Efficient computation of equilibria for extensive games

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Game tree (game in extensive form)
**Strategic (or normal) form**

**Strategy** of a player:
specifies a move for **every** information set of that player.

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**Reduced strategic form**

**Reduced strategy** of a player:

specifies a move for every information set of that player,
except for those information sets unreachable due to an own earlier move (where we write * instead of a move).

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number of pure strategies typically \textbf{exponential} in number of information sets.

\textbf{Example:}

number of information sets \(= \ell\),
number of pure strategies \(= 2^\ell\).

\textbf{Example [Kuhn]:} simplified poker game,
number of information sets \(= 13\),
number of pure strategies \(= 8192\).
Example: Game with (1) bounded number of moves per node, (2) no subgames (otherwise simplify by solving subgames first).

This tree with $n$ nodes: $\approx 2^{\sqrt{n}/2}$ strategies per player, reduced strategic form still (sub-)exponential in tree size.
Our result (sneak preview)

The **sequence form** is a strategic description of an extensive game with perfect recall that has the **same** size as the game tree, as opposed to **exponential** size of reduced strategic form.

The same known strategic-form algorithms for **finding equilibria** can be applied to the sequence form:
- linear programming (LP) for two-player zero-sum games,
- linear complementarity (LCP) for general two-player games,

Game tree of size $n$:
- sequence form size $n \times n$,
- reduced strategic form: possibly size $2\sqrt{n}$.
# Size of reduced strategic form versus sequence form

<table>
<thead>
<tr>
<th>tree depth (nodes)</th>
<th>tree size (nodes)</th>
<th>number of reduced strategies player 1</th>
<th>player 2</th>
<th>Reduced Strategic Form size</th>
<th>indep. SF variables pl. 1</th>
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Use behavior strategies

**Behavior strategy** = **local** randomization

Mixed strategy too redundant, use behavior strategy instead:

- only one variable per **move**:
  - player 1 chooses **L** with probability $X_L$
  - player 1 chooses **R** with probability $X_R$ . . .
  - player 2 chooses **a** with probability $Y_a$ . . .

- expected payoff = $5\, Y_a X_L + 10\, Y_a X_R Y_p X_U + 15\, Y_a X_R Y_p X_V + \cdots$

- problem: **nonlinear**!
Variable transformation

For each sequence $\sigma$ of moves of player 1 introduce new variable $x_\sigma$

- new variables replace products:
  
  if $\sigma = PQRS$ then $x_\sigma = X_P X_Q X_R X_S$

- Example:

  $$x_L = X_L$$
  $$x_{RU} = X_R X_U$$
  $$\ldots$$
  $$y_a = Y_a$$
  $$y_{ap} = Y_a Y_p$$
  $$\ldots$$

- expected payoff $= 5 \ x_L \ y_a + 10 \ x_{RU} \ y_{ap} + 15 \ x_{RV} \ y_{ap} + \ldots$

  is linear in variables of one player.
New paradigm: Sequences instead of pure strategies

Before:

pure strategy \( i \)
probability \( x_i \)
mixed strategy \( x \)
characterized by \( 1x = 1 \)
expected payoff \( x^\top A y \)

After:

sequence \( \sigma \)
realization probability \( x_\sigma \)
realization plan \( x \)
characterized by \( Ex = e \)
expected payoff \( x^\top A y \)
\[
\begin{align*}
x_0 &= 1 \\
x_L + x_R &= x_0 \\
x_{RU} + x_{RV} &= x_R
\end{align*}
\]
Realization plans

Realization plan \( x = (x_0, x_L, x_R, x_C, x_D, x_{RU}, x_{RV}) \)

(= vector of realization probabilities)
characterized by \( x \geq 0 \) and linear equalities

\[
x_0 = 1 \\
x_0 = x_L + x_R \\
x_0 = x_C + x_D \\
x_R = x_{RU} + x_{RV}
\]

written as \( E x = e \) with

\[
E = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{bmatrix}, \quad e = \begin{bmatrix}
1 \\
0 \\
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\end{bmatrix}
\]
The sequence form

Payoff matrix $A$

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expected payoff $x^\top A y$,

rows played with $x$ subject to $x \geq 0$, $Ex = e$,

columns played with $y$ subject to $y \geq 0$, $Fy = f$. 
How to play

**Given:** realization plan $x$ with $Ex = e$.

Move $L$ is last move of **unique** sequence, say $PQL$, where $x_{PQL} + x_{PQR} = x_{PQ}$.

$\implies$ behavior-probability $(L) = \frac{x_{PQL}}{x_{PQ}}$.

Required assumption of **perfect recall** [Kuhn 1953, Selten 1975]:
Each node in an information set is preceded by same sequence, here $PQ$, of the player’s **own** earlier moves.
Best responses – LP duality

1) Best response \( x \) against fixed \( y \) solves LP:

\[
\begin{align*}
\text{max} \quad & \quad x^\top (Ay) \\
\text{subject to} \quad & \quad Ex = e \\
& \quad x \geq 0
\end{align*}
\]

2) Consider the dual of this LP:

\[
\begin{align*}
\text{min} \quad & \quad e^\top u \\
\text{subject to} \quad & \quad E^\top u \geq Ay
\end{align*}
\]

LP duality \( \implies \) same optimal value (payoff to player 1).
Best responses – LP duality

2) Consider the dual of this LP:

\[
\begin{align*}
\min_u & \quad e^\top u \\
\text{subject to} & \quad E^\top u \geq Ay
\end{align*}
\]

LP duality \implies \text{same optimal value (payoff to player 1)},

3) minimized by player 2 if zero-sum game, \( B = -A \):

\[
\begin{align*}
\min_{u, y} & \quad e^\top u \\
\text{subject to} & \quad E^\top u \geq Ay \\
& \quad Fy = f \\
& \quad y \geq 0
\end{align*}
\]
Example

1) Best response LP

\[
\begin{align*}
\max_x \quad & x^\top (Ay) \\
\text{subject to} \quad & Ex = e \\
\quad & x \geq 0
\end{align*}
\]

\[
\begin{bmatrix}
x_0 \\
x_L \\
x_R \\
x_C \\
x_D
\end{bmatrix} \begin{bmatrix}
1 & -1 & 1 & 0 \\
1 & 0 & 0 & 2 \\
1 & 0 & 0 & 2 \\
1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
0 \\
2 \\
2 \\
1 \\
0 \\
\end{bmatrix}
\]

2) dual LP

\[
\begin{align*}
\min_u \quad & e^\top u \\
\text{subject to} \quad & E^\top u \geq Ay
\end{align*}
\]

\[
\begin{bmatrix}
u_0 \\
u_1 \\
u_2
\end{bmatrix} \begin{bmatrix}
1 & -1 & -1 & 0 \\
1 & 0 & 0 & 2 \\
1 & -1 & 1 & 2 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
0 \\
2 \\
2 \\
1 \\
0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} \rightarrow \min
\]
Example

2) dual LP

\[
\min_{u} \quad e^\top u \\
\text{subject to} \quad E^\top u \geq Ay
\]

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
1 & 1 & 1 & 2 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
1 0 0 \rightarrow \text{min}
\]

3) Treat \( y \) as a variable:

\[
\min_{u, y} \quad e^\top u \\
\text{subject to} \quad E^\top u \geq Ay \\
Fy = f \\
y \geq 0
\]

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
1 & 1 & 1 & 6 \\
1 & 0 & 0 & 0 \\
-1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_0 \\
y_a \\
y_b \\
y_c
\end{bmatrix} \geq 0
\]

\[
\begin{bmatrix}
1 & 2 & 4 & 3 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
1 0 0 \rightarrow \text{min}
\]
Input:
Two-person game tree with perfect recall.

Theorem:
A zero-sum game is solved via a Linear Program (LP) of linear size.

Theorem:
A non-zero-sum game is solved via a Linear Complementarity Problem (LCP) of linear size.

A sample equilibrium is found by Lemke’s algorithm.

This algorithm mimicks the Harsanyi–Selten tracing procedure and finds a normal form perfect equilibrium.
Consider a prior \((x, y)\), and a new variable \(z_0\) in the system

\[
\begin{align*}
\mathbf{r} &= \begin{bmatrix} E^\top u \\ F^\top v - B^\top x \end{bmatrix} \\
\mathbf{s} &= \begin{bmatrix} -Ay \\ -B^\top x \end{bmatrix}z_0 \geq 0
\end{align*}
\]

Equilibrium condition \(x^\top r = 0, \ y^\top s = 0, \ [z_0 = 0]\).

Initial solution \(z_0 = 1, \ x = 0, \ y = 0\).

Complementary pivoting:
\(x_\sigma \leftrightarrow r_\sigma, \ y_\tau \leftrightarrow s_\tau\), until \(z_0\) leaves the basis.