A characterization of single-peaked single-crossing domain

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based on joint work with Piotr Faliszewski and Piotr Skowron
Voters and Their Preferences

- n voters, m candidates
- Each voter has a complete ranking of the candidates (his preference order)
- Problem: with no assumption on preference structure
  - counterintuitive behavior may occur
  - computational problems are often hard
Single-Peaked Preferences

• **Definition**: a preference profile is single-peaked (SP) wrt an ordering $<$ of candidates (axis) if for each voter $v$ there exists a candidate $C$ such that:
  – $v$ ranks $C$ first
  – if $C < D < E$, $v$ prefers $D$ to $E$
  – if $A < B < C$, $v$ prefers $B$ to $A$

• **Example**:
  – voter 1: $C > B > D > E > F > A$
  – voter 2: $A > B > C > D > E > F$
  – voter 3: $E > F > D > C > B > A$
**Single-Crossing Preferences**

**Definition**: a profile is single-crossing (SC) wrt an ordering of voters \((v_1, ..., v_n)\) if for each pair of candidates \(A, B\) there exists an \(i \in \{0, ..., n\}\) such that voters \(v_1, ..., v_i\) prefer \(A\) to \(B\), and voters \(v_{i+1}, ..., v_n\) prefer \(B\) to \(A\).
Single-Peaked vs. Single-Crossing Preferences

• Similarities:
  – both are motivated by the idea that the society is aligned along a single axis
  – both can be checked in poly-time
  – both ensure existence of a Condorcet winner
  – both enable efficient algorithms for many social choice problems
  – both admit forbidden minor characterization

• Differences:
  – order on candidates vs. order on voters
Single-Peaked Profile That Is Not Single-Crossing

- \( v_1 \) and \( v_2 \) have to be adjacent (because of B, C)
- \( v_3 \) and \( v_4 \) have to be adjacent (because of B, C)
- \( v_1 \) and \( v_3 \) have to be adjacent (because of A, D)
- \( v_2 \) and \( v_4 \) have to be adjacent (because of A, D)

a contradiction
Single-Crossing Profile That Is Not Single-Peaked

Each candidate is ranked last exactly once
Can we characterize preference profiles that are simultaneously single-peaked and single-crossing?
1D-Euclidean Preferences

• Both voters and candidates are points in $\mathbb{R}$
• $v$ prefers $A$ to $B$ if $|v - A| < |v - B|$
• Observation: 1D-Euclidean preferences are
  – single-peaked (wrt ordering of candidates on the line)
  – single-crossing (wrt ordering of voters on the line)
1-Euclidean Preferences: Bad News

- Proposition: There exists a preference profile that is SP and SC, but not 1-Euclidean

\[ v_1: 1 \ A_1 A_2 A_3 \ 2 \ B_1 B_2 B_3 \ 3 \ C_1 C_2 C_3 \ D_1 D_2 D_3 \ 4 \ 5 \]
\[ v_2: A_2 A_1 A_3 \ 2 \ B_1 B_2 B_3 \ 3 \ 1 \ C_1 C_2 C_3 \ D_1 D_2 D_3 \ 4 \ 5 \]
\[ v_3: B_2 B_1 B_3 \ 3 \ C_1 C_2 C_3 \ D_1 D_2 D_3 \ 4 \ 2 \ A_3 A_2 A_1 \ 1 \ 5 \]
\[ v_4: C_2 C_1 C_3 \ D_1 D_2 D_3 \ 4 \ 3 \ B_3 B_2 B_1 \ 2 \ A_3 A_2 A_1 \ 1 \ 5 \]
\[ v_5: D_2 D_1 D_3 \ C_3 C_2 C_1 \ 4 \ 5 \ 3 \ B_3 B_2 B_1 \ 2 \ A_3 A_2 A_1 \ 1 \]
\[ v_6: 5 \ 4 \ D_3 D_2 D_1 \ C_3 C_2 C_1 \ 3 \ B_3 B_2 B_1 \ 2 \ A_3 A_2 A_1 \ 1 \]
A Different Angle

• A preference profile is called narcissistic is every candidate is ranked 1st at least once
• Proposition: Every narcissistic SC profile is SP (axis = 1st vote)
• Proof:
  – suffices to show that if $v_1$ prefers A to B to C, then no voter ranks B last out of A, B, and C
Pre-NSC Preferences

• Are all SP-SC profiles narcissistic?
  – obviously no: being SP and SC is robust to deletions, and being narcissistic is not

• Definition: a profile is called pre-NSC if it can be extended to a narcissistic SC profile by adding voters
  – every pre-NSC profile is SP and SC

• Main Theorem: the converse is also true
Characterization

• **Theorem**: every SP-SC profile is pre-NSC

• Proof idea:
  – constructive argument: extend a SP-SC profile to a narcissistic one
  – crucial lemma: given a SP-SC profile $V = (v_1, ..., v_n)$, there is a vote $v_0$ such that $(v_0, v_1, ..., v_n)$ is SP and SC, and $v_0$ is an axis witnessing that $V$ is SP
  – by the lemma, can assume that the profile is SP wrt 1\textsuperscript{st} vote
  – use 1\textsuperscript{st} vote as a guiding order to insert votes
Lemma: Proof Idea

- Lemma: given a SP-SC profile \( V = (v_1, \ldots, v_n) \), there is a vote \( v_0 \) such that \( (v_0, v_1, \ldots, v_n) \) is SP and SC and \( v_0 \) is an axis witnessing that \( V \) is SP

- Proof idea:
  - try to add an arbitrary axis witnessing that \( V \) is SP
  - if this fails, pick a "minimal" pair of candidates that is at fault
  - modify the axis by swapping tails
  - argue that tail swap can be performed \( \leq m \) times
Algorithmic Perspective

• Our proof implies a **polynomial-time** algorithm for
  (1) checking whether a given profile $V$ is **pre-NSC**
  (2) finding a **narcissistic** profile extending it

• A simpler algorithm for (2) given (1):
  – for each missing candidate $A$, find possible
    positions in $V$ to insert a vote $v_A$ that ranks $A$ first
  – turns out that there is $\leq 1$ position for each candidate
  – if $v_A$ is the only vote to be inserted between $v_i$ and $v_{i+1}$,
    construct $v_A$ by moving $A$ to the top of $v_i$
  – if both $v_A$ and $v_B$ need to be inserted between $v_i$ and $v_{i+1}$,
    $v_A$ precedes $v_B$ iff $A$ precedes $B$ in $v_i$
Applications to Fully Proportional Representation: Monroe’s Rule

- n voters, m candidates
- **Task**: elect a k-member parliament
- **Constraints**:
  - candidates are *explicitly assigned* to voters
  - each elected candidate represents $\approx n/k$ voters
  - voter’s *dissatisfaction* is determined by the *rank* of his representative in his vote (via a *scoring rule*)
- **Objective**: minimize
  - sum of voters’ dissatisfaction (Monroe$^+$), or
  - maximum dissatisfaction (Monroe$^{\text{max}}$)
- Both Monroe$^+$ and Monroe$^{\text{max}}$ are NP-hard for general preferences
Monroe’s Rule: Example

- $k = 2$, scoring rule = Borda
- A can be assigned to at most 5 voters
- For $\text{Monroe}^+$, we can assign B to $v_1 - v_4$ or C to $v_2 - v_5$
- For $\text{Monroe}^{\text{max}}$, the only solution is to assign C to 4 arbitrary voters

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Single-Peaked Trajectories

• above \((A, i)\): \# of candidates \(v_i\) ranks above \(A\)

• **Definition**: a profile is said to have **single-peaked trajectories property (SPTP)** if for every candidate \(A\) there exists a voter \(v_i\) such that
  – above \((A, j)\) \(\geq\) above\((A, k)\) whenever \(j < k \leq i\)
  – above \((A, j)\) \(\geq\) above\((A, k)\) whenever \(j > k \geq i\)

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• **Claim**: pre-NSC profiles have **SPTP**
Monroe^{\text{max}} and SPTP

• **Claim**: if a profile has SPTP, then the set of voters matched to an elected candidate under Monroe^{\text{max}} is a contiguous segment of $V$

• **Corollary**: for pre-NSC preferences Monroe^{\text{max}} admits a very efficient DP algorithm

• [Betzler, Slinko, Uhlmann’13]: for single-peaked preferences Monroe^{\text{max}} admits a DP algorithm (but a much slower one)
Comment: Single-Peaked and Single-Crossing Profiles and SPTP

• **Observation:**
  a single-crossing profile may fail to have **SPTP**

• **Observation:**
  a single-peaked profile may fail to have **SPTP** (wrt natural order of the voters)
Future Work: Other Applications

• Are there algorithmic problems that are
  – hard for single-peaked preferences
  – hard for single-crossing preferences
  – easy for pre-NSC preferences?

• I.e., the problem admits an algorithm that relies on SPTP

• Candidate problems:
  – manipulation of STV
  – certain questions about control and bribery
Future Work: Extensions

• **Generalization**: profiles that are single-peaked/single-crossing on a tree

• **Definition**: a profile \( V \) is single-peaked on a tree \( T \) if candidates can be matched to vertices of \( T \) so that the restriction of \( V \) to every path in \( T \) is single-peaked

• **Definition**: a profile \( V \) is single-crossing on a tree \( T \) if voters can be matched to vertices of \( T \) so that the restriction of \( V \) to every path in \( T \) is single-crossing

• **Question**: given \( T_1 \) and \( T_2 \), can we characterize elections that are single-peaked on \( T_1 \) and single-crossing on \( T_2 \)?