Algebraic Combinatorics and the Parity Argument

PPA membership of Combinatorial Nullstellensatz and related problems

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Contents

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Let $\mathbb{F}$ be an arbitrary field, and let $f \in \mathbb{F}[x_1, \ldots x_m]$ be an m-variable polynomial. Suppose that the degree of f is $\sum_{j=1}^{n} t_j$, where each t_j is a nonnegative integer, and that the coefficient of $\prod_{j=1}^{m} x_j^{t_j}$ is nonzero. Then, if $S_1, S_2, \ldots, S_m$ are subsets of $\mathbb{F}$ with $|S_j| > t_j$ for all $j = 1, \ldots, m$, then there exists an $(s_1, s_2, \ldots, s_m) \in S_1 \times S_2 \times \cdots \times S_m$ such that $f(s_1, s_2, \ldots, s_m) \neq 0$.

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The proofs of its applications are algebraic, and hence non-constructive in the sense that they supply no efficient algorithm for solving the corresponding algorithmic problems.

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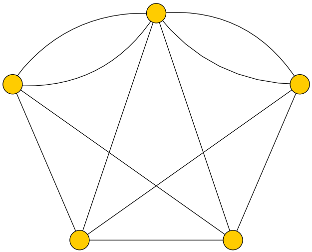

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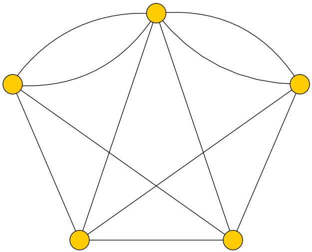

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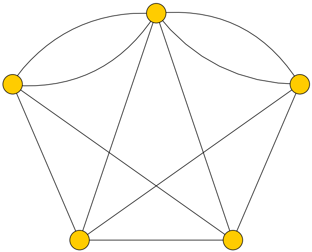

$n = 5$ vertices, $11 > 5 \cdot (3-1)$ edges $\Longrightarrow$ there exists a 3-divisible subgraph.

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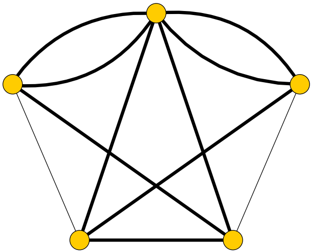

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Useful corollary of Combinatorial Nullstellensatz

Let p be an arbitrary prime. Let us be given some m-variable polynomials $f_1, f_2, \ldots, f_n$ over $\mathbb{F}_p$ with no constant terms. If

$$m > (p-1) \cdot \sum_{i=1}^{n} \deg(f_i),$$

then there exists a vector $\mathbf{0} \neq \mathbf{x} \in \{0,1\}^m$ such that $f_i(\mathbf{x}) = 0$ for all i.

$f_A(\mathbf{x}) = x_1 + x_2 + x_3 + x_4$

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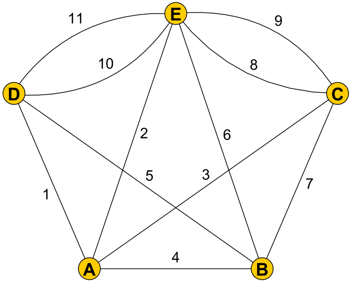

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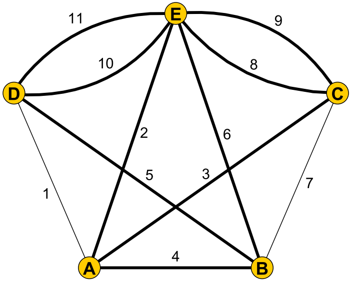

p^d-divisible subgraphs

In a previous paper, Alon, Friedland and Kalai answered the analogous question modulo prime powers with no use of Combinatorial Nullstellensatz.

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The analogous theorem about k-divisible subgraphs is not known, if k is not a prime power, but one can prove that if the graph has sufficiently large number of edges, there exists a k-divisible subgraph.

Our main results

New proofs via Combinatorial Nullstellensatz

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Suppose that $f_1, f_2, \ldots, f_n$ are m-variable polynomials over $\mathbb{Z}$ without constant terms. Then, if

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Theorem

Finding a 2^d-divisible subgraph and Combinatorial Nullstellensatz MOD 2 belong to PPA.

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This example can be extended to any polynomial f and prime power p^d.

Combinatorial Nullstellensatz MOD 2

In the rest of this presentation, we focus on PPA and the complexity of Combinatorial Nullstellensatz MOD 2.

Theorem (Combinatorial Nullstellensatz MOD 2)

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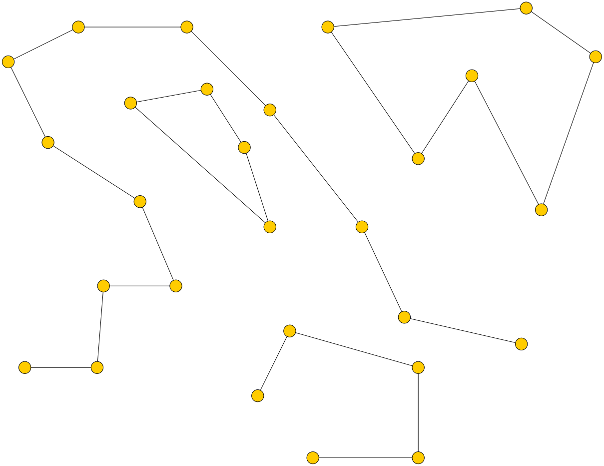
- we verify this conjecture

Reminder about Polynomial Parity Argument

In '94, Papadimitriou defined the complexity class Polynomial Parity Argument.

A problem is in PPA if and only if it is reducible to the End Of The Line.



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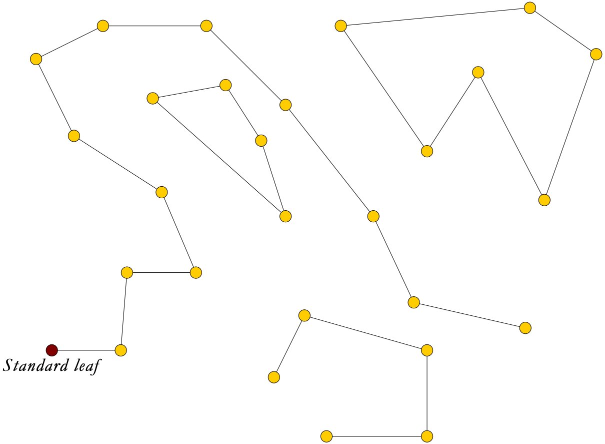

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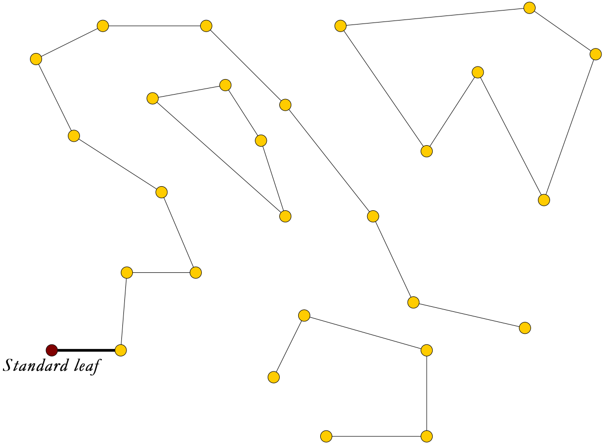

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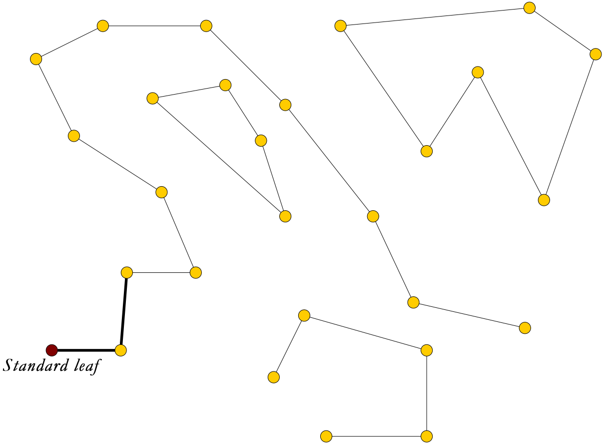

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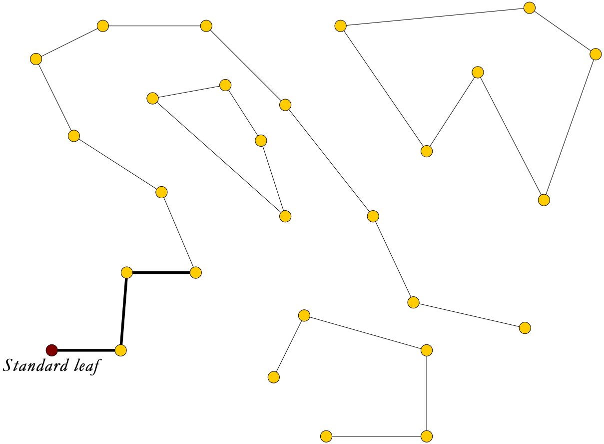

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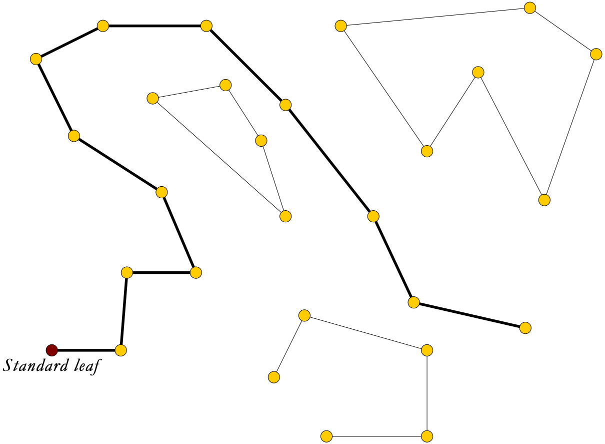

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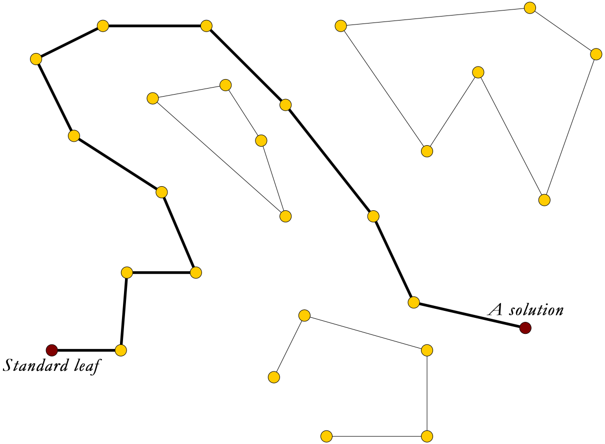

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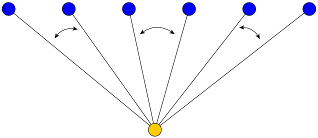
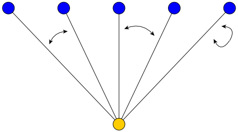


For an odd-degree node:



PPA membership of Chévalley's theorem

Theorem (Chévalley)

Let $p_1, p_2, \ldots, p_n$ be polynomials in m variables over $\{0, 1\}$. Suppose that $\sum_{i=1}^{n} \deg(p_i) < m$. Then, the number of common solutions of the polynomial equation system $p_i(x_1, \ldots, x_m) = 0$ $(i = 1 \ldots n)$ is even. In particular, if there is a solution, there exists another.

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Chévalley MOD 2

Input: polynomials $p_1, p_2, \ldots, p_n$ over $\{0,1\}$ such that $\sum_{i=1}^n \deg(p_i) < m$. Also, we are given a root $(c_1, c_2, \ldots, c_m) \in \{0,1\}^m$ of the equation system $p_i(\mathbf{x}) = 0$ $(i = 1, \ldots, n)$

Find: another root of the equation system $p_i(\mathbf{x}) = 0$ $(i = 1, \ldots, n)$.

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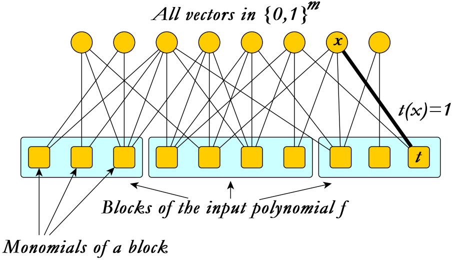
Papadimitriou showed that Chévalley MOD 2 belongs to PPA. Our following proof about Combinatorial Nullstellensatz is based on his proof but it requires trickier pairing function.

The construction of End Of The Line graph



The input polynomial: $f = \sum_{i=1}^{k} \left(\prod_{j=1}^{m_i} p_{ij} \right)$.

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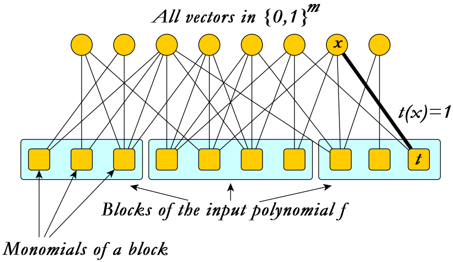

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It has k blocks: $\prod_{j=1}^{m_1} p_{1j}, \prod_{j=1}^{m_2} p_{2j}, \ldots, \prod_{j=1}^{m_k} p_{kj}$.

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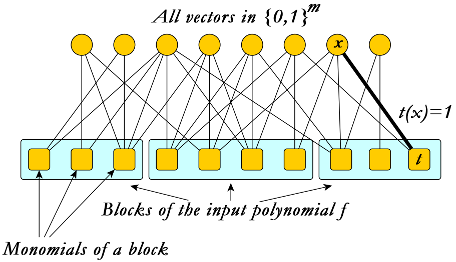

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A monomial (term) in the ith block can be represented by an $(m_i + 1)$-tuple of integers: $(i, a_{i,1}, \ldots, a_{i,m_i})$. $a_{i,j}$ shows that the term is the product of $a_{i,j}$th monomials of p_{ij}.

E.g. in $(1 + x_1)(1 + x_2)$ $(i, 1, 1) \sim 1, (i, 1, 2) \sim x_2, (i, 2, 1) \sim x_1, (i, 2, 2) \sim x_1 x_2$.

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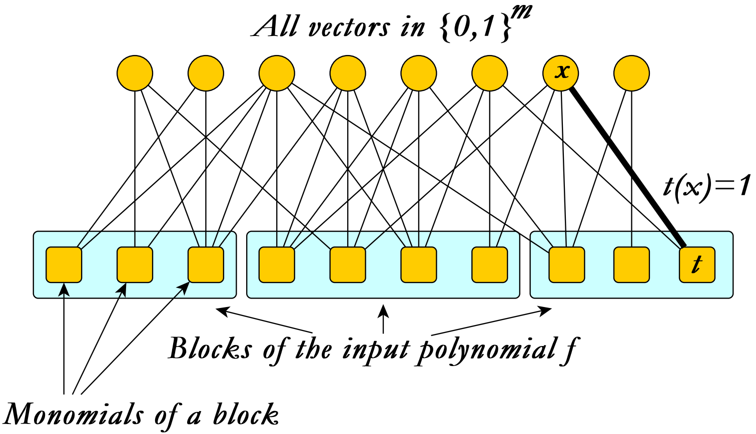

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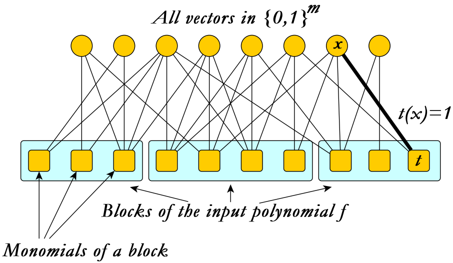
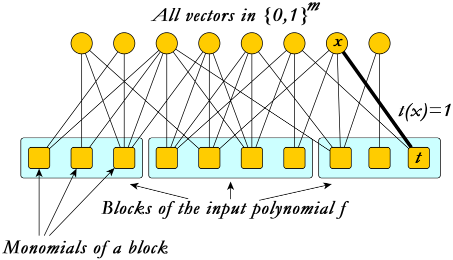


Edges:

A vector x is connected to a term t if and only if the value of t is 1 at x.

The construction of End Of The Line graph



For a vector $(s_1, s_2, \ldots, s_m)$, $f(s_1, s_2, \ldots, s_m) \neq 0$ holds if and only if in the constructed graph its degree is odd.

The degree of a term $t(\mathbf{x}) \neq x_1 \ldots x_m$ is even because there exists a variable x_i not appearing in t. The degree of term term $x_1 \ldots x_m$ is odd because it is connected only to the vector $(1, 1, \ldots, 1)$.

The standard leaf is the term $x_1 \ldots x_m$. Another leaf is a solution.

Pairing for a term t

However, the nodes of this graph have exponentially large degrees, and therefore we must exhibit a pairing function between the edges out of a node.

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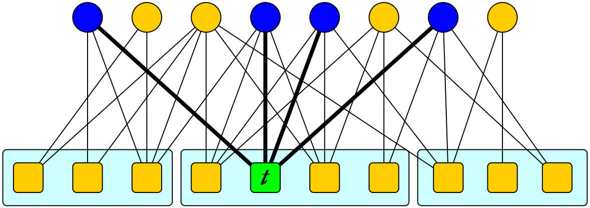
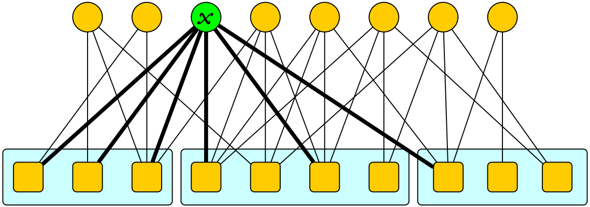


For a term t

For a node corresponding to the term $t(\mathbf{x}) \neq x_1 x_2 \ldots x_m$, we pair up the vector $\mathbf{x}$ via the variable x_l is such that does not appear in t.

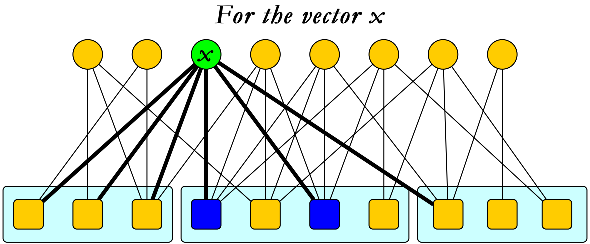
Pairing at a vector x

For the vector x



Suppose that $f(\mathbf{x}) = 0$. The case $f(\mathbf{x}) = 1$ can be checked similarly.

Pairing at a vector x



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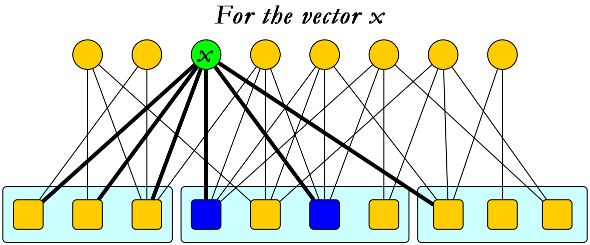
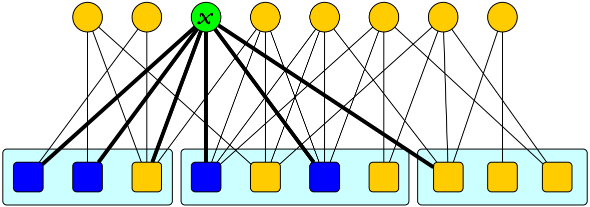


For a block $g = \prod_{j=1}^{m_i} p_{ij}$ such that $g(\mathbf{x}) = 0$, then there is an index j such that $p_{ij}(\mathbf{x}) = 0$. Pick the smallest such j. There is an even number of monomials of p_{ij} such that $p_{ij}(\mathbf{x}) = 1$. We pair these monomials by a pairing function ϕ_i. Then the mate of term $(i, a_{i1}, \ldots, a_{ij}, \ldots, a_{i,m_i})$ is $(i, a_{i1}, \ldots, \phi_i(a_{ij}), \ldots, a_{i,m_i})$.

Pairing at a vector x

For the vector x



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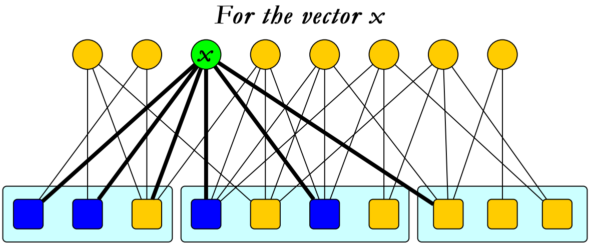
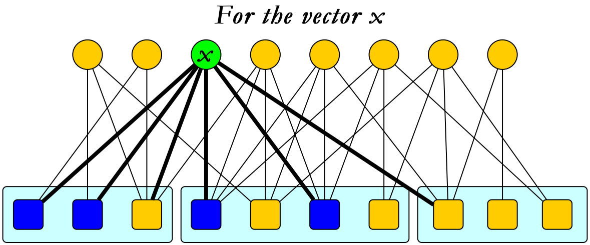


For a block $g = \prod_{j=1}^{m_i} p_{ij}$ such that $g(\mathbf{x}) = 1$, then for all index j, that $p_{ij}(\mathbf{x}) = 1$ holds. We can pair all but one monomials of p_{ij} with $p_{ij}(\mathbf{x}) = 1$ by a pairing function ϕ_{ij}. One of them does not have a mate, denote its index by ω_{ij}.

Pairing at a vector x



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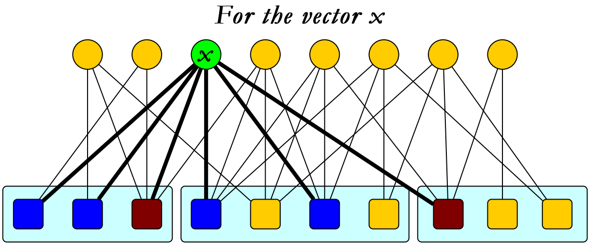
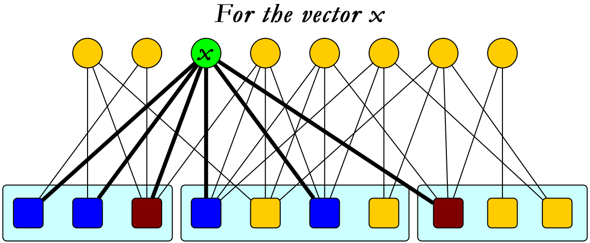
If there exists an index j such that $a_{ij} \neq \omega_{ij}$, pick the smallest such j. Then the mate of $(i, a_{i1}, \ldots, a_{i,m_i})$ is $(i, a_{i1}, \ldots, \phi_{ij}(a_{ij}), \ldots, a_{i,m_i})$.

Pairing at a vector x



What about the term t if it is represented by $(i, \omega_{i1}, \ldots, \omega_{i,m_i})$?

Pairing at a vector x



What about the term t if it is represented by $(i, \omega_{i1}, \ldots, \omega_{i,m_i})$?

Since $f(\mathbf{x}) = 0$, there is an even number of blocks that are 1 at $\mathbf{x}$. We pair these blocks by a pairing function ϕ.

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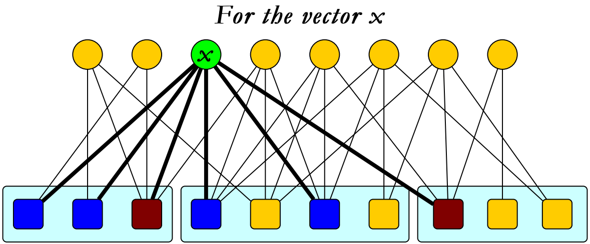

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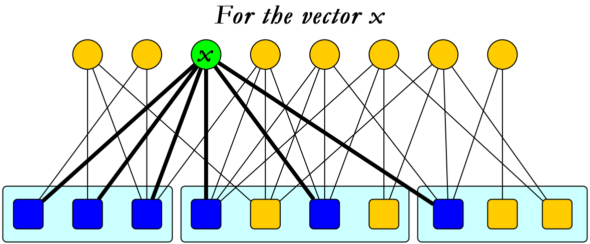
Then, the mate of $(i, \omega_{i1}, \ldots, \omega_{i,m_i})$ is $(\phi(i), \omega_{\phi(i),1}, \ldots, \omega_{\phi(i),m_{\phi(i)}})$.

Pairing at a vector x



The key idea is this upper-level pairing function which pair up such blocks.

Pairing at a vector x

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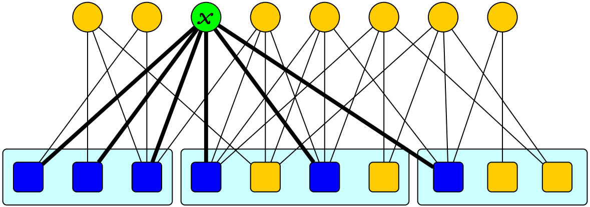

The key idea is this upper-level pairing function which pair up such blocks.

So, we presented a polynomial algorithm that computes the mate of an edge out of a node, and therefore we reduced Combinatorial Nullstellensatz MOD 2 to the End Of The Line, so the proof is complete.

2^d-divisible subgraph.

Input: a positive integer d and a graph $G = (V, E)$, where $|V| = n$, $|E| = m$ and $m > n \cdot (2^d - 1) - 2^{d-1}$.

Find: a 2^d-divisible subgraph, that is, an $\emptyset \neq F \subseteq E$ such that for every $v \in V$, the number of incident edges of F is divisible by 2^d.

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Theorem

Finding a 2^d-divisible subgraph is polynomially reducible to Combinatorial Nullstellensatz, hence it belongs to PPA.

Thank you for your attention!

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