Continuous Network Design
Hardness and Approximation

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Wardrop equilibrium

Definition Multi-commodity flow $f$ is Wardrop equilibrium if commodities use shortest paths.

$Wardrop\ equilibrium$

\[ d_1 = 4 \quad d_2 = 6 \]
Wardrop equilibrium - example

\[ d_1 = 4 \quad d_2 = 6 \]

Total travel time = 200
Wardrop equilibrium - example

\[ d_1 = 4 \quad d_2 = 6 \]

Total travel time = 150
A network design problem

d_1 = 4  \quad d_2 = 6
A network design problem

BPR function

\[
c_e(x) = t_e \left( 1 + b_e \left( \frac{f_e}{z_e} \right)^4 \right)
\]

- \( t_e \): free flow travel time
- \( b_e \): bias
- \( z_e \): capacity

\( d_1 = 4 \) \hspace{1cm} \( d_2 = 6 \)
“Die Philosophen haben die Welt nur verschieden interpretiert; es kommt aber darauf an, sie zu verändern.”

“Philosophers have hitherto only interpreted the world in various ways; the point is to change it.”
Problem definition

- Directed graph $G = (V, E)$
  - Set $K$ of commodities with $(s_k, t_k, d_k) \in V \times V \times \mathbb{R}_{\geq 0}$
  - Construction price (per unit) $l_e$
  - Cost functions $c_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, mapping $\nu_e/z_e$ to latency $c_e(\nu_e/z_e)$
    (cont. differentiable, semi-convex, $c_e(x)$ and $x^2 c_e(x)$ non-decreasing and unbounded)

- Objective: Find a vector of capacities $(z_e)_{e \in E}$ minimizing
  \[
  \sum_{e \in E} \left( c_e(\nu_e/z_e) \nu_e + z_e \cdot l_e \right) \quad \text{s.t.: } \nu = (\nu_e)_{e \in E} \text{ is a Wardrop equilibrium}
  \]
State-of-the-art

- Large body of work on heuristic approaches
  
  e.g. [Dafermos, TR '69][Dantzig et al., TR '79][Marcotte, MP '85]

- Approximation Algorithm
  
  - 5/4 for affine cost functions $c(v/z) = a + b(v/z)$
  
  - Closed formula for monomials $c(v/z) = a + b(v/z)^d$
    tending to 2 as $d$ goes to $\infty$.

- “one of the most important, difficult and challenging problems in transport” [Yang and Bell, TRev '98]

- Discrete network design problem
  (Decide which edges to remove from the network)
  
  - Strongly NP-hard [Roughgarden, JCSS '06]
A hard instance

\[ c(v/z) = \frac{v}{z}, \quad l = 1 \Rightarrow v^* = z^* \]

\[ c(v/z) = 0, \]

\[ c(v/z) = 4, \]

- \( \neg x, \neg y, \neg z \) satisfies the formula
- buy capacity for \( x, y, z \)
- total cost: \( 4 \#cl + 2\#cl \cdot \#var \)
Solving a relaxation

- Find a vector of capacities \((z_e)_{e \in E}\) minimizing
  \[
  \sum_{e \in E} \left( c_e \left( \frac{v_e}{z_e} \right) v_e + z_e \cdot l_e \right)
  \]
  s.t.: \(v = (v_e)_{e \in E}\) is a Wardrop equilibrium

\[
(CNDP)
\]

[Marcotte, MP '85]

**Lemma** \((CNDP')\) can be solved in polynomial time.

**Proof:**
- \(\partial / \partial z_e \left( c_e \left( \frac{v_e}{z_e} \right) v_e + z_e \cdot l_e \right) = 0 \Leftrightarrow l_e = \left( \frac{v_e}{z_e} \right)^2 c'_e \left( \frac{v_e}{z_e} \right)\)
- If \(u_e\) solves \(l_e = (x_e)^2 c'_e(x_e)\), then \(u_e = \frac{v_e}{z_e}\)
- \(\min \sum_{e \in E} \left( c_e(u_e) + \frac{l_e}{u_e} \right) v_e\) s.t. \(v\) is flow
- compute a shortest path for each commodity
Approximation algorithm

Solve the relaxation

\((v^*, z^*)\)

Choose \(z\) such that \((v^*, z)\) is a Wardrop equilibrium

\((v^*, z)\)

Let \(\lambda > 0\).

Solve the relaxation

\((v^*, z^*)\)

Compute a Wardrop equilibrium \(v\) for \(\lambda z^*\)

\((v, \lambda z^*)\)

take the best
The parameter $p$

Let $(v^*, z^*)$ solve the relaxed problem (CNDP').

$$C(v^*, z^*) = \sum_{e \in E} c_e(v^*_e/z^*_e) \cdot v^*_e + \sum_{e \in E} z^*_e \cdot l_e$$

Routing costs \quad Construction costs

$C^R(v^*, z^*) := p \cdot C(v^*, z^*)$ \quad $C^Z(v^*, z^*) := (1-p) \cdot C(v^*, z^*)$
Analysis of the approximation algorithm

**Definition** For a set $\mathcal{C}$ of cost functions, let

$$
\mu(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x \geq 0} \max_{\gamma \in [0,1]} \gamma \left(1 - \frac{c(\gamma x)}{c(x)}\right)
$$

and $\gamma(\mathcal{C})$ be the value for which the inner max is attained.

- **Example:** $\mathcal{C} = \{c : c(x) = a \, x + b\}$
  - $\mu(\mathcal{C}) = \max_{\gamma \in [0,1]} \gamma \left(1 - \gamma\right)$
Choosing the best of the two, we obtain a guarantee of

$$\max_{p \in [0,1]} \min \left\{ 1 + \gamma(1-p), \left( \sqrt{p} + \sqrt{\mu(1-p)} \right)^2 \right\}$$

The approximation guarantee is

$$\frac{(\gamma+\mu+1)^2}{(\gamma+\mu+1)^2 - 4\mu\gamma}$$
Further analysis

- Our algorithm has
  \[ \rho^* = \frac{(\gamma + \mu + 1)^2}{(\gamma + \mu + 1)^2 - 4\gamma \mu} < \frac{9}{5} \]

- Marcotte’s algorithm has
  \[ \rho = 1 + \mu < 2 \]
  \[ > \rho^* \]
Conclusion

‣ Hardness for the continuous network design problem

‣ First approximation guarantee for arbitrary cost functions

‣ Improved approximation guarantees for monomials

Thank you.