffs in Fig. 18, with additional circled facet labels for unplayed strategies, and equilibrium strategy pairs.

In Fig. 19, the same pictures are shown more abstractly where u and v are the expected payoffs to player 1 and 2, respectively, which are required to be at least as large as the expected payoff for every pure strategy. In a general $m \times n$ bimatrix game, let a_{ij} and b_{ij} be the payoffs to player 1 and 2, respectively, for each pure strategy pair i, j. Analogous to X, let Y be the mixed strategy simplex of player 2 given by $Y = \{y \in \mathbb{R}^n \mid y_j \geq 0,\ \sum_{j=1}^n y_j = 1\}$. Then P and Q are the *best response polyhedra*

$$\begin{aligned} P &= \{(x,v) \in X \times \mathbb{R} \mid \textstyle\sum_{i=1}^m b_{ij} x_i \leq v,\ 1 \leq j \leq n\}, \\ Q &= \{(y,u) \in Y \times \mathbb{R} \mid \textstyle\sum_{j=1}^n a_{ij} y_j \leq u,\ 1 \leq i \leq m\}. \end{aligned} \tag{1}$$

In Fig. 19, Q is shown on the left and refers to the set of mixed strategies y of player 2 together with the best responses of player 1 and corresponding payoff u to player 1.

Consider one of the inequalities $\sum_{j=1}^{n} a_{ij} y_j \leq u$ in the definition of Q for some pure strategy i of player 1, which in the example is either T or B. When this inequality is *tight*, that is, holds as equality, then i is clearly a best response to y. This tight inequality defines a *face* (intersection with a valid hyperplane) of the polyhedron, and is often a facet (a face of maximum dimension), as here for both strategies T and B; however, for the polyhedron P on the right in Fig. 19 the tight inequality $\sum_{i=1}^{m} b_{ij} x_i = v$ when j is the pure strategy l of player 2 does not define a facet because it only holds for the single point where $\text{prob}(B) = 0$ and $v = 3$, that is, for $(x, v) = ((1,0), 3) \in P$.

We *label* each point (y, u) of Q with the strategy i of player 1 whenever i is a best response to y, that is, when $\sum_{j=1}^{n} a_{ij} y_j = u$. In addition, (y, u) is labeled with strategy j of player 2 if $y_j = 0$. The labels for these *unplayed* pure strategies are shown with circles around them in Fig. 19. So any point $((1,0), u)$ where $\text{prob}(r) = 0$ (the left edge of Q) is labeled with r, and any point $((0,1), u)$ where $\text{prob}(r) = 1$ and hence $\text{prob}(l) = 0$ (the right edge of Q) is labeled with l. Similarly, in the right picture the left and right edges of P have label B and T, respectively. The bottom edge of P is labeled with the pure strategy r, and the left vertex $(x, v) = ((1,0), 3)$ has label l in addition to the labels B and r.

With this labeling, an equilibrium (x, y) of the game with payoffs u and v to player 1 and 2 is given by a pair $((x, v), (y, u))$ in $P \times Q$ that is *completely labeled*, that is, each pure strategy, here T, B, l, r, of either player appears as a label of (x, v) or (y, u). Otherwise, a *missing label* represents a pure strategy that has positive probability but is not a best response, which does not hold in an equilibrium because it contradicts the best response condition.

In Fig. 19, one equilibrium (x, y), indicated by the small disks to mark the two points (x, v) and (y, u), is given by $x = y = (0, 1)$ with $u = v = 2$, which is the pure strategy pair (B, r). Here the point (y, u) of Q has labels B and l and the point (x, v) of P has labels r and T, so these two points together have all labels T, B, l, r. A second equilibrium component, indicated by the rectangles, is given by any point (y, u) on the entire edge of Q that has label T and $(x, v) = ((1,0), 3)$ in P which has the three labels r, l, B. These are also the two equilibrium components in the GTE output in Fig. 17. The edge of Q with label T is the convex hull of the two vertices $((1,0), 1)$ with labels T and r, and $((1/2, 1/2), 1)$ with labels T and B. Each of these vertices of Q, together with the vertex $((1,0), 3)$ of P, forms a completely labeled *vertex pair*. For these two vertex pairs of $P \times Q$, one label (r or B) appears twice and represents a best-response pure strategy that is played with probability zero, which is allowed in equilibrium.

The *extreme equilibria* computed by GTE are in fact all vertex pairs of $P \times Q$ that are completely labeled, which are then further processed so as to detect the equilibrium components. We give an outline of how these computations work in GTE, described in detail and compared with other approaches by Avis et al. (2010).

Vertex enumeration, that is, listing all vertices of a polyhedron defined by linear inequalities, is a well-studied problem where we use the *lexicographic reverse search* method *lrs* by Avis (2000; 2006). This method reverses the steps of the simplex algorithm for linear programming for a certain deterministic pivoting rule. It starts with a vertex v_0 of the polyhedron and a linear objective function that is maximized at that vertex. The simplex algorithm for maximizing this linear function computes from every vertex a path of *pivoting steps* to v_0. With a deterministic pivoting rule, that path is unique. In lrs, the pivoting rule chooses as entering variable the variable with the least index (i.e., smallest subscript) that improves the objective function, and the leaving variable via a lexicographic rule. The unique paths of simplex steps from the vertices to v_0 define a tree with root v_0. The lrs algorithm explores this tree by traversing the tree edges in the reverse direction using a depth-first search.

In lrs, as in the simplex algorithm, the vertices of $P \times Q$ are represented by *basic feasible solutions* to the inequalities in (1) when they are represented in equality form via slack variables. Here, these are vectors s in $\mathbb{R}^n$ and r in $\mathbb{R}^m$ so that the constraints that define P and Q are written as

$$\sum_{i=1}^{m} b_{ij}x_i + s_j = v, \qquad r_i + \sum_{j=1}^{n} a_{ij}y_j = u, \qquad x_i, s_j, r_i, y_j \geq 0 \tag{2}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$, and $\sum_{i=1}^{n} x_i = 1$ and $\sum_{j=1}^{n} y_j = 1$ so that $x \in X$ and $y \in Y$. A feasible solution x, s, v, r, y, u to (2) defines a point in $P \times Q$ and vice versa. A tight inequality corresponds to a slack variable that is zero. In fact, the labels for a strategy pair x, y, where u and v are chosen minimally so that (2) holds, which determines r and s, are the pure strategies i so that $x_i = 0$ or $r_i = 0$ for player 1 and $y_j = 0$ or $s_j = 0$ for player 2. The equilibrium condition of being completely labeled is equivalent to the *complementarity* condition

$$x_i r_i = 0 \quad (1 \leq i \leq m), \qquad y_j s_j = 0 \quad (1 \leq j \leq n). \tag{3}$$

In a basic feasible solution to (2), the labels correspond to the indices i or j of the nonbasic variables, and to basic variables that happen to have value zero.

Basic solutions where basic variables have value zero are called *degenerate*, and games where this happens are also called degenerate. It is easy to see (see von Stengel, 2002) that this corresponds to a mixed strategy of a player that has more best responses than the size of its support; in terms of labels, this defines a point of P with more than m labels or a point of Q with more than n labels. An example is the degenerate game in Fig. 17 where the pure strategy T (which is a mixed strategy $x = (1, 0)$ with a support of size one) has two best responses, which is the point $((1, 0), 3)$ of P in Fig. 19 with three labels B, l, r. For extensive games, the strategic form is typically degenerate, as the examples in Fig. 3 and 17 demonstrate. In lrs, all arithmetic computations are done in *arbitrary precision arithmetic* (rather than floating point arithmetic with possible rounding errors) in order to recognize with certainty that a basic variable is zero. In addition, lrs uses the economical *integer pivoting* method where basic feasible solutions are represented with integers rather than rational numbers; see, for example, von Stengel (2007, Section 3.5).

After enumerating all vertices of P and Q with lrs, those pairs that fulfill the complementarity condition (3) are the extreme equilibria of the game. Avis et al. (2010) also describe an improved method *lrsNash*, which is used in GTE, where only the vertices of one polytope, say P, are generated, and the set L of labels *missing* from each such vertex (x, v) is used to identify the face $Q(L)$ of the other polytope that has exactly the labels in L. If this face is not empty, then each of its vertices

(y, u) defines an extreme equilibrium (x, y). An alternative method to enumerate all extreme equilibria is the EEE method of Audet et al. (2001) which performs a depth-first search that chooses one tight inequality for either player in each search step. In Avis et al. (2010), EEE is implemented with exact integer arithmetic and compared with *lrsNash*, and performs better for larger games from size 15×15 onwards.

Given the list of extreme equilibria, the set of all equilibria is completely described as follows (for details see Avis et al., 2010). Consider the bipartite graph R with edges (x, y) that are the extreme equilibria. Each connected component C of this graph defines a topologically connected component of equilibria, as it is output by GTE. For each C, identify the maximal bipartite "cliques" in R of the form $U \times V \subseteq C$, that is, every pair (x, y) in $U \times V$ is an extreme equilibrium. Then any convex combination of U paired with any convex combination of V is also a Nash equilibrium, and every equilibrium of the game can be represented in this way. There may be several such cliques that define a component C, as in Fig. 14 and 15 above. In GTE, the maximal cliques $U \times V$ are computed with an extension of the fast clique enumeration algorithm by Bron and Kerbosch (1973).

For a two-player game in strategic form, GTE computes by default all Nash equilibria, as just described. However, the state-of-the-art algorithms *lrsNash* and EEE have exponential running time due the typically exponential growth of the number of vertices of $P \times Q$. These equilibrium enumeration algorithms therefore take prohibitively long for games with more than 25, certainly 30 pure strategies for each player. In addition, it is NP-hard to decide if a bimatrix game has more than one Nash equilibrium (Gilboa and Zemel, 1989; Conitzer and Sandholm, 2008), so one cannot expect to answer that question quickly for larger games.

In practice, a finite game may represent a discretization of an infinite game, where it may be useful to know its equilibria when each player has several hundred strategies. In that case, it is still possible to use methods that find one equilibrium, such as the classic algorithm by Lemke and Howson (1964). This is a path-following algorithm that follows a sequence of pivoting steps, similar to the simplex algorithm for linear programming (for expositions see Shapley, 1974, or von Stengel, 2002; 2007). Like the simplex algorithm, it can take exponentially many steps on certain worst-case instances (Savani and von Stengel, 2006). However, these seem to be rare in practice, and the algorithm is typically fast for random games.

The Lemke–Howson algorithm has a free parameter, which is a pure strategy of a player that together with the best response to that strategy defines a "starting point" of the algorithm in the space of mixed strategies $X \times Y$. Varying this starting point over all pure strategies may lead to different equilibria (however, if that is not the case, the game may still have equilibria that are elusive to the algorithm).

GTE uses a related but different method that is implemented using the algorithm by Lemke (1965), which allows for starting points that can be arbitrary mixed strategy pairs $(\bar{x}, \bar{y})$ (von Stengel, van den Elzen, and Talman, 2002). In addition, the computation can be interpreted as the "tracing procedure" by Harsanyi and Selten (1988), where players' strategies are best responses to a weighted combination of their *prior* $(\bar{x}, \bar{y})$ and the actual strategies that they are playing. Initially, the players only react to the prior, whose weight is then decreased; equilibrium is reached when the weight of the prior becomes zero. If starting from sufficiently many random priors gives always the same equilibrium, this does not prove that this equilibrium is unique, but may be considered as sufficient reason to suggest this as a plausible way to play the game.

The computational experiments of von Stengel, van den Elzen, and Talman (2002) show that for random games, the method finds an equilibrium in a small number of pivoting steps, and that typically many equilibria are found by varying the prior by choosing it uniformly at random from the strategy simplices. For a systematic investigation of this method in GTE, it would be useful to generate larger games as discretizations of games defined by arithmetic formulas for payoff functions; this is a future programming project where a closer integration with Gambit is envisaged.

6 Equilibrium computation for games in extensive form

The standard approach to finding equilibria of an extensive game is to convert it to strategic form, and to apply the corresponding algorithms. However, the number of pure strategies grows typically *exponentially* in the size of the game tree. In contrast, the *sequence form* is a strategic description that has the same size as the game tree, which is used in GTE. We sketch the main ideas here; for more details see von Stengel (1996; 2002) or von Stengel, van den Elzen, and Talman (2002).

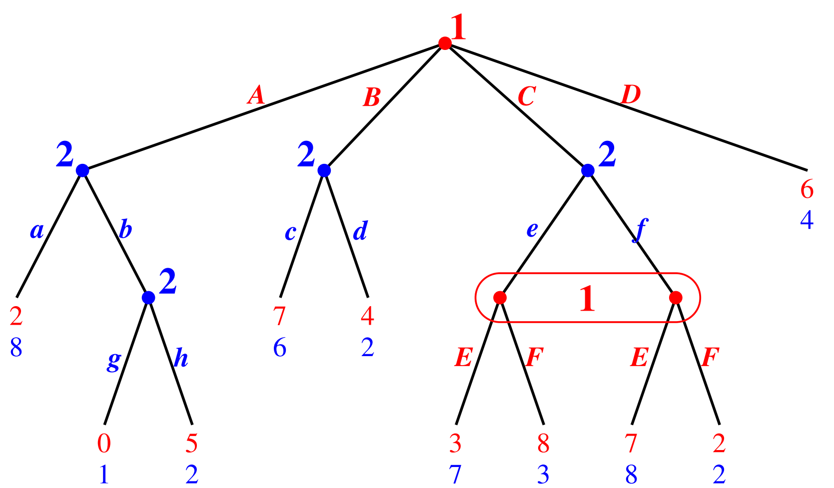


Figure 20 Extensive game with perfect information for player 2 and no information for player 1.

Fig. 20 shows an extensive game where player 1 has four moves at the root and two moves at his second information set. Player 2 has perfect information and four singleton information sets with two moves each. A pure strategy of a player defines a move for every information set, so player 1 has 8 and player 2 has 16 pure strategies. Some of these pure strategies can be identified because they describe the same "plan" of playing the game. A *reduced* pure strategy does not specify a move at an information set that is unreached due to an earlier own move of the player. For example, after move *a* player 2 does not need to specify move *g* or *h* because the information set with that move cannot be reached. We represent such an unspecified move by an asterisk " * " at the place where the move would be listed, so *a*ce* is a reduced strategy where * stands for either *g* or *h*, as in the strategy *bgce* where no reduction is possible. The reduced strategic form is shown in Fig. 21, which is a 5×12 game. Each cell shows the payoffs that result when the two strategies meet.

1 \ 2	*a*ce*	*a*cf*	*a*de*	*a*df*	*bgce*	*bgcf*	*bgde*	*bgdf*	*bhce*	*bhcf*	*bhde*	*bhdf*
*A**	2, 8	2, 8	2, 8	2, 8	0, 1	0, 1	0, 1	0, 1	5, 2	5, 2	5, 2	5, 2
*B**	7, 6	7, 6	4, 2	4, 2	7, 6	7, 6	4, 2	4, 2	7, 6	7, 6	4, 2	4, 2
CE	3, 7	7, 8	3, 7	7, 8	3, 7	7, 8	3, 7	7, 8	3, 7	7, 8	3, 7	7, 8
CF	8, 3	2, 2	8, 3	2, 2	8, 3	2, 2	8, 3	2, 2	8, 3	2, 2	8, 3	2, 2
*D**	6, 4	6, 4	6, 4	6, 4	6, 4	6, 4	6, 4	6, 4	6, 4	6, 4	6, 4	6, 4

Figure 21 Reduced strategic form of the game in Fig. 20. In a reduced strategy, any move at an information set that cannot be reached due to an earlier own move is left unspecified and replaced by "*".

In this game, a *mixed strategy* requires four independent probabilities for player 1 (the fifth is then determined because probabilities sum to one) and eleven for player 2. In general, the number of reduced strategies is *exponential* in the size of the game tree, so there is a large number of mixed strategy probabilities to compute.

1 \ 2	$\emptyset$	*a*	*bg*	*bh*	*c*	*d*	*e*	*f*
A		2, 8	0, 1	5, 2				
B					7, 6	4, 2		
CE							3, 7	7, 8
CF							8, 3	2, 2
D	6, 4							

Figure 22 Sequence form payoffs of the game in Fig. 20. All empty cells have zero entries. The rows are played with a probability distribution, whereas the columns are played with weights y_σ subject to the equations $y_\emptyset = 1 = y_a + y_{bg} + y_{bh} = y_c + y_d = y_e + y_f$.

The *sequence form* is a compact strategic description of the game that has the *same* size as the game tree. Instead of strategies, it uses *sequences* of moves of a given player along a path from the root to a leaf. The payoff matrices are sparse and contain payoffs for those pairs of sequences of the two players that lead to a leaf, as shown in Fig. 22 for the game in Fig. 20. This table is evidently more compact, and vastly more so for larger games. In this game, sequences and reduced strategies coincide for player 1, because that player does not get any information about the moves of the other player. For player 2, her sequences σ are played with probabilities y_σ that are not distributions over the set of all sequences, but are subject to three separate equations $y_a + y_{bg} + y_{bh} = 1$, $y_c + y_d = 1$, $y_e + y_f = 1$. These equations express the revealed information to player 2 following the respective moves A, B, and C of player 1, and can be derived systematically from the structure of the information sets (with one equation per information set, where we have substituted some equations such as $y_b = y_{bg} + y_{bh}$ for the unused nonterminal sequence b; also, player 2's empty sequence $\emptyset$ leads here to the rightmost leaf and has constant probability $y_\emptyset = 1$).

The twelve reduced strategies of player 2 are in effect the combinations of one of the sequences a, bg, bh combined with one of c, d and one of e, f, respectively. In the sequence form, these sequences are randomized *independently*. This randomization translates to a *behavior strategy*, where the player randomizes locally over his moves at each information set rather than globally over his pure strategies as in a mixed strategy. The underlying assumption about the information sets is that of *perfect recall* which says that a player does not forget what he knew or did earlier. Using the sequence form implies the theorem of Kuhn (1953) that a player with perfect recall can replace a mixed strategy by an equivalent behavior strategy.

In GTE, the sequence form is implemented with the path-following algorithm of Lemke (1965) mentioned at the end of the previous section, as described by von Stengel, van den Elzen, and Talman (2002), which starts from an arbitrary "prior" as a starting vector and finds one equilibrium. It can be applied to relatively large extensive games where the strategic form is hopelessly large to be used. For smaller extensive games, an enumeration of all Nash equilibria based on the sequence form is implemented as described by Huang (2011, Chapter 2); for a related approach see Audet, Belhaiza, and Hansen (2009). The number of independent probabilities in the sequence form is at most that number in the reduced strategic form. For example, player 2 in Fig. 22 has four such independent probabilities, like y_a, y_{bg}, y_c, y_e, as opposed to eleven for her reduced strategies; player 1 has four independent probabilities for both sequences and reduced strategies. Huang's approach only uses these independent probabilities and thus keeps a low dimension of the suitably defined best-response polyhedra, which is important for the employed vertex enumeration algorithms that have exponential running time.

7 Software architecture and development

In this final section, we describe the architecture of the GTE software on the server and client computers, and its development. The GTE program is open-source and part of the Gambit project, with the goal of further integration with the existing Gambit modules. The development is time intensive and relies on the efforts of volunteers, where we strongly invite any interested programmer to contribute.

The GTE software is accessed by a web browser. Behind its web address runs a *server* computer, which delivers a webpage and the game-drawing software of GTE to the user's *client* computer. The user can then create games, and store them on his client computer. The *game solvers* for equilibrium computation reside on the server. They are invoked by sending a description of the game from the client to the server over the internet. When the algorithm has terminated, the server sends the text with the computed equilibria back to the client where they are displayed.

The client program with the graphical user interface (GUI) is written in a variant of the JavaScript programming language called ActionScript which is displayed with the “Flash Player”. Flash is a common software that runs as a “plug-in” on most web browsers. We chose Flash for GTE in 2010 because of the predictability and speed of its graphical display. However, Flash does not work on the iPad and iPhone mobile devices (it does work on regular Apple computers), with the more universally accepted HTML5 standard as its suggested replacement. In the future, we plan to replace the ActionScript programs by regular JavaScript along with streamlining and further improving the GUI part of GTE.

For security reasons, the client program requires active permission of the user when writing or reading files on the client computer. When storing or loading a game as a local file, this is anyhow initiated by the user and therefore causes no additional delay. We have designed a special XML format for games in extensive or strategic form, where XML stands for “extensible markup language” similar to the HTML language for web pages. The tree is described by its logical structure. For example, the leftmost leaf in Fig. 17 reached by move T is encoded as follows:

```
<outcome move="T">
   <payoff player="1">1</payoff>
   <payoff player="2">3</payoff>
</outcome>
```

In addition, the file contains parameters for graphical settings such as the orientation of the game tree. The files can be converted to and from the file format used by Gambit. The XML description is also used to encode the game for sending it to the server for running a game solver.

Communication with the server requires identifying the user when there are different simultaneous client sessions, and the use of the relevant internet protocols. These are standard methods used for typical internet transactions, which we use as freely available routines in the Java programming language in which we have written the server programs.

The first version of GTE was written on a publically provided server (the Google App engine). For security reasons, all programs on that platform have to be written in Java, and are restricted in their ability to access programs written in other programming languages, called “native” code. In particular, we could not use the C program lrs for vertex enumeration, which was therefore rewritten in Java. However, this requires to duplicate in Java any improvement of lrs such as the *lrsNash* program mentioned in Section 5, which is inefficient and prone to errors. In addition, it is difficult to incorporate other game solvers. For that reason, the web server for GTE is now provided on a university computer, and uses the original lrs code of Avis (2006).

GTE allows to compute all Nash equilibria of a two-player game, but the running time increases exponentially with the size of the game. This means that there is a relatively narrow range of input size (currently about 15 to 20 strategies per player) where the efficiency of the implementation (such as a factor of ten in running time) matters, and beyond which the computation takes an impractically long time. For small games, the game solvers would run satisfactorily on the client using JavaScript, although at present there are no good arbitrary precision arithmetic routines for this programming language, apart from having to implement the algorithms separately.

For games of "medium" size (below the "exponential barrier"), it is best to use a state-of-the-art algorithm on the server, which works equally well for small games.

However, one problem is that computations of, say, one hour also use the computational resources of the server, which degrade its performance, and which will be slow if there are multiple such computations at the same time. Here, we provide the option of installing the server on any local computer, which then also works independently of an internet connection. This requires downloading and compiling the GTE code, together with the necessary components of a Java compiler, a lightweight server program called "Jetty" to communicate locally with the web browser, and the (free) software for compiling the ActionScript part of GTE into an executable Flash program. This requires time and patience, probably more so than an installation of Gambit, but offers the same interface as that known from the web, so that the user may find the investment worth it.

All GTE software is open source and free to use and alter under the GNU General Public Licence (which requires derived software to be free as well). The software repository is at `https://github.com/gambitproject/gte/wiki` (including documentation), as part of the Gambit project which has long been open-source. The version control system and collaborative tool "git" stores the current and all previous versions of the project. Git efficiently stores snapshots of the entire project with all modules and documentation, which can be branched off and re-merged in decentralized development branches. The `github` repository for storing projects developed with git is free to use for open-source software, and offers a convenient web interface for administration and documentation.

Open-source software is the appropriate way to develop an academic project such as GTE which has limited commercial use. Academic software should be publically available for making computations reproducible, which is increasingly recognized as a need for validating scientific results that rely on computation (Barnes, 2010).

On the algorithm side, one of the next steps is to integrate other existing game solvers. Foremost among these are the algorithms already implemented in Gambit, which should be easily incorporated via their existing file formats for input and output. These include (see McKelvey, McLennan, and Turocy, 2010) algorithms for finding Nash equilibria of more than two players with polynomial systems of equations, or via iterated polymatrix approximation (Govindan and Wilson, 2004), or simplicial subdivision (van der Laan, Talman, and van der Heyden, 1987), and the recent implementation of "action-graph games" (Jiang, Leyton-Brown, and Bhat, 2011). Moreover, larger games should not be created manually but automatically. One focus of the current development of Gambit is to provide better facilities for the automatic generation of games using the Python programming language.

So far, the focus of GTE development has been to design a robust and attractive graphical "front end" for creating and manipulating games. Such an implementation does not count as scholarly research, so it is difficult to fund it directly with a research grant, or to let a PhD student invest a lot of time in it. As a topic for an MSc thesis or undergraduate programming project, usually too much time is needed to become familiar with the existing code or the necessary background in game theory. The Gambit project and GTE were sponsored by studentships from the "Google Summer of Code" in 2011 and 2012. Here, the most successful projects for GTE were those where the students could bring in their technical expertise on web graphics, for example, to complement our background in game-theoretic algorithms.

As with open-source software in general, the development of GTE relies on the efforts, competence, and enthusiasm of volunteers, where we were lucky to get help from excellent students listed in the Acknowledgments. We invite and encourage any interested programmer to contribute to this project.

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References

Audet, C., S. Belhaiza, and P. Hansen (2009), A new sequence form approach for the enumeration of all extreme Nash equilibria for extensive form games. International Game Theory Review 11, 437–451.

Audet, C., P. Hansen, B. Jaumard, and G. Savard (2001), Enumeration of all extreme equilibria of bimatrix games. SIAM Journal on Scientific Computing 23, 323–338.

Avis, D. (2000), lrs: a revised implementation of the reverse search vertex enumeration algorithm. In: Polytopes–Combinatorics and Computation, eds. G. Kalai and G. Ziegler, DMV Seminar Band 29, Birkhäuser, Basel, pp. 177–198.

Avis, D. (2006), User's Guide for lrs. `http://cgm.cs.mcgill.ca/~avis`.

Avis, D., G. Rosenberg, R. Savani, and B. von Stengel (2010), Enumeration of Nash equilibria for two-player games. Economic Theory 42, 9–37.

Bagwell, K. (1995), Commitment and observability in games. Games and Economic Behavior 8, 271–280.

Balthasar, A. V. (2009), Geometry and Equilibria in Bimatrix Games. PhD Thesis, London School of Economics.

Barnes, N. (2010), Publish your computer code: it is good enough. Nature 467, 753–753.

Belhaiza, S. J., A. D. Mve, and C. Audet (2010), XGame-Solver Software. `http://faculty.kfupm.edu.sa/MATH/slimb/XGame-Solver-Webpage/index.htm`

Bron, C., and J. Kerbosch (1973), Finding all cliques of an undirected graph. Communications of the ACM 16, 575–577.

Conitzer, V., and T. Sandholm (2008), New complexity results about Nash equilibria. Games and Economic Behavior 63, 621–641.

Datta, R. S. (2010), Finding all Nash equilibria of a finite game using polynomial algebra. Economic Theory 42, 55–96.

Gilboa, I., and E. Zemel (1989), Nash and correlated equilibria: some complexity considerations. Games and Economic Behavior 1, 80–93.

Govindan, S., and R. Wilson (2004), Computing Nash equilibria by iterated polymatrix approximation. Journal of Economic Dynamics and Control 28, 1229–1241.

Harsanyi, J. C., and R. Selten (1988), A General Theory of Equilibrium Selection in Games. MIT Press, Cambridge, MA.

Hauk, E., and S. Hurkens (2002), On forward induction and evolutionary and strategic stability. Journal of Economic Theory 106, 66–90.

Huang, W. (2011), Equilibrium Computation for Extensive Games. PhD Thesis, London School of Economics.

Jiang, A. X., K. Leyton-Brown, and N. A. R. Bhat (2011), Action-graph games. Games and Economic Behavior 71, 141–173.

Kohlberg, E., and J.-F. Mertens (1986), On the strategic stability of equilibria. Econometrica 54, 1003–1037.

Kuhn, H. W. (1953), Extensive games and the problem of information. In: Contributions to the Theory of Games II, eds. H. W. Kuhn and A. W. Tucker, Annals of Mathematics Studies 28, Princeton Univ. Press, Princeton, pp. 193–216.

Langlois, J.-P. (2006), GamePlan, a Windows application for representing and solving games. `http://userwww.sfsu.edu/langlois/`

Lemke, C. E. (1965), Bimatrix equilibrium points and mathematical programming. Management Science 11, 681–689.

Lemke, C. E., and J. T. Howson, Jr. (1964), Equilibrium points of bimatrix games. Journal of the Society for Industrial and Applied Mathematics 12, 413–423.

McKelvey, R. D., A. M. McLennan, and T. L. Turocy (2010), Gambit: Software Tools for Game Theory, Version 0.2010.09.01. `http://www.gambit-project.org`

Nash, J. (1951), Noncooperative games. Annals of Mathematics 54, 286–295.

Osborne, M. J. (2004), An Introduction to Game Theory. Oxford University Press, Oxford.

Quint, T., and M. Shubik (1997), A theorem on the number of Nash equilibria in a bimatrix game. International Journal of Game Theory 26, 353–359.

Savani, R. (2005), Solve a bimatrix game. Interactive website at `http://banach.lse.ac.uk/`

Savani, R., and B. von Stengel (2004), Exponentially Many Steps for Finding a Nash Equilibrium in a Bimatrix Game. CDAM Research Report LSE-CDAM-2004-03.

Savani, R., and B. von Stengel (2006), Hard-to-solve bimatrix games. Econometrica 74, 397-429.

Shapley, L. S. (1974), A note on the Lemke–Howson algorithm. Mathematical Programming Study 1: Pivoting and Extensions, 175–189.

van der Laan, G., A. J. J. Talman, and L. van der Heyden (1987), Simplicial variable dimension algorithms for solving the nonlinear complementarity problem on a product of unit simplices using a general labelling. Mathematics of Operations Research 12, 377–397.

von Stengel, B. (1996), Efficient computation of behavior strategies. Games and Economic Behavior 14, 220–246.

von Stengel, B. (1999), New maximal numbers of equilibria in bimatrix games. Discrete and Computational Geometry 21, 557–568.

von Stengel, B. (2002), Computing equilibria for two-person games. In: Handbook of Game Theory, Vol. 3, eds. R. J. Aumann and S. Hart, North-Holland, Amsterdam, pp. 1723–1759.

von Stengel, B. (2007), Equilibrium computation for two-player games in strategic and extensive form. In: Algorithmic Game Theory, eds. N. Nisan et al., Cambridge Univ. Press, Cambridge, pp. 53–78.

von Stengel, B. (2012), Rank-1 games with exponentially many Nash equilibria. arXiv:1211.2405.

von Stengel, B., A. H. van den Elzen, and A. J. J. Talman (2002), Computing normal form perfect equilibria for extensive two-person games. Econometrica 70, 693–715.