Strategic Characterization of the Index of an Equilibrium

Arndt von Schemde

Bernhard von Stengel

Department of Mathematics
London School of Economics

Battle of the sexes

he \ she	football	opera
football	2 4	0 0
opera	0 0	4 2

Battle of the sexes with in–laws

he \ she	football	opera	opera with her mom
football	2 4	0 0	1 1
opera	0 0	4 2	5 0

Battle of the sexes with in–laws

she / he	football	opera	opera with her mom
football	2 4	0 0	1 1
opera	0 0	4 2	5 0

Battle of the sexes with in–laws

he \ she	football	opera	opera with her mom
football	4, 2	0, 0	1, 1
opera	0, 0	2, 4	0, 5

Battle of the sexes with in–laws

he \ she	football	opera	opera with her mom
football	2 4	0 0	1 1
opera	0 0	4 2	5 0

Make NE unique by adding strategies

Given:

nondegenerate (A,B), Nash equilibrium (NE) (x,y).

Question:

Is there a game G *extending* (A,B) by adding strategies so that (x,y) is the **unique** NE of G?

e.g.: G obtained from (A,B) by adding *columns*, (x,y) becoming $(x,[y, 0,0,\ldots,0])$ for G.

Make NE unique by adding strategies

Given:

nondegenerate (A,B), Nash equilibrium (NE) (x,y).

Question:

Is there a game G *extending* (A,B) by adding strategies so that (x,y) is the **unique** NE of G?

e.g.: G obtained from (A,B) by adding *columns*, (x,y) becoming (x,[y, 0,0,...,0]) for G.

- can be done for **pure**-strategy NE
- but not for **mixed** NE of “battle of the sexes”

Strategic characterization of the index

We will show a conjecture by Josef Hofbauer:

Theorem:

For nondegenerate (A,B), Nash equilibrium (x,y):

$\quad$ index (x,y) = +1

$\Leftrightarrow$ $\exists$ game G *extending* (A,B)
so that (x,y) is the **unique** equilibrium of G.

for m x n game:
G obtained from (A,B) by adding 3m columns.

Sub-matrices of equilibrium supports

Given: nondegenerate (A,B), $A>0$, $B>0$,
Nash equilibrium (x,y).

$A = (a_{ij})$, $B = (b_{ij})$

$A_{xy} = (a_{ij})_{\, i\in \mathrm{supp}(x),\, j\in \mathrm{supp}(y)}$
$B_{xy} = (b_{ij})_{\, i\in \mathrm{supp}(x),\, j\in \mathrm{supp}(y)}$

A_{xy}, B_{xy} have **full rank** $|\mathrm{supp}(x)|$,
nonzero determinants.

Index of an equilibrium (Shapley 1974)

Given: nondegenerate game (A,B), $A>0$, $B>0$,

Nash equilibrium (x,y).

$$\text{index}\,(x,y) = (-1)^{|\text{supp}(x)|+1}\ \text{sign}\ \det(A_{xy}\ B_{xy})$$

$$\in \{ +1, -1 \}$$

Properties of the index

- independent of
  - positive constant added to all payoffs
  - order of pure strategies
  - pure strategy payoffs outside equilibrium support

- pure-strategy equilibria have index +1

- sum of indices over all equilibria is +1

- the two endpoints of any *Lemke-Howson path* are equilibria of opposite index.

Definition of symmetric index

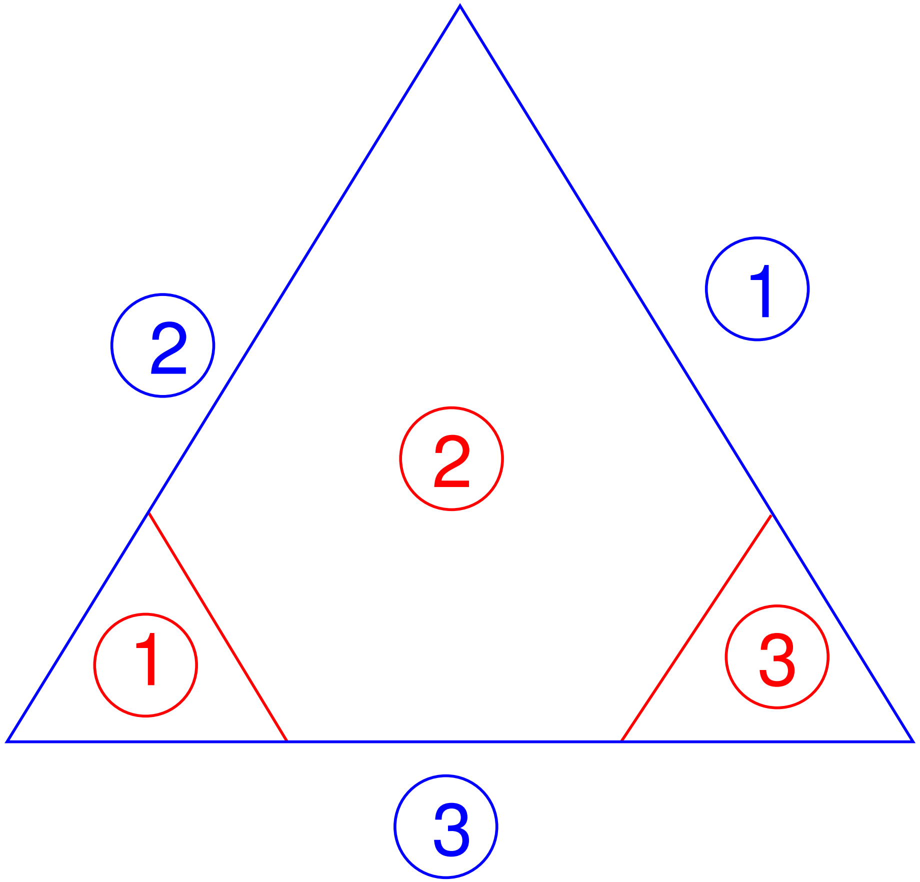
Given: nondegenerate symmetric game (B^T, B), $B > 0$,

SNE (symmetric Nash equilibrium) (x, x).

symmetric index $(x, x) = (-1)^{|\mathrm{supp}(x)|+1}$ sign det(B_{xx})

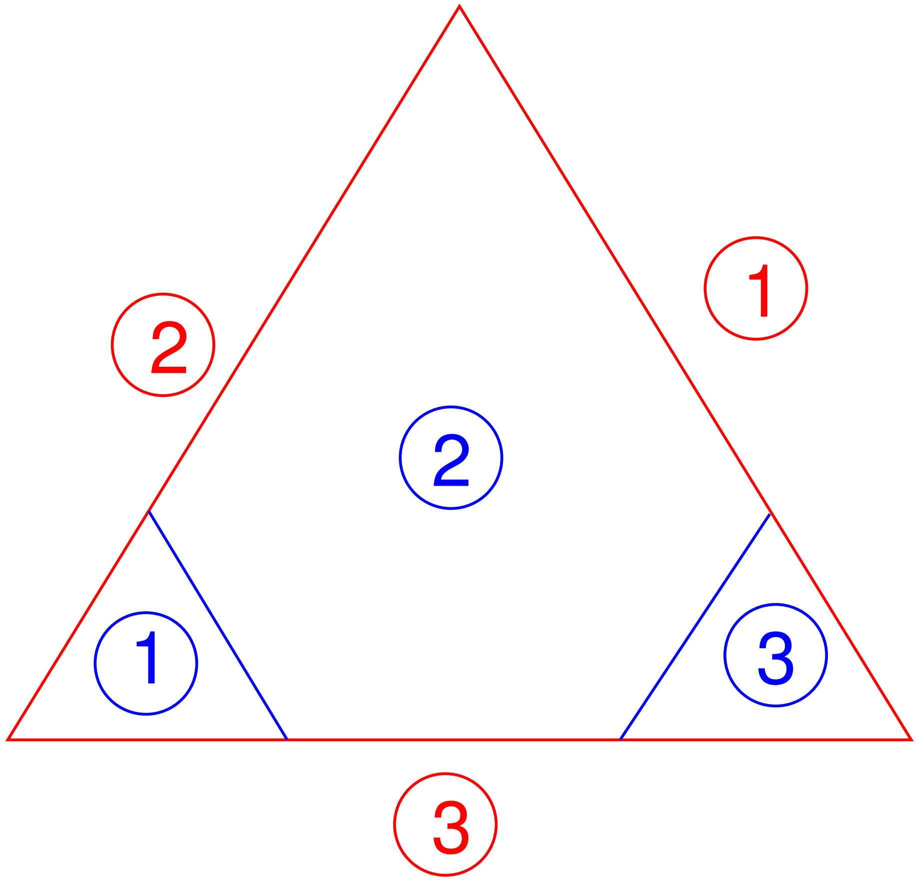
$\in \{ +1, -1 \}$

Symmetric NE of symmetric games



	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

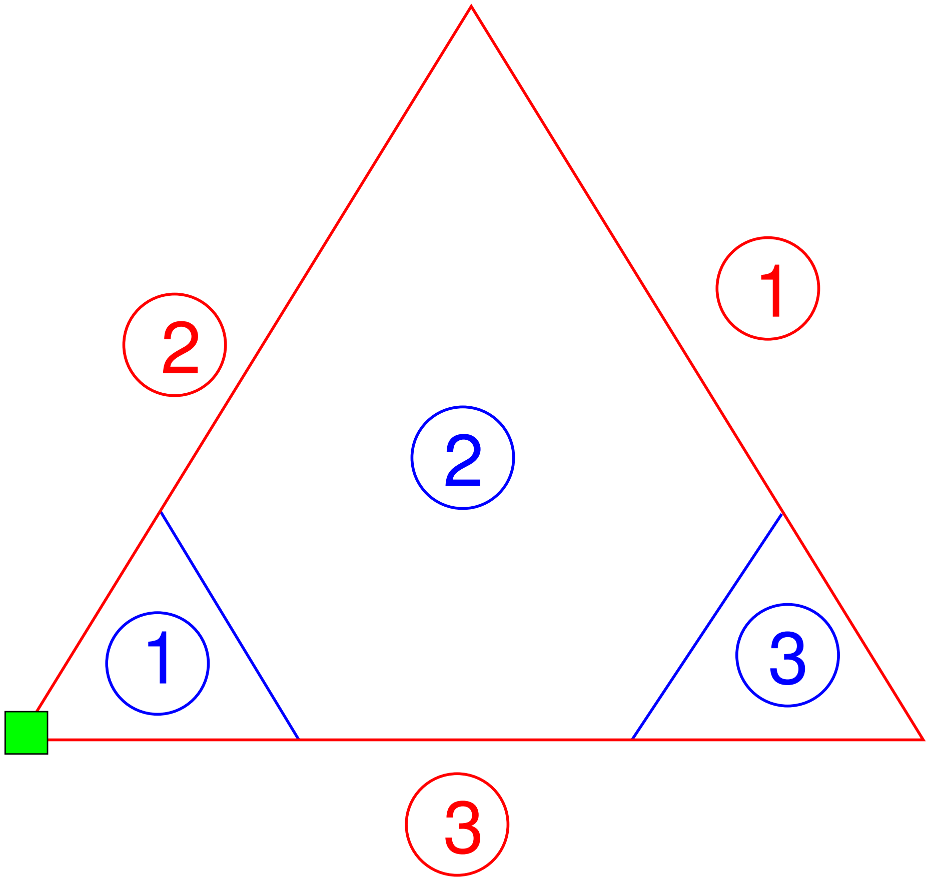
best–reply regions



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

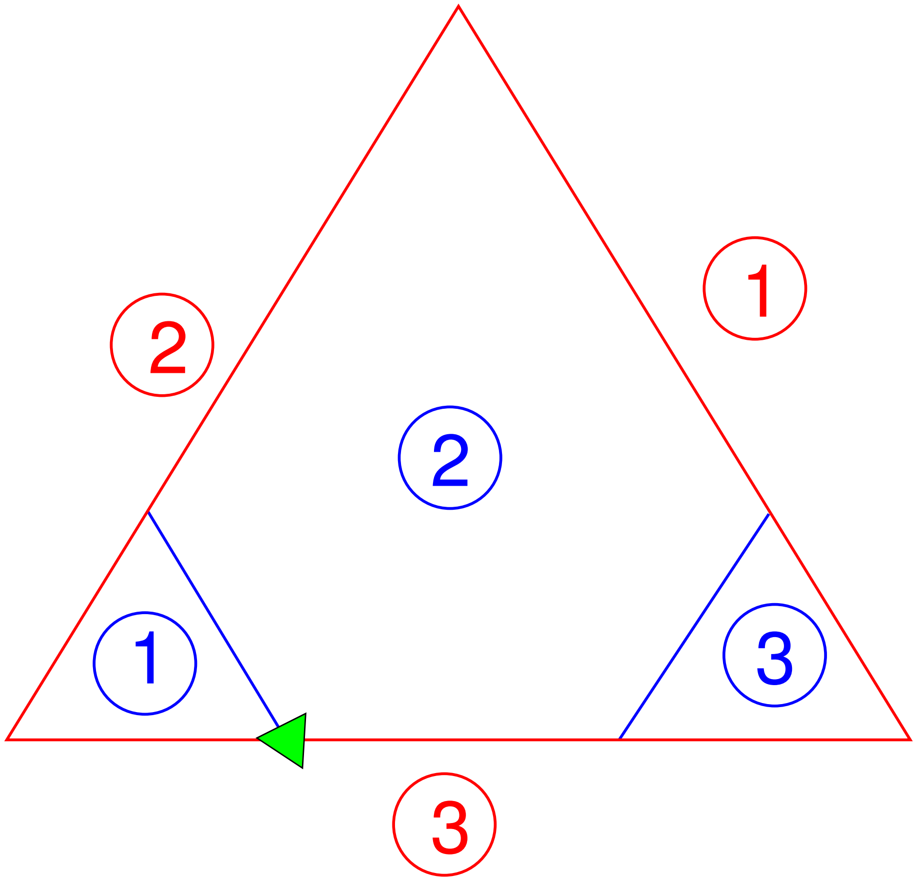
symmetric equilibria



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

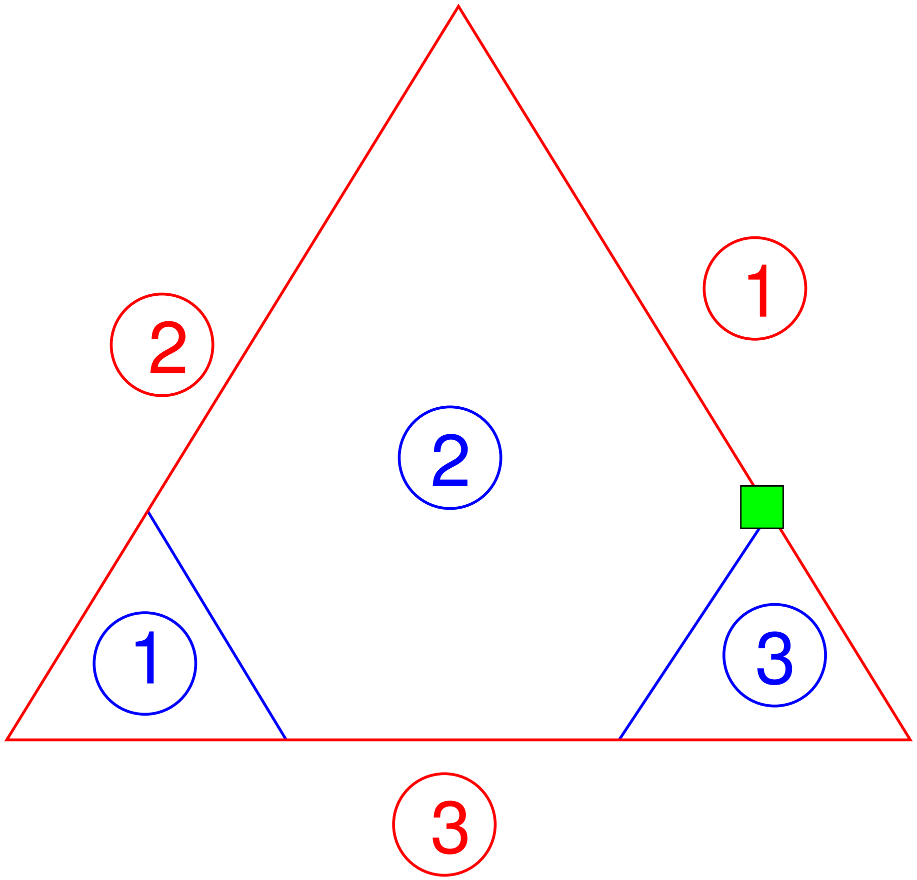
symmetric equilibria



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

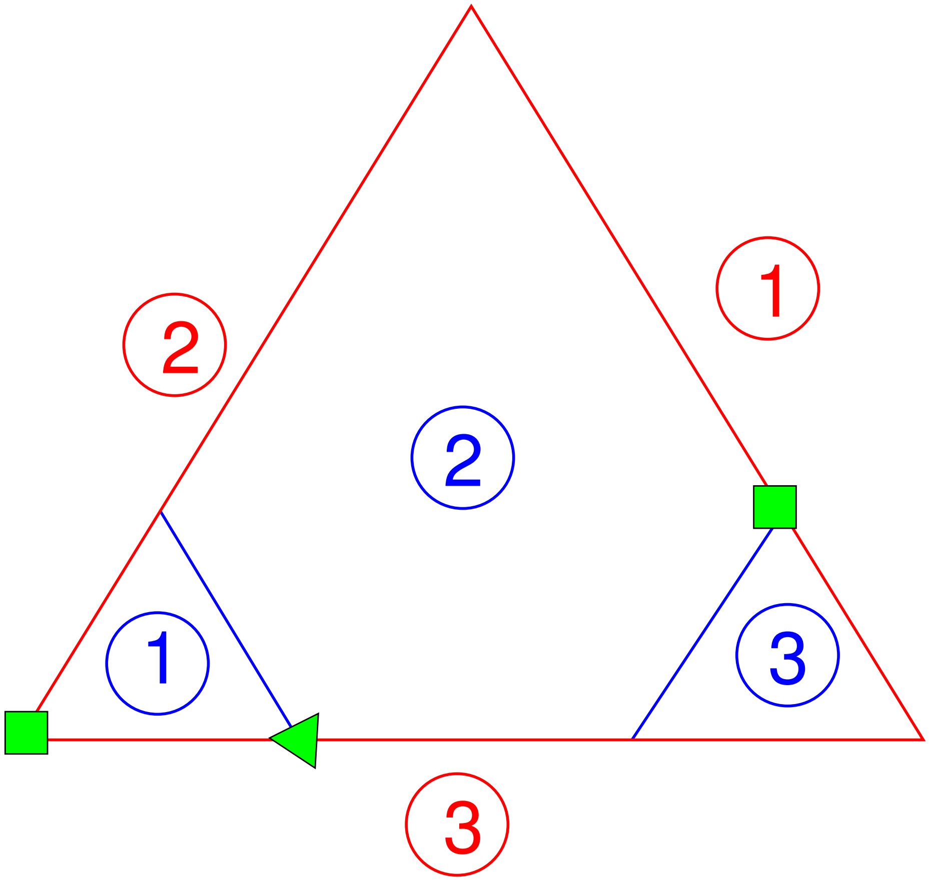
symmetric equilibria



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

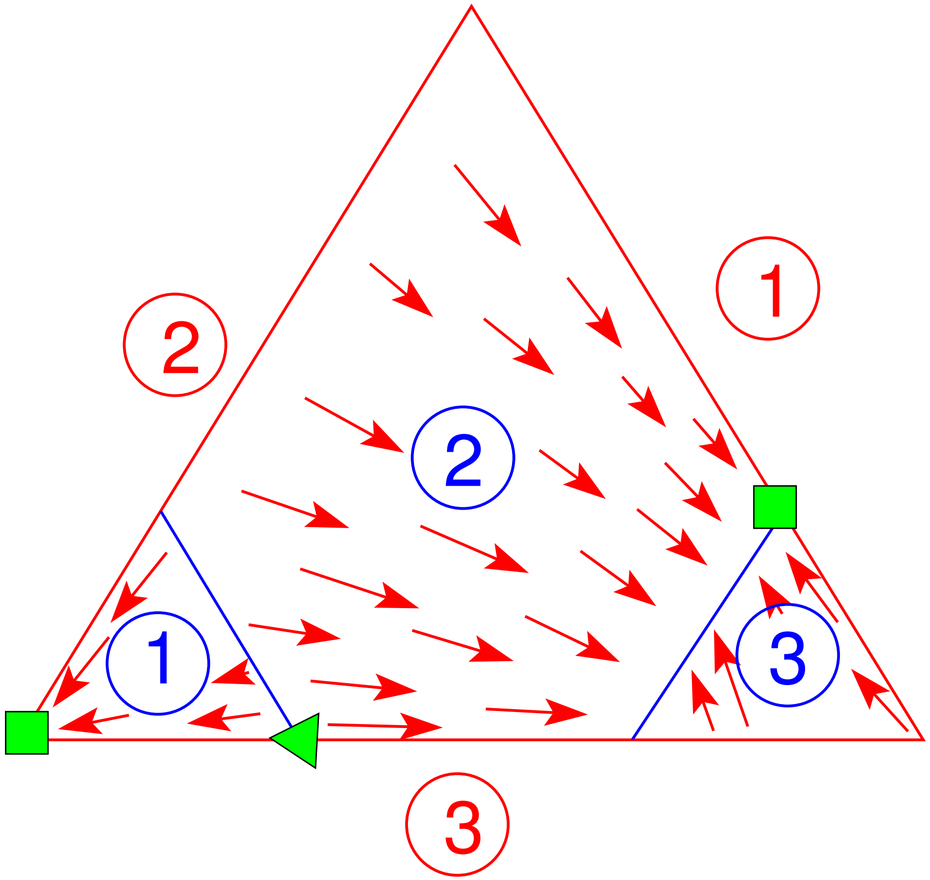
symmetric equilibria



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

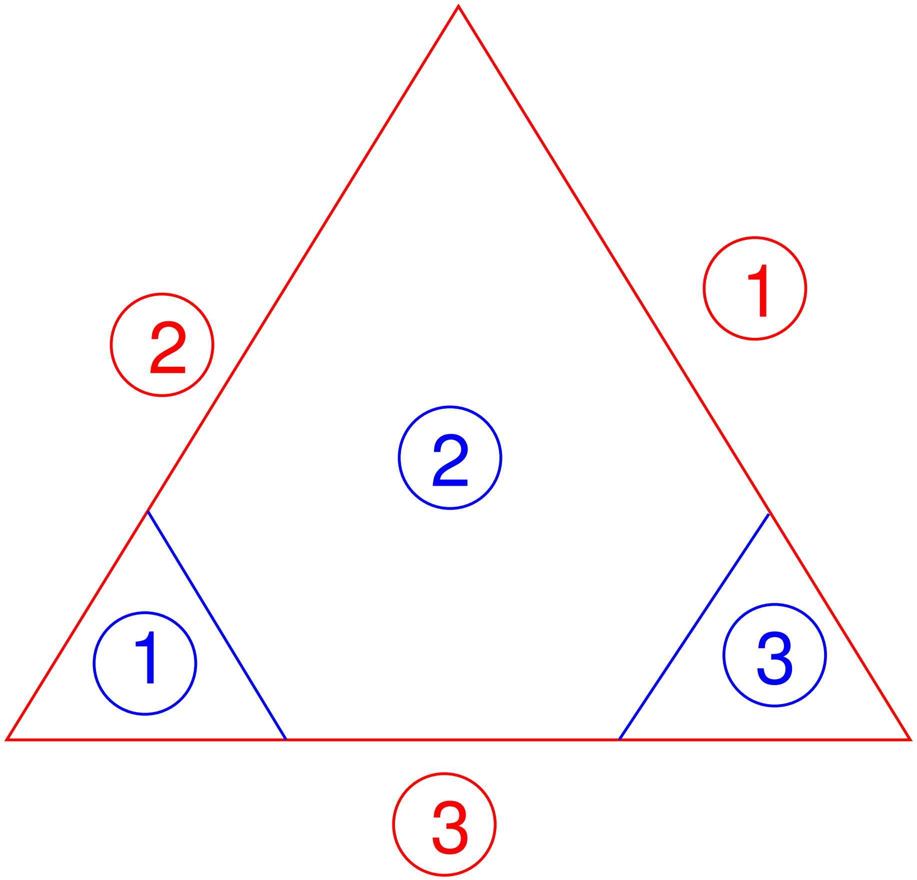
Only ■ dynamically stable, ◀ not



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

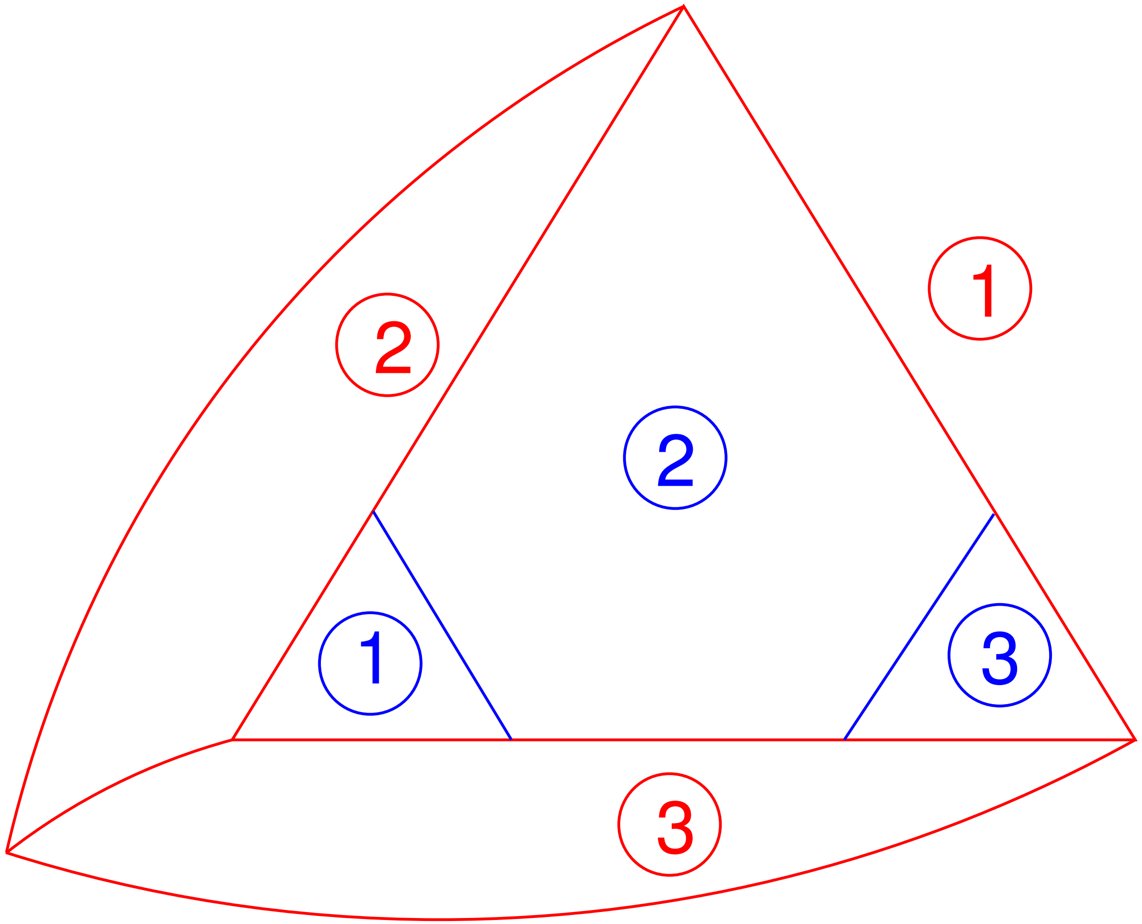
best–reply regions



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

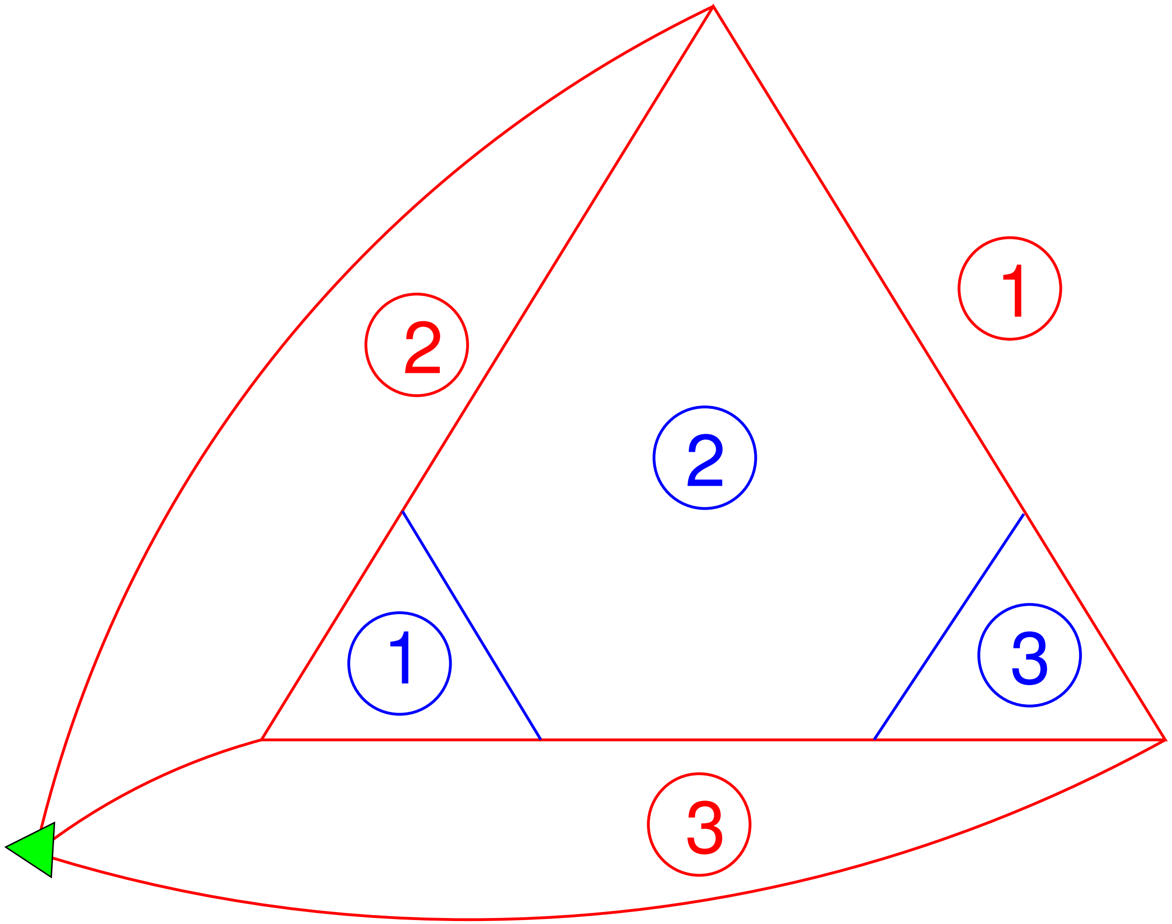
best–reply regions



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

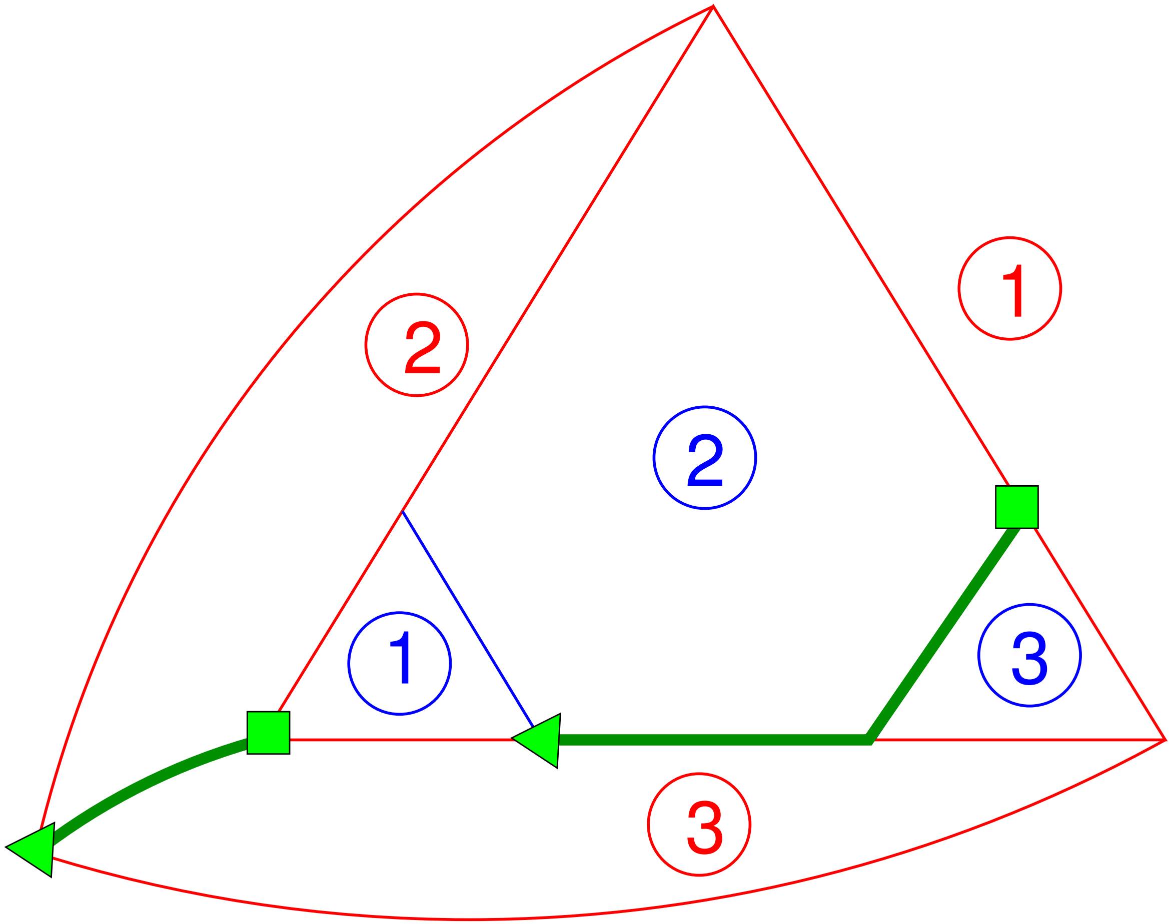
best–reply regions



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

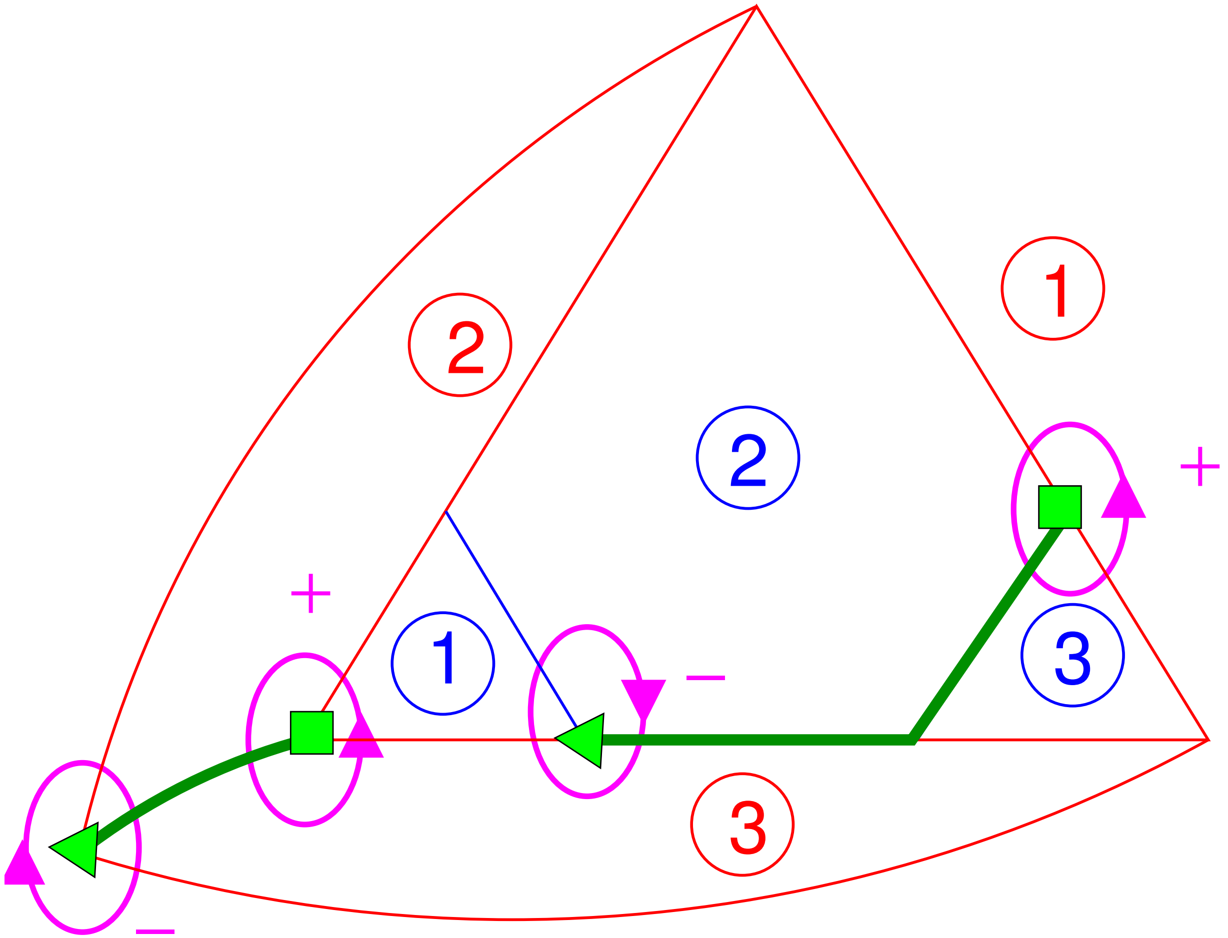
symmetric Lemke–Howson paths



	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

symmetric Lemke–Howson paths



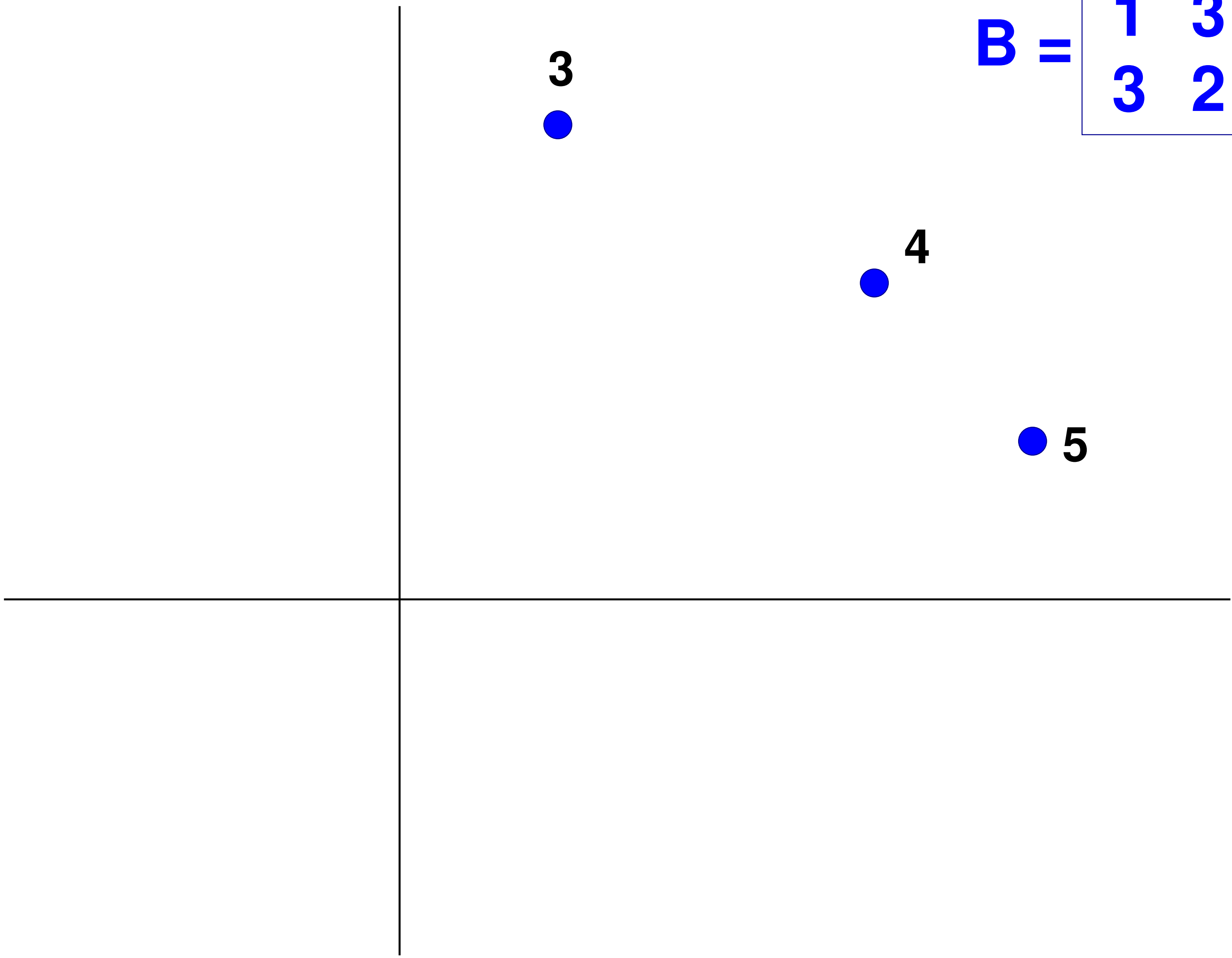
	1	2	3
1	**3**	**2**	**0**
2	**0**	**2**	**3**
3	**0**	**2**	**0**

= B

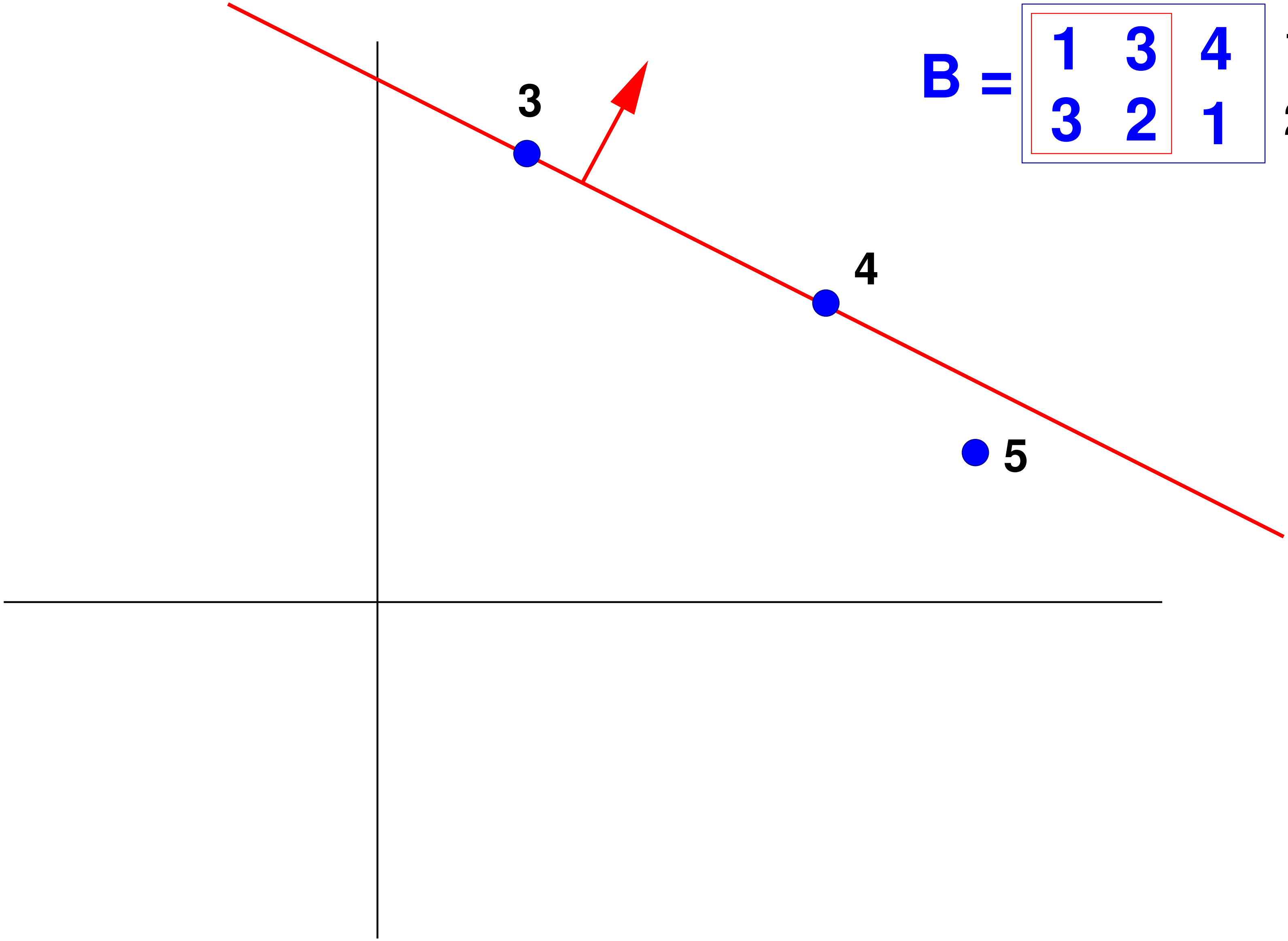
Points instead of best–reply regions

B =

	3	4	5	
	1	3	4	1
	3	2	1	2



Mixed strategies of player 1



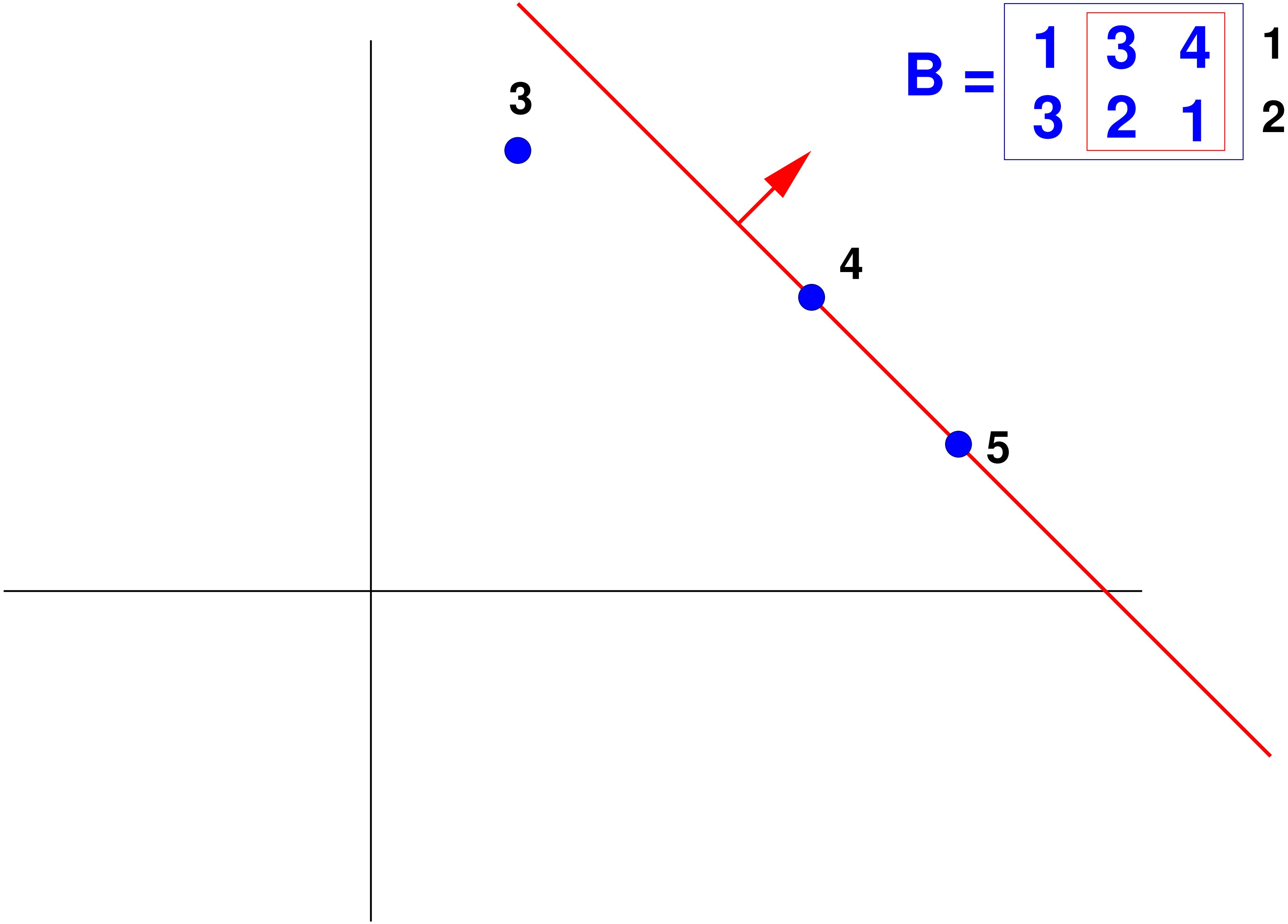
$B =$

	3	4	5
1	1	3	4
2	3	2	1

Mixed strategies of player 1

$B =$

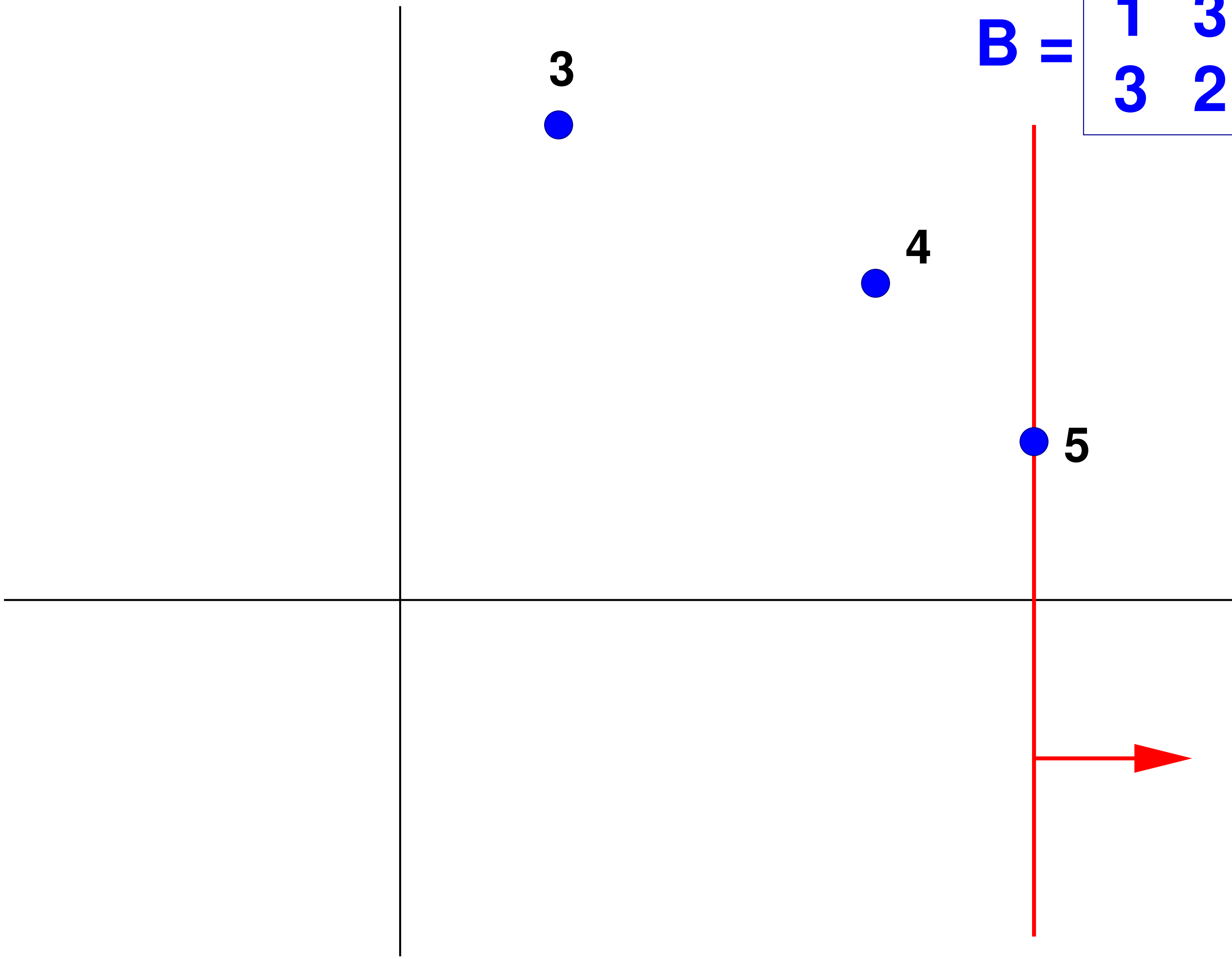
	3	4	5	
	1	3	4	1
	3	2	1	2



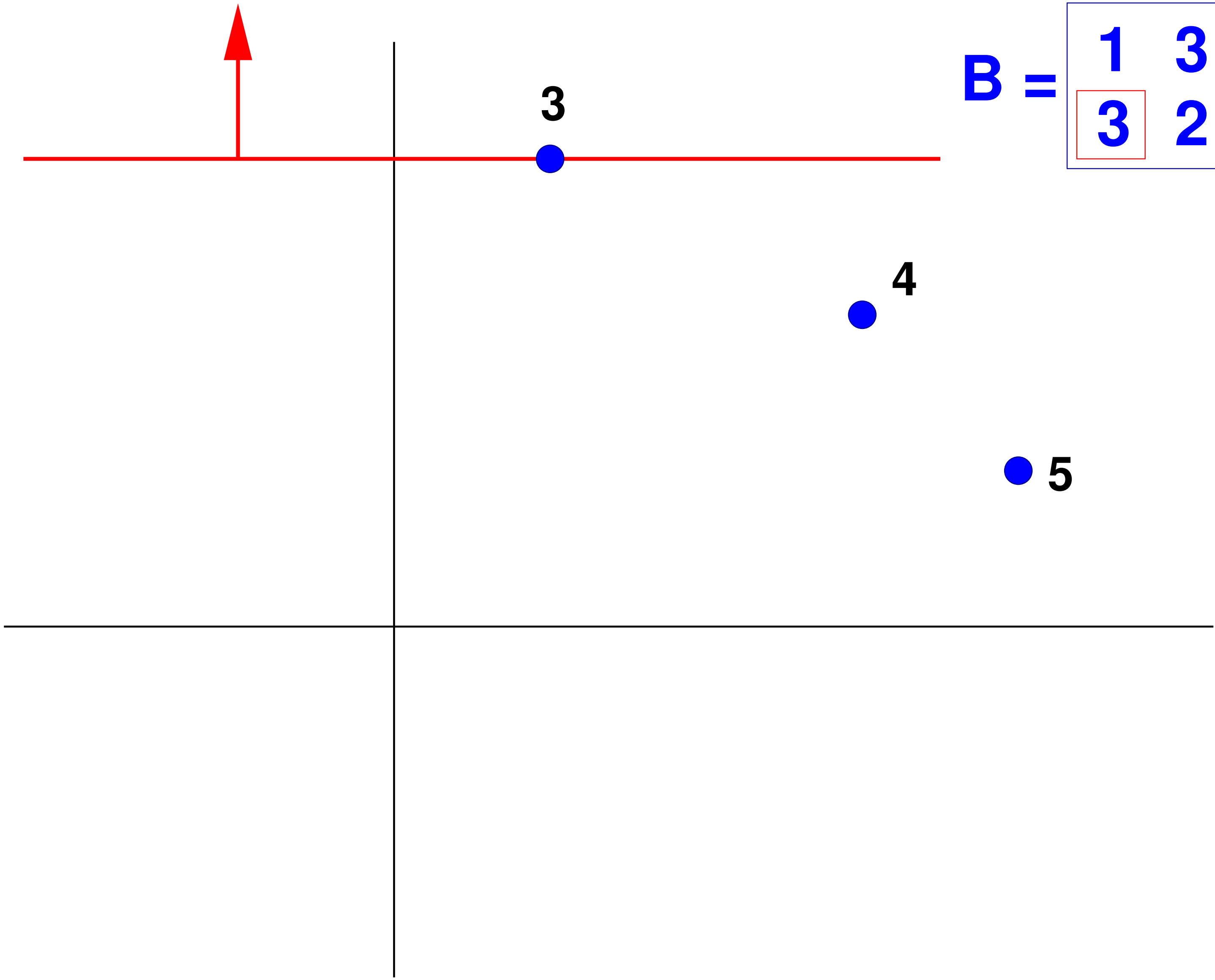
Mixed strategies of player 1

B =

	3	4	5	
	1	3	4	1
	3	2	1	2



Mixed strategies of player 1



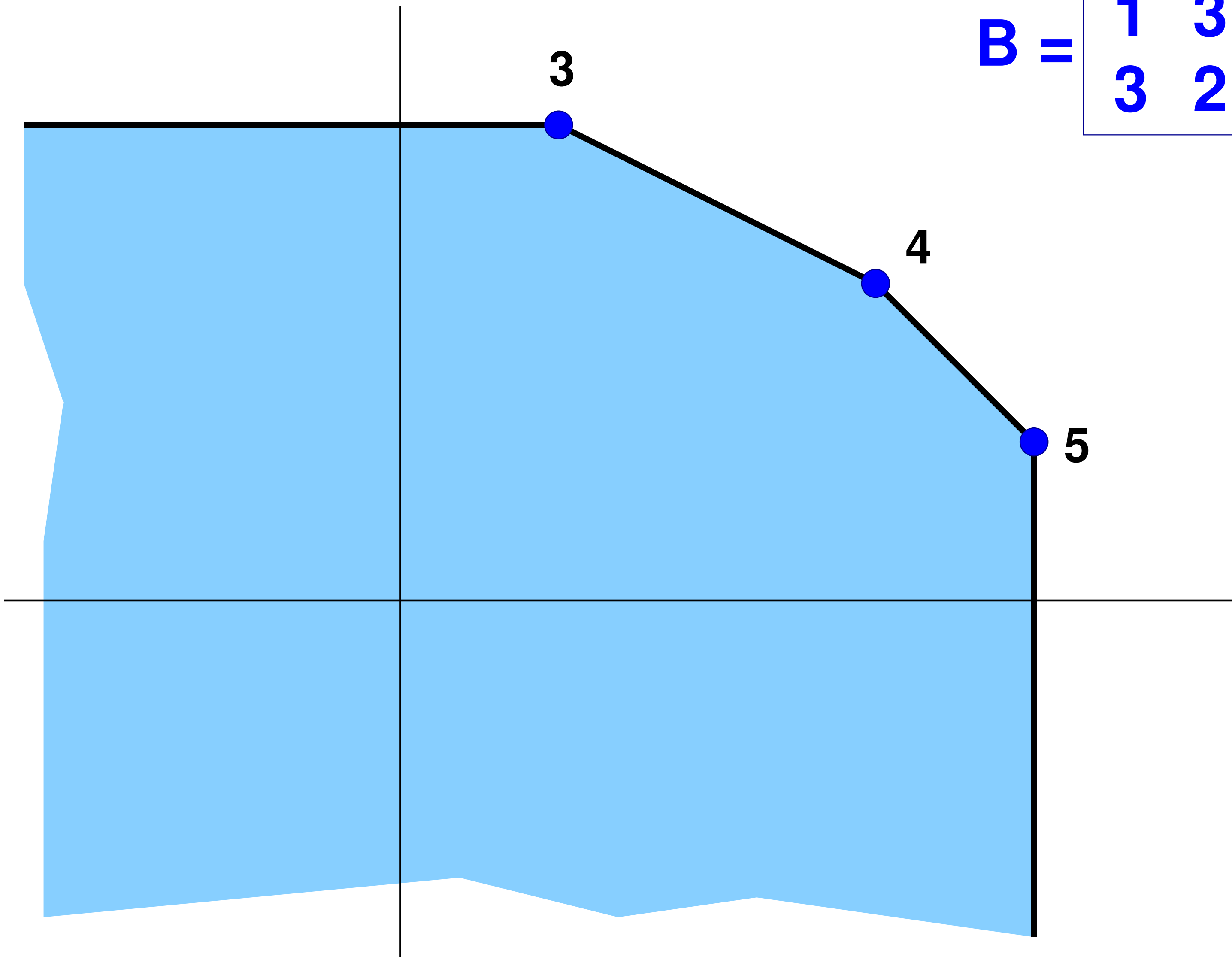
B =

	3	4	5
1	1	3	4
2	3	2	1

Non–negative convex hull

B =

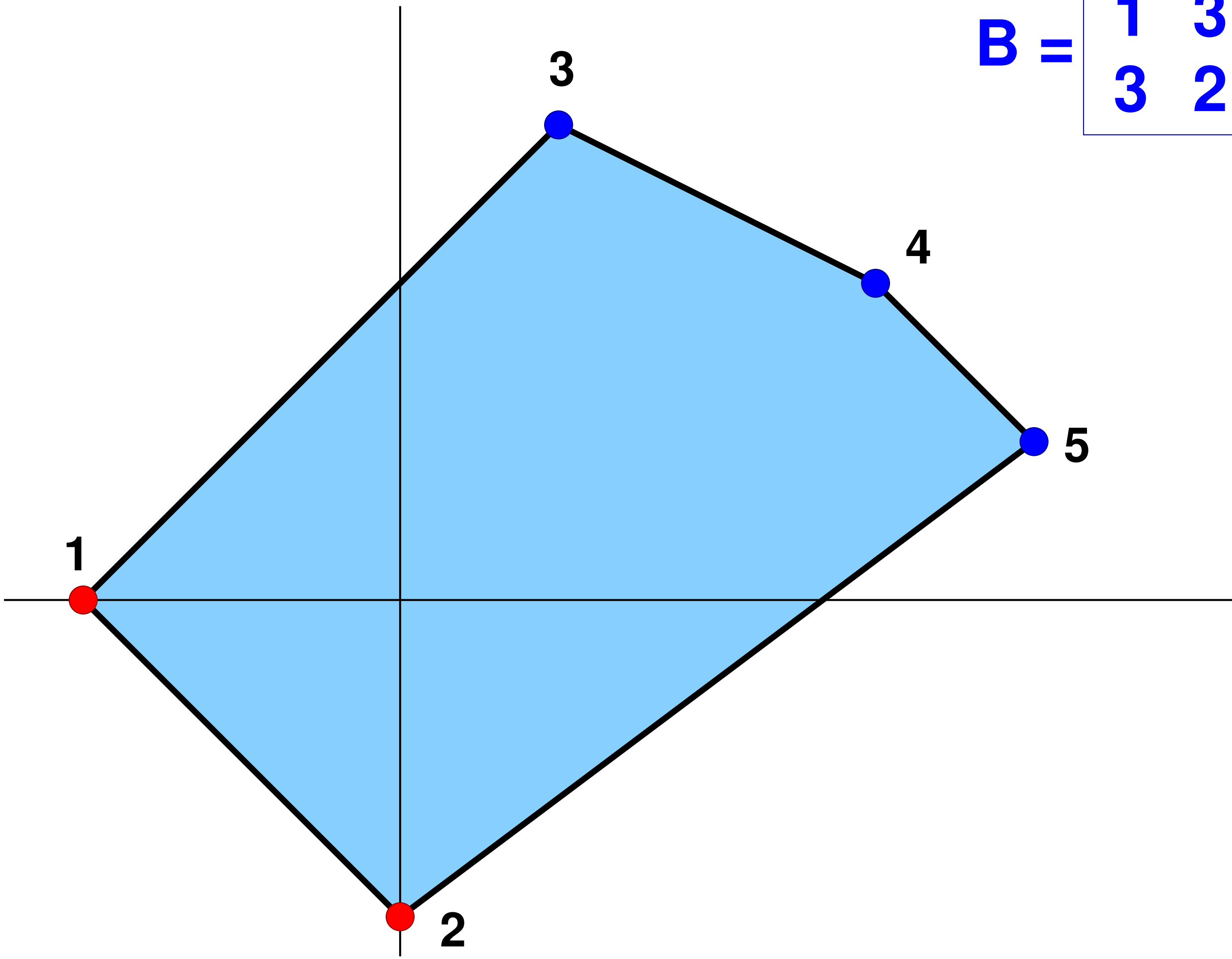
	3	4	5
1	1	3	4
2	3	2	1



Use negative unit vectors instead

B =

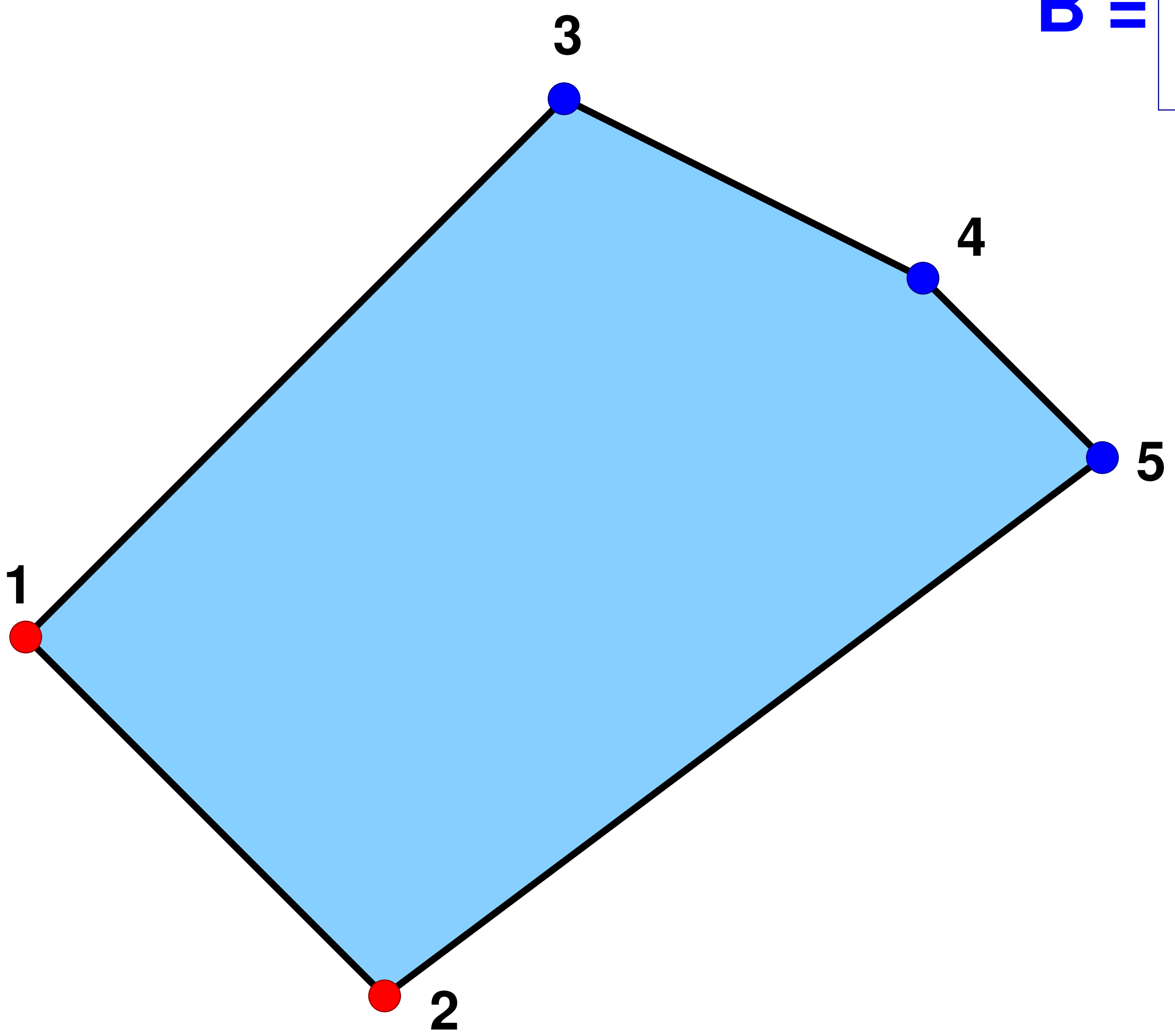
	3	4	5
1	1	3	4
2	3	2	1



Use negative unit vectors instead

B =

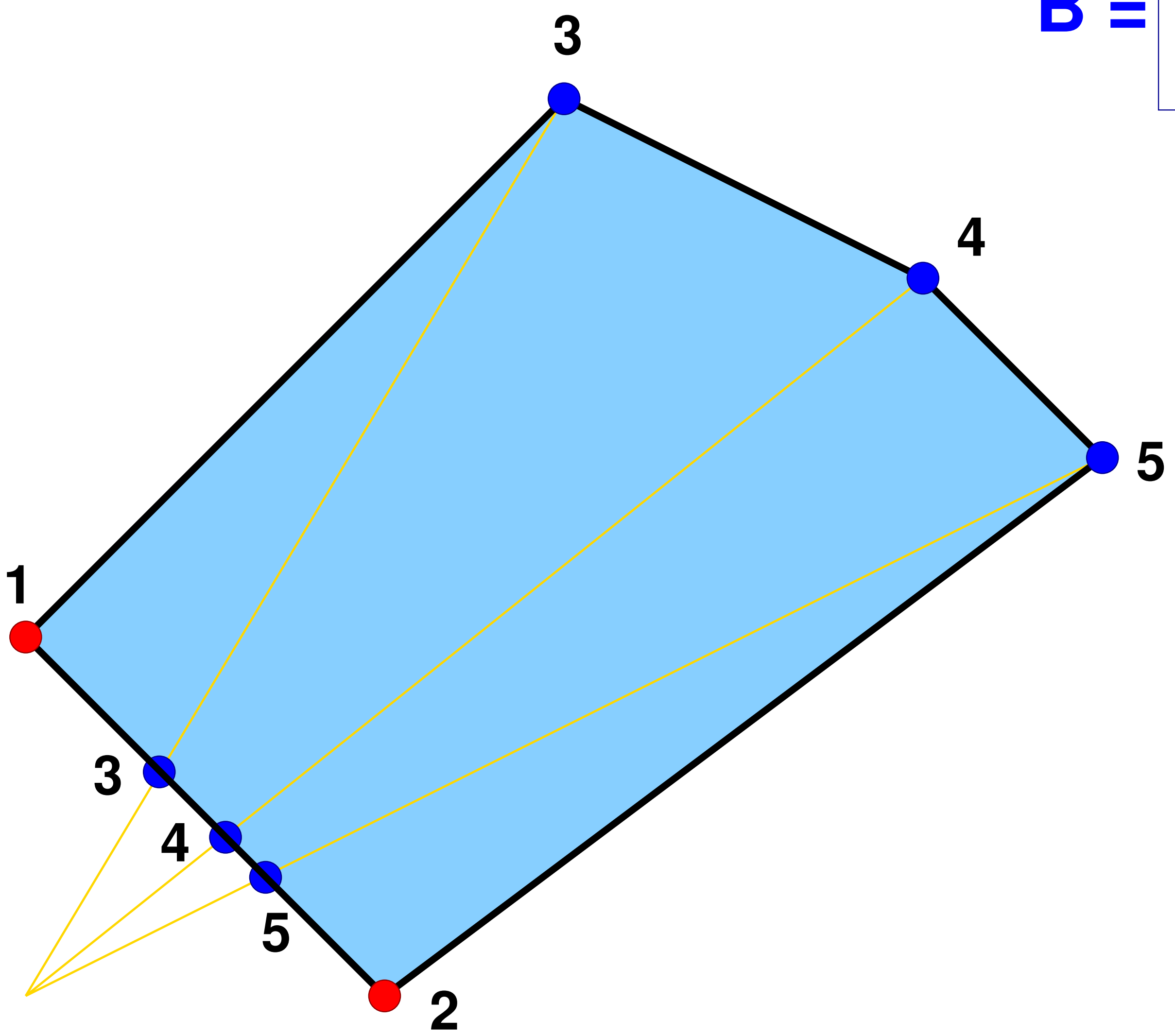
	3	4	5
1	1	3	4
2	3	2	1



Project to get Schlegel diagram

B =

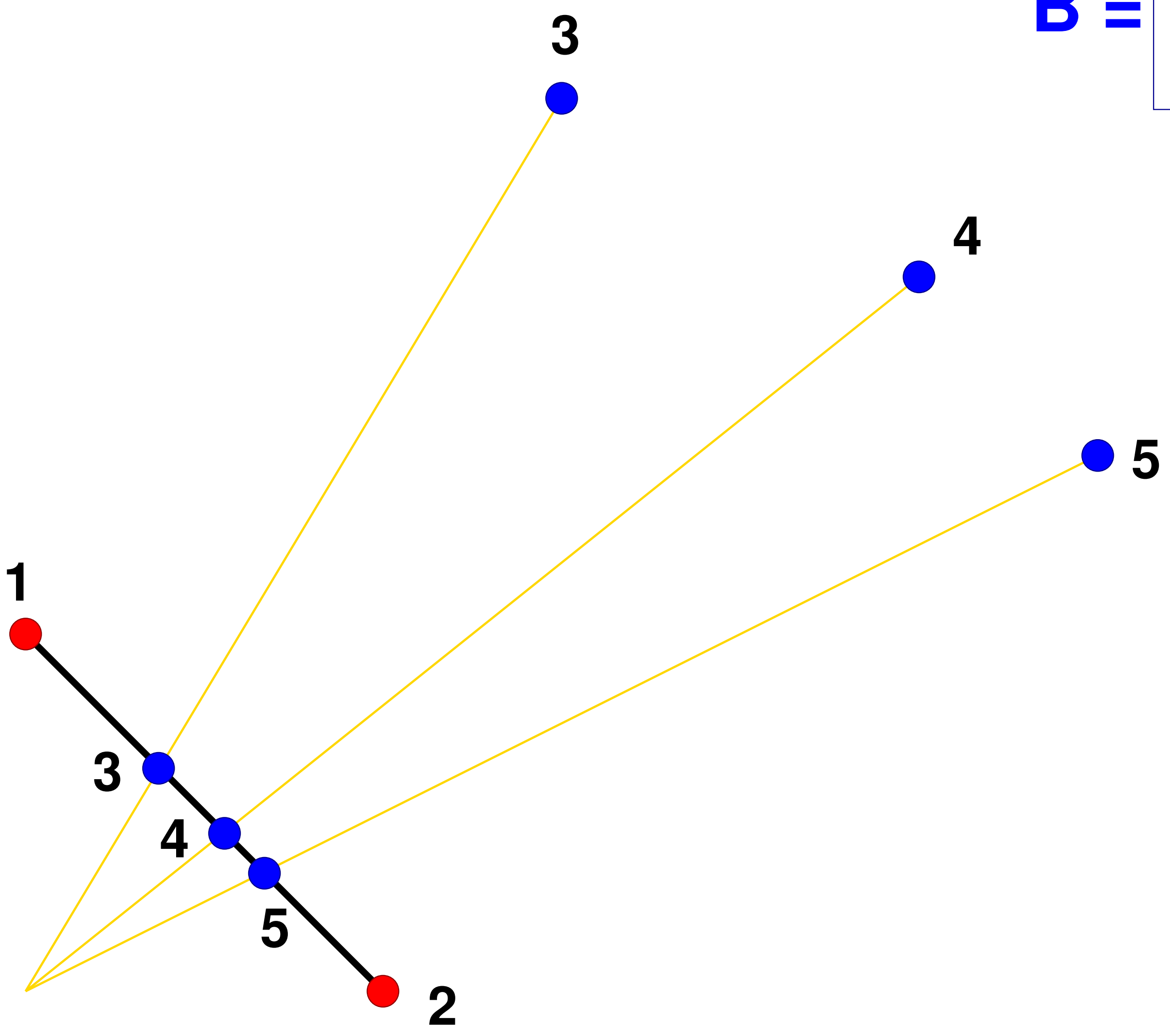
	3	4	5
1	1	3	4
2	3	2	1



Project to get Schlegel diagram

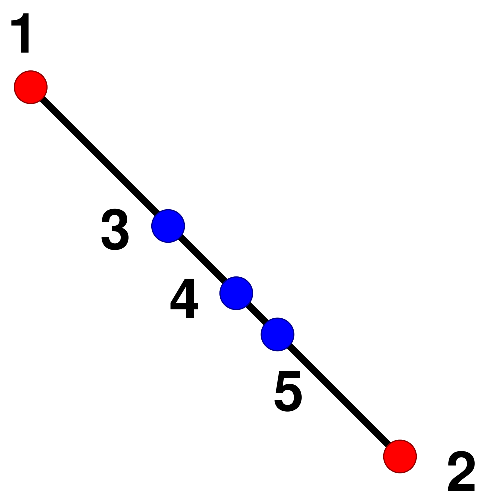
B =

	3	4	5
1	1	3	4
2	3	2	1



Project to get Schlegel diagram

$$B = \begin{array}{c} \\ \end{array}\begin{array}{ccc} 3 & 4 & 5 \\ \hline 1 & 3 & 4 \\ 3 & 2 & 1 \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}$$



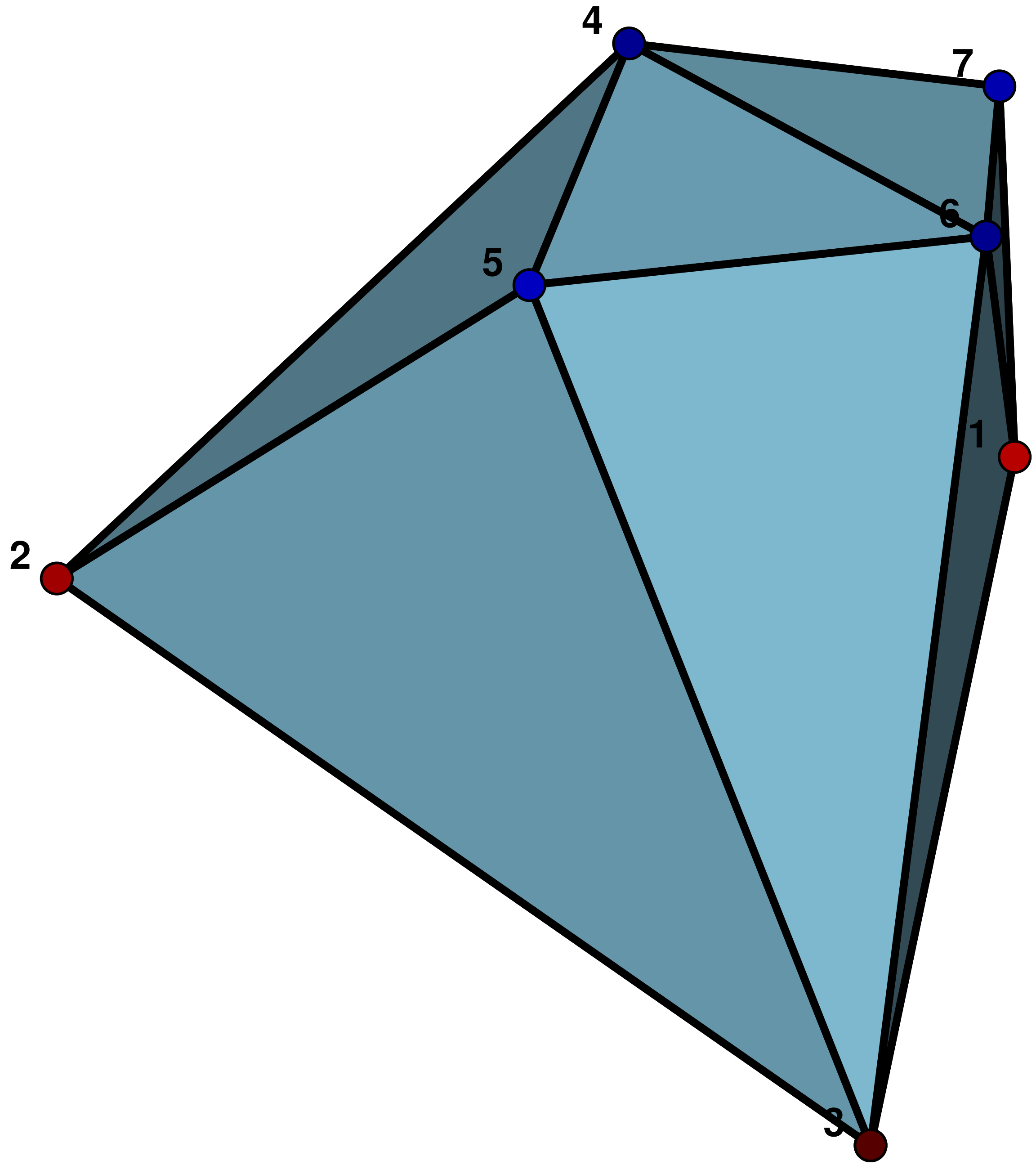
Example: m=3 strategies for player 1

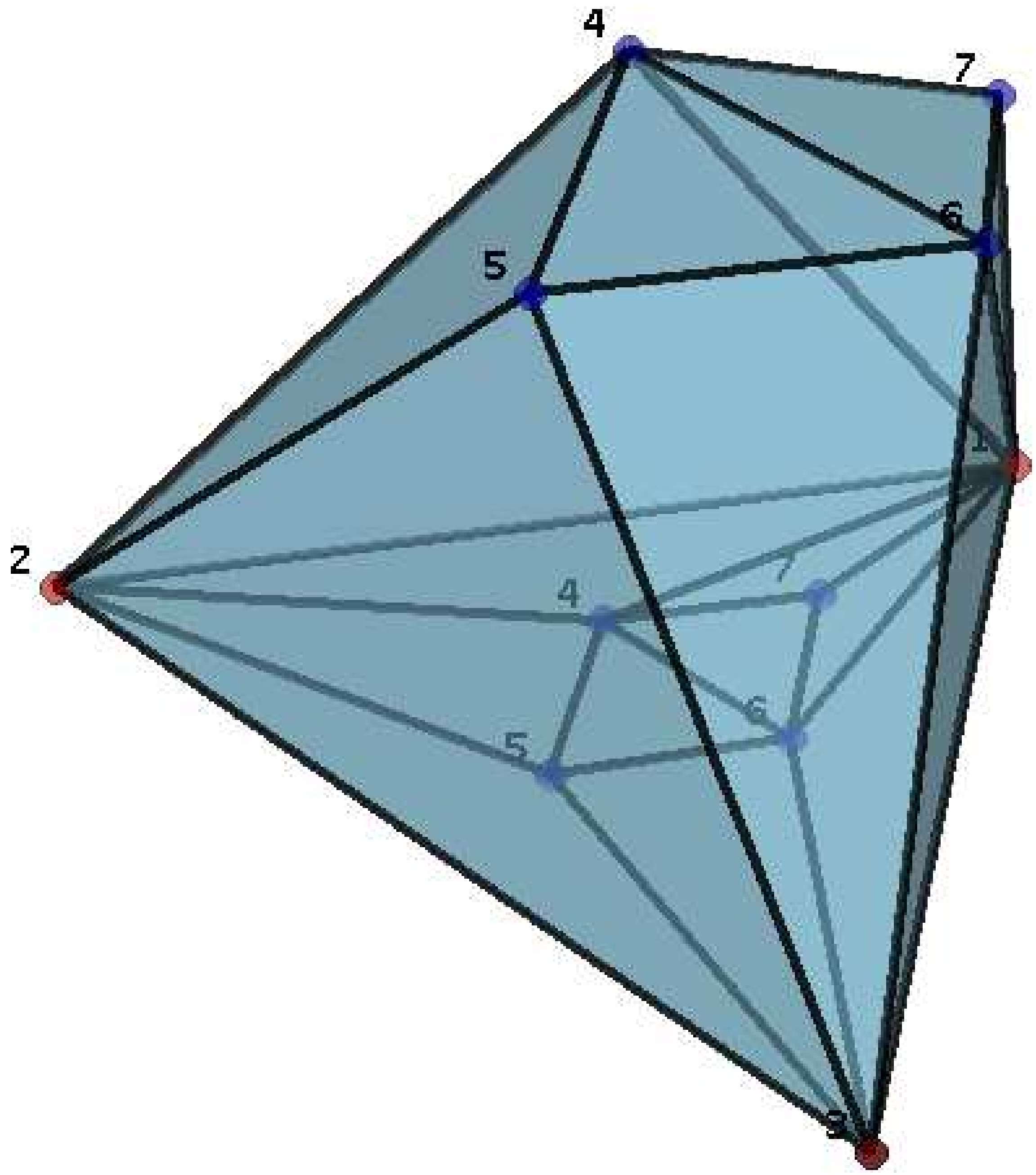
$$A = \begin{pmatrix} 0 & 10 & 0 & 10 \\ 10 & 0 & 0 & 0 \\ 8 & 0 & 10 & 8 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 10 & 0 & -10 \\ 0 & 0 & 10 & 8 \\ 10 & 0 & 0 & 8 \end{pmatrix}$$

$$P^{\Delta} = \mathrm{conv}\left(\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -M & 0 & 0 & 0 & 10 & 0 & -10 \\ 0 & -M & 0 & 0 & 0 & 10 & 8 \\ 0 & 0 & -M & 10 & 0 & 0 & 8 \end{array} \right),$$

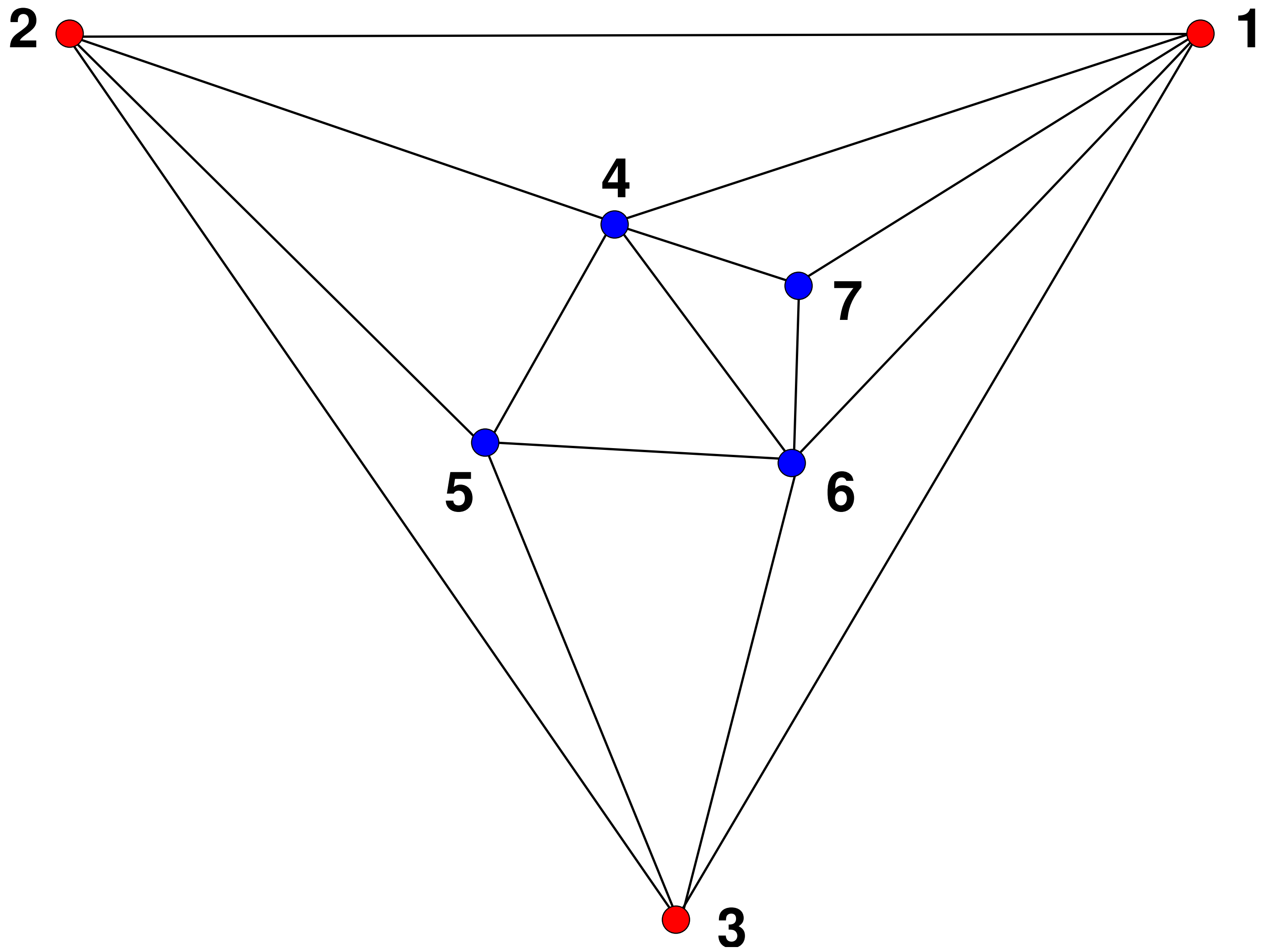
M large


4
7
6
5
1
2
3

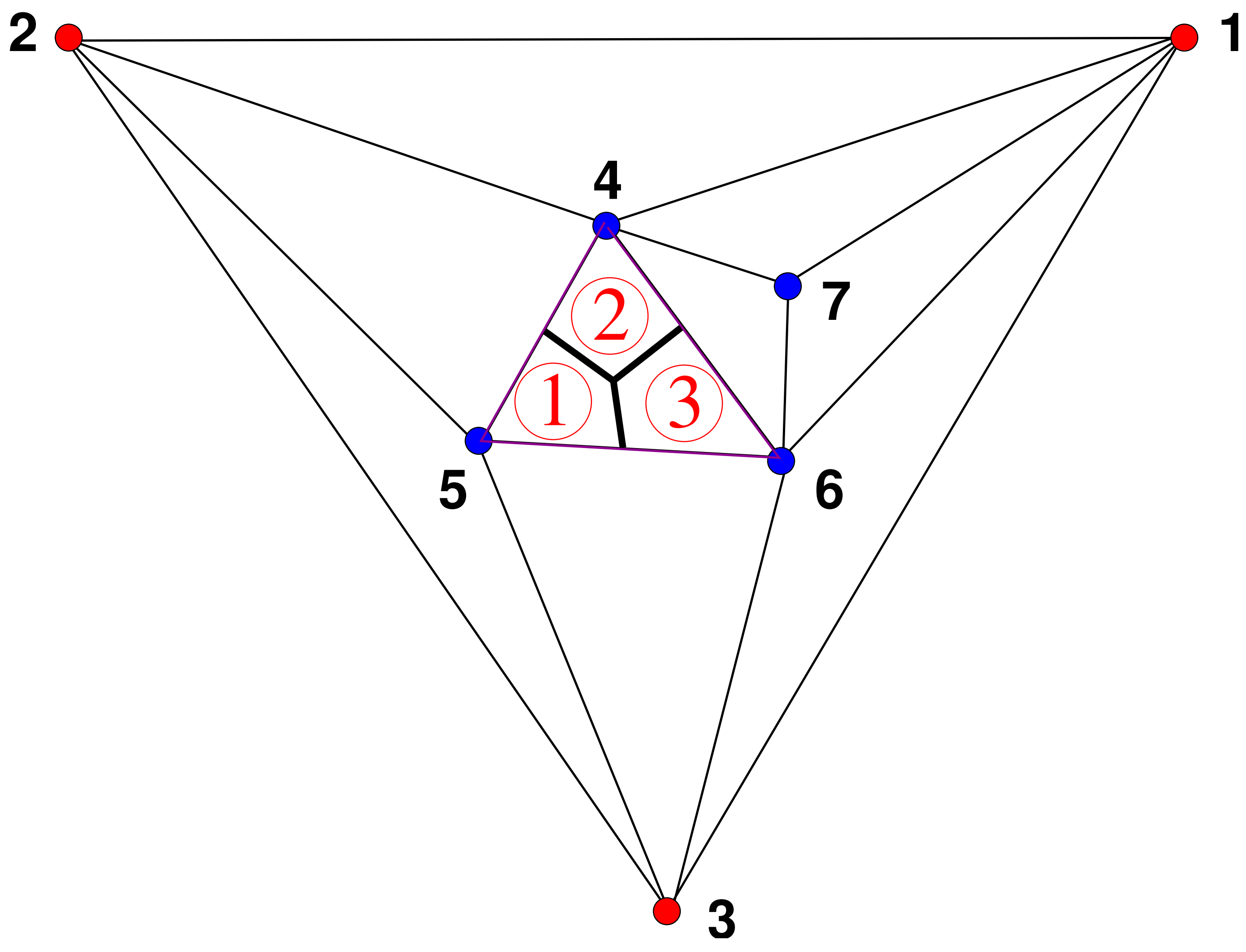


4
7
6
5
1
2
4
7
6
5
3

Schlegel diagram for P^{Δ}



subdivide facet: player 1's best responses



Facets of P^{Δ} = potential equilibrium strategies of player 1

Each **facet** of P^{Δ}

= **simplex** spanned by m **columns** of [$-M\ I_m$, B]

Normal vector of facet

= **mixed strategy** of player 1

m **columns**

= best responses of player 2

or unused strategies of player 1

Subdivide facets into special best response regions of player 1

use matrix B:

P^{Δ} = conv(

1	2	3	4	5	6	7
-M	0	0	0	10	0	-10
0	-M	0	0	0	10	8
0	0	-M	10	0	0	8

)

use matrix A:

1	2	3	4	5	6	7
1	0	0	0	10	0	10
0	1	0	10	0	0	0
0	0	1	8	0	10	8

Subdivide facets into special best response regions of player 1

P^{Δ} = conv(...) facet

1	2	3	4	5	6	7
-M	0	0	0	10	0	-10
0	-M	0	0	0	10	8
0	0	-M	10	0	0	8

subdivide into best response regions

1	2	3	4	5	6	7
1	0	0	0	10	0	10
0	1	0	10	0	0	0
0	0	1	8	0	10	8

Subdivide facets into special best response regions of player 1

P^{Δ} = conv(

1	2	3	4	5	6	7
-M	0	0	0	10	0	-10
0	-M	0	0	0	10	8
0	0	-M	10	0	0	8

)

facet with **unplayed strategy:**

1	2	3	4	5	6	7
1	0	0	0	10	0	10
0	1	0	10	0	0	0
0	0	1	8	0	10	8

unit vector for **slack variable**

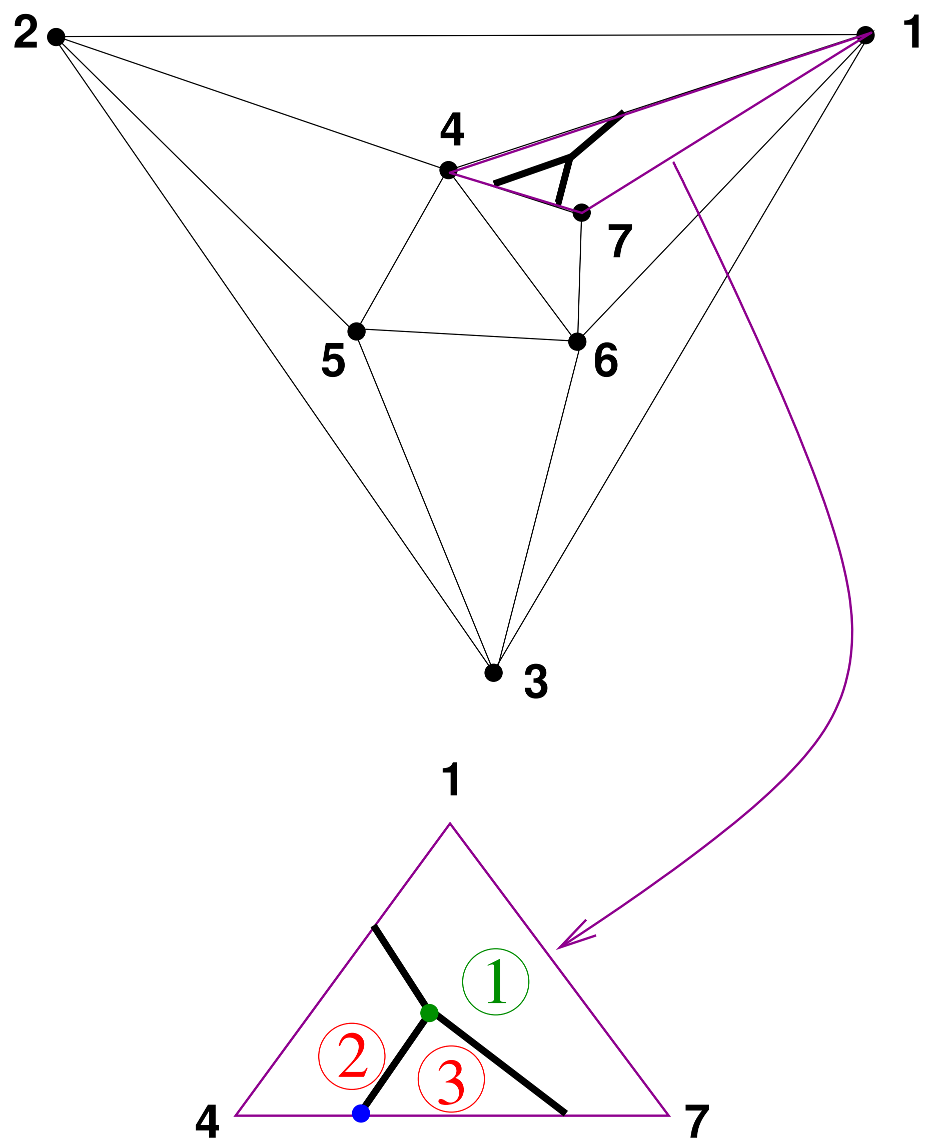
subdivide facet using slack variables

equilibrium iff

all labels 1...m

4/5 1/5 0

	4	7	
1	0	10	2
2	10	0	8
3	8	8	8



subdivide facet using slack variables

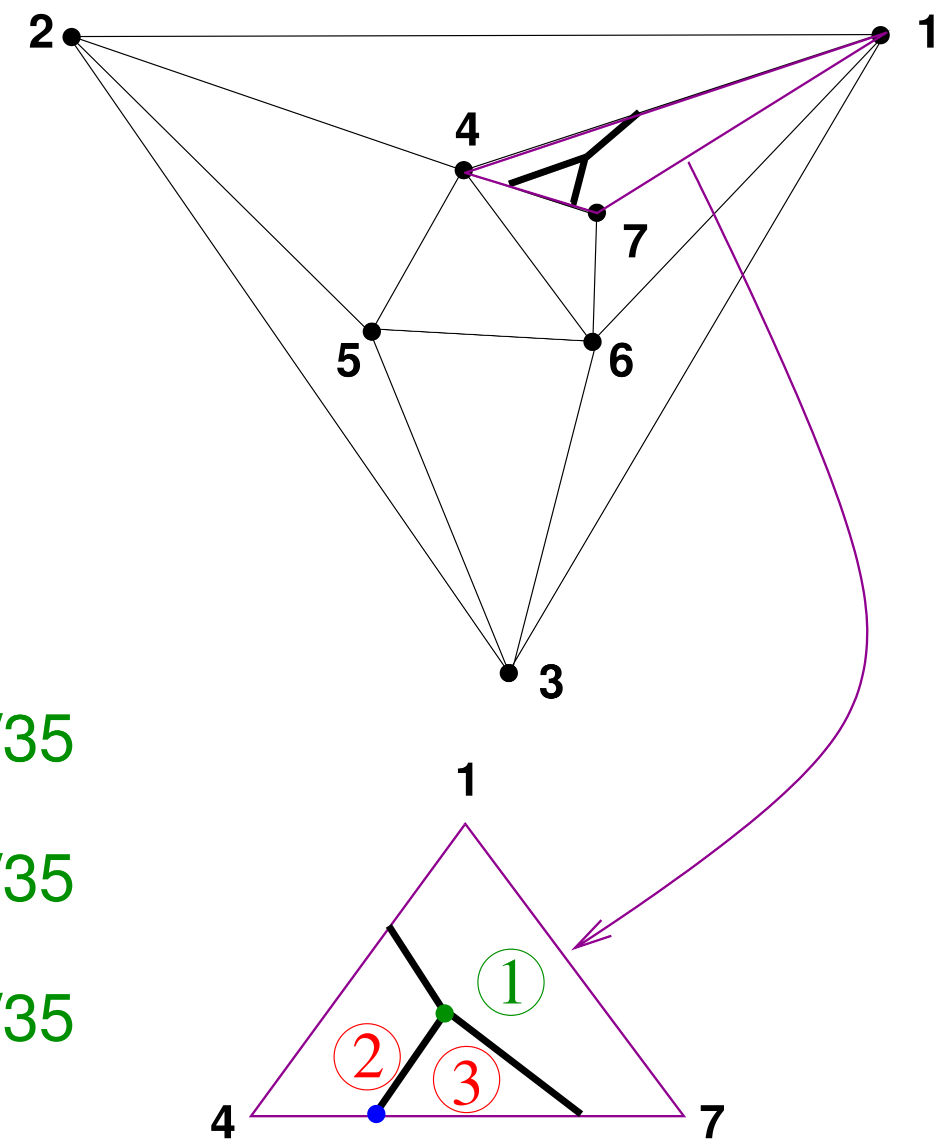
equilibrium iff

all labels 1...m

4/35	1/35	30/35

4/5	1/5	0

	(4)	(7)	(1)		
(1)	0	10	1	2	40/35
(2)	10	0	0	8	40/35
(3)	8	8	0	8	40/35



The full dual construction

Given: nondegenerate $m \times n$ game (A,B), $m \leq n$.

Let $P^{\Delta} = \mathrm{conv}\, [-M\, I_m, B]$, simplicial polytope.

Subdivide surface of P^{Δ} into regions with labels $1,\ldots,m$ using columns of $[I_m, A]$ that correspond to facets.

The full dual construction

Given: nondegenerate $m \times n$ game (A,B), $m \leq n$.

Let P^Δ = conv [$-M\ I_m$, B], simplicial polytope.

Subdivide surface of P^Δ into regions with labels $1,\ldots,m$ using columns of [I_m, A] that correspond to facets.

Equilibrium (x,y)
= **point** $f(y)$ on facet corresponding to x,
vertices = best responses player 2

with **all labels** $1,\ldots,m$ =
best responses player 1

The full dual construction

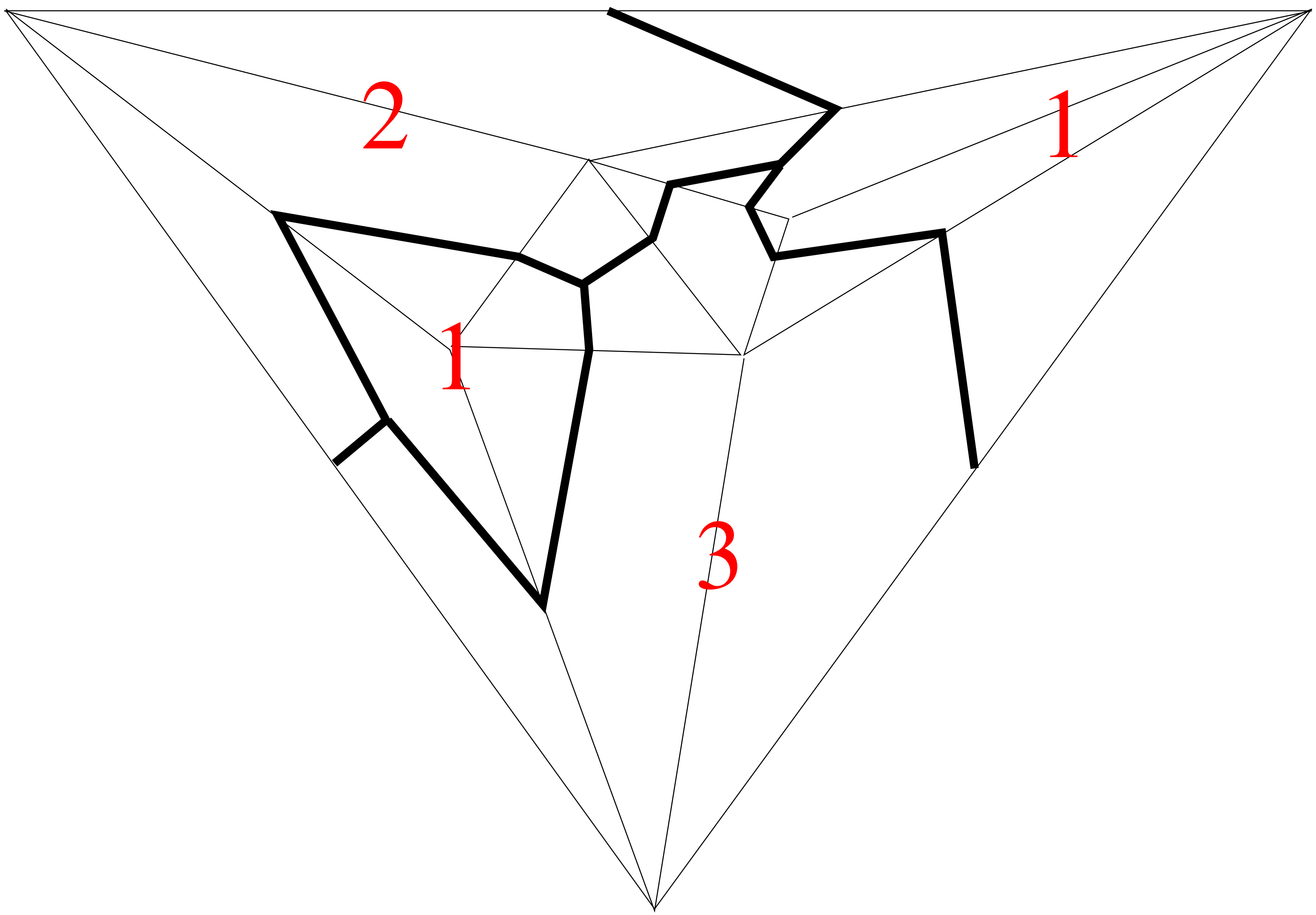
Given: nondegenerate $m \times n$ game (A,B), $m \leq n$.

Let P^Δ = conv [$-M\, I_m$, B], simplicial polytope.

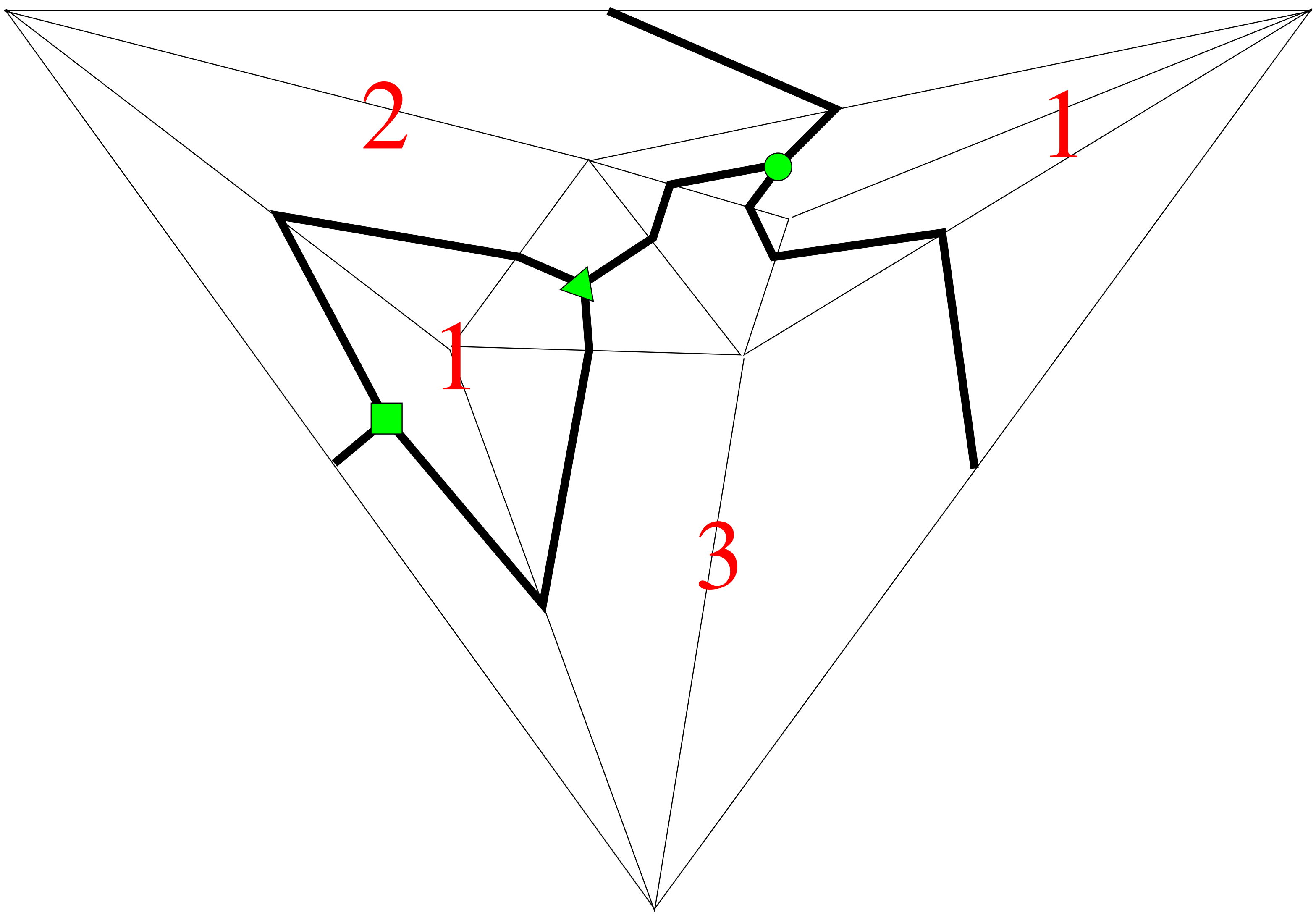
Subdivide surface of P^Δ into regions with labels 1,...,m
using columns of [I_m, A] that correspond to facets.

Equilibrium (x,y)
= **point** $f(y)$ on facet corresponding to x,
vertices = best responses player 2,
unused strategies player 1
with **all labels** 1,...,m =
best responses player 1,
slacks (=worse pay) for unused strategies.

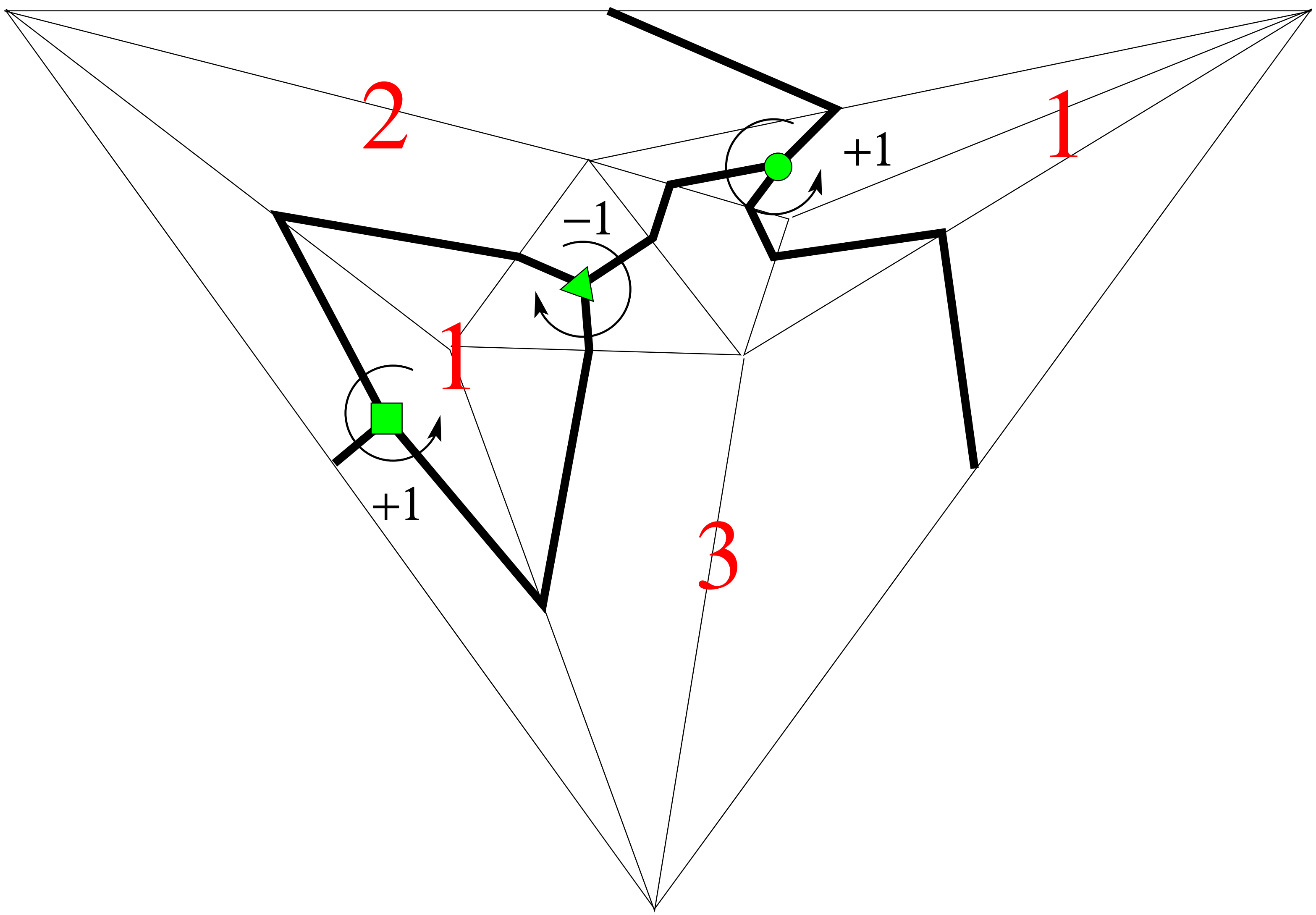
The full dual construction



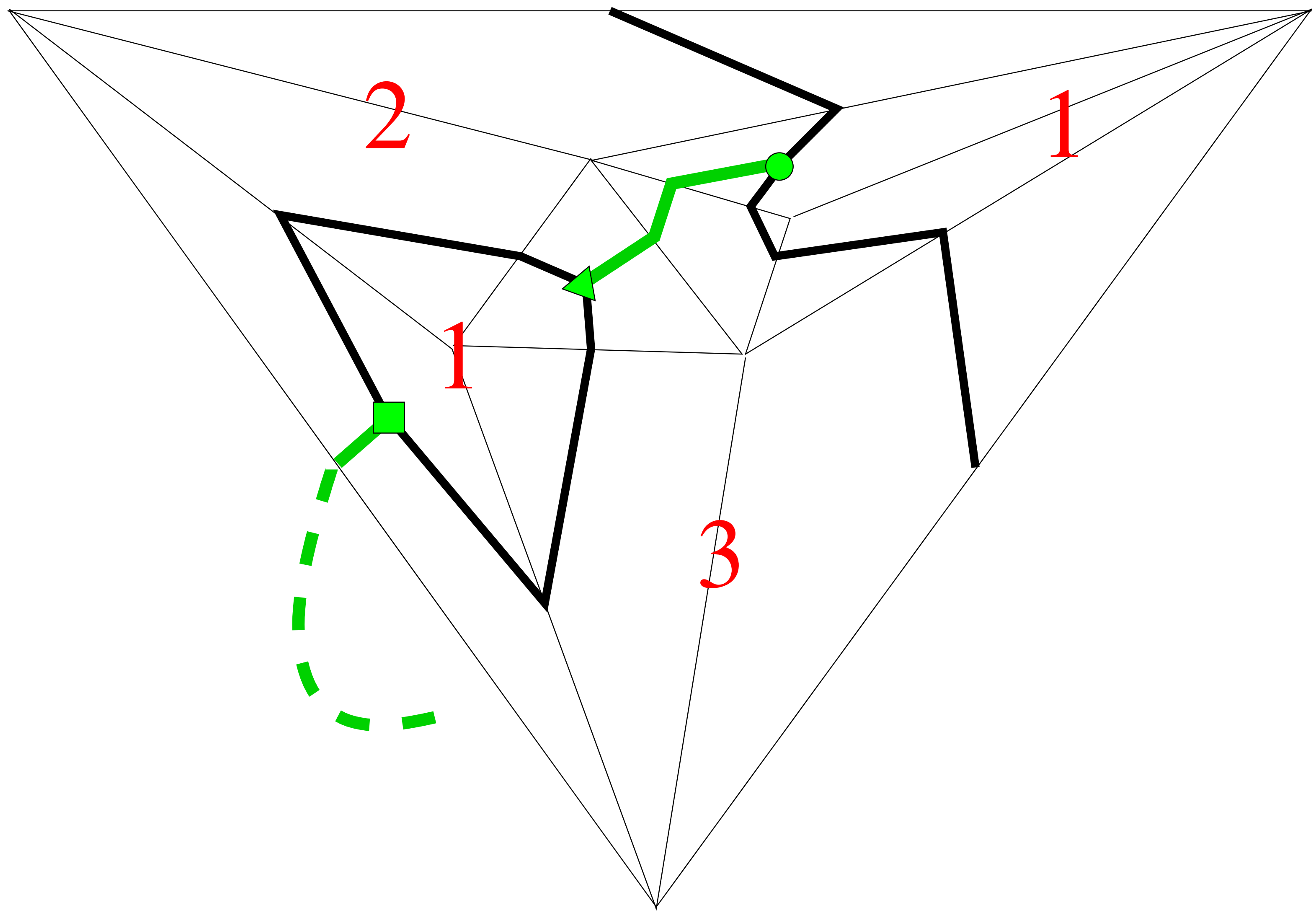
Equilibria have all m labels



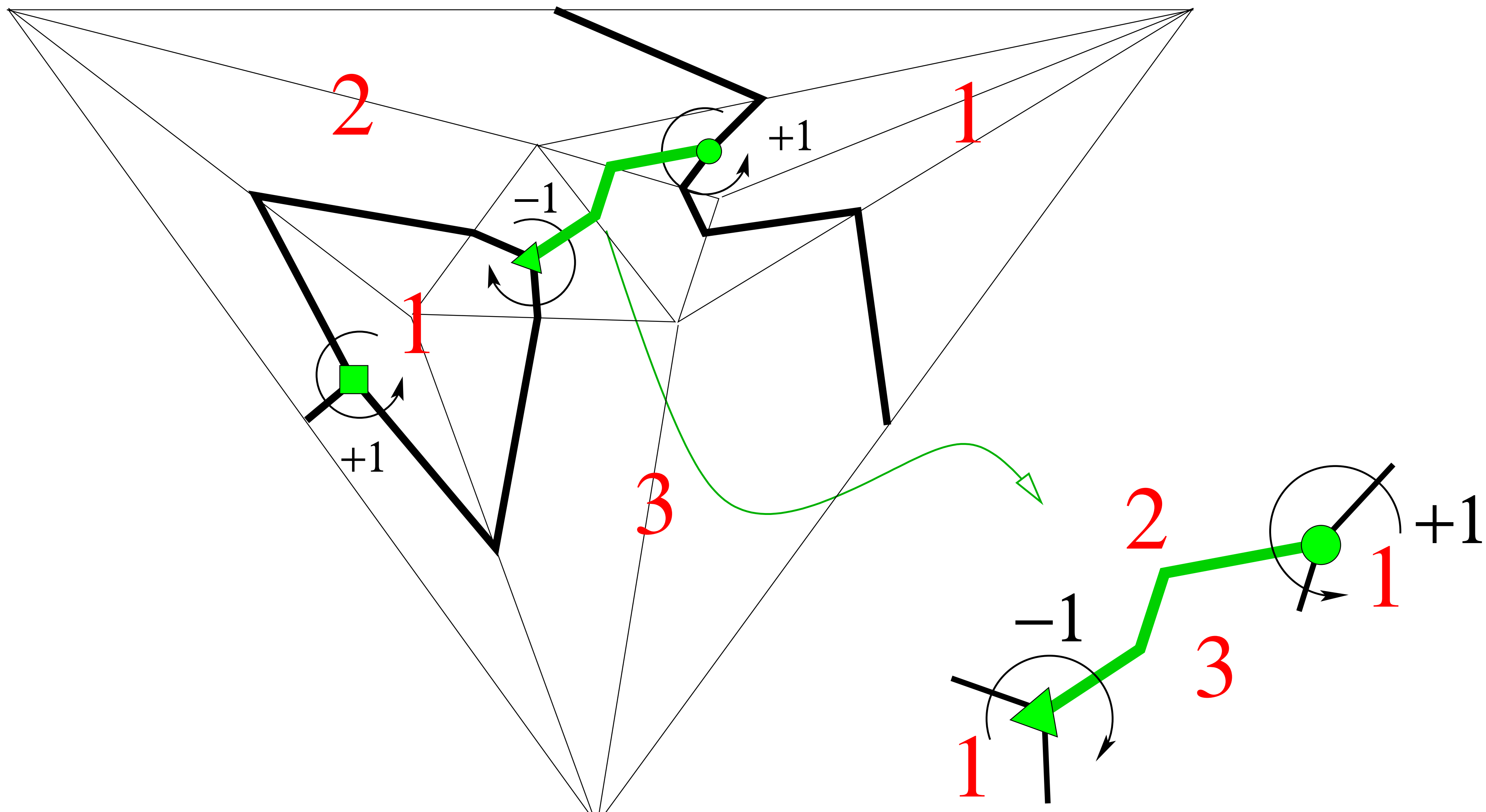
Index = orientation



Lemke–Howson paths



Opposite index of endpoints



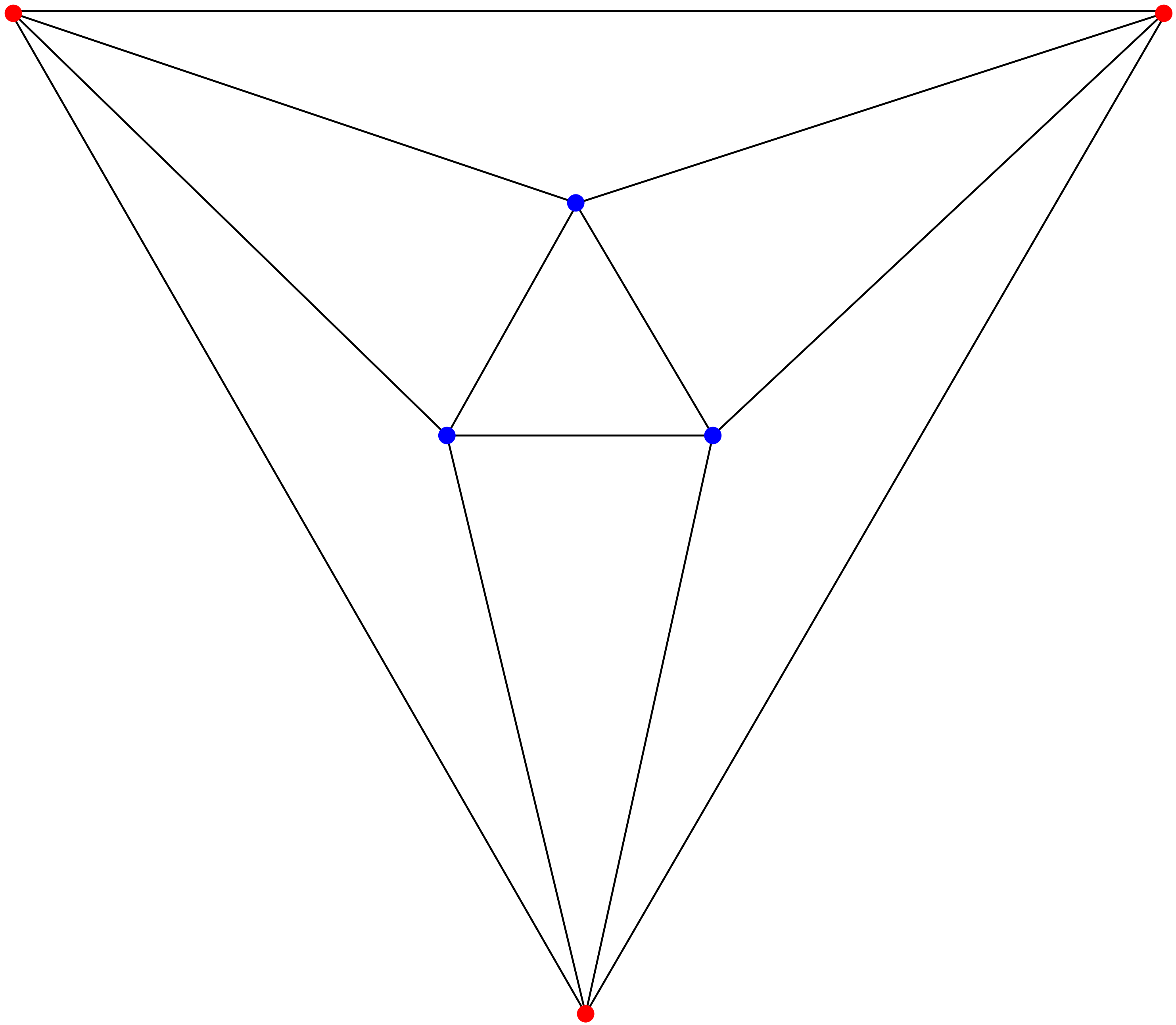
Completely mixed NE of 3x3 coordination game made unique

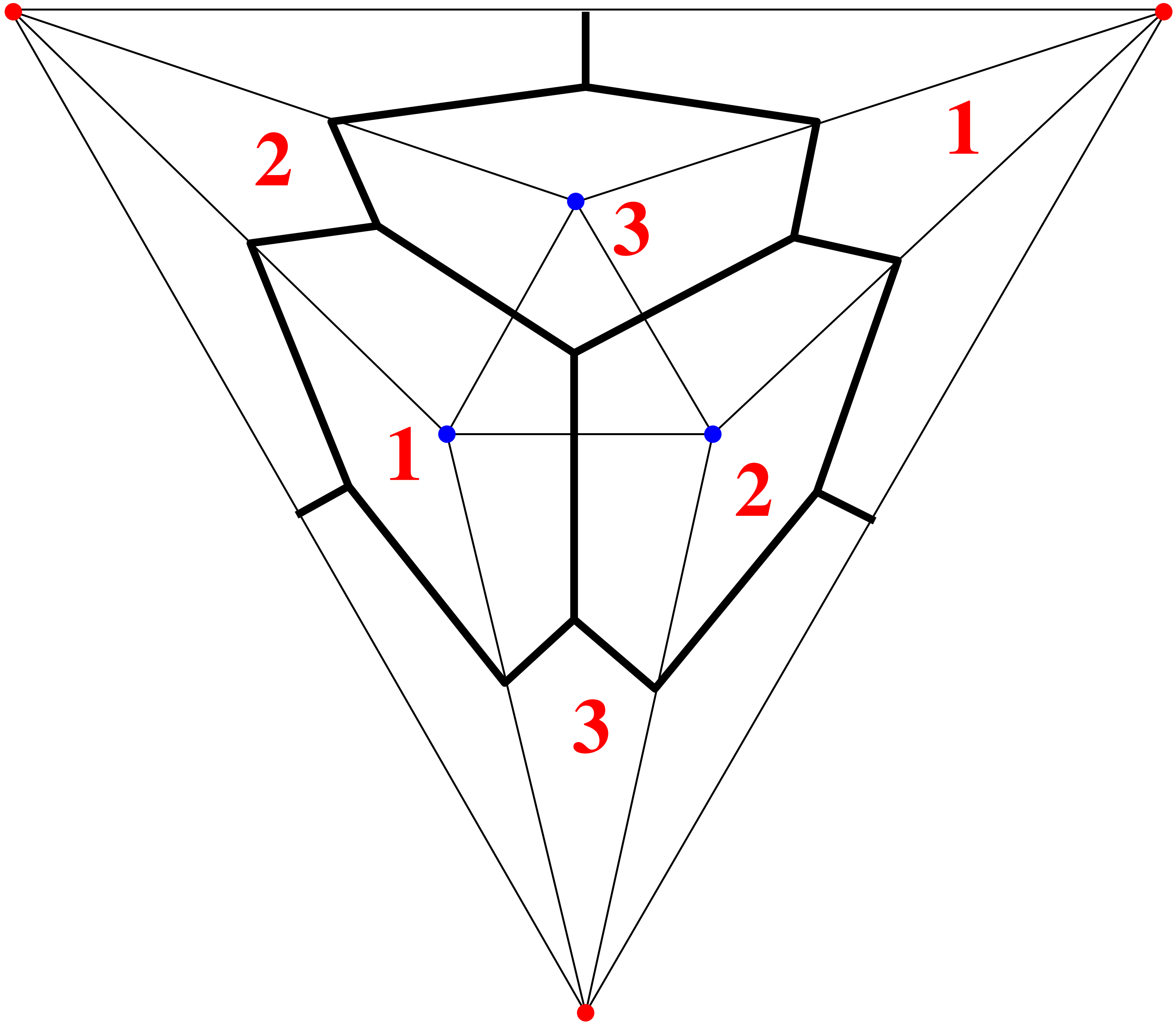
A =

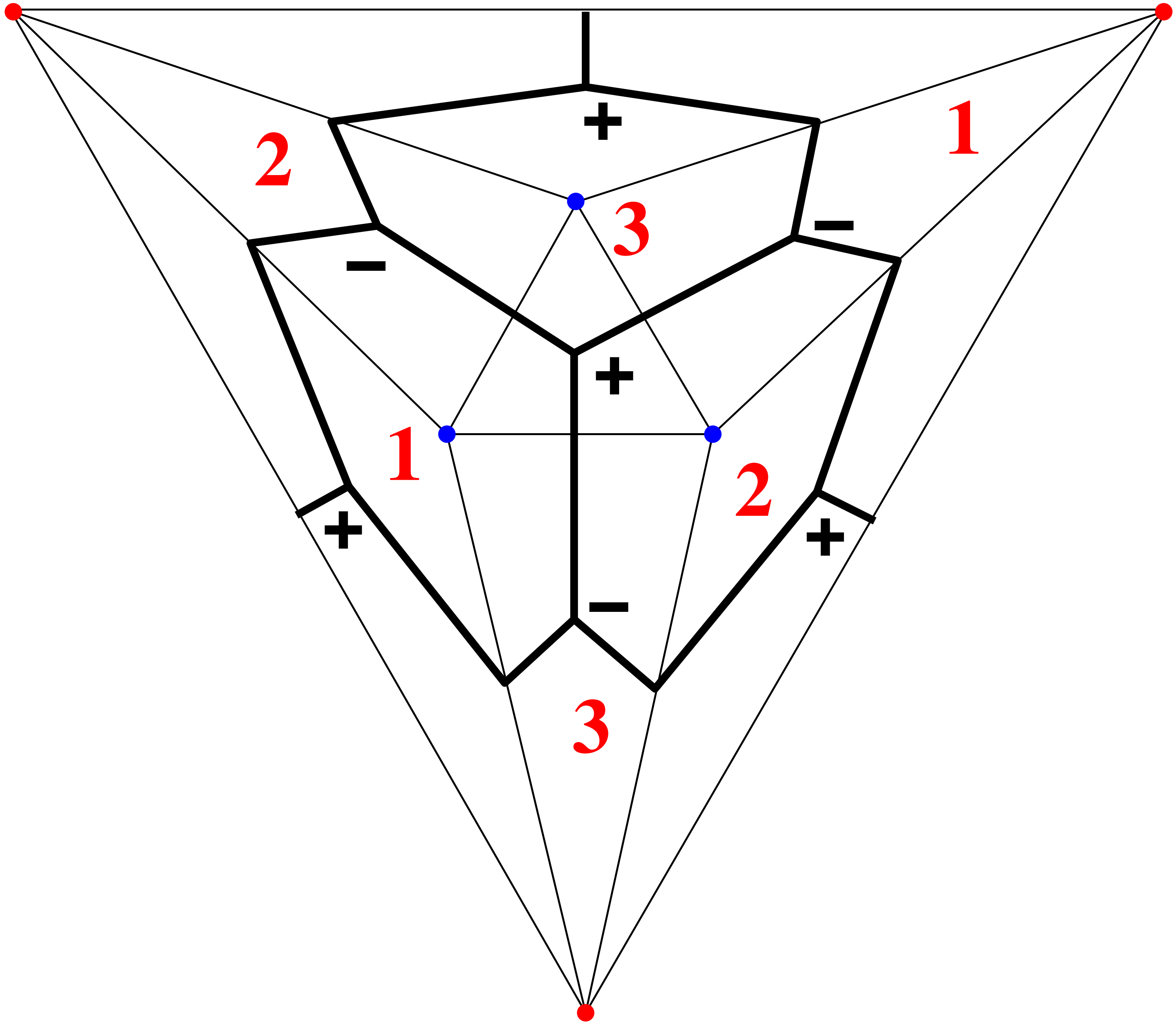
2	1	1	0	0	1
1	2	1	1	0	0
1	1	2	0	1	0

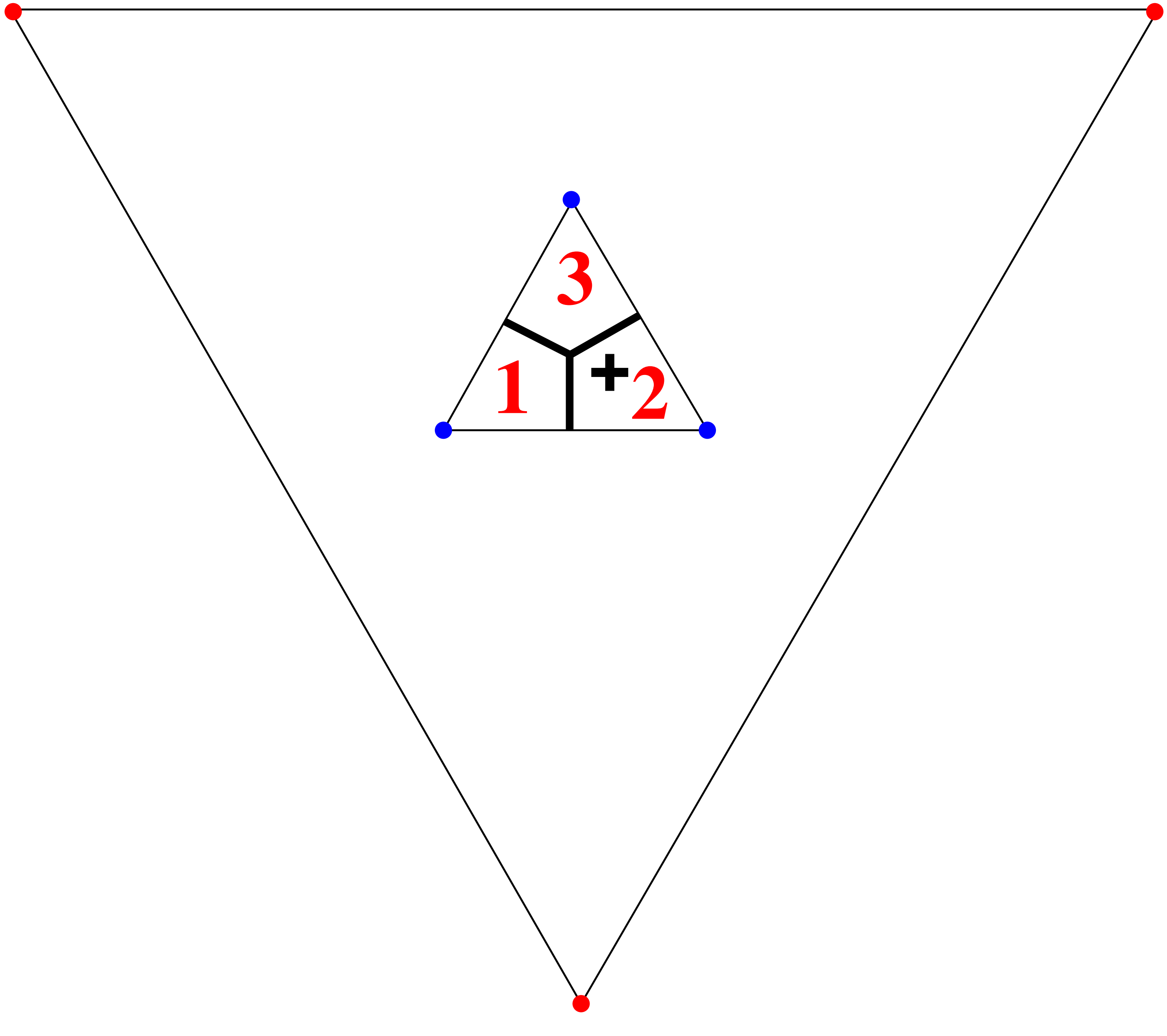
B =

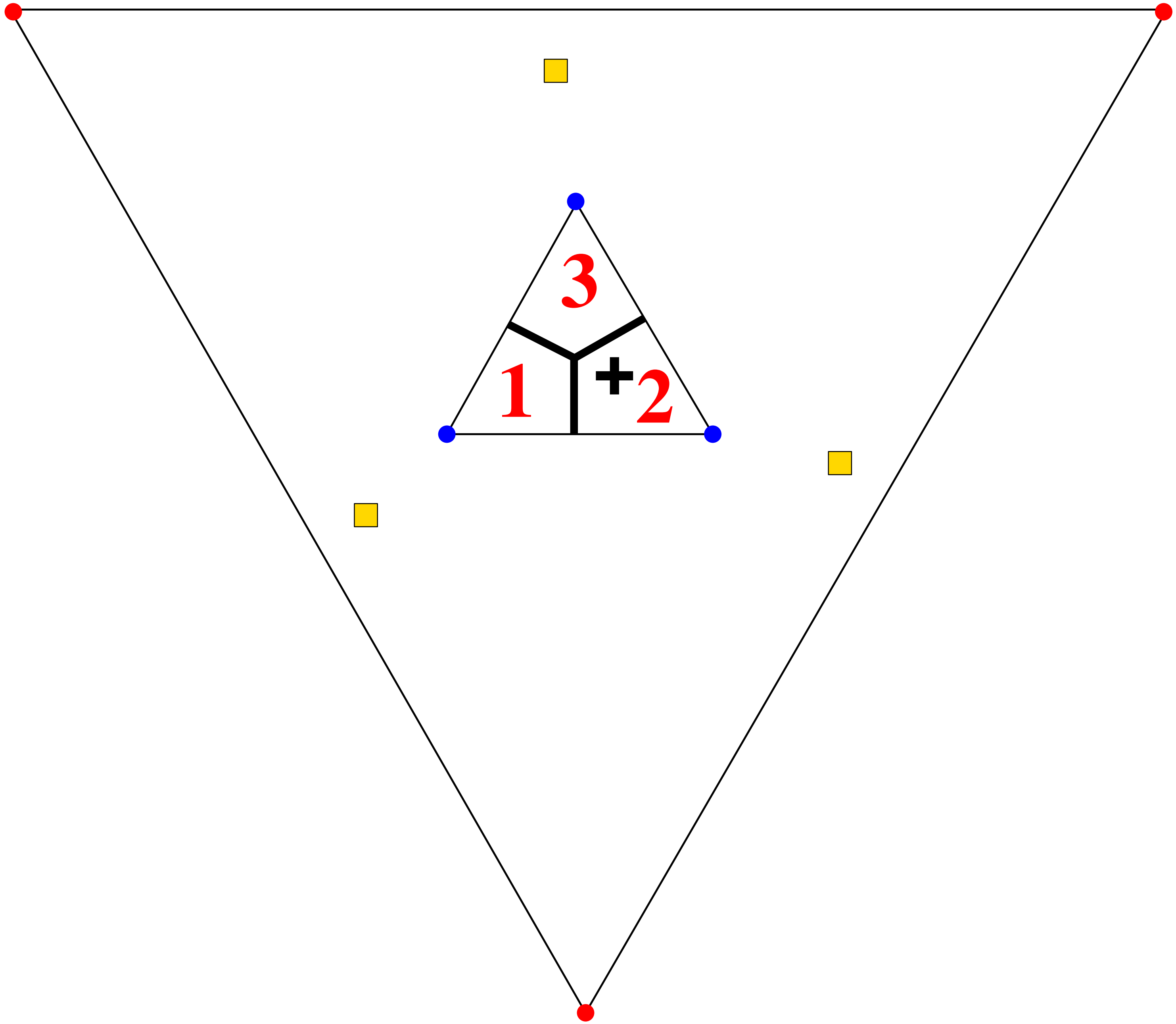
2	1	1	2.8	0.8	-0.4
1	2	1	-0.4	2.8	0.8
1	1	2	0.8	-0.4	2.8



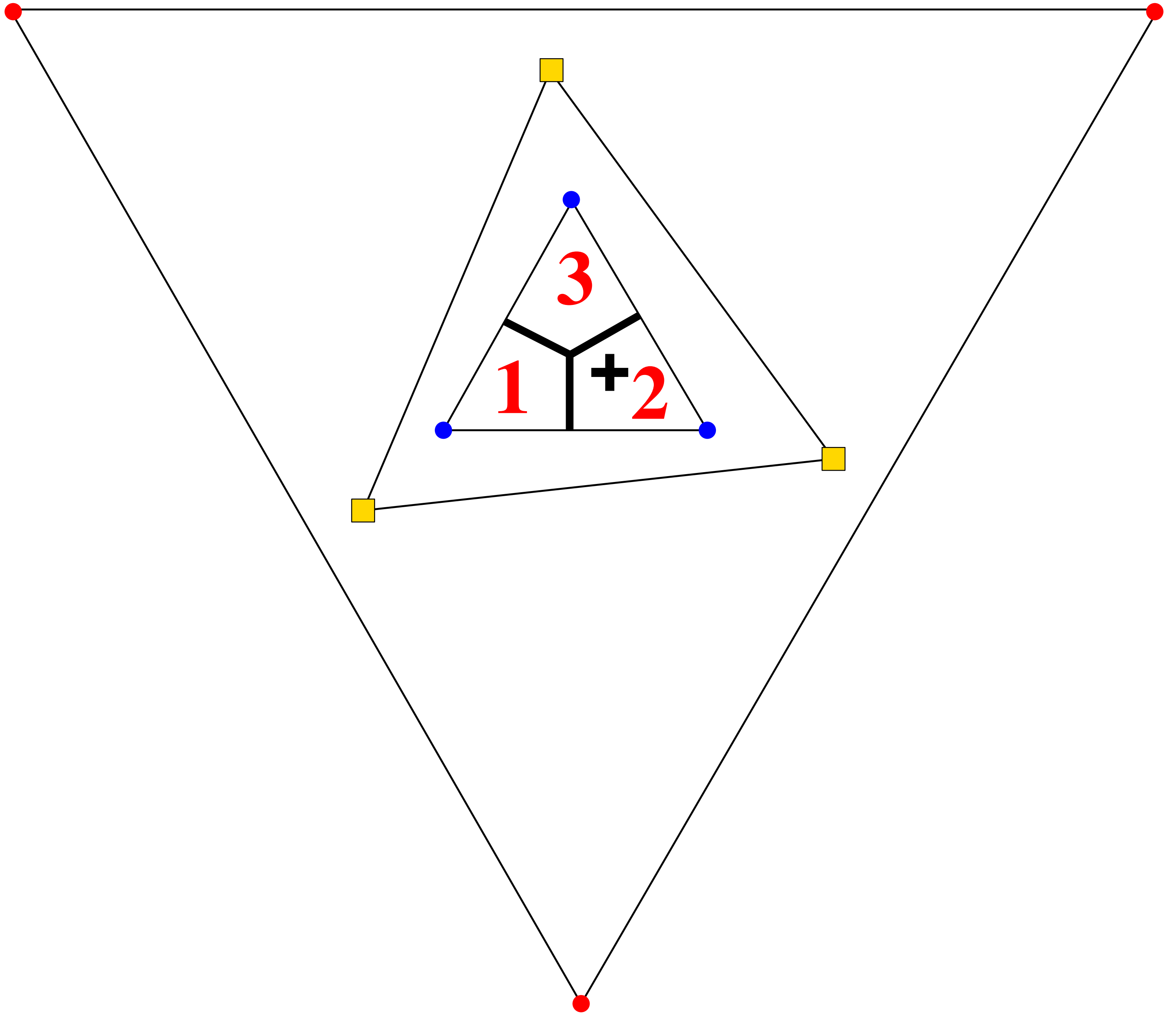

2
1
3
1
2
3

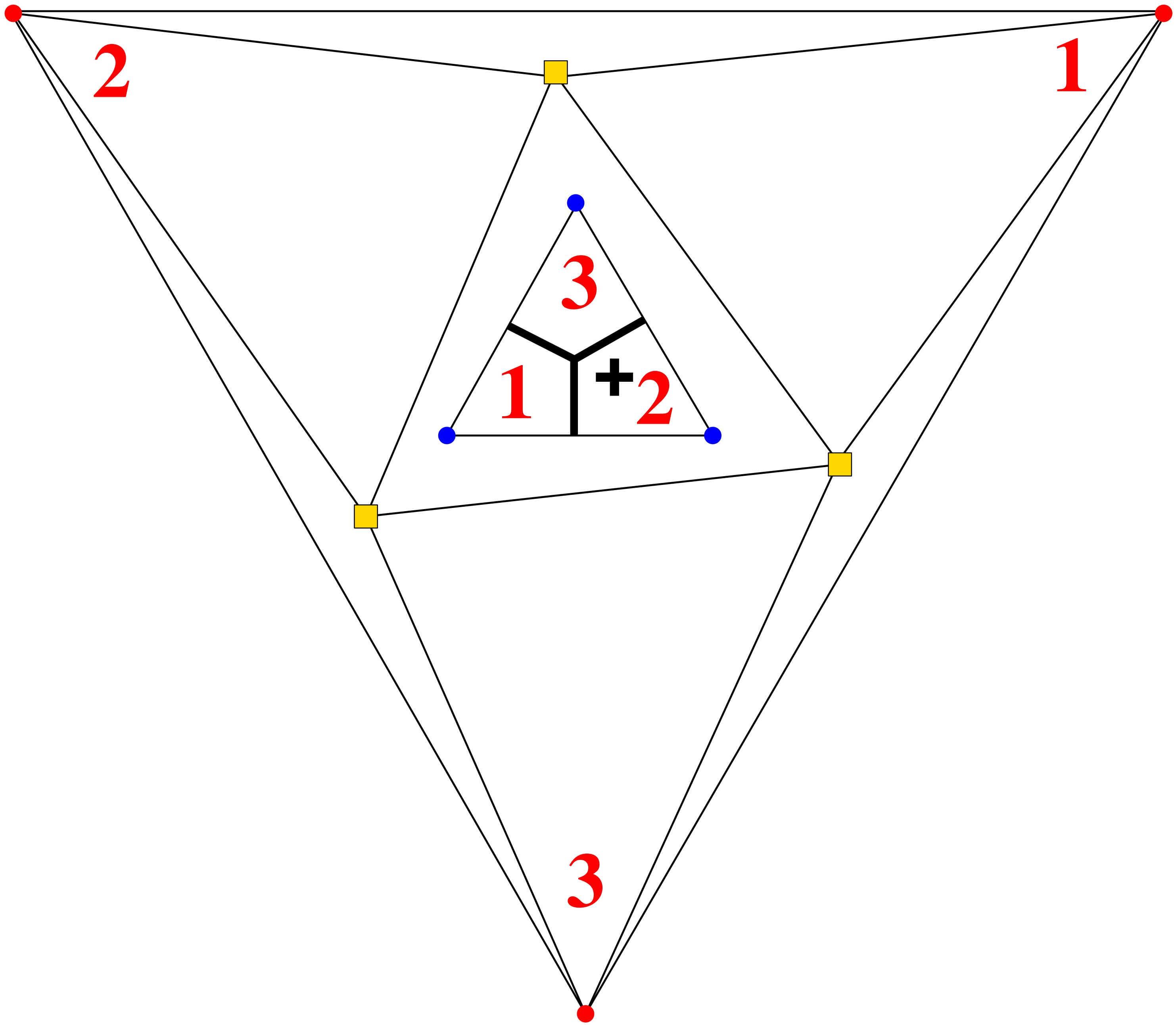

+
2
1
3
−
−
+
1
2
+
+
−
3

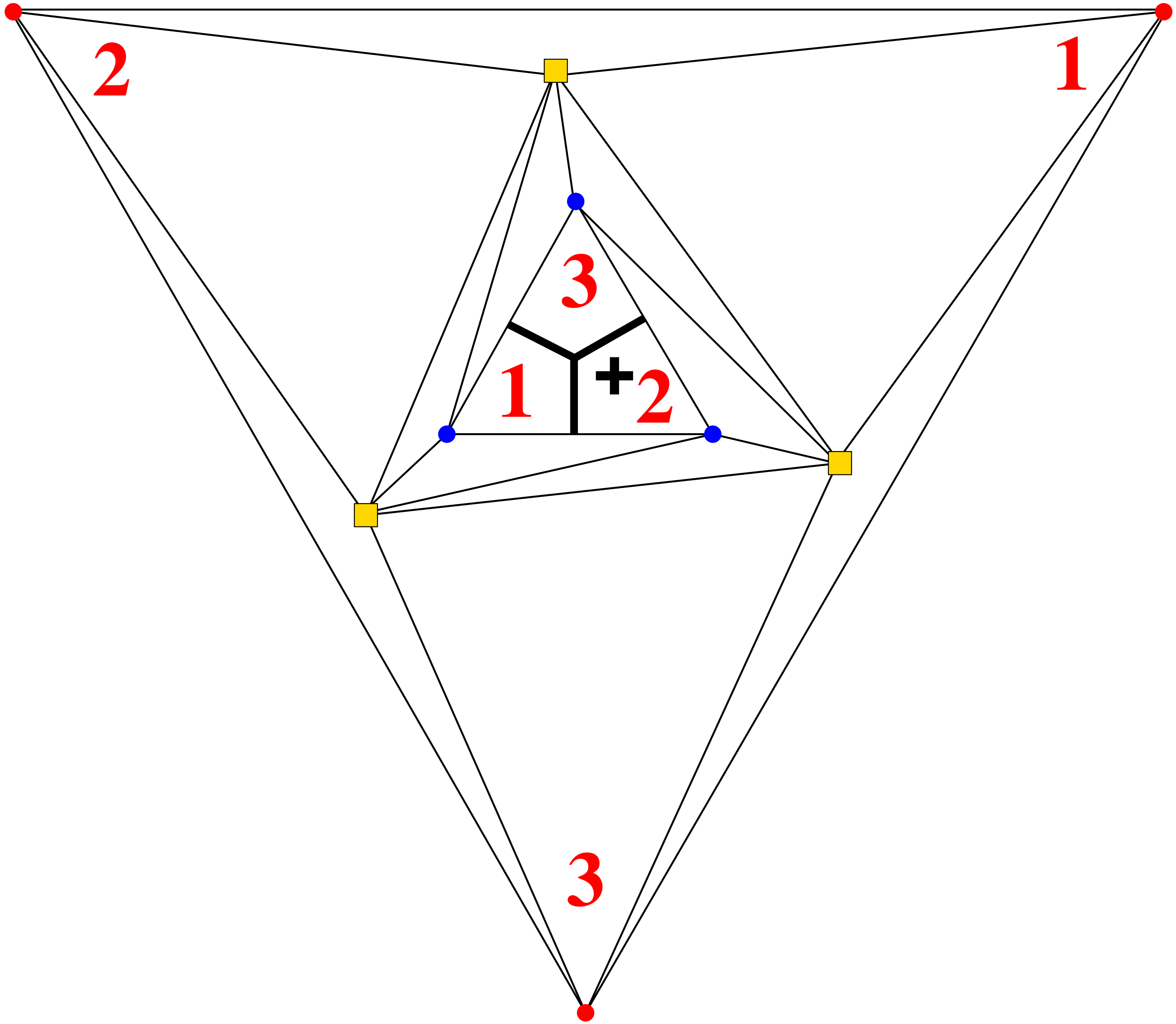

3
1
2



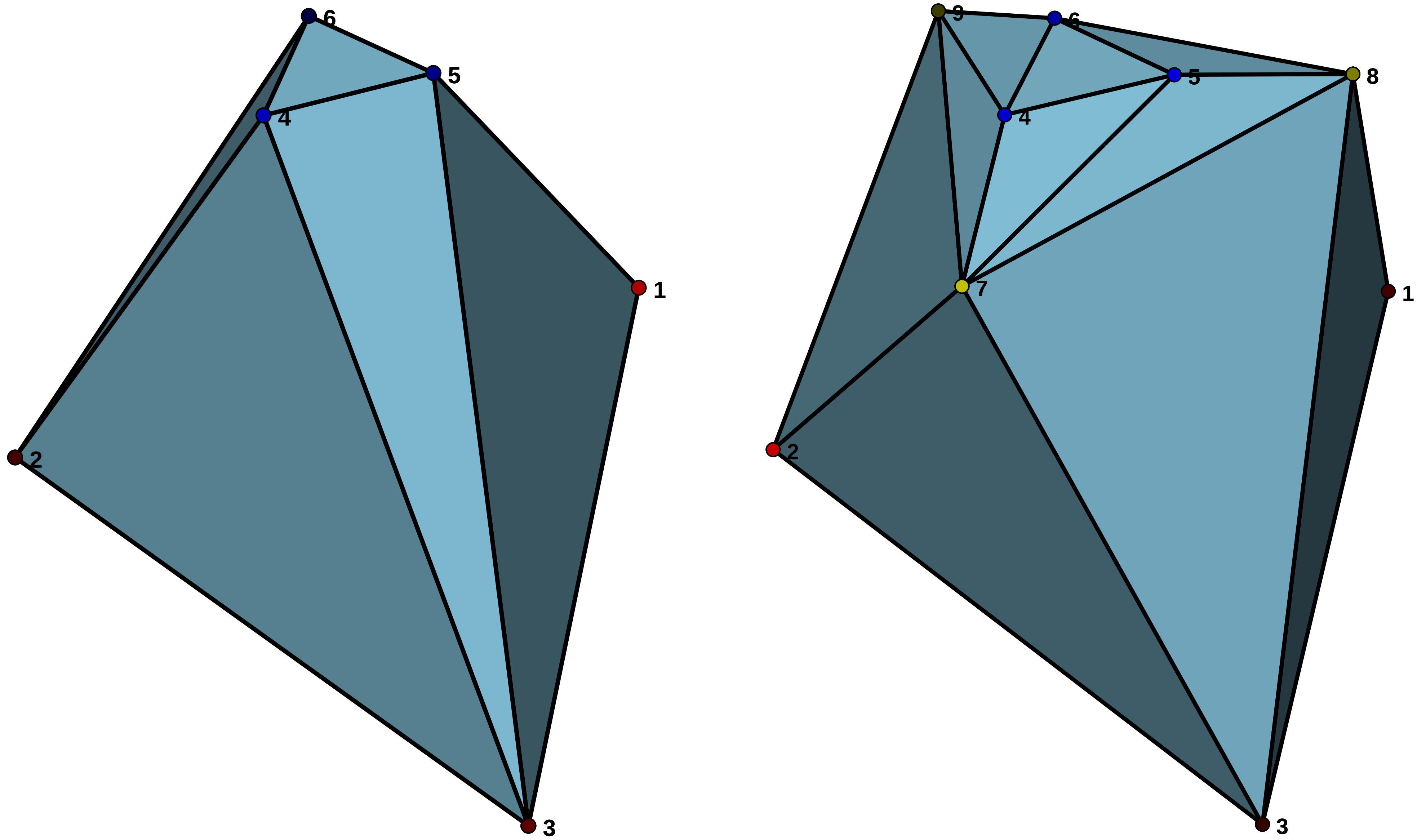
3
1
2

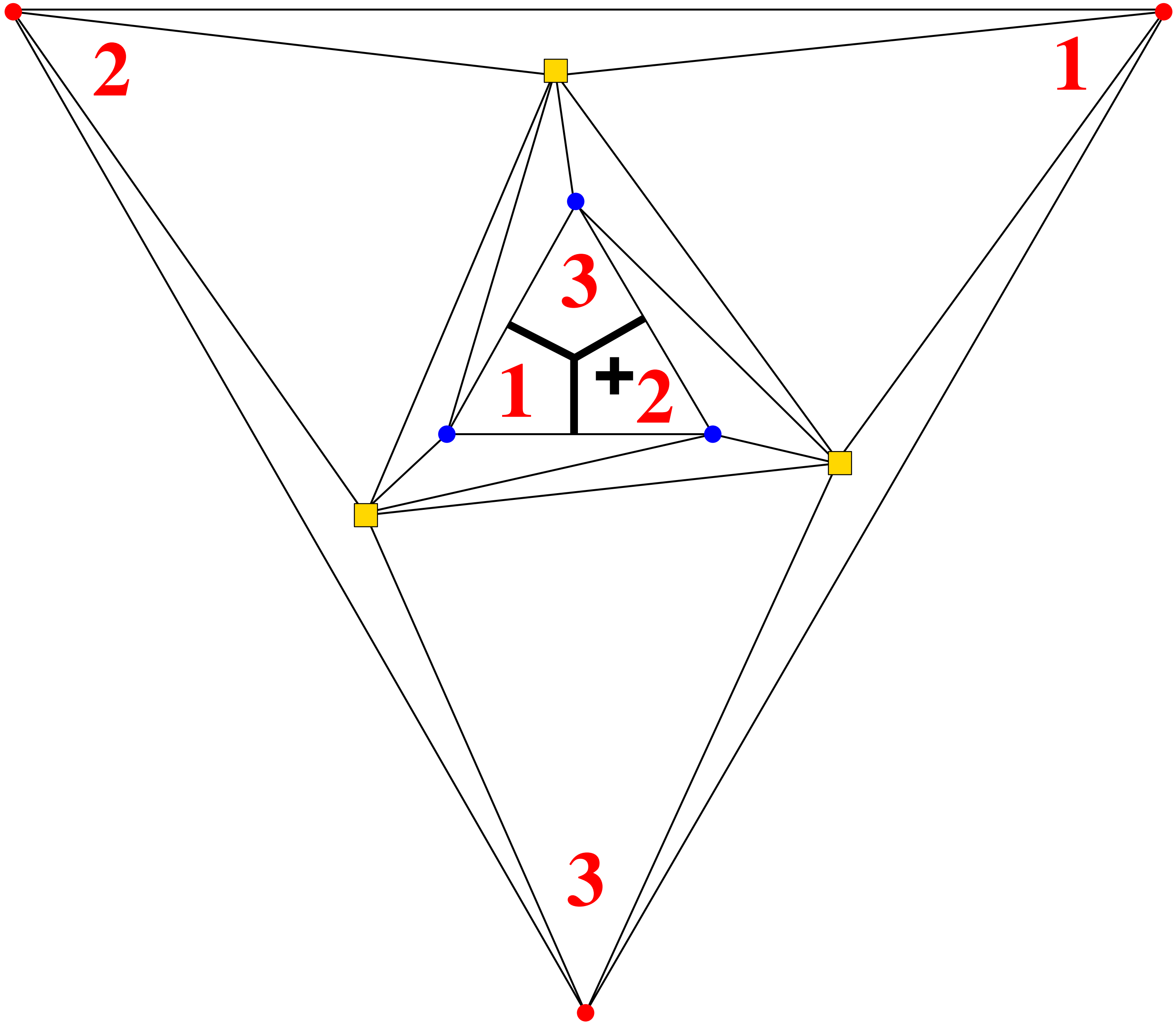

3
1
2
+

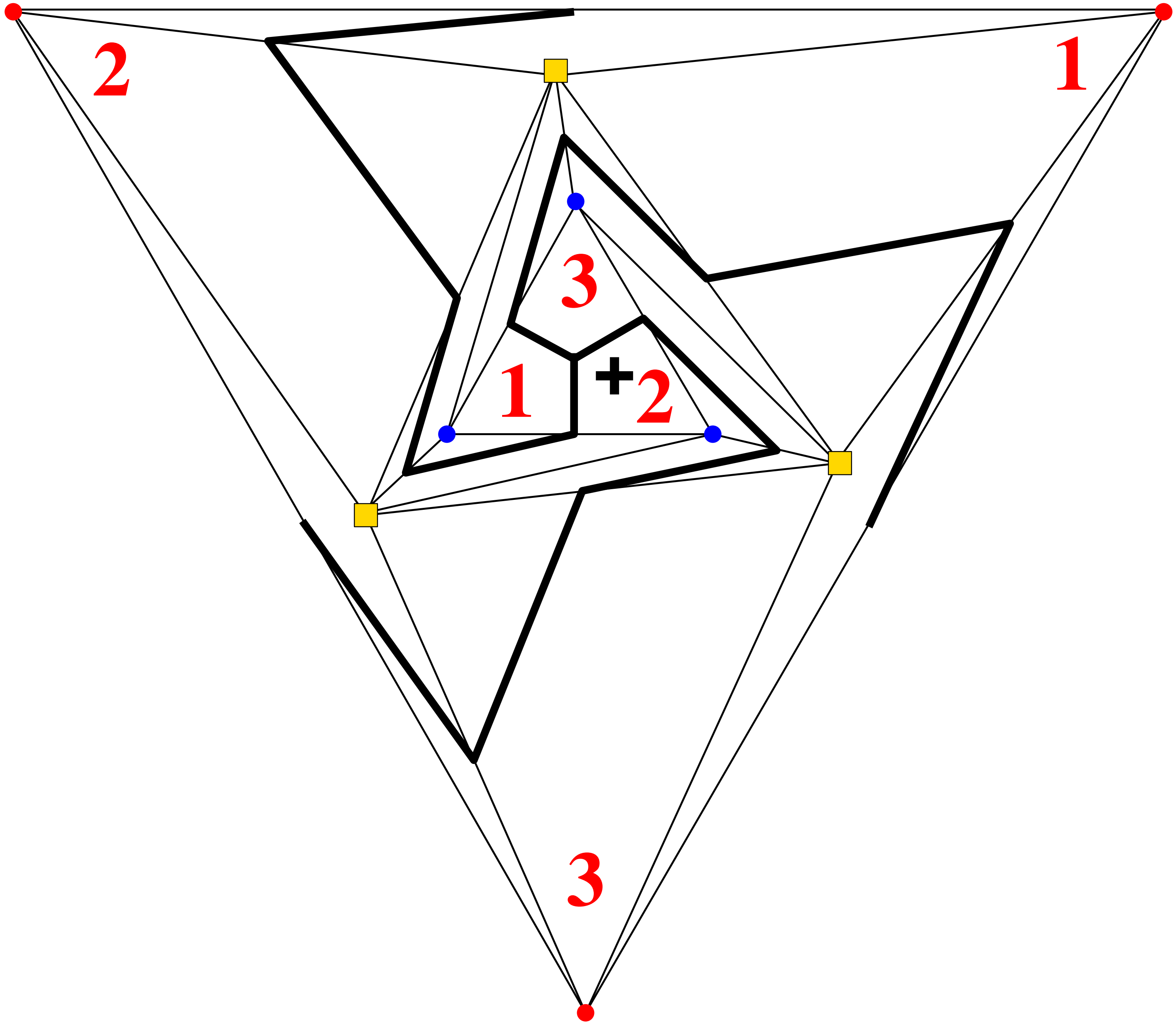

2
1
3
1
+
2
3


2
1
3
+
1
2
3

Adding new columns: polytope view




2
1
3
1
+
2
3



2
1
3
1
+
2
3

Summary

Dual construction

- Points = payoff columns (replies) of player 2
- Facet subdivision = player 1's best replies
- Visualization and **characterization of index**
- Illustration of L-H algorithm
- Low dimension $m-1$, for any n

Other applications

- Components of equilibria, hyperstability
- Fixed point theorems, Sperner's lemma
  via bimatrix games

Summary

Dual construction

- Points = payoff columns (replies) of player 2
- Facet subdivision = player 1's best replies
- Visualization and **characterization of index**
- Illustration of L-H algorithm
- Low dimension $m-1$, for any n

Other applications

- Components of equilibria, hyperstability
- Fixed point theorems, Sperner's lemma
  via bimatrix games (use for PPAD completeness?)