(Approximately) Optimal Impartial Selection

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(joint work with Max Klimm, TU Berlin)

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Impartial Selection

- Select member of a set of agents based on nominations by agents from the same set
- Applications
  - selection of representatives
  - award of a prize
  - assignment of responsibilities
  - peer review: papers, research proposals, ...
- Assumption: agents are impartial to the selection of *other* agents
  - will reveal their opinion truthfully...
  - as long as it does not affect their own chance of selection
- Goal: preserve impartiality, select agent with many nominations

A Formal Model

- Set $\mathcal{G}$ of graphs (N, E) without self loops
  vertices represent agents, $(i, j) \in E$ means i nominates j
- $\delta^-_S(i, G) = \left|\left\{(j, i) \in E : G = (N, E), j \in S\right\}\right|$
  number of nominations $i \in N$ receives (indegree) from $S \subseteq N$
- selection mechanism: maps each $G \in \mathcal{G}$ to distribution on N
- f is impartial if

$$\big(f((N, E))\big)_i = \big(f((N, E'))\big)_i \quad \text{if} \quad E \setminus (\{i\} \times V) = E' \setminus (\{i\} \times V)$$

- f is α-optimal, for $\alpha \leq 1$, if for all $G \in \mathcal{G}$,

$$\frac{\mathbb{E}_{i \sim f(G)}[\delta^-_N(i, G)]}{\Delta(G)} \geq \alpha,$$

where $\Delta(G) = \max_{i \in N} \delta^-_N(i, G)$

Related Work

- Impartial Nominations for a Prize (Moulin, Holzman)
  - plurality, deterministic mechanisms, axiomatic study
- Strategyproof Selection from the Selectors (Alon et al.)
  - approval, deterministic and randomized mechanisms, selection of k agents with large number of nominations
- Impartial Division of a Dollar (de Clippel et al.)
  - more general than randomized mechanisms, axiomatic study
- Plurality: one nomination per agent (outdegree one)
- Approval: zero or more nominations (arbitrary outdegree)

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randomized	$[1/4, 1/2]$	?

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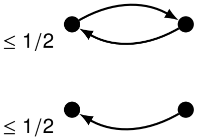

Outline and Results

- 1/2-optimal mechanism for approval
- same mechanism is 7/12-optimal for plurality
  (may actually be 2/3-optimal, but not better)
- upper bound for plurality of roughly 3/4
- Lower bounds from
  - better analysis of the mechanism of Alon et al.
  - generalization of the analysis to a (fairly) natural generalization of the mechanism
- Upper bound from optimization approach to finding mechanisms

The 2-Partition Mechanism (Alon et al.)

- Randomly partition N into (S_1, S_2)
- Select $i \in \arg\max_{i' \in S_2} \delta^-_{S_1}(i', G)$ uniformly at random
- 1/4-optimal
  - consider any $G \in \mathcal{G}$ and vertex i^* with degree $\Delta = \Delta(G)$
  - $i^* \in S_2$ with probability $1/2$
  - $\mathbb{E}[\delta^-_{S_1}(i, G) \mid i^* \in S_2] = \Delta/2$
  - vertex selected when $i^* \in S_2$ has at least this degree
- Tight for graph with a single edge
- Not obvious how to improve this, and by which analysis

The 2-Partition Mechanism (Revisited)

- Consider vertex i^* with degree Δ
- Randomly partition $N \setminus \{i^*\}$ into (S_1, S_2)
- Based on (S_1, S_2) adversary chooses $d = \max_{i \in S_2} \delta^-_{S_1}(i, G)$
- i^* goes to S_1 or S_2 with probability $1/2$ each
- Depending on $d^* = \delta^-_{S_1}(i^*, G)$, adversary will
  - set d to 0 and let i^* win with probability $1/2$
  - set d to d^* and beat i^* (assume ties broken against i^*)
- Selected vertex has expected degree $\min\{\Delta/2, d^*\}$
- Sum over distribution of d^*

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- Parameterized lower bound $\alpha(\Delta)$ in closed form
  - non-decreasing in Δ
  - $\alpha(1) = 1/4$
  - $\alpha(2) = 3/8$

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The k-Partition Mechanism

randomly partition N into $(S_1, \dots, S_k)$, denote $S_{<j} = \bigcup_{i<j} S_i$

$i^* := \bot$, $d^* := 0$

for $j = 2, \dots, k$

  if $\max_{i \in S_j} \delta^-_{S_{<j} \setminus \{i^*\}}(i, G) \geq d^*$

    choose $i \in \arg\max_{i' \in S_j} \delta^-_{S_{<j}}(i', G)$ uniformly at random

    $i^* := i$, $d^* := \delta^-_{S_{<j}}(i, G)$

select i^*

- Goal: parameterized lower bound $\alpha_k(\Delta)$

The k-Partition Mechanism

- Consider vertex i^* with degree Δ
- Randomly partition $N \setminus \{i^*\}$ into $(S_1, \ldots, S_k)$
- For $j = 2, \ldots, k$, adversary decides to beat i^* or let it win if $i^* \in S_j$
- i^* goes to each S_j with probability $1/k$
- Only rightmost alternative to beat i^* matters, as that alternative (or i^*) is selected
- For fixed $(S_1, \ldots, S_k)$, selected vertex has expected degree

$$\min_{j=1,\ldots,k} \left\{ \delta^-_{S_{<j}}(i^*, G) + \frac{k-j}{k}\Big(\Delta - \delta^-_{S_{<j}}(i^*, G)\Big) \right\}$$

- Sum over distribution of $\left(\delta^-_{S_j}(i^*, G)\right)_{j=1,\ldots,k}$

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select i^*

- Parameterized lower bound $\alpha_k(\Delta)$
  - for every $k \geq 2$, non-decreasing in $\Delta = \Delta(G)$
  - $\alpha_k(1) = (k-1)/(2k)$
  - $\alpha_k(2) = 7/12 - 3/8k^{-1} - 1/12k^{-2}$

The Permutation Mechanism

pick random permutation $(\pi_1, \ldots, \pi_n)$ of N, denote $\pi_{<j} = \bigcup_{i<j} \{\pi_i\}$

$i^* := \perp$, $d^* := 0$

for $j = 2, \ldots, k$

  if $\delta^-_{\pi_{<j} \setminus \{i^*\}}(\pi_j) \geq d^*$

    $i^* := \pi_j$, $d^* := \delta^-_{\pi_{<j}}(\pi_j)$

return i^*

- Limit of k-partition mechanism as $k \to \infty$
- 1/2-optimal for approval, 7/12-optimal for plurality
- k-partition for fixed k may be more desirable, allows more anonymous processing of ballots

Upper Bound for Plurality

- For any α-optimal impartial selection mechanism for plurality,

$$\alpha \leq \begin{cases} 5/6 & \text{if } n = 3, \\ (6n-1)/8n & \text{if } n \geq 6 \text{ even, and} \\ 3/4 & \text{otherwise} \end{cases}$$

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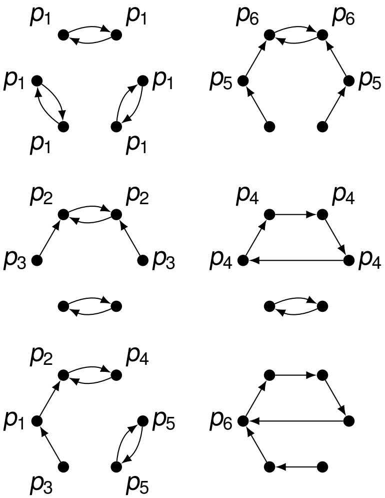
- Optimal mechanisms via linear optimization and graph isomorphism
- Number of constraints linear in number of graphs
- Upper bound for small graphs from dual, then generalize

Upper Bound for Plurality, $n \geq 6$ even



W.l.o.g., only consider symmetric mechanisms

$$np_1 = 1$$
$$2p_2 + 2p_3 \leq 1$$
$$p_1 + p_2 + p_3 + p_4 + (n-4)p_5 = 1$$
$$2p_5 + 2p_6 \leq 1$$
$$4p_4 \leq 1$$

$$p_6 \leq 1/2 - 1/(4n)$$

$$\alpha \leq \frac{2p_6 + (1-p_6)}{2} = \frac{p_6 + 1}{2}$$

Thank you!