Convex programmes for linear Arrow-Debreu markets

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joint work with
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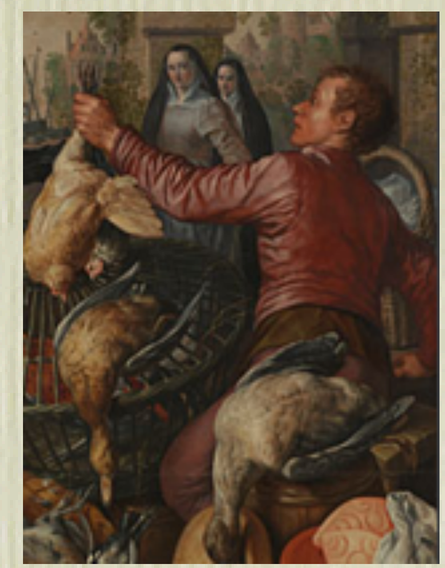
Linear exchange markets

Léon Walras, 1874



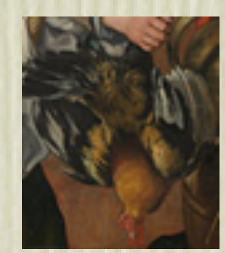
Linear exchange markets

- Set of agents A arriving to the market with an initial endowment of divisible goods.
- Without loss of generality: 1 agent ↔ 1 good.
- U_{ij} utility of agent i on the entire unit of good j.
- Market equilibrium: prices p_i and allocations of goods to agents x_{ij} such that every agent spends exactly her income in a way that maximises her utility for the given prices.

$U_{11}=4$
$p_{11}=2$
2 ✓

$U_{12}=4$
$p_{12}=3$
1.33 ✗

$U_{13}=6$
$p_{13}=3$
2 ✓



$U_{14}=0$
$p_{14}=1$
0 ✗

Arrow-Debreu Theorem 1954

- Market equilibrium exists even in the more general case of convex utilities
- Linear Exchange Market = Linear Arrow-Debreu market
- Proof based on fixed point theorems.
- Basic question of equilibrium computation: when is it possible to compute an equilibrium efficiently?

Our contribution

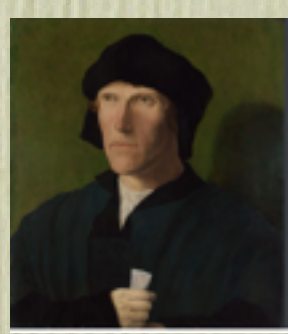
- We formulate a new, rational convex programme that describes equilibria for linear Arrow-Debreu markets. It gives
  - simple proof of existence,
  - simple proof of rationality,
  - establishes links to known programmes for linear Fisher markets.

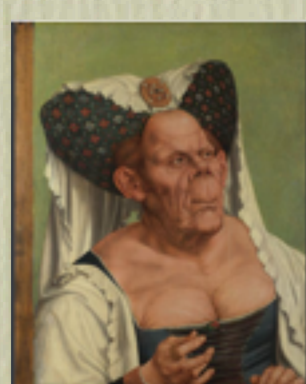
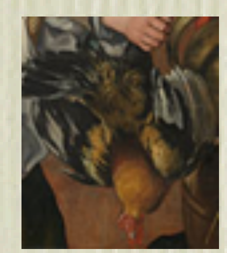
Linear Fisher markets

Irving Fisher, 1891

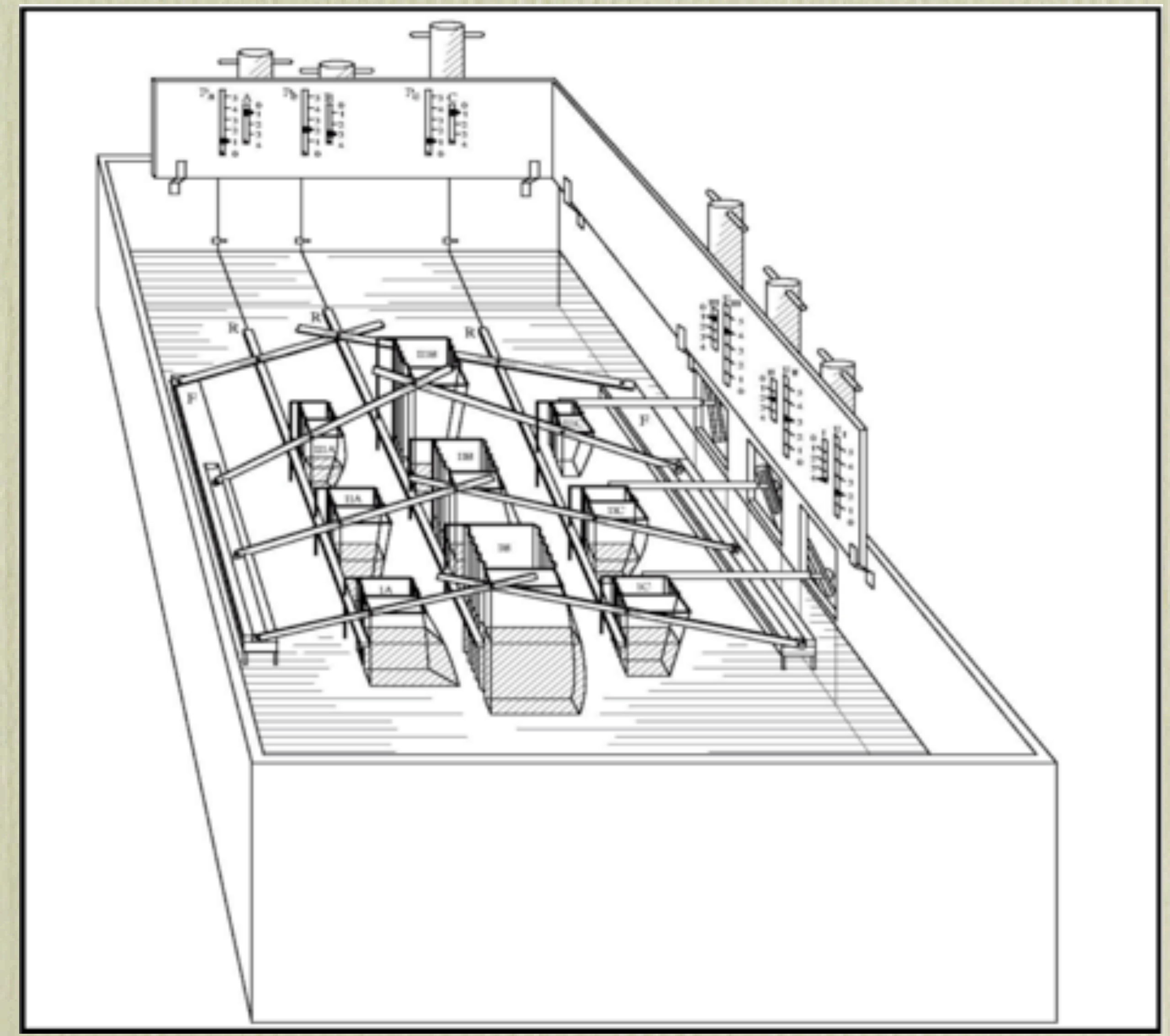
Reduction: add banker agent with special good corresponding to money

- Special case of the linear Arrow-Debreu market
- B: buyers and G: goods
- Buyer i has a budget m_i, and 1 divisible unit of each good j
- U_{ij}: utility of buyer i on good j
- **Market equilibrium:** prices p_i and allocations x_{ij} such that
  - everything is sold
  - all the money is spent
  - every buyer maximises her utility w.r.t the given prices.




Fisher's method to compute equilibrium



Eisenberg-Gale convex programme, 1959

$$\max \sum_{i \in B} m_i \log U_i$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

prices: optimal Lagrange multipliers

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$

- Optimal solutions correspond to equilibrium prices.
- There exists a rational optimal solution.

Different convex programme
Shymrev; Devanur 2009

y_{ij} : amount of money payed by agent i for good j

$$\min \sum_{i \in G} p_j \log p_j - \sum_{ij \in E} y_{ij} \log U_{ij}$$

$$\sum_{j \in G} y_{ij} = m_i \quad \forall i \in B$$

$$\sum_{i \in B} y_{ij} = p_j \quad \forall j \in G$$

$$y \geq 0$$

- Optimal solutions correspond to equilibrium prices.
- There exists a rational optimal solution.

Combinatorial algorithms

- Devanur, Papadimitriou, Saberi, Vazirani '02: polynomial time combinatorial algorithm using max-flow techniques
- Several extensions studied over the last decade
- Strongly polynomial algorithms: Orlin '10, V. '12b
- Rational convex programmes (Vazirani): convex programme with rational optimum

General frameworks (V. '12a&12b)

Eisenberg-Gale

x_{ij} : amount of good j purchased by i

$$\max \sum_{i \in B} m_i \log U_i$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x \geq 0$$

Concave generalised flow

V12a: polynomial time combinatorial algorithm

Shmyrev

y_{ij} : amount of money payed by agent i for good j

$$\min \sum_{i \in G} p_j \log p_j - \sum_{ij \in E} y_{ij} \log U_{ij}$$

$$\sum_{j \in G} y_{ij} = m_i \quad \forall i \in B$$

$$\sum_{i \in B} y_{ij} = p_j \quad \forall j \in G$$

$$y \geq 0$$

Flow with separable convex objective

V12b: strongly poly algorithm under certain assumptions

Shymrev's convex programme

y_{ij} : amount of money payed by agent i for good j

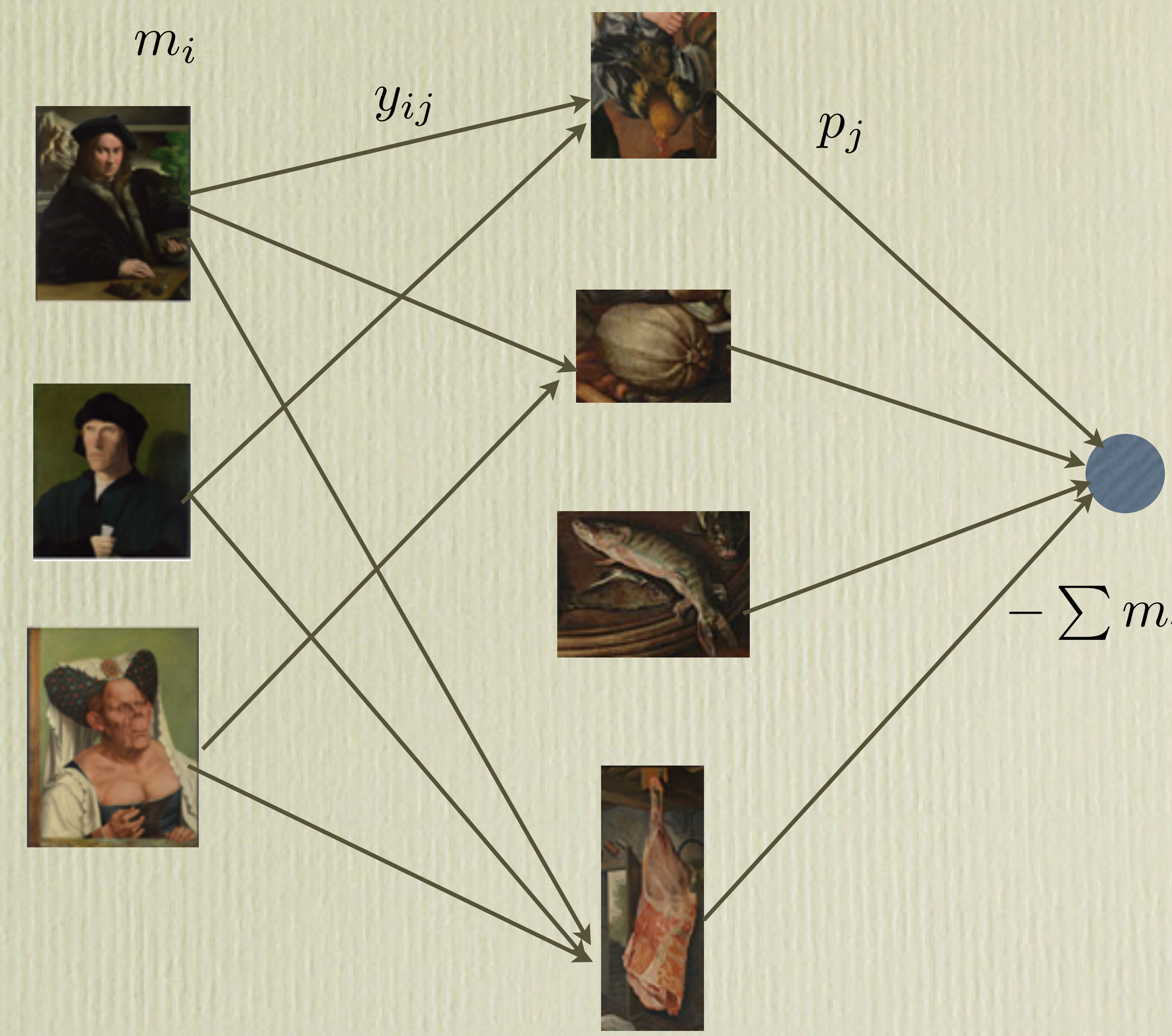
$$\min \sum_{i \in G} p_j \log p_j - \sum_{ij \in E} y_{ij} \log U_{ij}$$

$$\sum_{j \in G} y_{ij} = m_i \quad \forall i \in B$$

$$\sum_{i \in B} y_{ij} = p_j \quad \forall j \in G$$

$$y \geq 0$$

Flow with separable convex cost



Linear Arrow-Debreu markets

- Set of agents *A*
- Every agent owns the full 1 unit of one of the goods






<u>NOTE:</u> Money can be rescaled. If $(p_1,p_2,\ldots,p_n)$ is an equilibrium, then $(tp_1,tp_2,\ldots,tp_n)$ is also an equilibrium for any $t>0$.

Linear Arrow-Debreu markets: early history

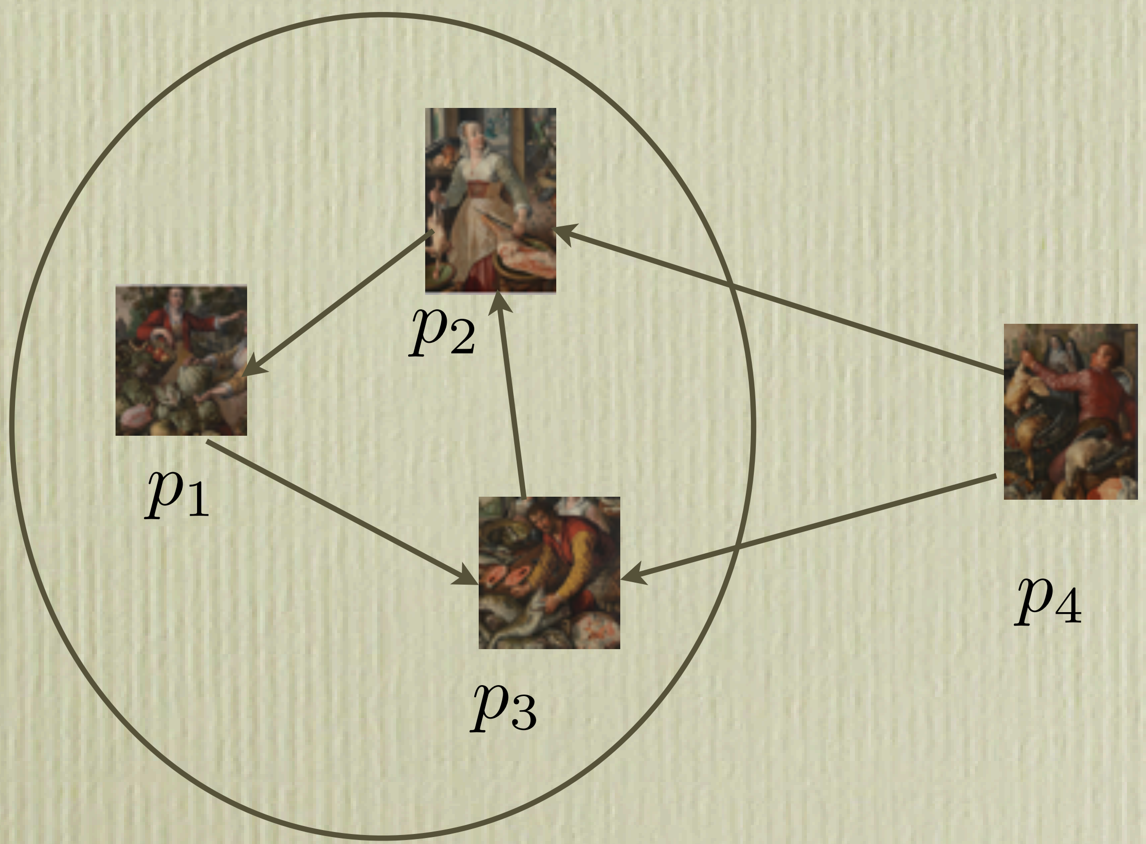
- No convex programme/polynomial time algorithm was known for a long time, existence only based on fixed point theorems
- Gale '76: sufficient and necessary conditions on existence
- Eaves '76: Lemke-type path following algorithm to compute equilibrium
  - proves that there exists a rational optimal solution!

Conditions for existence of equilibrium

G=(A,E): directed graph of arcs *ij* for $U_{ij}>0$

<u>THEOREM</u> (Gale 1957)
If G is strongly connected then there exists an equilbirium.

<u>THEOREM</u> (Gale 1976; simplified)
There exists an equilibrium if and only if every singleton strongly connected component of *G* has a self-loop.



<u>Proof.</u> (necessity)
In an equilibrium, $p_i>0$ for all agents

$$\text{inflow} = p_1 + p_2 + p_3 + p_4 > p_1 + p_2 + p_3 = \text{outflow}$$

Convex programmes - a convoluted history...

- Jain '2004: feasibility convex programme
- Polynomial time algorithm based on the Ellipsoid method

$$q_i - q_j \le \log\left(\sum_{k:ik\in E} U_{ik}x_{ik}\right) - \log U_{ij} \quad \forall ij \in E$$

$$\sum_{j:ji\in E} x_{ji} = 1 \quad \forall i \in A$$

$$x \ge 0$$

$$\frac{U_{ij}}{p_j} \le \frac{U_i}{p_i}$$

For a fe

equili

Question (Vazirani): is there a rational convex programme?

Disadvantages:

- Does not prove existence of feasible solution: needs theorem on existence
- Does not prove rationality.
- The same was already given by Nenakov&Primak in Russian in 1983!

A surprising discovery...

Cornet, 1989 (unpublished tech.report)

$$\max\ t$$

$$U_{ij} e^{q_i - q_j} + t \leq \sum_{k:ik \in E} U_{ik} x_{ik} \quad \forall ij \in E$$

$$\sum_{j:ji \in E} x_{ji} \leq 1 \quad \forall i \in A$$

$$x \geq 0$$

Remarks:

- First (?) convex programme to prove existence of equilibrium.
- Proof uses nontrivial argument on Lagrangian duality.
- We get Jain's programme for t=0.

THEOREM I

If this programme is bounded, then the optimum value is 0 and we get an equilibrium with assignments x_{ij} and prices e^{q_i}.

THEOREM II

If G is strongly connected then this programme is bounded.

Our new convex programme

$$\min \sum_{i \in A} p_i \log \frac{p_i}{\beta_i} - \sum_{ij \in E} y_{ij} \log U_{ij}$$

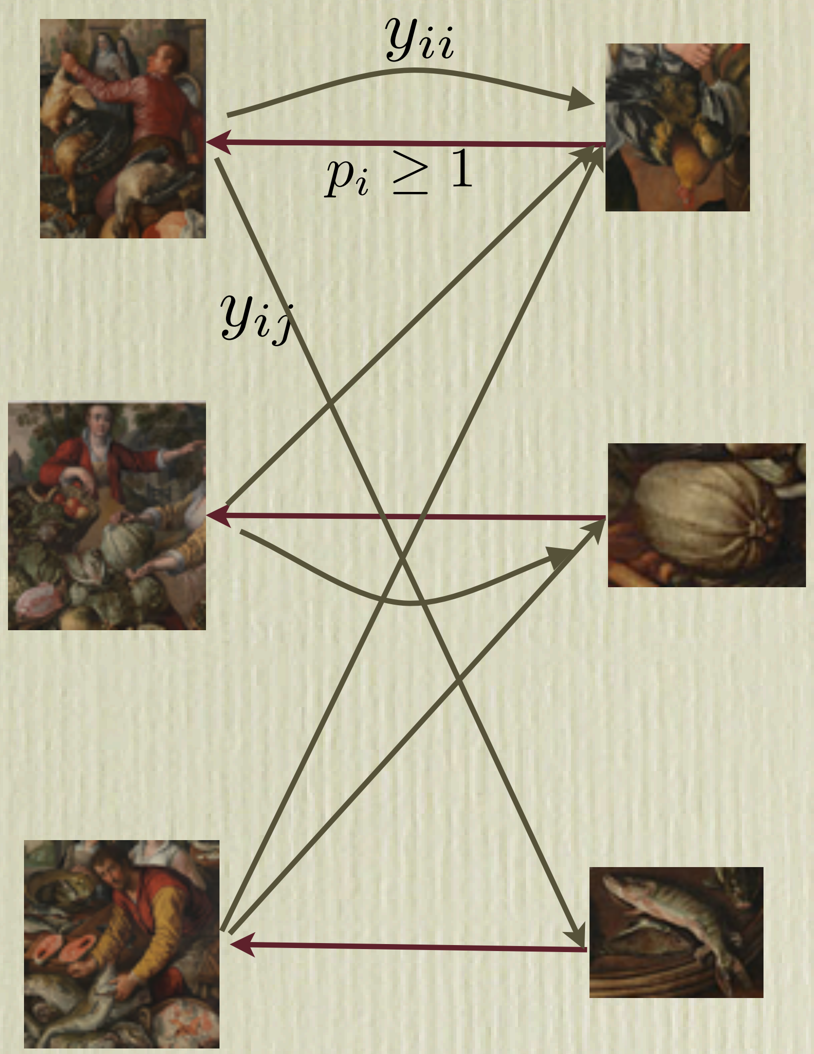
$$\sum_{i:ij \in E} y_{ij} = p_j \quad \forall j \in A$$

$$\sum_{j:ij \in E} y_{ij} = p_i \quad \forall i \in A$$

$$U_{ij}\beta_i \leq p_j \quad \forall ij \in E$$

$$p_i \geq 1 \quad \forall i \in A$$

$$y, \beta \geq 0$$



Circulation polyhedron

THEOREM:

- *Feasible if and only if there exists an equilibrium (Gale's condition).*
- *Optimum value = 0.*
- *Every optimal solution corresponds to an equilibrium and vice versa (up to scaling).*

Comparison with programmes for Fisher market

$$\min \sum_{i \in A} p_i \log \frac{p_i}{\beta_i} - \sum_{ij \in E} y_{ij} \log U_{ij}$$

$$\frac{p_i}{\beta_i} = U_i$$

$$\sum_{i:ij \in E} y_{ij} = p_j \quad \forall j \in A$$

$$m_j$$

$$\sum_{j:ij \in E} y_{ij} = p_i \quad \forall i \in A$$

$$U_{ij}\beta_i \leq p_j \quad \forall ij \in E$$

$\beta_i = 1/$best bang per buck

$$p_i \geq 1 \quad \forall i \in A$$

$$y, \beta \geq 0$$

$$\max \sum_{i \in B} m_i \log U_i$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x \geq 0$$

Eisenberg-Gale

$$\min \sum_{i \in G} p_j \log p_j - \sum_{ij \in E} y_{ij} \log U_{ij}$$

$$\sum_{j \in G} y_{ij} = m_i \quad \forall i \in B$$

$$\sum_{i \in B} y_{ij} = p_j \quad \forall j \in G$$

$$y \geq 0$$

Shmyrev

Our new convex programme

$$\min \sum_{i\in A} p_i \log \frac{p_i}{\beta_i} - \sum_{ij\in E} y_{ij} \log U_{ij}$$
$$\sum_{i:ij\in E} y_{ij} = p_j \quad \forall j \in A$$
$$\sum_{j:ij\in E} y_{ij} = p_i \quad \forall i \in A$$
$$U_{ij}\beta_i \le p_j \quad \forall ij \in E$$
$$p_i \ge 1 \quad \forall i \in A$$
$$y, \beta \ge 0$$

THEOREM:

- *Feasible if and only if there exists an equilibrium (Gale's condition).*
- *Optimum value = 0.*
- *Every optimal solution corresponds to an equilibrium and vice versa (up to scaling).*

LEMMA:
The objective value is 0 if and only if the solution describes a market equilibrium.

Proof:

$$-\log U_{ij} \ge \log \beta_i - \log p_j$$

$$\sum_{i\in A} p_i \log \frac{p_i}{\beta_i} - \sum_{ij\in E} y_{ij} \log U_{ij} \ge$$
$$\sum_{i\in A} p_i \log \frac{p_i}{\beta_i} + \sum_{ij\in E} y_{ij} (\log \beta_i - \log p_j) =$$
$$\sum_{i\in A} p_i \log \frac{p_i}{\beta_i} + \sum_{i\in A} p_i \log \beta_i - \sum_{j\in A} p_j \log p_j = 0$$

Comparison with Cornet's programme

Our new programme	Cornet
• Feasible region nonempty ⇔ every singleton strongly connected component of G has a loop ⇔ there exists an equilibrium.	• Always feasible. Proof of boundedness only if G is strongly connected.
• Linear feasible region	• Nonlinear constraints.
• Rationality: there always exists an optimal extreme point solution.	• No proof of rationality.

answers Vazirani's question on the rational convex programme.

Simple corollaries

- In any two equilibrium solutions, each player gets the same amount of utility. (Gale '76, Cornet '89).
- The set of equilibrium prices is a convex polyhedral cone (Mertens '03, Florig '04) - both > 10 pages proofs
- The set of equilibrium money transfers is convex. In contrast, Cornet showed that the convexity of the allocations of goods.

Lagrangian duality

$$\min \sum_{i \in A} p_i \log \frac{p_i}{\beta_i} - \sum_{ij \in E} y_{ij} \log U_{ij}$$

$$\sum_{i:ij \in E} y_{ij} = p_j \quad \forall j \in A$$

$$\sum_{j:ij \in E} y_{ij} = p_i \quad \forall i \in A$$

$$U_{ij}\beta_i \le p_j \quad \forall ij \in E$$

$$p_i \ge 1 \quad \forall i \in A$$

$$y, \beta \ge 0$$

$$\min \sum_{i \in A} \tau_i$$

$$-\delta_j + \gamma_i \le -\log U_{ij} \quad \forall ij \in E$$

$$\tau_i + \delta_i - \gamma_i + \sum_{j:ji \in E} w_{ji} \le 1 + \log \sum_{j:ij \in E} U_{ij} w_{ij} \quad \forall i \in A$$

$$\tau, w \ge 0$$

- Dual: similar to Cornet's, but different
- "Self duality": a market equilibrium provides optimum solution to both programmes
- Proof of the main theorem: nontrivial argument on the KKT-conditions.

Polynomial time algorithms

- Jain: Ellipsoid method using the convex programme.
- Ye '08: efficient interior point algorithm using Jain's programme
- Duan&Mehlhorn '13: combinatorial algorithm
  - based on DPSV algorithm for linear Fisher
  - doesn't rely on convex programmes
- No strongly polynomial algorithm is known

Future work

- Develop a strongly polynomial algorithm
  - Our programme might be a useful tool
  - Identify a more general class of convex programmes where such an algorithm could work:
    - V '12b: strongly polynomial algorithm for minimum cost flows with separable convex objectives (under some oracle assumptions).

Thank you for your attention!