Geometric Views of Linear Complementarity Algorithms and Their Complexity

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LCP - Definition

Given: $q \in \mathbf{R}^n$, $M \in \mathbf{R}^{n \times n}$

Find: $z \in \mathbf{R}^n$ so that

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz \geq \mathbf{0}$$

$\perp$ means orthogonal:

$$z^T w = 0$$

$\Leftrightarrow$ $z_{\mathbf{i}} w_{\mathbf{i}} = 0$ all $\mathbf{i} = 1, \ldots, n$.

LP in inequality form

primal: **max** $c^T x$

subject to $Ax \leq b$

$x \geq \mathbf{0}$

dual: **min** $y^T b$

subject to $y^T A \geq c^T$

$y \geq \mathbf{0}$

Weak duality: x, y feasible (fulfilling constraints)

$\Rightarrow$ $c^T x \leq y^T A x \leq y^T b$

Strong duality: primal and dual are feasible

$\Rightarrow$ $\exists$ feasible x, y: $c^T x = y^T b$ (x, y optimal)

LCP generalizes LP

LCP encodes the complementary slackness of strong duality:

$$c^T x = y^T A x = y^T b$$

$$\Leftrightarrow \quad \underset{\geq \mathbf{0}}{(y^T A - c^T)}\,\underset{\geq \mathbf{0}}{x} = 0, \quad \underset{\geq \mathbf{0}}{y^T}\,\underset{\geq \mathbf{0}}{(b - Ax)} = 0.$$

LP $\Leftrightarrow$ LCP

$$\begin{array}{cc} x \geq \mathbf{0} & \perp \\ y \geq \mathbf{0} & \perp \end{array} \quad \boxed{\begin{array}{r} -\,c \\ b \end{array}} \; \boxed{\begin{array}{lr} & +\,A^T y \\ -\,Ax & \end{array}} \; \begin{array}{l} \geq \mathbf{0} \\ \geq \mathbf{0} \end{array}$$

Symmetric equilibria of symmetric games

Given: $n \times n$ **payoff matrix** A for row player
A^T for column player

mixed strategy x = probability distribution on $\{1,\ldots,n\}$

$$\Leftrightarrow \quad x \ge \mathbf{0}\,, \quad \mathbf{1}^T x = 1$$

equilibrium (x, x)

$\Leftrightarrow$ x **best response** to x

Remark: As general as $m \times n$ games (A, B).

Best responses

Given: $n \times n$ **payoff matrix** A,
mixed strategy y of column player

Ay = vector of **expected payoffs** against y,
components $(Ay)_i$

x **best response** to y

$\Leftrightarrow$ x maximizes expected payoff $x^T Ay$

best response condition:

$\Leftrightarrow$ $\forall \mathbf{i}: \ x_i > 0 \ \Rightarrow \ (Ay)_i = u = \max_k \, (Ay)_k$

Symmetric equilibria as LCP solutions

equilibrium (x, x) of game with payoff matrix A

$\Leftrightarrow$ x **best response** to x

$\Leftrightarrow$ $\mathbf{1}^\top x = 1$,

$x \geq \mathbf{0} \quad \perp \quad Ax \leq \mathbf{1}u$

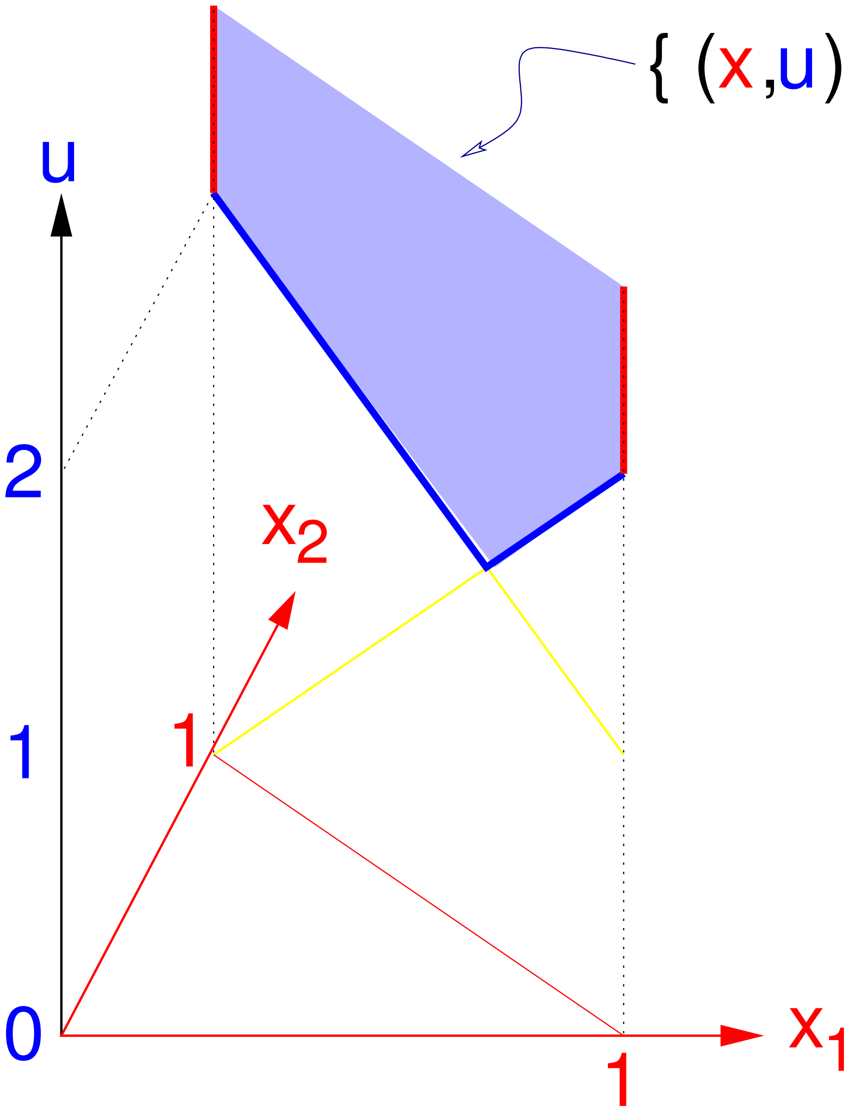
w.l.o.g. $A > 0 \quad \Rightarrow \quad u > 0$,

equilibrium (x, x)

$\Leftrightarrow$ $z = (1/u)\, x \qquad (1/u = \mathbf{1}^\top z)$,

$$\boxed{z \geq \mathbf{0} \quad \perp \quad Az \leq \mathbf{1} \qquad \text{"equilibrium } z\text{"}}$$

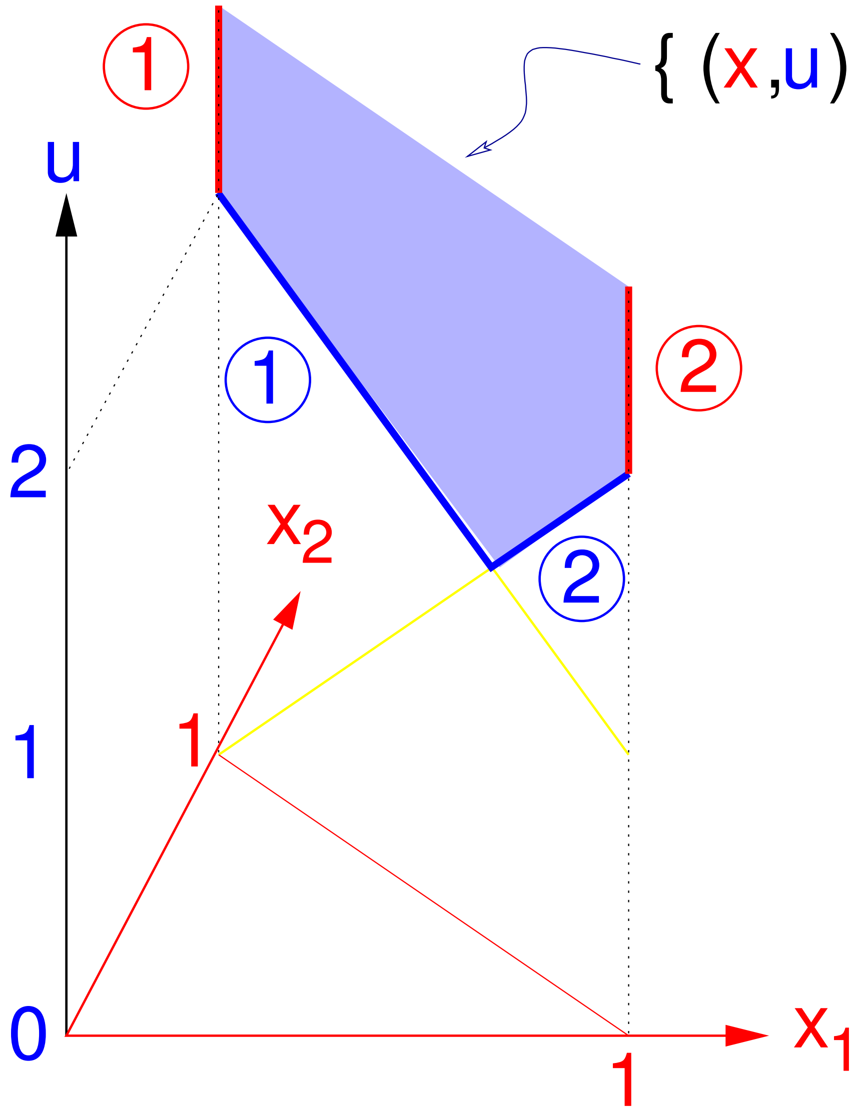
Best response polyhedron



$\{ (x, u) \mid x \geq \mathbf{0},\ \mathbf{1}^T x = 1,\ Ax \leq \mathbf{1}u \}$

$$A = \boxed{\begin{matrix} 1 & 2 \\ 2 & 0 \end{matrix}}$$

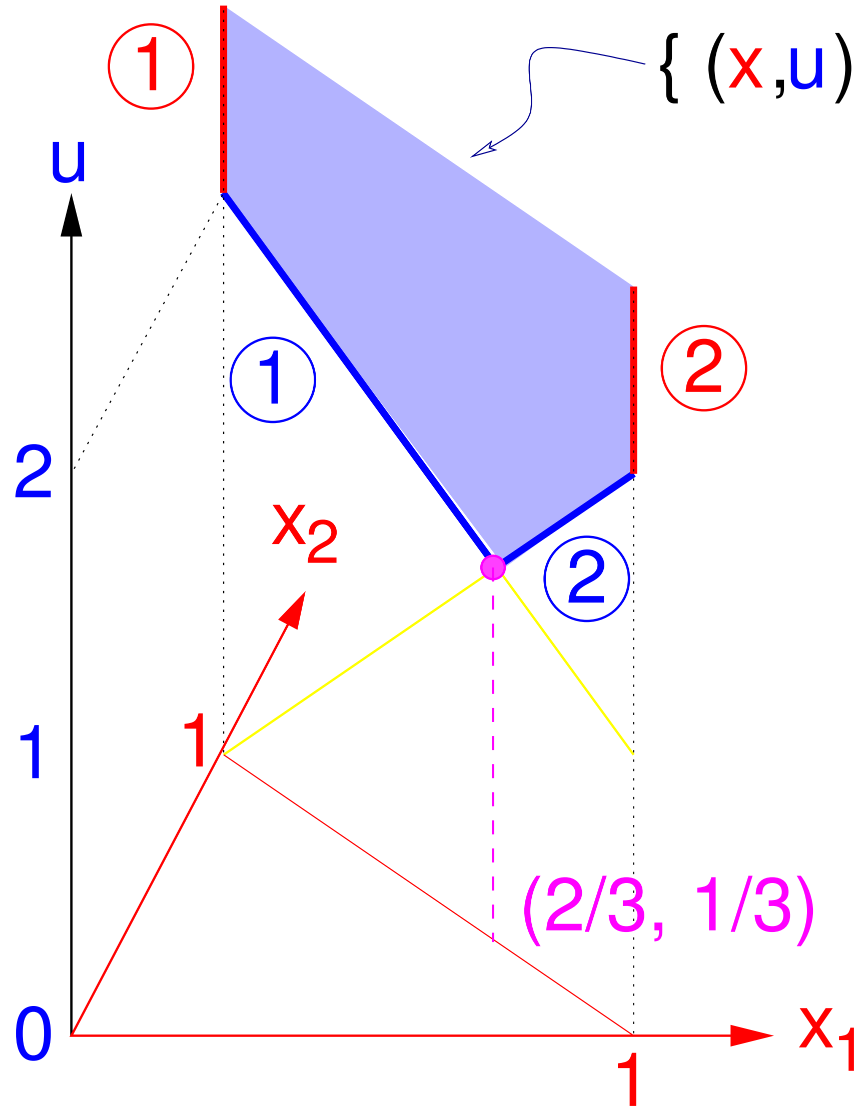
Best response polyhedron



$\{\ (x,u)\ \mid\ x \ge \mathbf{0},\ \ \mathbf{1}^T x = 1,\ Ax \le \mathbf{1}u\ \}$

$$A = \begin{array}{cc|c} ① & ② & \\ \hline 1 & 2 & ① \\ 2 & 0 & ② \end{array}$$

Best response polyhedron

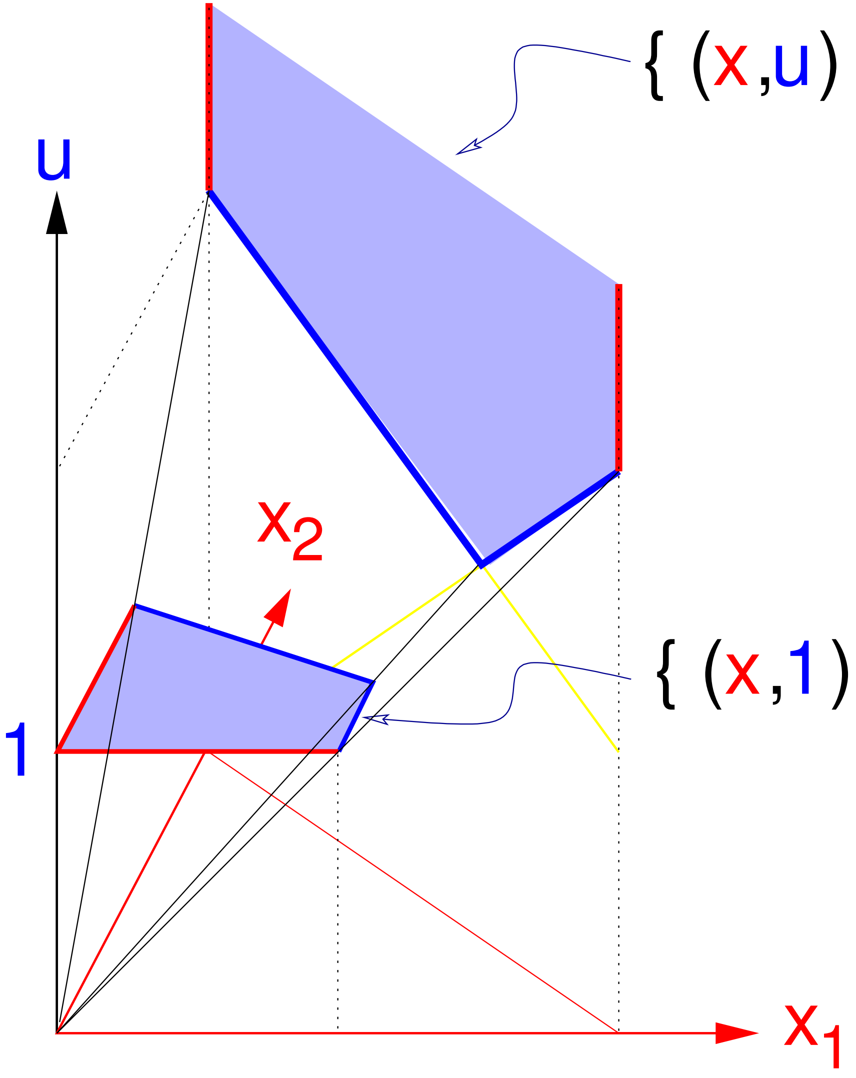


$$\{\, (x,u) \mid x \ge \mathbf{0},\ \mathbf{1}^\top x = 1,\ Ax \le \mathbf{1}u \,\}$$

$$A = \begin{array}{cc|c} \textcircled{1} & \textcircled{2} & \\ \hline 1 & 2 & \textcircled{1} \\ 2 & 0 & \textcircled{2} \end{array}$$

equilibrium
(completely labeled)

Projective transformation

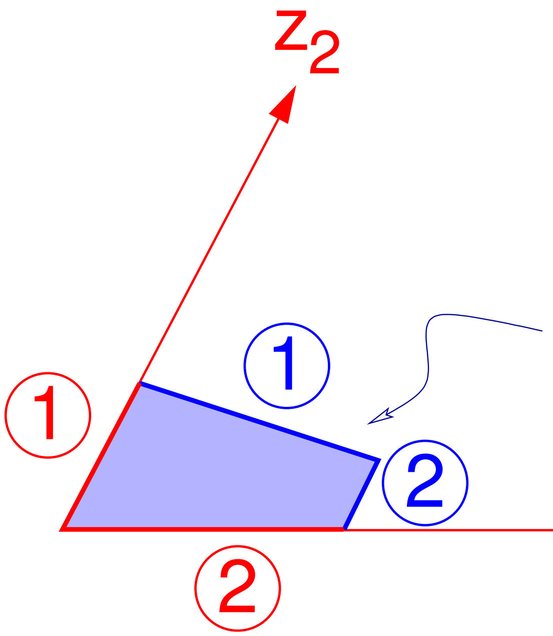


$\{\ (x,u)\ \mid\ x \geq \mathbf{0},\ \ \mathbf{1}^T x = 1,\ Ax \leq \mathbf{1}u\ \}$

$\{\ (x,1)\ \mid\ x \geq \mathbf{0},\ Ax \leq \mathbf{1}\ \}$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

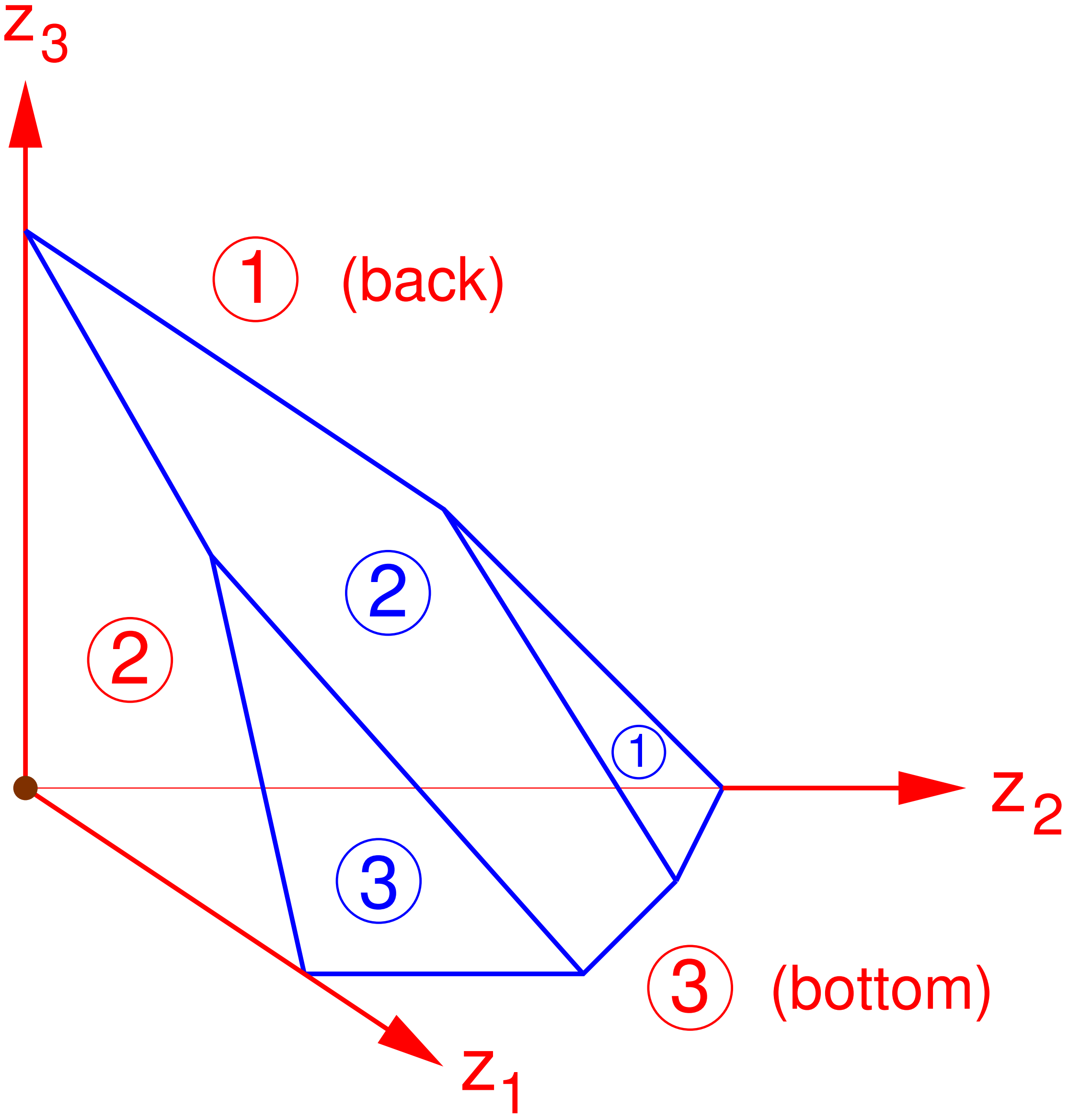
Best response polytope



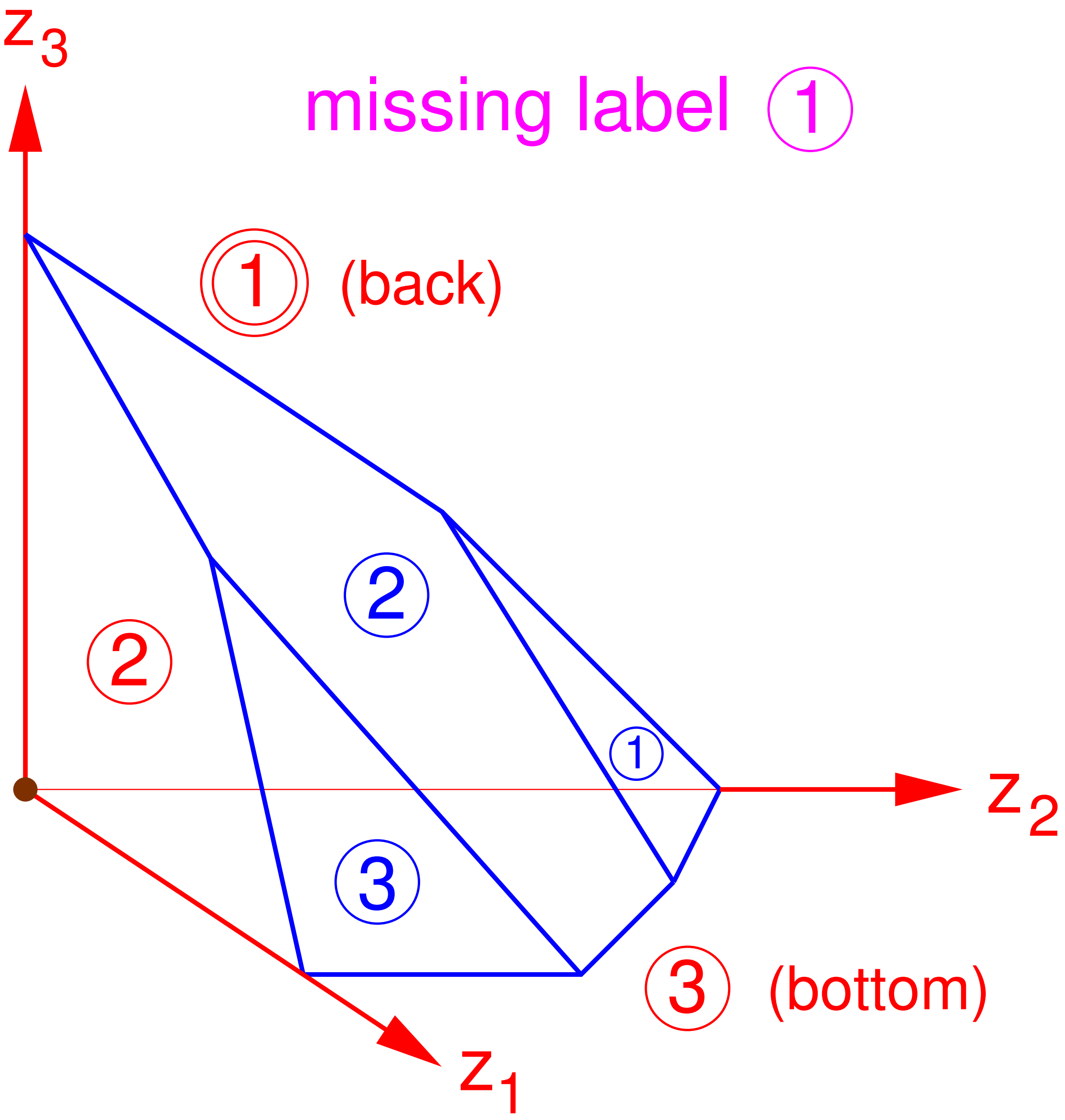
$$\{\, z \mid z \ge \mathbf{0},\ A z \le \mathbf{1} \,\}$$

$$A = \boxed{\begin{matrix} 1 & 2 \\ 2 & 0 \end{matrix}}$$

Symmetric Lemke–Howson algorithm

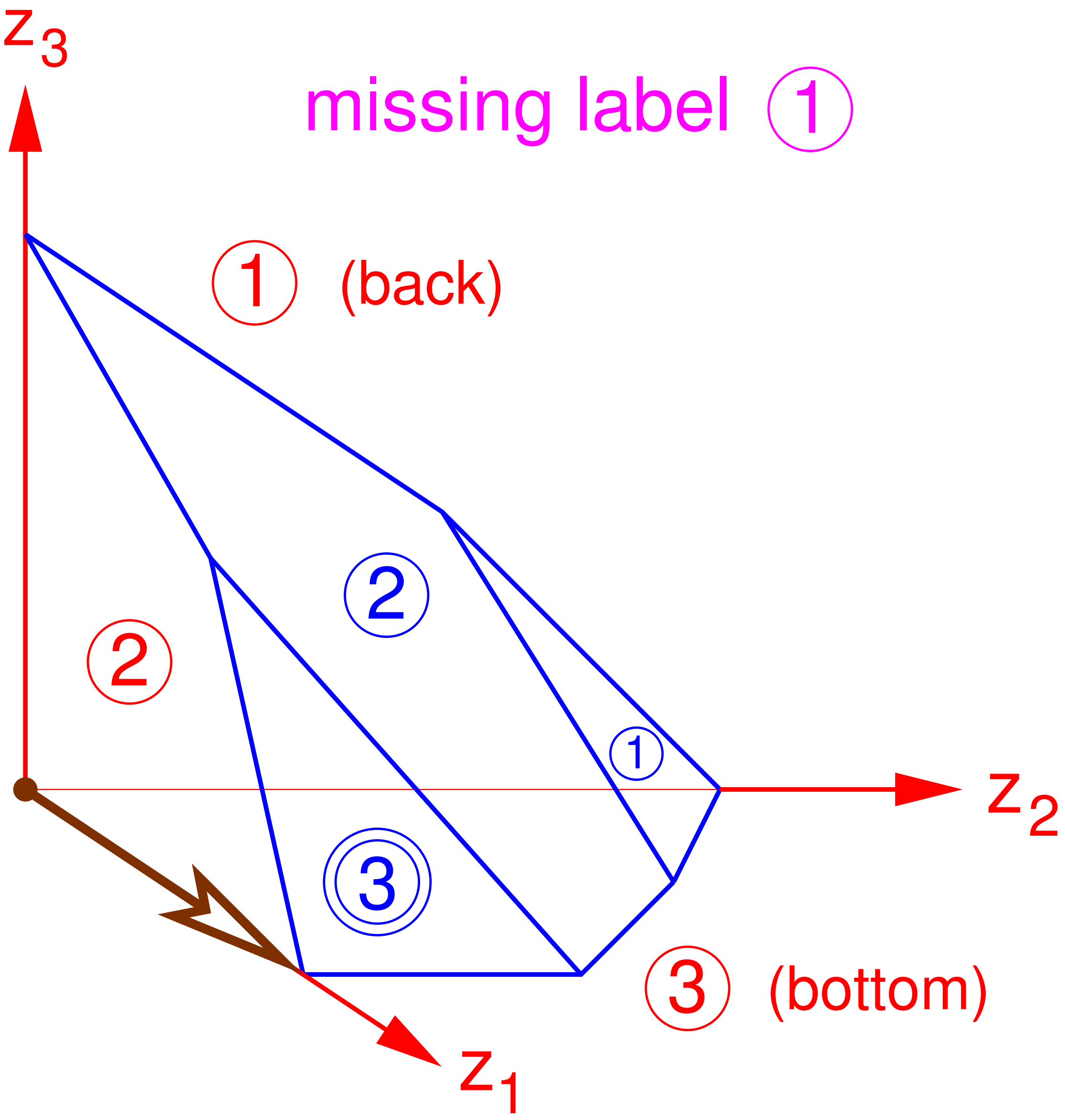


Symmetric Lemke–Howson algorithm

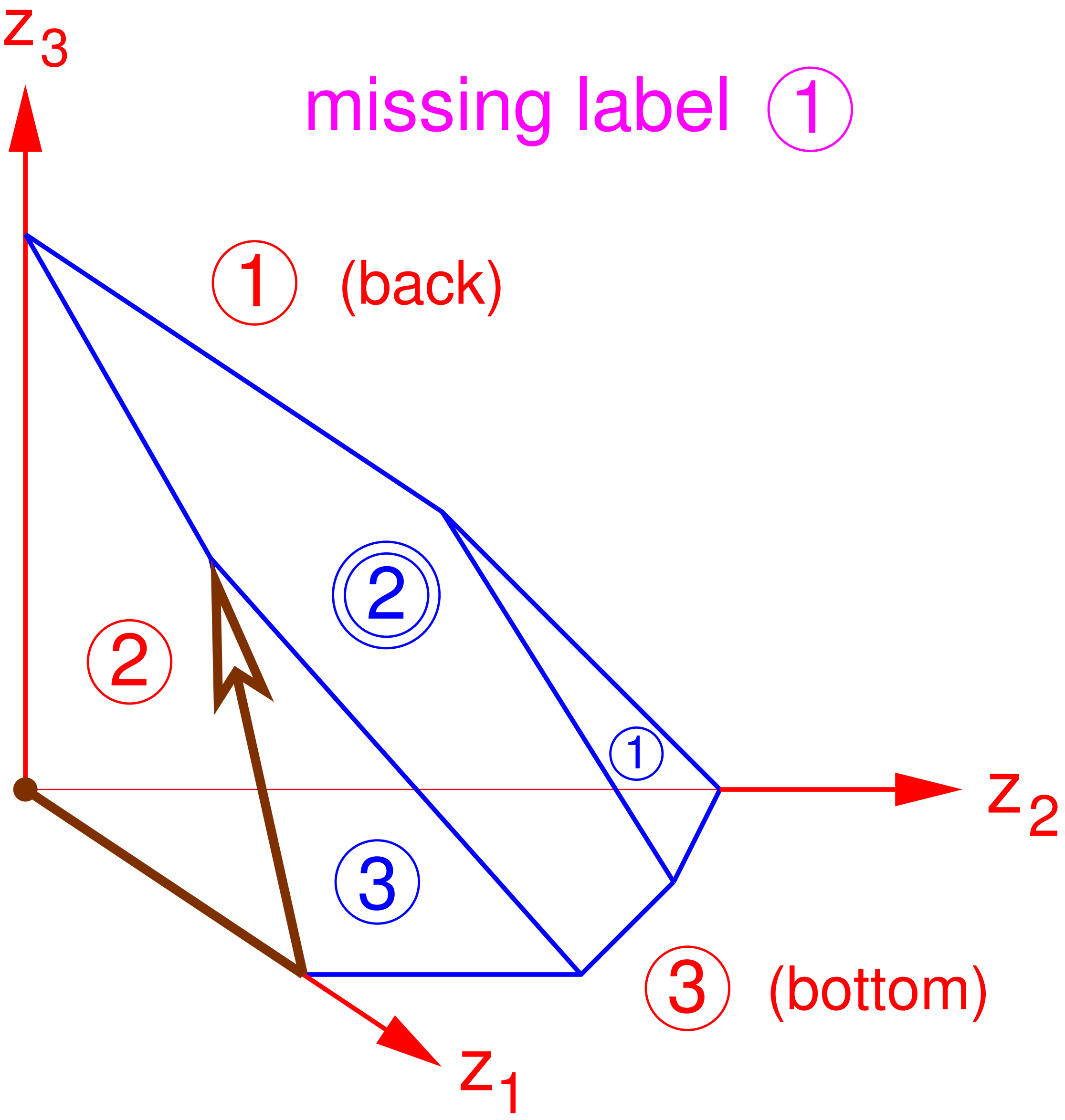


missing label ①

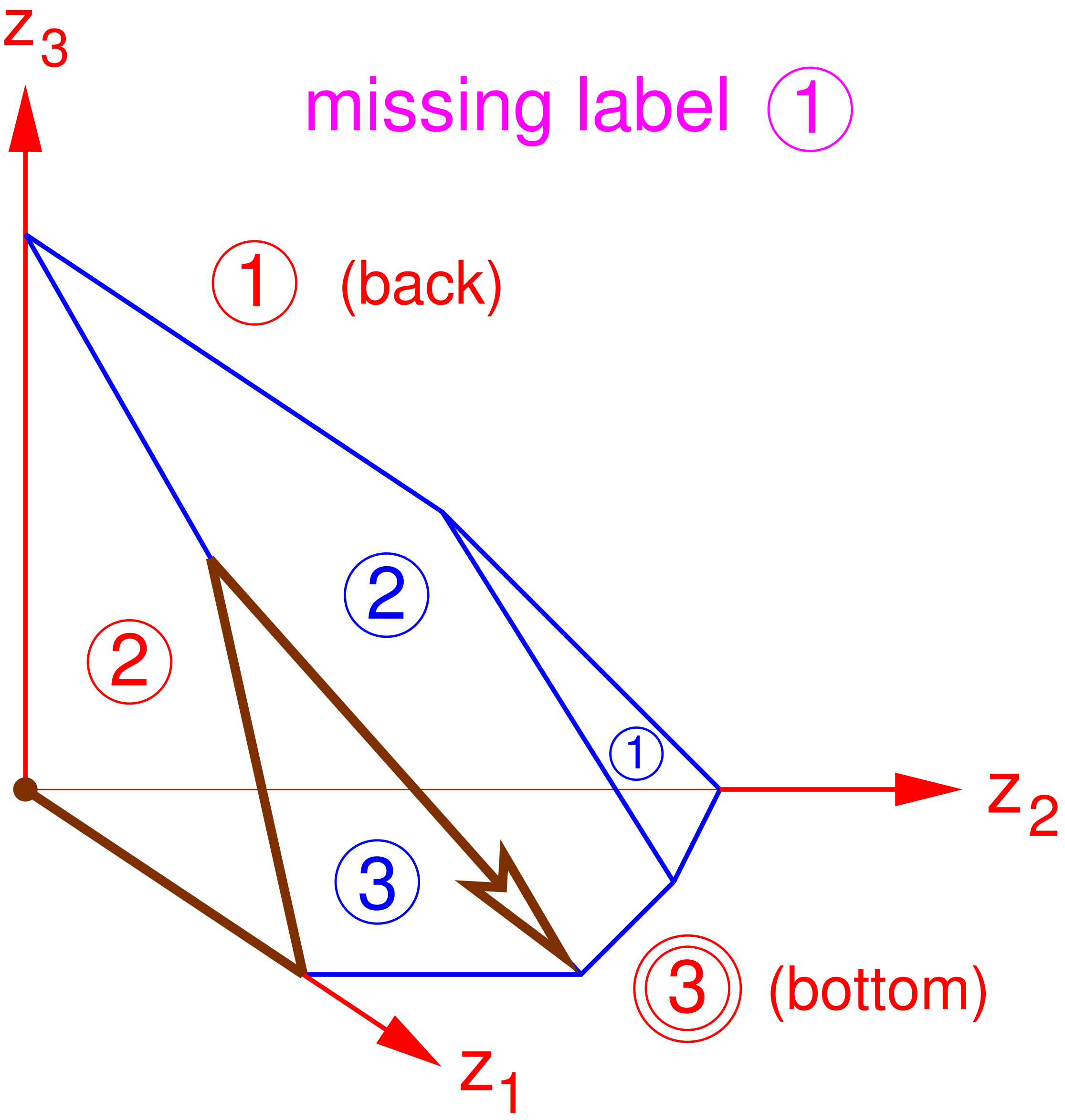
Symmetric Lemke–Howson algorithm



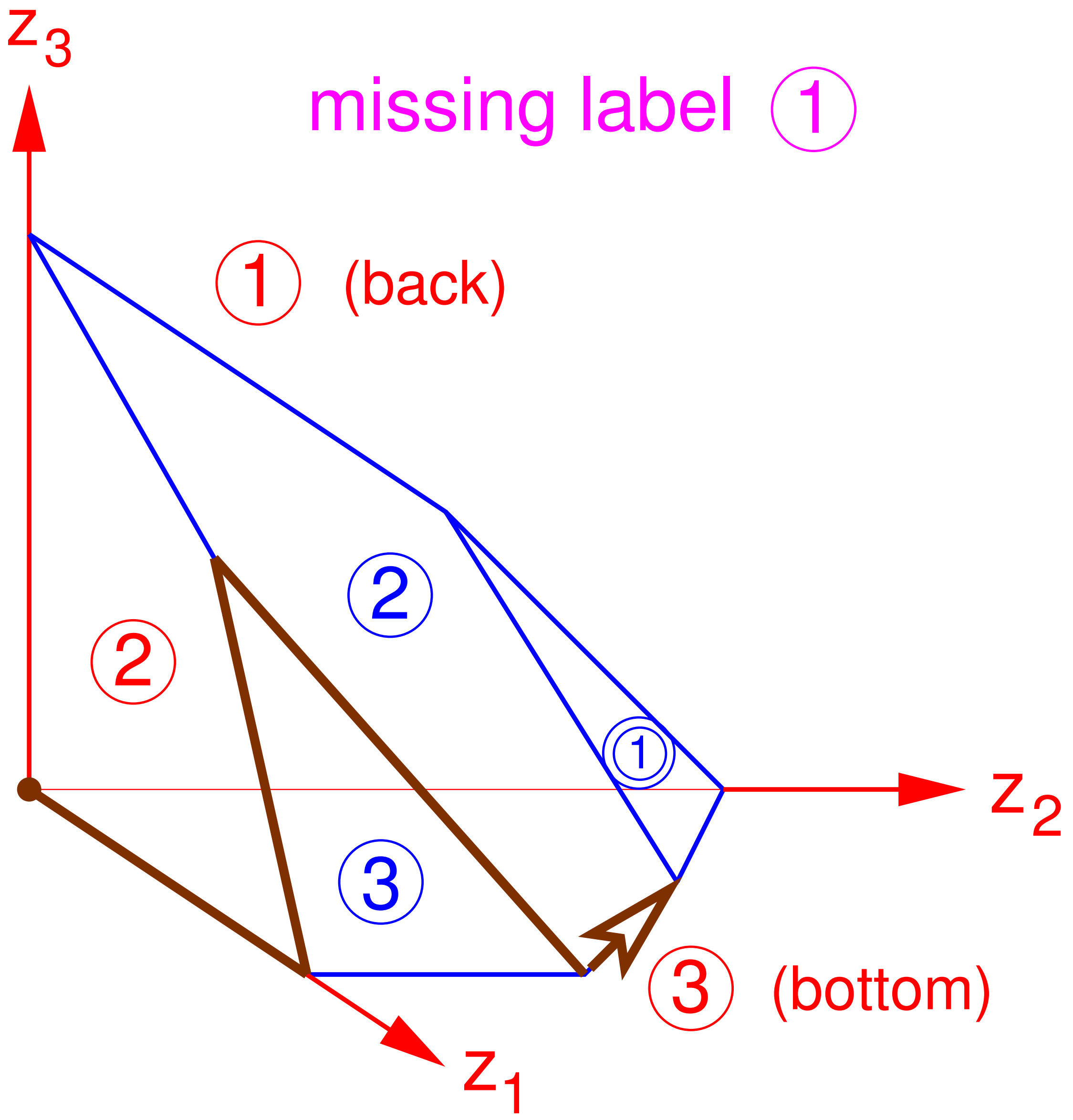
Symmetric Lemke–Howson algorithm



Symmetric Lemke–Howson algorithm

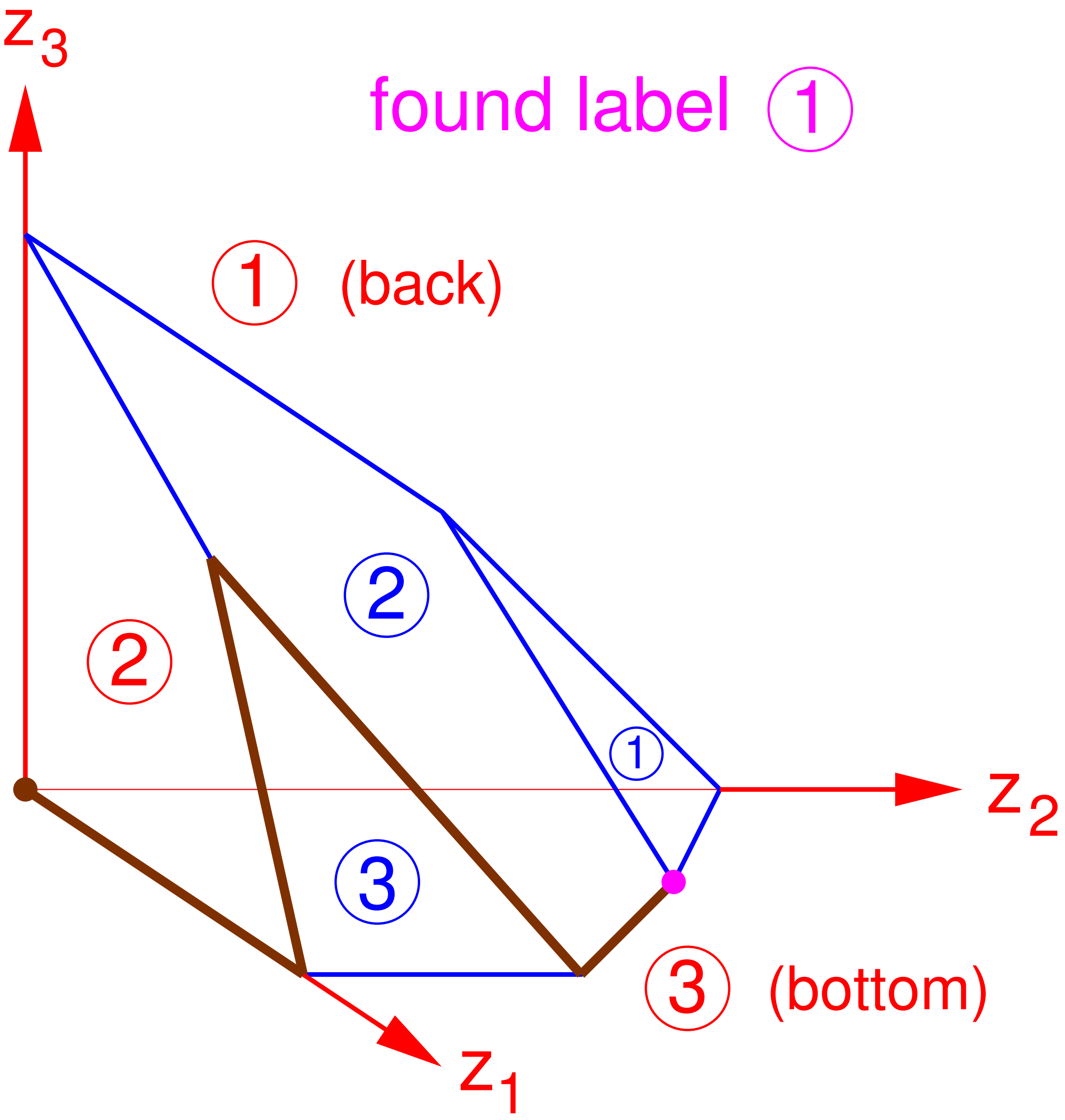


Symmetric Lemke–Howson algorithm



missing label ①

Symmetric Lemke–Howson algorithm



Why Lemke-Howson works

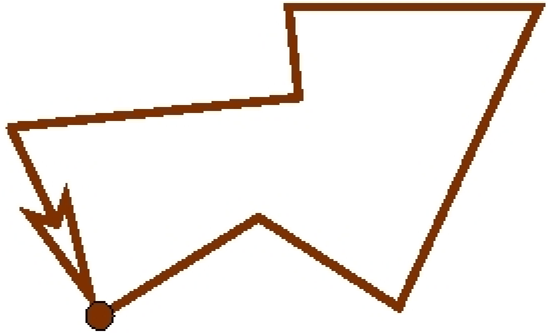
LH finds at least one Nash equilibrium because

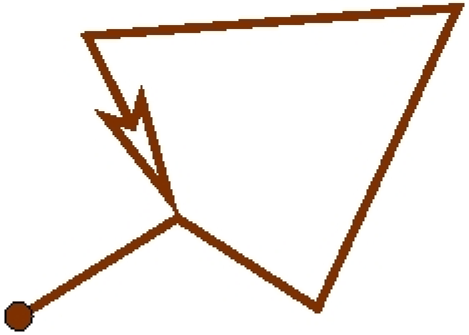
- finitely many "vertices"

for nondegenerate (generic) games:

- **unique** starting edge given missing label
- **unique** continuation

⇒ precludes "coming back" like here:





Costs instead of payoffs

$$\boxed{\begin{array}{cc} 1 & 2 \\ 2 & 0 \end{array}} \quad \rightarrow \quad \boxed{\begin{array}{cc} 2 & 1 \\ 1 & 3 \end{array}}$$

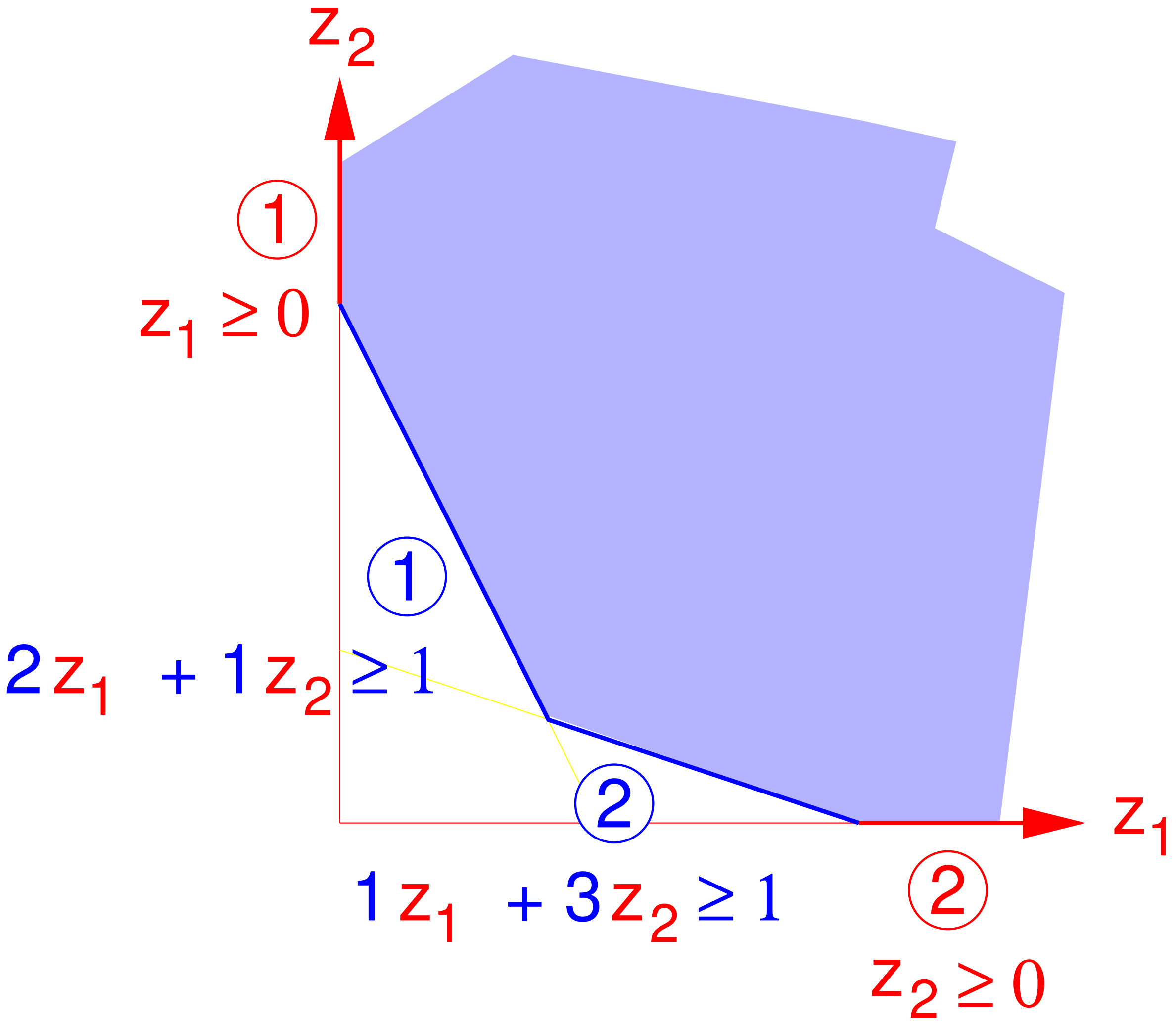
a_{ik}
payoff

$3 - a_{ik}$
cost

with new cost matrix $A > 0$:

$$\boxed{\text{equilibrium } z \quad \Leftrightarrow \qquad z \geq \mathbf{0} \quad \perp \quad Az \geq \mathbf{1}}$$

Polyhedral view



Lemke's algorithm

given LCP

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz \qquad \geq \mathbf{0}$$

Lemke's algorithm

augmented LCP

$$\mathbf{z} \geq \mathbf{0} \quad \perp \quad \mathbf{w} = \mathbf{q} + M\mathbf{z} + \mathbf{d}z_0 \geq \mathbf{0}$$

$$z_0 \geq 0$$

Lemke's algorithm

augmented LCP

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz + \mathbf{d}z_{\mathbf{0}} \geq \mathbf{0}$$

$$z_{\mathbf{0}} \geq 0$$

where

$\mathbf{d} > \mathbf{0}$ **covering vector**

$z_{\mathbf{0}}$ extra variable

$z_{\mathbf{0}} = 0$ $\Leftrightarrow$ $z \perp w$ solves original LCP

Lemke's algorithm

augmented LCP

$$z \geq \mathbf{0} \quad \perp \quad w = q + Mz + \mathbf{d}z_{\mathbf{0}} \geq \mathbf{0}$$

$$z_{\mathbf{0}} \geq 0$$

Initialization:

$$\boxed{z = \mathbf{0} \quad \perp \quad w = q \qquad\qquad + \mathbf{d}z_{\mathbf{0}} \geq \mathbf{0}}$$

$z_{\mathbf{0}} \geq 0$ minimal $\Rightarrow$ $w_{\mathbf{i}} = 0$ for some **i**

pivot $z_{\mathbf{0}}$ in, $w_{\mathbf{i}}$ out,

$\Rightarrow$ can increase $z_{\mathbf{i}}$ while maintaining $z \perp w$.

Lemke's algorithm = complementary pivoting

***n* equations:** $$\boldsymbol{w} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{z} + \boldsymbol{d}\boldsymbol{z}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \boldsymbol{z}_0$$

maintain $$\boldsymbol{w} \geq \mathbf{0} \;\perp\; \boldsymbol{z} \geq \mathbf{0}$$

	in out

Lemke’s algorithm = complementary pivoting

***n* equations:** $\boldsymbol{w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0}$

maintain $\boldsymbol{w \geq 0 \;\perp\; z \geq 0}$

	in	*out*
$\boldsymbol{w_1 = -1 + 2z_1 + z_2 + 2z_0}$		
$\boldsymbol{w_2 = -1 + z_1 + 3z_2 + z_0}$		

Lemke's algorithm = complementary pivoting

***n* equations:** $\quad w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $\quad w \geq 0 \perp z \geq 0$

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
$w_2 = -1 + z_1 + 3z_2 + \boxed{z_0}$	z_0	

Lemke's algorithm = complementary pivoting

***n* equations:** $w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $w \geq 0 \perp z \geq 0$

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
$w_2 = -1 + z_1 + 3z_2 + \boxed{z_0}$	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2

Lemke's algorithm = complementary pivoting

***n* equations:** $$w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$$

maintain $$w \geq 0 \perp z \geq 0$$

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2

Lemke's algorithm = complementary pivoting

***n* equations:** $w = q + Mz + dz_0 = \begin{bmatrix}-1\\-1\end{bmatrix} + \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix} z + \begin{bmatrix}2\\1\end{bmatrix} z_0$

maintain $w \geq 0 \perp z \geq 0$

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2

Lemke's algorithm = complementary pivoting

***n* equations:** $\quad w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $\quad w \geq 0 \perp z \geq 0$

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
$2z_0 = 2 + 2w_2 - 2z_1 - 6z_2$		

Lemke's algorithm = complementary pivoting

***n* equations:** $\quad w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $\quad w \geq 0 \ \perp \ z \geq 0$

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
$2z_0 = 2 + 2w_2 - 2z_1 - 6z_2$		
$w_1 = 1 + 2w_2 - 5z_2$		

Lemke's algorithm = complementary pivoting

***n* equations:** $$w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$$

maintain $$w \geq 0 \perp z \geq 0$$

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
$w_1 = 1 + 2w_2 - 5z_2$		

Lemke's algorithm = complementary pivoting

***n* equations:** $w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $w \geq 0 \perp z \geq 0$

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
$w_1 = 1 + 2w_2 - \boxed{5z_2}$	z_2	

Lemke's algorithm = complementary pivoting

***n* equations:** $w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $w \geq 0 \perp z \geq 0$

$$\begin{array}{rcllllll|ll}
 & & & & & & & & in & out \\
\hline
 & & & & & & & & z_0 & \\
z_0 & = & 1 & & + & w_2 - & z_1 & - 3z_2 & & w_2 \\
w_1 & = & 1 & & + & 2w_2 & & - \boxed{5z_2} & z_2 & \\
z_2 & = & 0.2 & - 0.2w_1 & + & 0.4w_2 & & & & w_1
\end{array}$$

Lemke's algorithm = complementary pivoting

***n* equations:** $\boldsymbol{w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0}$

maintain $\boldsymbol{w \geq 0 \perp z \geq 0}$

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
	z_2	
$z_2 = 0.2 - 0.2w_1 + 0.4w_2$		w_1

Lemke's algorithm = complementary pivoting

***n* equations:** $\boldsymbol{w} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{z} + \boldsymbol{d}z_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $\boldsymbol{w} \geq \boldsymbol{0} \;\perp\; \boldsymbol{z} \geq \boldsymbol{0}$

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
	z_2	
$z_2 = 0.2 - 0.2w_1 + 0.4w_2$		w_1
$-3z_2 = -0.6 + 0.6w_1 - 1.2w_2$		

Lemke’s algorithm = complementary pivoting

***n* equations:** $w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $w \geq 0 \perp z \geq 0$

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
	z_2	
$z_2 = 0.2 - 0.2w_1 + 0.4w_2$		w_1
$-3z_2 = -0.6 + 0.6w_1 - 1.2w_2$		
$z_0 = 0.4 + 0.6w_1 - 0.2w_2 - z_1$		

Lemke's algorithm = complementary pivoting

***n* equations:** $w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $w \geq 0 \perp z \geq 0$

	in	*out*
	z_0	
		w_2
	z_2	
$z_2 = 0.2 - 0.2w_1 + 0.4w_2$		w_1
$z_0 = 0.4 + 0.6w_1 - 0.2w_2 - z_1$		

Lemke's algorithm = complementary pivoting

***n* equations:** $\quad w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $\quad w \geq 0 \perp z \geq 0$

	in	*out*
	z_0	
		w_2
	z_2	
$z_2 = \quad 0.2 - 0.2w_1 + 0.4w_2$		w_1
$z_0 = \quad 0.4 + 0.6w_1 - 0.2w_2 - \boxed{z_1}$	z_1	

Lemke's algorithm = complementary pivoting

***n* equations:** $w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0$

maintain $w \geq 0 \;\perp\; z \geq 0$

	in	*out*
	z_0	
		w_2
	z_2	
$z_2 = 0.2 - 0.2w_1 + 0.4w_2$		w_1
$z_0 = 0.4 + 0.6w_1 - 0.2w_2 - \boxed{z_1}$	z_1	
$z_1 = 0.4 + 0.6w_1 - 0.2w_2 \qquad - z_0$		z_0

Lemke’s algorithm = complementary pivoting

***n* equations:** $\boldsymbol{w = q + Mz + dz_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} z_0}$

maintain $\boldsymbol{w \geq 0 \ \perp \ z \geq 0}$

	in	***out***
	z_0	
		w_2
	z_2	
$z_2 = 0.2 - 0.2w_1 + 0.4w_2$		w_1
	z_1	
$z_1 = 0.4 + 0.6w_1 - 0.2w_2 \quad - \quad z_0$		z_0

Potential difficulties of Lemke

Complementary pivoting: $\boldsymbol{w_i}$ out → $\boldsymbol{z_i}$ in, $\boldsymbol{z_i}$ out → $\boldsymbol{w_i}$ in.

- degeneracy: nonunique leaving variable
- numerical stability: must follow **unique** path
- **ray termination**: no leaving variable (analogous to unbounded objective function in simplex algorithm)
- worst-case exponential complexity

Lexicographic degeneracy resolution

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Lexicographic degeneracy resolution

$$\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{b} \\ \boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} &= \boldsymbol{b} \end{aligned}$$

Lexicographic degeneracy resolution

$$\begin{aligned}
\boldsymbol{Ax} &= \boldsymbol{b} \\
\boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} &= \boldsymbol{b} \\
\boldsymbol{A_B x_B} &= \boldsymbol{b} - \boldsymbol{A_N x_N}
\end{aligned}$$

Lexicographic degeneracy resolution

$$\begin{aligned}
Ax &= b \\
A_B x_B + A_N x_N &= b \\
A_B x_B &= b - A_N x_N \\
x_B &= A_B^{-1} b - A_B^{-1} A_N x_N
\end{aligned}$$

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ with powers of small $\varepsilon > \mathbf{0}$, $\vec{\varepsilon} = (\mathbf{1}, \varepsilon, \varepsilon^{\mathbf{2}}, \ldots \varepsilon^{\boldsymbol{n}})^\top$

$$\begin{aligned}
\boldsymbol{Ax} &= \boldsymbol{b} \\
\boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} &= \boldsymbol{b} \\
\boldsymbol{A_B x_B} &= \boldsymbol{b} - \boldsymbol{A_N x_N} \\
\boldsymbol{x_B} &= \boldsymbol{A_B^{-1}}\, \boldsymbol{b} - \boldsymbol{A_B^{-1} A_N x_N}
\end{aligned}$$

Lexicographic degeneracy resolution

perturb ***b*** with powers of small $\varepsilon > \mathbf{0}$, $\qquad \vec{\varepsilon} = (\mathbf{1}, \varepsilon, \varepsilon^{\mathbf{2}}, \ldots \varepsilon^{\boldsymbol{n}})^\top$

$$\begin{aligned}
\boldsymbol{Ax} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} \\
\boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} \\
\boldsymbol{A_B x_B} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} - \boldsymbol{A_N x_N} \\
\boldsymbol{x_B} &= \boldsymbol{A_B^{-1}}[\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} - \boldsymbol{A_B^{-1} A_N x_N}
\end{aligned}$$

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ with powers of small $\varepsilon > 0$, $\vec{\varepsilon} = (1, \varepsilon, \varepsilon^2, \ldots \varepsilon^n)^\top$

$$\begin{aligned}
\boldsymbol{Ax} &= [\, \boldsymbol{b} \mid \boldsymbol{I}\,]\, \vec{\varepsilon} \\
\boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} &= [\, \boldsymbol{b} \mid \boldsymbol{I}\,]\, \vec{\varepsilon} \\
\boldsymbol{A_B x_B} &= [\, \boldsymbol{b} \mid \boldsymbol{I}\,]\, \vec{\varepsilon} \; - \; \boldsymbol{A_N x_N} \\
\boldsymbol{x_B} &= \boldsymbol{A_B^{-1}} [\, \boldsymbol{b} \mid \boldsymbol{I}\,]\, \vec{\varepsilon} \; - \; \boldsymbol{A_B^{-1} A_N x_N} \\
\boldsymbol{x_B} &= [\, \boldsymbol{A_B^{-1} b} \mid \boldsymbol{A_B^{-1}}\,]\, \vec{\varepsilon} \; - \; \boldsymbol{A_B^{-1} A_N x_N}
\end{aligned}$$

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ with powers of small $\varepsilon > \mathbf{0}$, $\qquad \vec{\varepsilon} = (\mathbf{1}, \varepsilon, \varepsilon^{\mathbf{2}}, \ldots \varepsilon^{\boldsymbol{n}})^\top$

$$\begin{array}{rcrcr}
\boldsymbol{Ax} & = & [\ \boldsymbol{b} \mid \boldsymbol{I}\ ]\,\vec{\varepsilon} & & \\
\boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} & = & [\ \boldsymbol{b} \mid \boldsymbol{I}\ ]\,\vec{\varepsilon} & & \\
\boldsymbol{A_B x_B} & = & [\ \boldsymbol{b} \mid \boldsymbol{I}\ ]\,\vec{\varepsilon} & - & \boldsymbol{A_N x_N} \\
\boldsymbol{x_B} & = & \boldsymbol{A_B^{-1}}[\ \boldsymbol{b} \mid \boldsymbol{I}\ ]\,\vec{\varepsilon} & - & \boldsymbol{A_B^{-1} A_N x_N} \\
\boldsymbol{x_B} & = & [\ \boldsymbol{A_B^{-1} b} \mid \boldsymbol{A_B^{-1}}\ ]\,\vec{\varepsilon} & - & \boldsymbol{A_B^{-1} A_N x_N}
\end{array}$$

nondegeneracy $\Leftrightarrow$ $\boldsymbol{x_B} > \mathbf{0}$ for small $\varepsilon > \mathbf{0}$ $\Leftrightarrow$ $[\ \boldsymbol{A_B^{-1} b} \mid \boldsymbol{A_B^{-1}}\ ]$ **lexico-positive** (first nonzero element in each row $> \mathbf{0}$).

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ with powers of small $\varepsilon > 0$, $\quad \vec{\varepsilon} = (1, \varepsilon, \varepsilon^2, \dots \varepsilon^n)^\top$

$$\begin{aligned}
Ax &= [\,b \mid I\,]\,\vec{\varepsilon} \\
A_B x_B + A_N x_N &= [\,b \mid I\,]\,\vec{\varepsilon} \\
A_B x_B &= [\,b \mid I\,]\,\vec{\varepsilon} - A_N x_N \\
x_B &= A_B^{-1}[\,b \mid I\,]\,\vec{\varepsilon} - A_B^{-1} A_N x_N \\
x_B &= [\,A_B^{-1} b \mid A_B^{-1}\,]\,\vec{\varepsilon} - A_B^{-1} A_N x_N
\end{aligned}$$

nondegeneracy $\Leftrightarrow$ $x_B > 0$ for small $\varepsilon > 0$ $\Leftrightarrow$ $[\,A_B^{-1} b \mid A_B^{-1}\,]$ **lexico-positive** (first nonzero element in each row > 0).

Example:

$$\begin{bmatrix} 1 & -9 & 4 & 0 \\ 0 & 3 & -100 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \vec{\varepsilon} = \begin{bmatrix} 1 - 9\varepsilon + 4\varepsilon^2 \\ 3\varepsilon - 100\varepsilon^2 + 2\varepsilon^3 \\ 5\varepsilon^3 \end{bmatrix}$$

Numerical stability with integer pivoting

	in	*out*

Numerical stability with integer pivoting

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
$w_2 = -1 + z_1 + 3z_2 + z_0$		

Numerical stability with integer pivoting

$$\begin{array}{rcrcrcrcr|cc}
 & & & & & & & & & \textit{in} & \textit{out} \\
\hline
w_1 & = & -1 & + & 2z_1 & + & z_2 & + & 2z_0 & & \\
w_2 & = & -1 & + & z_1 & + & 3z_2 & + & \boxed{z_0} & z_0 &
\end{array}$$

Numerical stability with integer pivoting

$$
\begin{array}{rcrrrrrrrrr|cc}
 & & & & & & & & & & & \textit{in} & \textit{out} \\
\hline
w_1 & = & -1 & & & + & 2z_1 & + & z_2 & + & 2z_0 & & \\
w_2 & = & -1 & & & + & z_1 & + & 3z_2 & + & \boxed{z_0} & z_0 & \\
z_0 & = & 1 & + & w_2 & - & z_1 & - & 3z_2 & & & & w_2
\end{array}
$$

Numerical stability with integer pivoting

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2

Numerical stability with integer pivoting

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2

Numerical stability with integer pivoting

	in	*out*
$w_1 = -1 + 2z_1 + z_2 + 2z_0$		
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
$2z_0 = 2 + 2w_2 - 2z_1 - 6z_2$		

Numerical stability with integer pivoting

$$
\begin{array}{rcrrrrrr|cc}
 & & & & & & & & \textit{in} & \textit{out} \\
\hline
w_1 & = & -1 & & +\ 2z_1 & +\ z_2 & +\ 2z_0 & & & \\
 & & & & & & & & z_0 & \\
z_0 & = & 1 & +\ w_2 & -\ z_1 & -\ 3z_2 & & & & w_2 \\
2z_0 & = & 2 & +\ 2w_2 & -\ 2z_1 & -\ 6z_2 & & & & \\
w_1 & = & 1 & +\ 2w_2 & & -\ 5z_2 & & & & \\
\end{array}
$$

Numerical stability with integer pivoting

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
$w_1 = 1 + 2w_2 - 5z_2$		

Numerical stability with integer pivoting

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
$w_1 = 1 + 2w_2 - \boxed{5z_2}$	z_2	

Numerical stability with integer pivoting

$$
\begin{array}{rcl|cc}
 & & & \textit{in} & \textit{out} \\ \hline
 & & & z_0 & \\
z_0 & = & 1 + w_2 - z_1 - 3z_2 & & w_2 \\
w_1 & = & 1 + 2w_2 - \boxed{5z_2} & z_2 & \\
5z_2 & = & 1 - w_1 + 2w_2 & & w_1
\end{array}
$$

Numerical stability with integer pivoting

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
	z_2	
$5z_2 = 1 - w_1 + 2w_2$		w_1

Numerical stability with integer pivoting

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
	z_2	
$5z_2 = 1 - w_1 + 2w_2$		w_1
$5z_0 = 5 + 5w_2 - 5z_1 - 15z_2$		

Numerical stability with integer pivoting

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
	z_2	
$5z_2 = 1 - w_1 + 2w_2$		w_1
$5z_0 = 5 + 5w_2 - 5z_1 - 15z_2$		
$-15z_2 = -3 + 3w_1 - 6w_2$		

Numerical stability with integer pivoting

	in	*out*
	z_0	
$z_0 = 1 + w_2 - z_1 - 3z_2$		w_2
	z_2	
$5z_2 = 1 - w_1 + 2w_2$		w_1
$5z_0 = 5 + 5w_2 - 5z_1 - 15z_2$		
$-15z_2 = -3 + 3w_1 - 6w_2$		
$5z_0 = 2 + 3w_1 - w_2 - 5z_1$		

Numerical stability with integer pivoting

	in	*out*
	z_0	
		w_2
	z_2	
$5z_2 = \quad 1 - \quad w_1 + 2w_2$		w_1
$5z_0 = \quad 2 + 3w_1 - \quad w_2 - 5z_1$		

Numerical stability with integer pivoting

	in	*out*
	z_0	
		w_2
	z_2	
$5z_2 = 1 - w_1 + 2w_2$		w_1
$5z_0 = 2 + 3w_1 - w_2 - \boxed{5z_1}$	z_1	

Numerical stability with integer pivoting

	in	*out*
	z_0	
		w_2
	z_2	
$5z_2 = 1 - w_1 + 2w_2$		w_1
$5z_0 = 2 + 3w_1 - w_2 - \boxed{5z_1}$	z_1	
$5z_1 = 2 + 3w_1 - w_2 \qquad - 5z_0$		z_0

Numerical stability with integer pivoting

	in	*out*
	z_0	
		w_2
	z_2	
$5z_2 = 1 - w_1 + 2w_2$		w_1
	z_1	
$5z_1 = 2 + 3w_1 - w_2 \qquad - 5z_0$		z_0

Ray termination

can be excluded for many “matrix classes”,

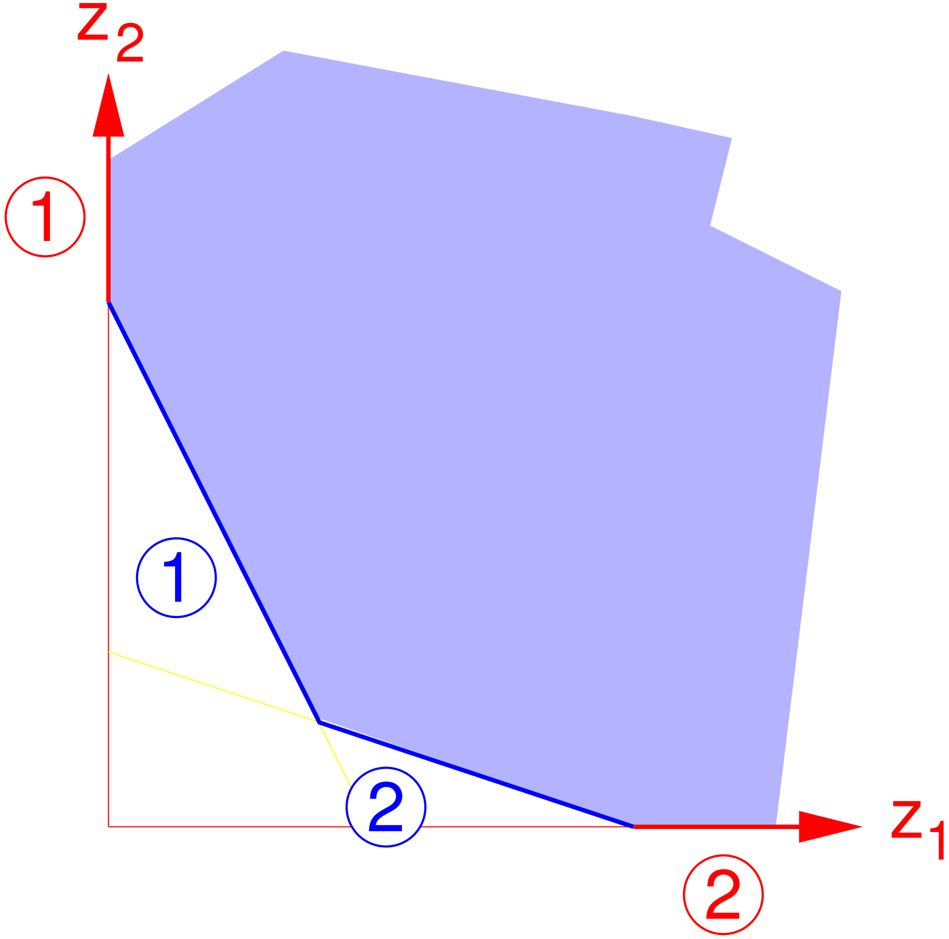
e.g. if $\boldsymbol{M}$ is copositive, i.e. $\boldsymbol{x} \geq \boldsymbol{0} \;\Rightarrow\; \boldsymbol{x}^\top \boldsymbol{M}\boldsymbol{x} \geq \boldsymbol{0}$

Polyhedral view of Lemke

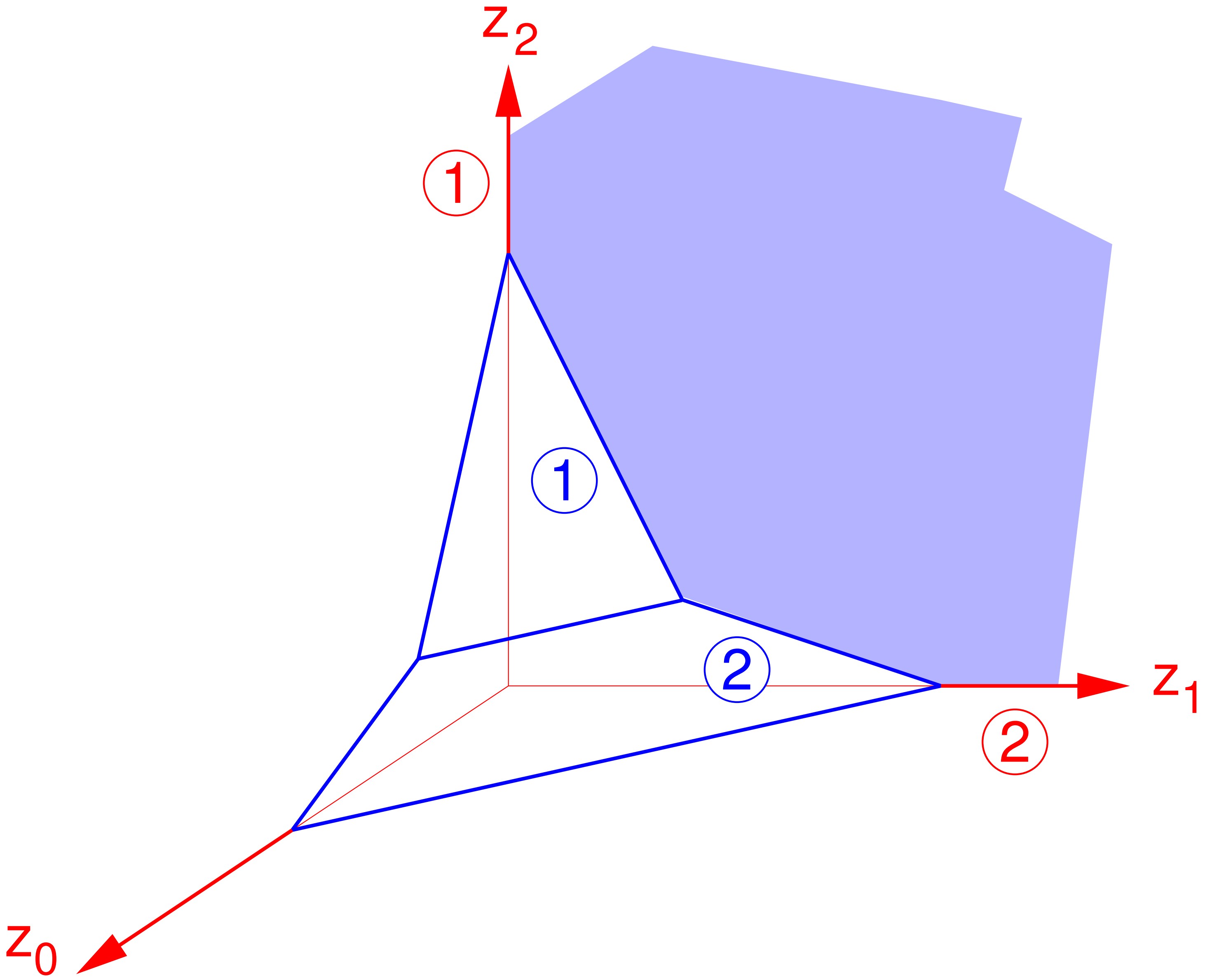
Polyhedral view of Lemke

Polyhedral view of Lemke

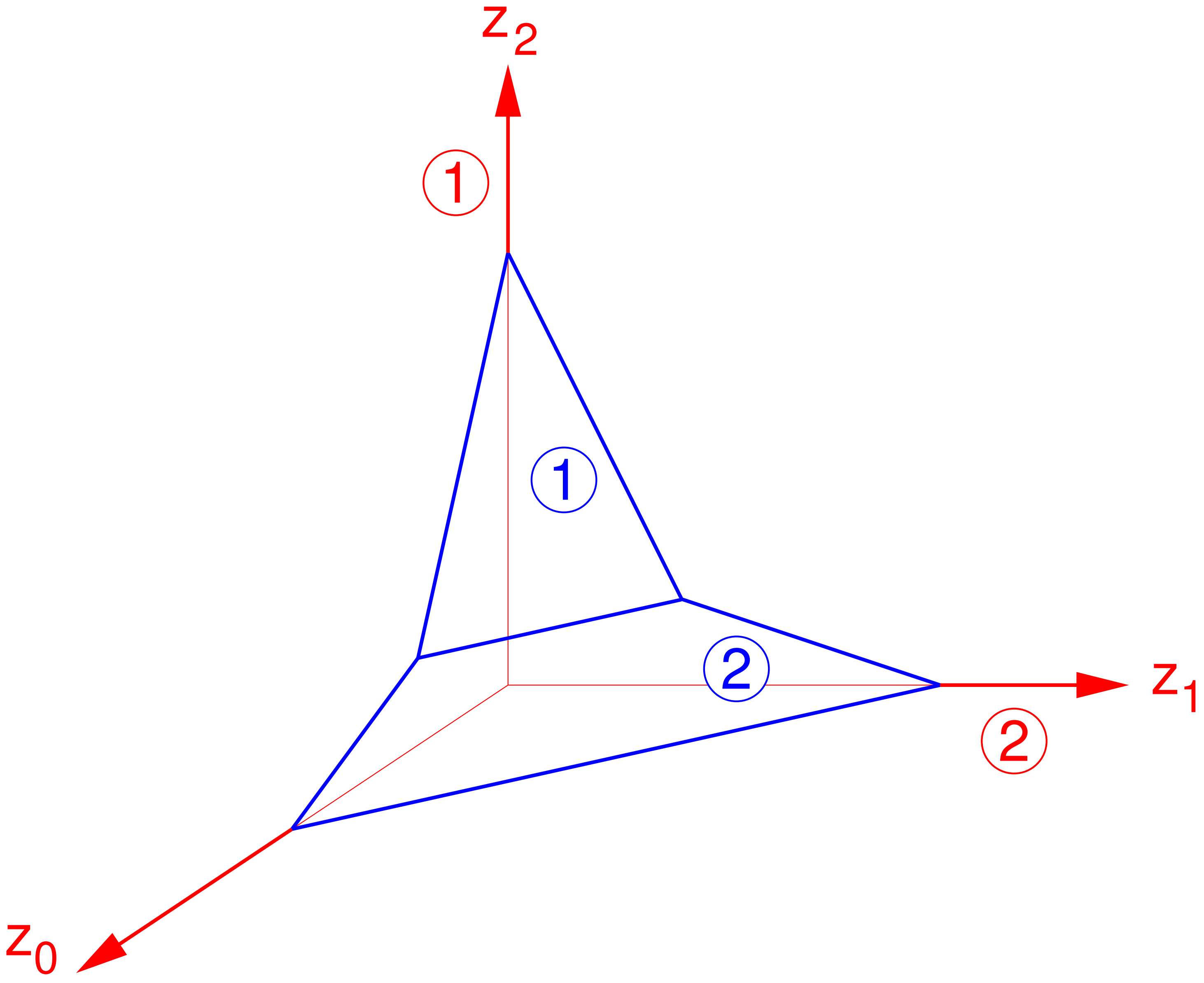
Polyhedral view of Lemke



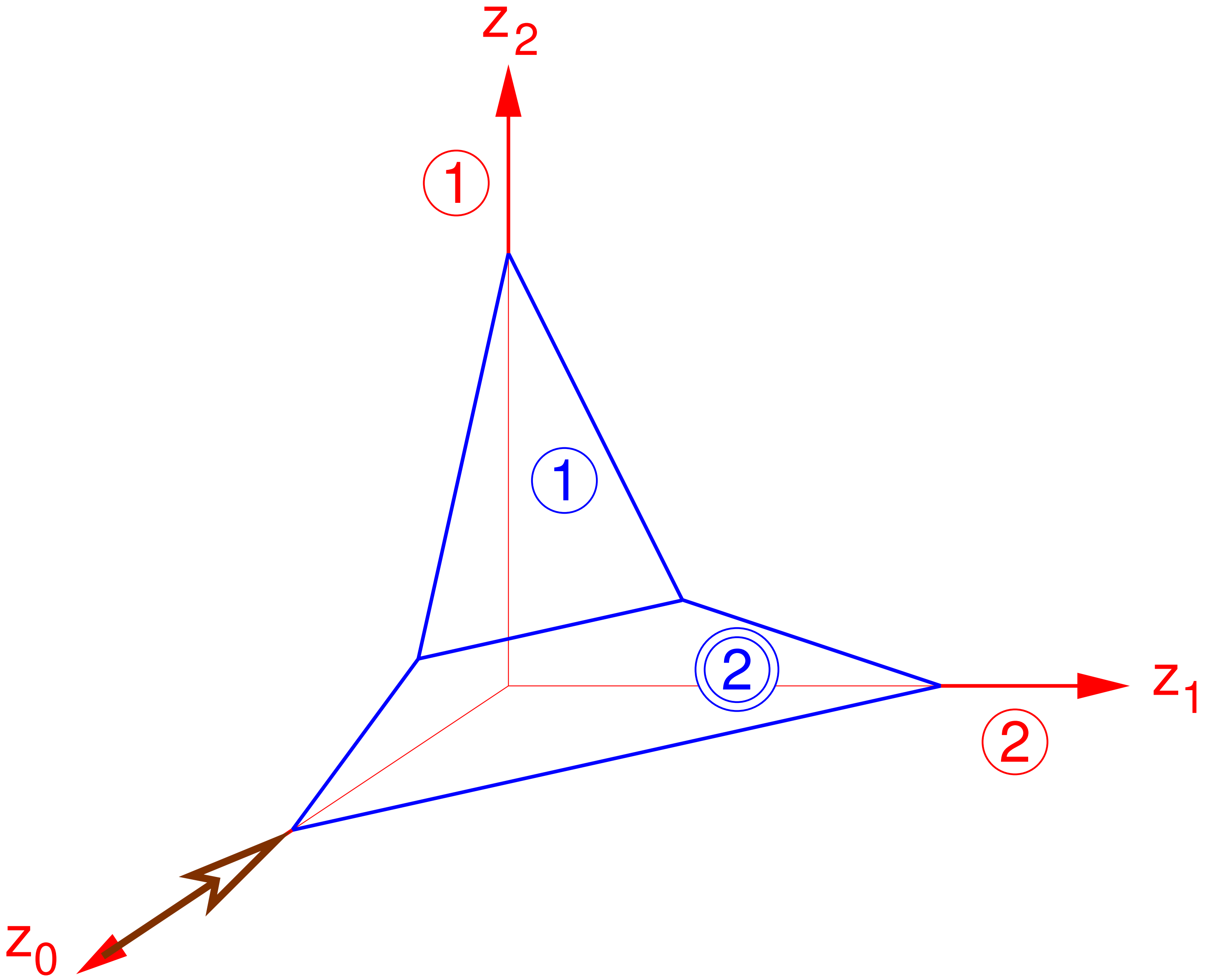
Polyhedral view of Lemke



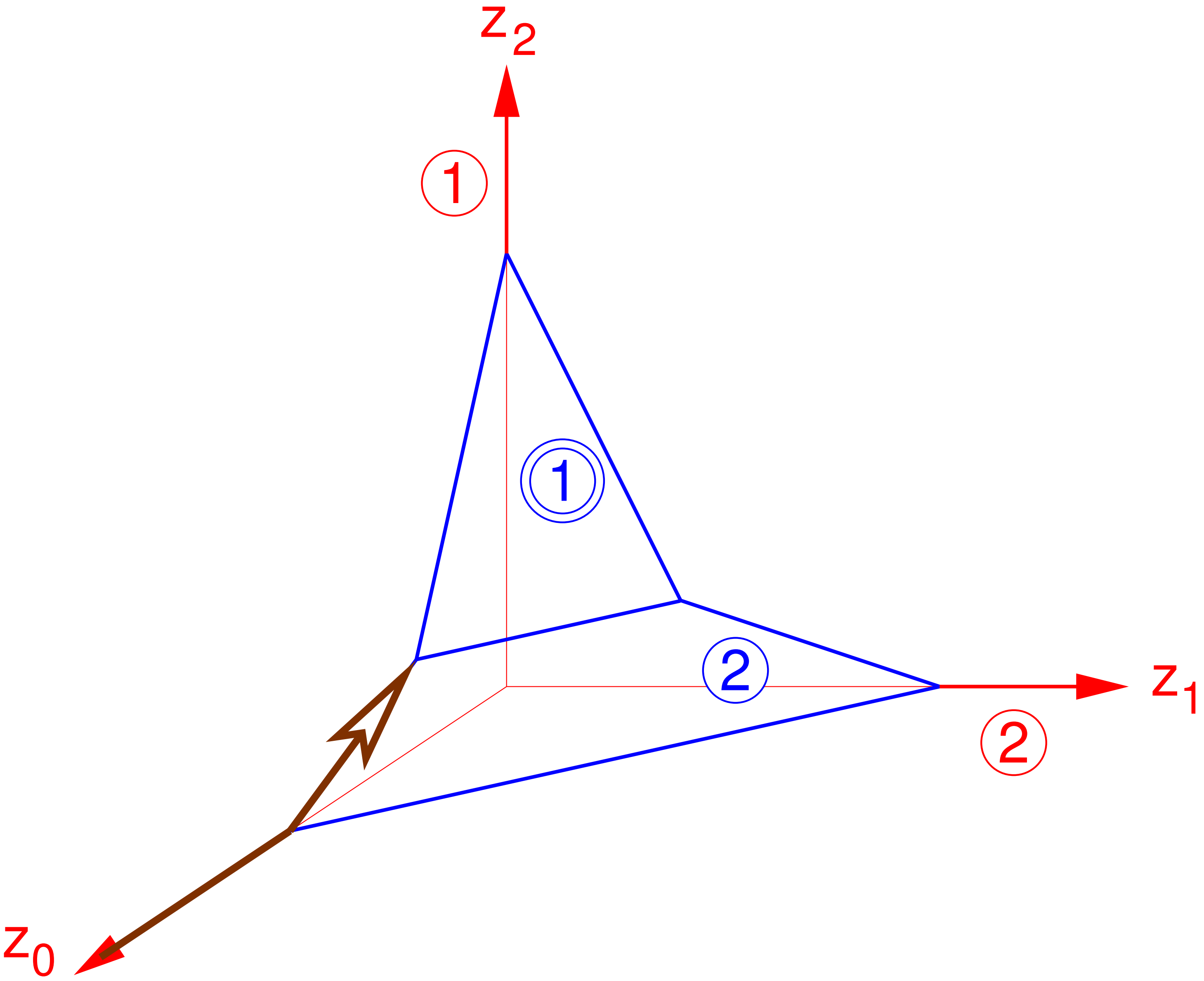
Polyhedral view of Lemke



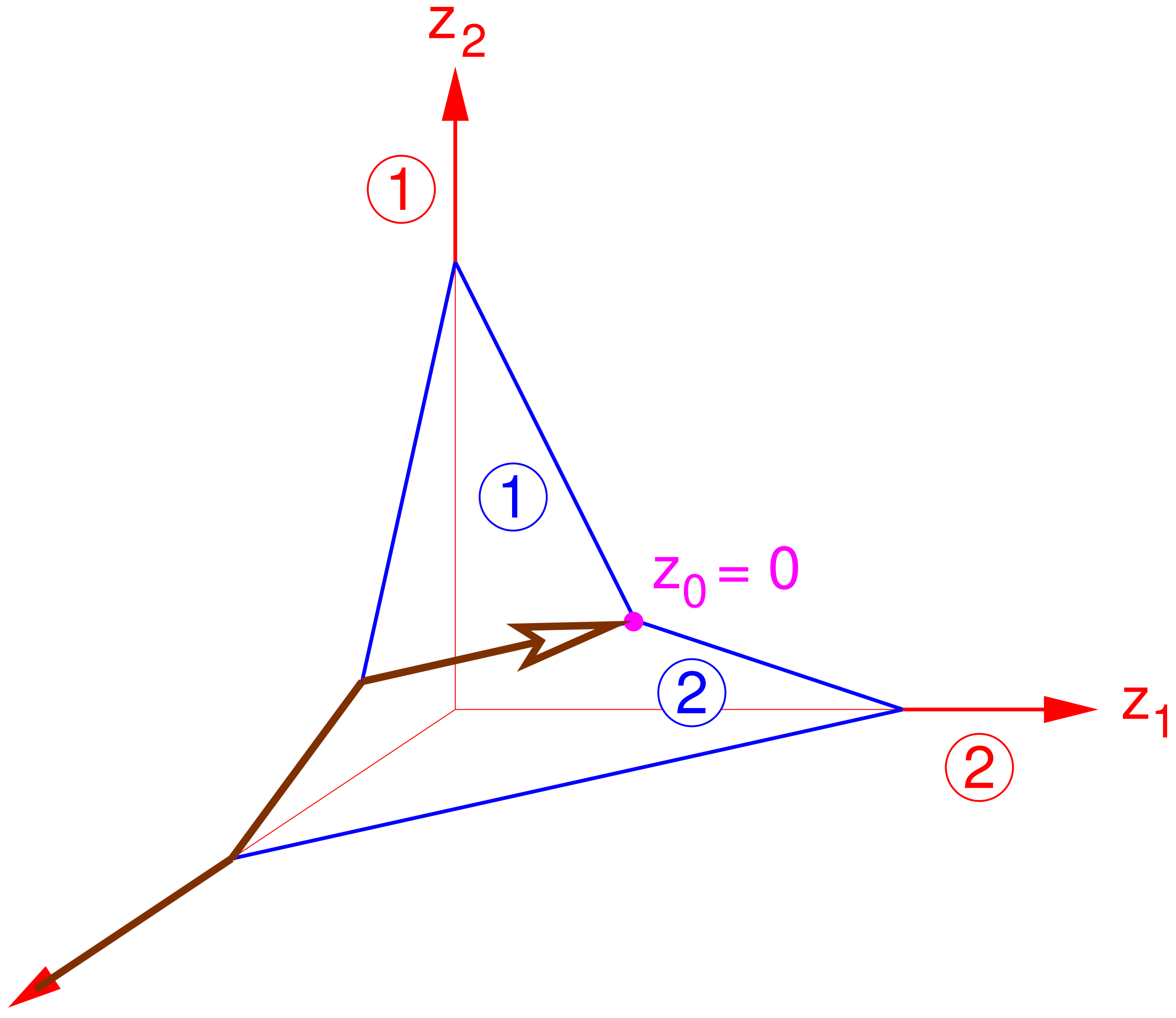
Polyhedral view of Lemke



Polyhedral view of Lemke



Polyhedral view of Lemke



Complementary cones

LCP $\quad z \geq \mathbf{0} \quad \perp \quad w = q + Mz \geq \mathbf{0}$

$\Leftrightarrow \quad z \geq \mathbf{0} \quad \perp \quad w \geq \mathbf{0}, \quad \boxed{-q = Mz - w}$

$\Leftrightarrow \quad -q$ belongs to a **complementary cone**:

$$\boxed{-q \in \mathbf{C}(\alpha) = \textbf{cone}\,\{\, M_{\mathbf{i}}\,,\ -e_{\mathbf{j}} \mid \mathbf{i} \in \alpha,\ \mathbf{j} \notin \alpha \,\}}$$

for some $\alpha \subseteq \{1,\ldots,n\}$, $\quad M = [M_1\, M_2 \cdots M_n]$

$\alpha = \{\, \mathbf{i} \mid z_{\mathbf{i}} > 0 \,\}$

Polyhedra versus cones

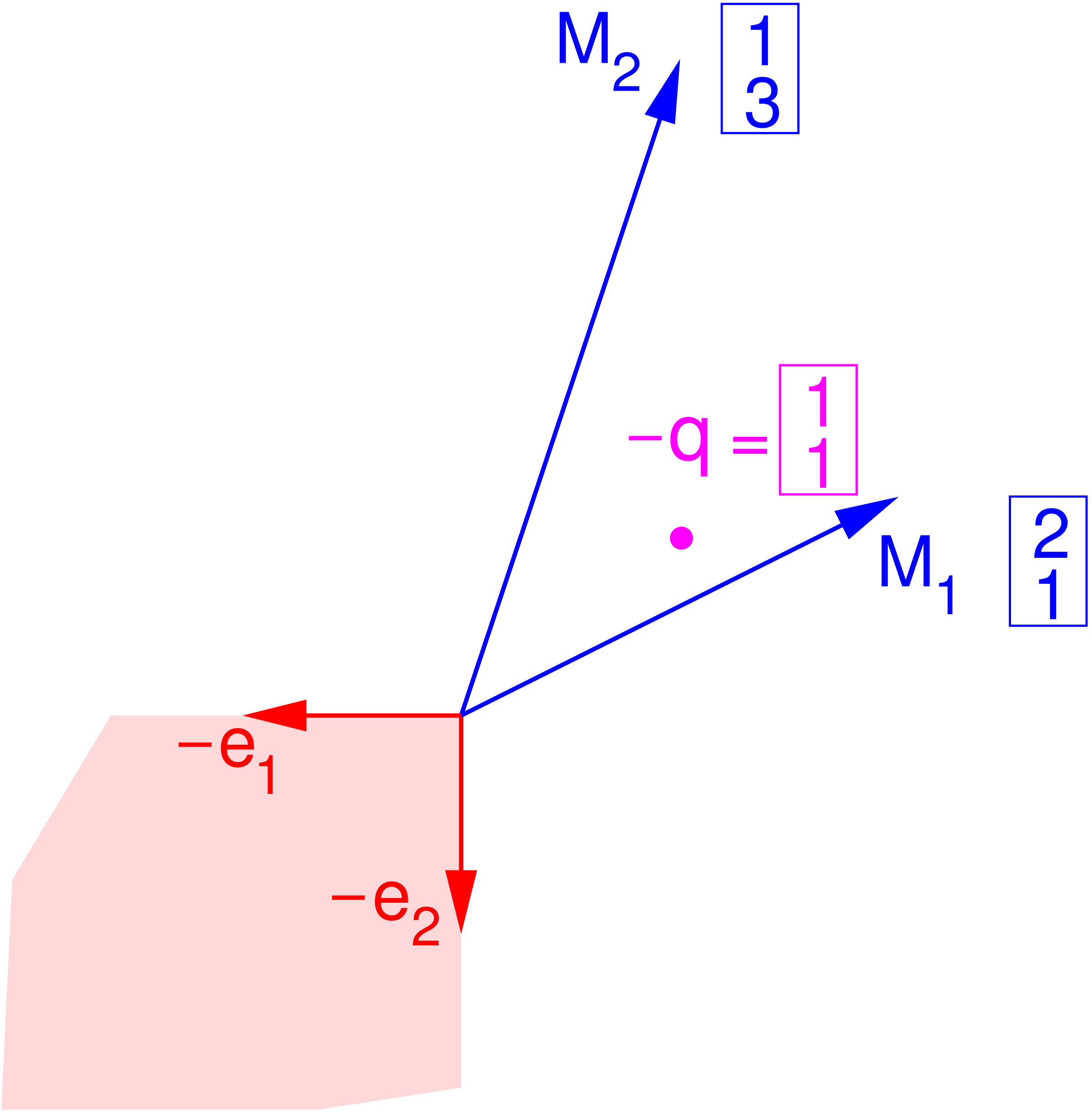
polyhedral view :

– gives feasibility,
want complementary **vertex**

complementary cones :

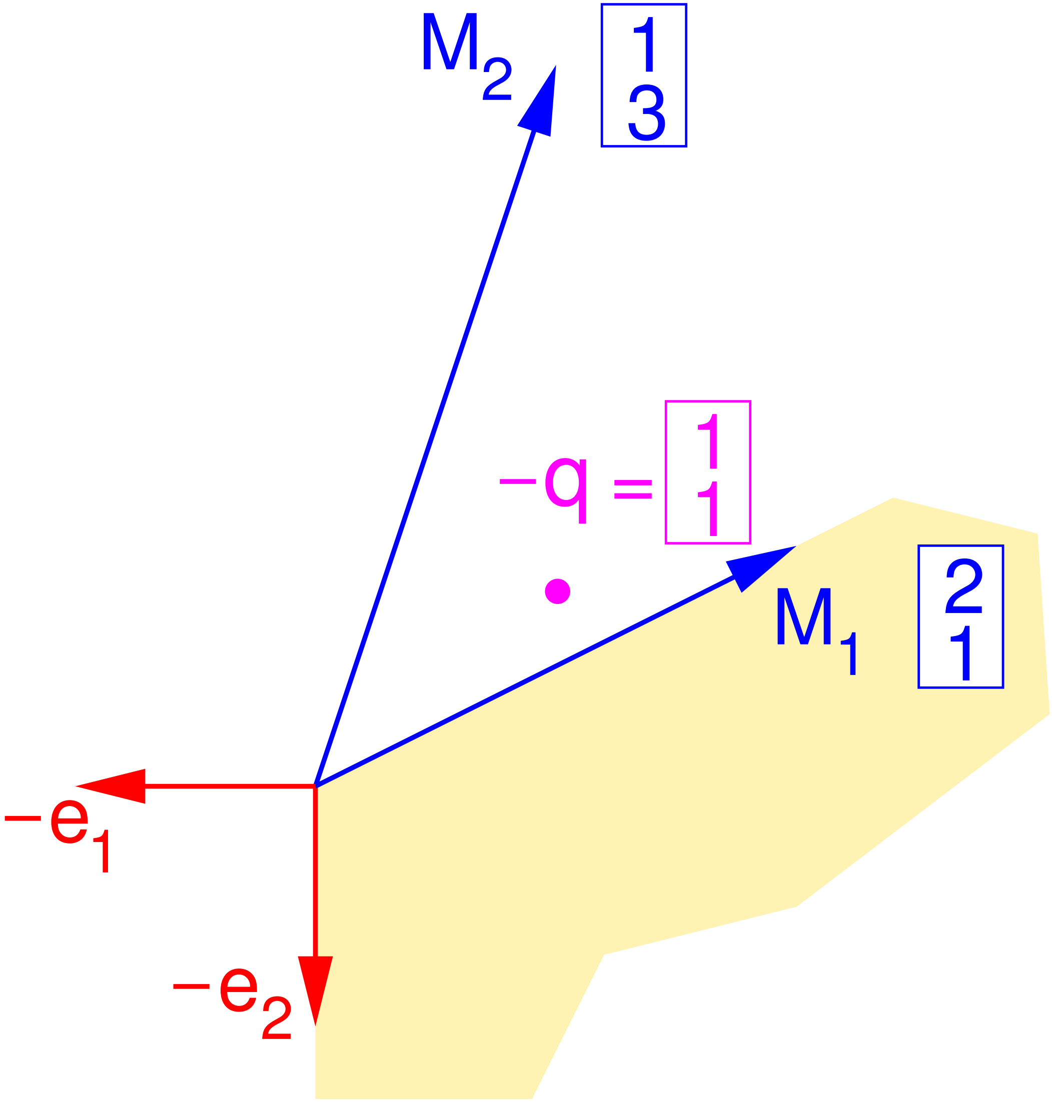
– gives complementarity and feasibility,
want α giving **cone** **C**(α) containing $-q$

Complementary cone C({})

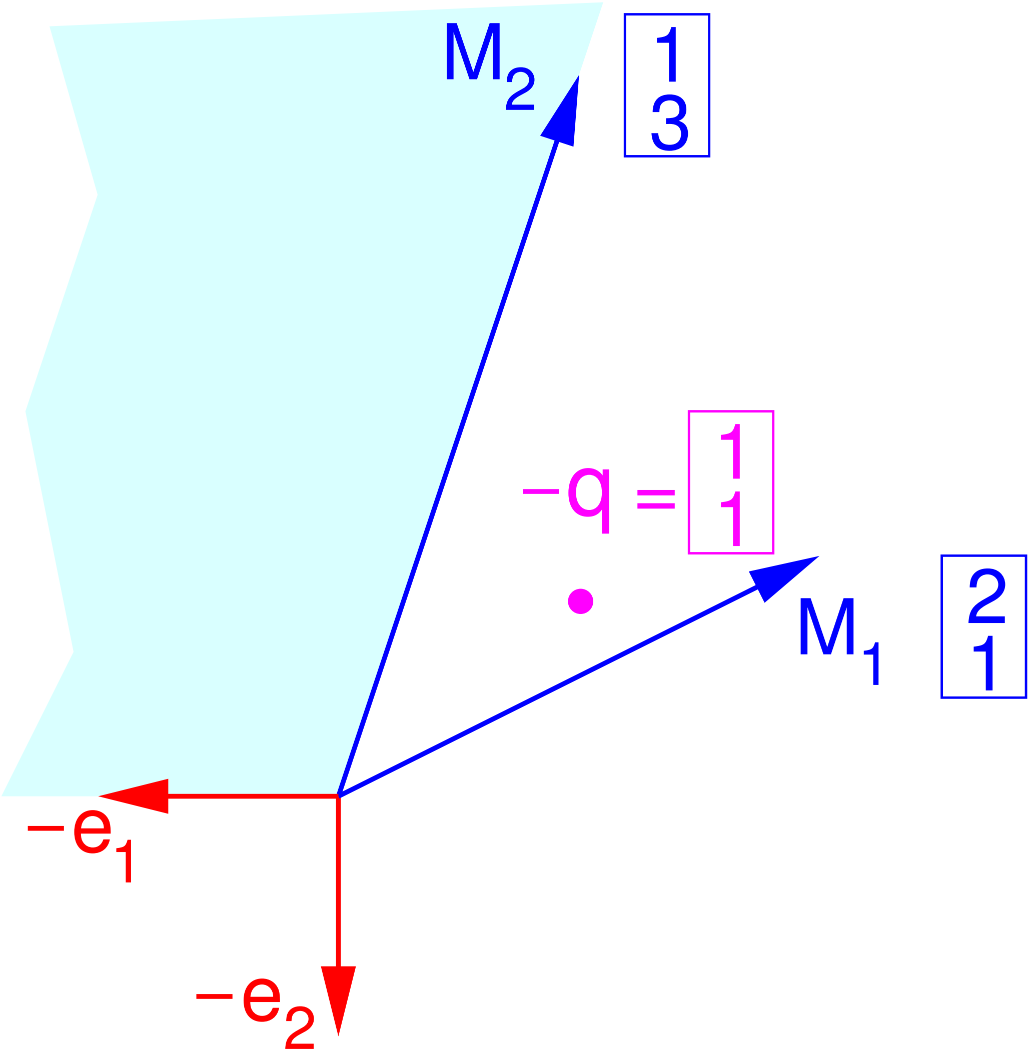


M_2 $\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_1 $\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$

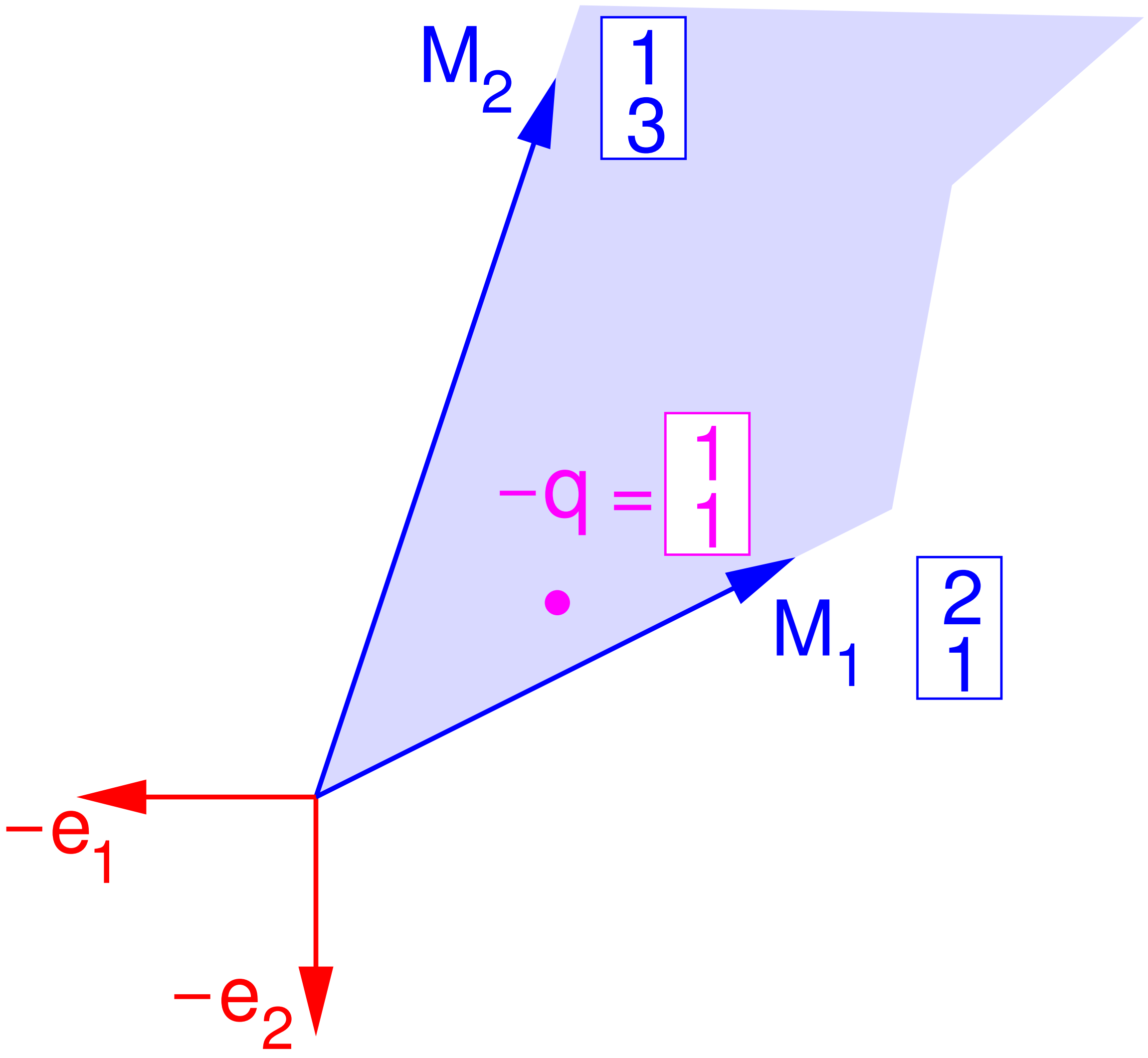
Complementary cone C({1})


M_2 $\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_1 $\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$

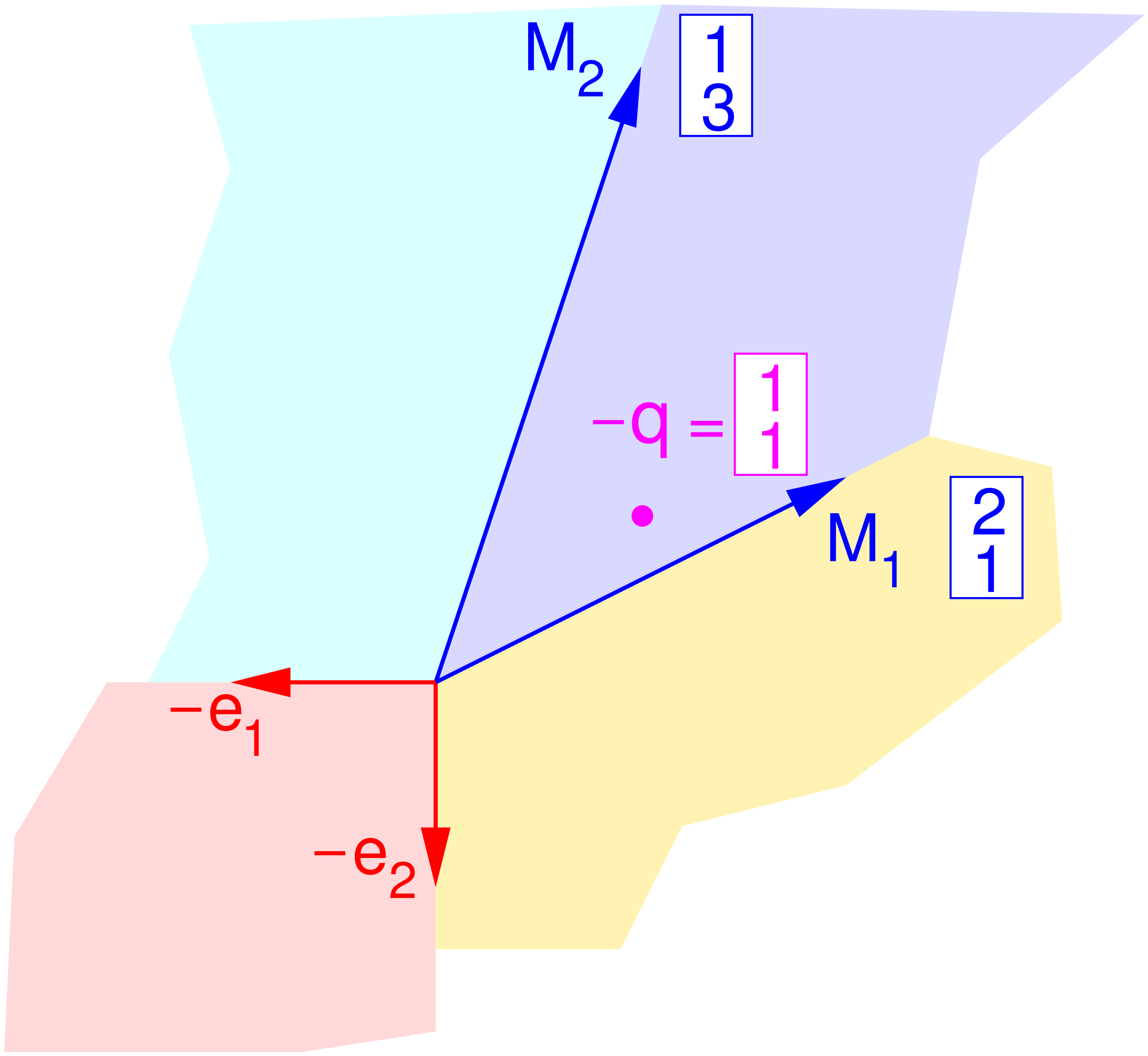
Complementary cone C({2})


M_2
$\begin{bmatrix}1\\3\end{bmatrix}$
$-q = \begin{bmatrix}1\\1\end{bmatrix}$
M_1
$\begin{bmatrix}2\\1\end{bmatrix}$
$-e_1$
$-e_2$

Complementary cone C({1,2})


M_2
$\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_1
$\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$

All complementary cones


M_2
$\begin{bmatrix}1\\3\end{bmatrix}$
$-q = \begin{bmatrix}1\\1\end{bmatrix}$
M_1
$\begin{bmatrix}2\\1\end{bmatrix}$
$-e_1$
$-e_2$

LCP map

Let $\alpha \subseteq \{1,\dots,n\}$,

α**-orthant** = **cone** $\{ e_i , \ -e_j \mid i \in \alpha, \ j \notin \alpha \}$,

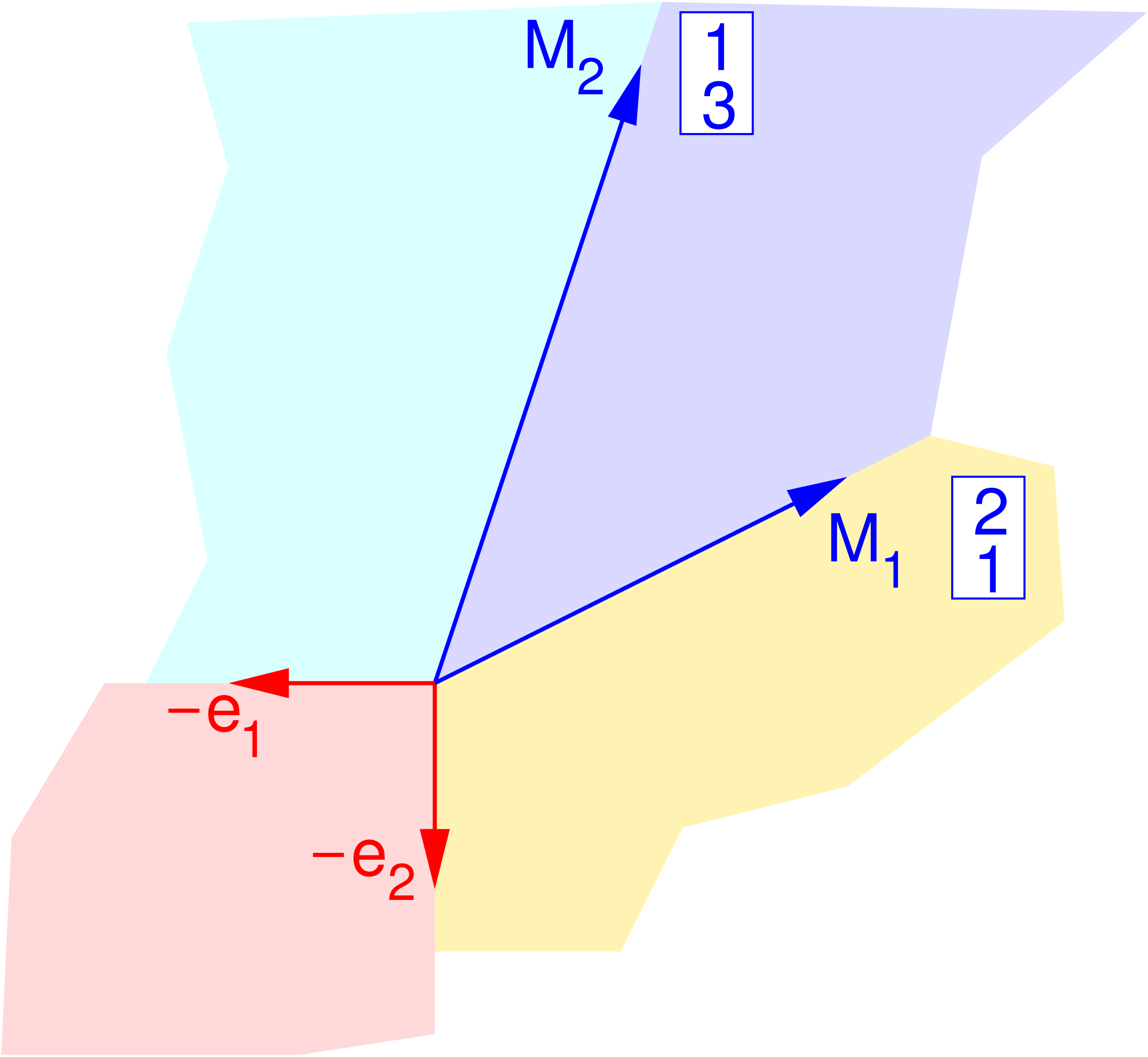
C(α) = **cone** $\{ M_i , -e_j \mid i \in \alpha, \ j \notin \alpha \}$,

$x_i^+ = \max (x_i , 0)$, $x_i^- = \min (x_i , 0)$

LCP map: $\boxed{F(x) = Mx^+ + x^-}$

so $\boxed{F(\alpha\text{-orthant}) = \mathbf{C}(\alpha)}$

Bijective LCP map F


M_2
$\begin{matrix}1\\3\end{matrix}$
M_1
$\begin{matrix}2\\1\end{matrix}$
$-e_1$
$-e_2$

P-matrix

P-matrix

$\Leftrightarrow$ every **principal minor** is positive:

$\det(M_{\alpha\alpha}) > 0$ for all $\alpha \subseteq \{1,\ldots,n\}$

e.g. $\boxed{\begin{matrix} 2 & 1 \\ 1 & 3 \end{matrix}}$

$\det(M_{1,1}) = 2 > 0$

$\det(M_{2,2}) = 3 > 0$

$\det(M_{12,12}) = \det(M) = 5 > 0$

P-matrix

P-matrix

$\Leftrightarrow$ every **principal minor** is positive:

$\det(M_{\alpha\alpha}) > 0$ for all $\alpha \subseteq \{1,\ldots,n\}$

P-matrix

$\Leftrightarrow$ **F** bijective

$\Leftrightarrow$ $\forall\, q \in \mathbf{R}^n \;\; \exists!\, z$ s.t. $z \geq \mathbf{0} \;\; \perp \;\; Mz \geq -q$

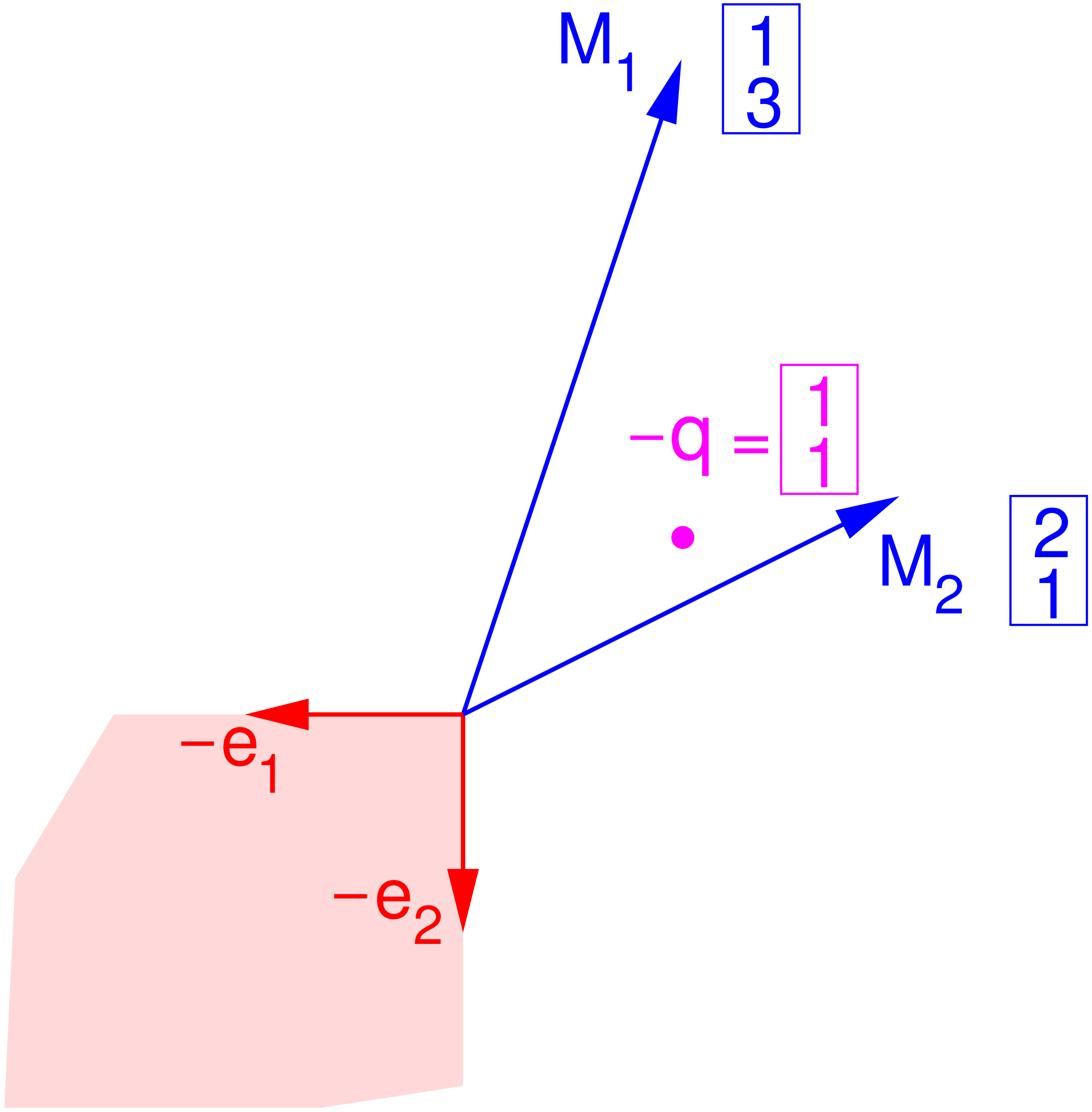
Not a P-matrix

Example:

$$\begin{array}{|cc|}\hline 1 & 2 \\ 3 & 1 \\ \hline\end{array}$$

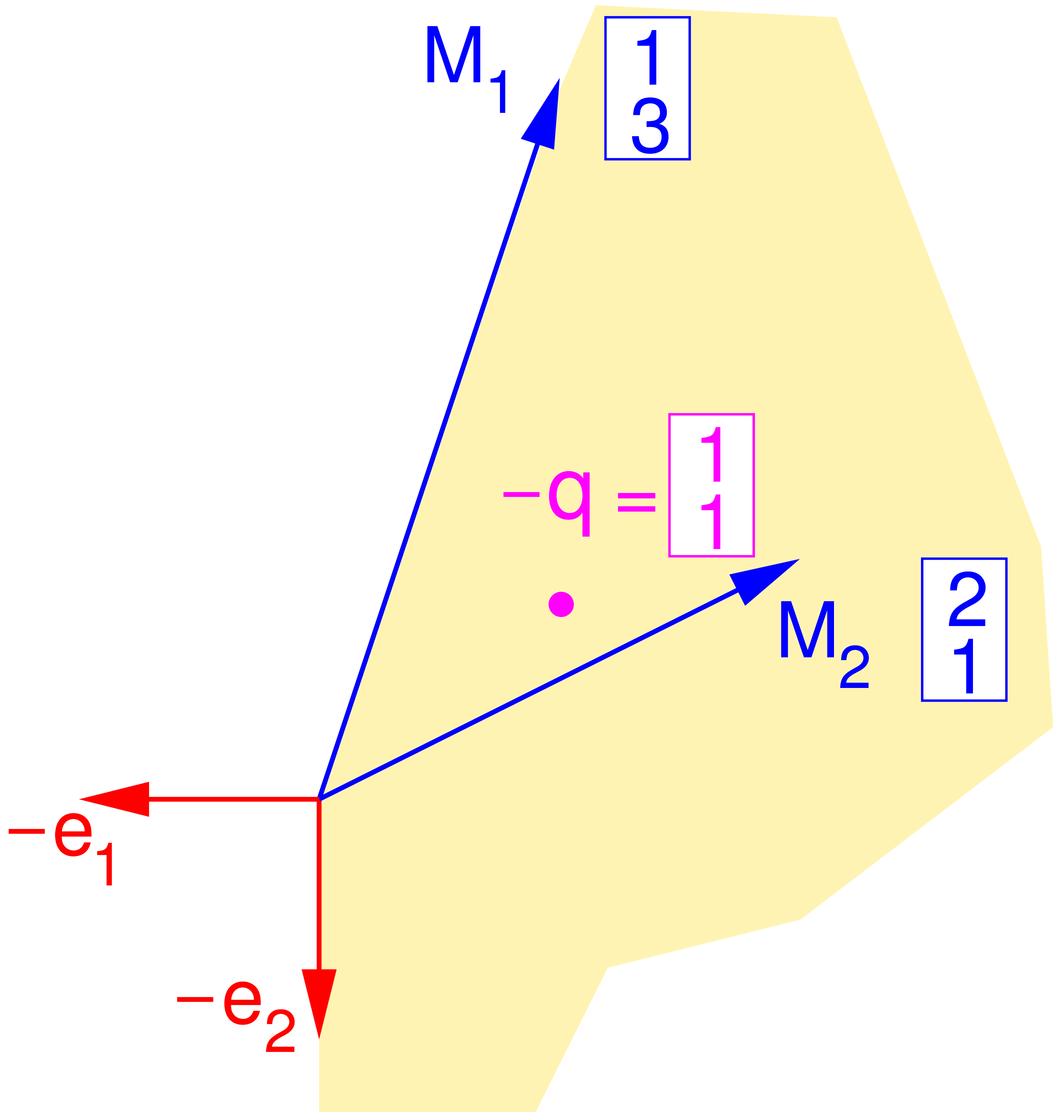
$$\det(M_{12,12}) = \det(M) = -5 < 0$$

Complementary cone C({})

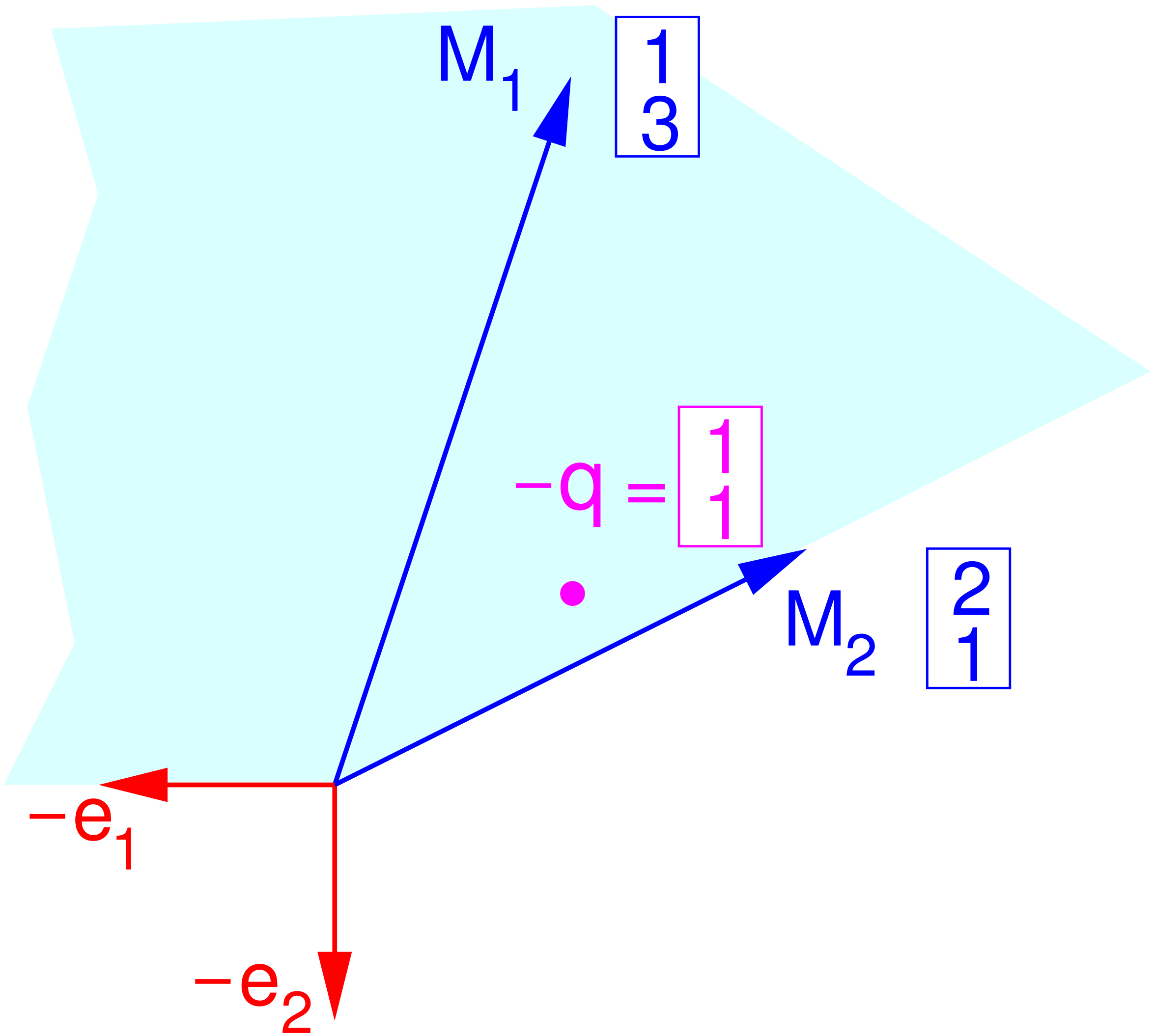


M_1 $\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_2 $\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$

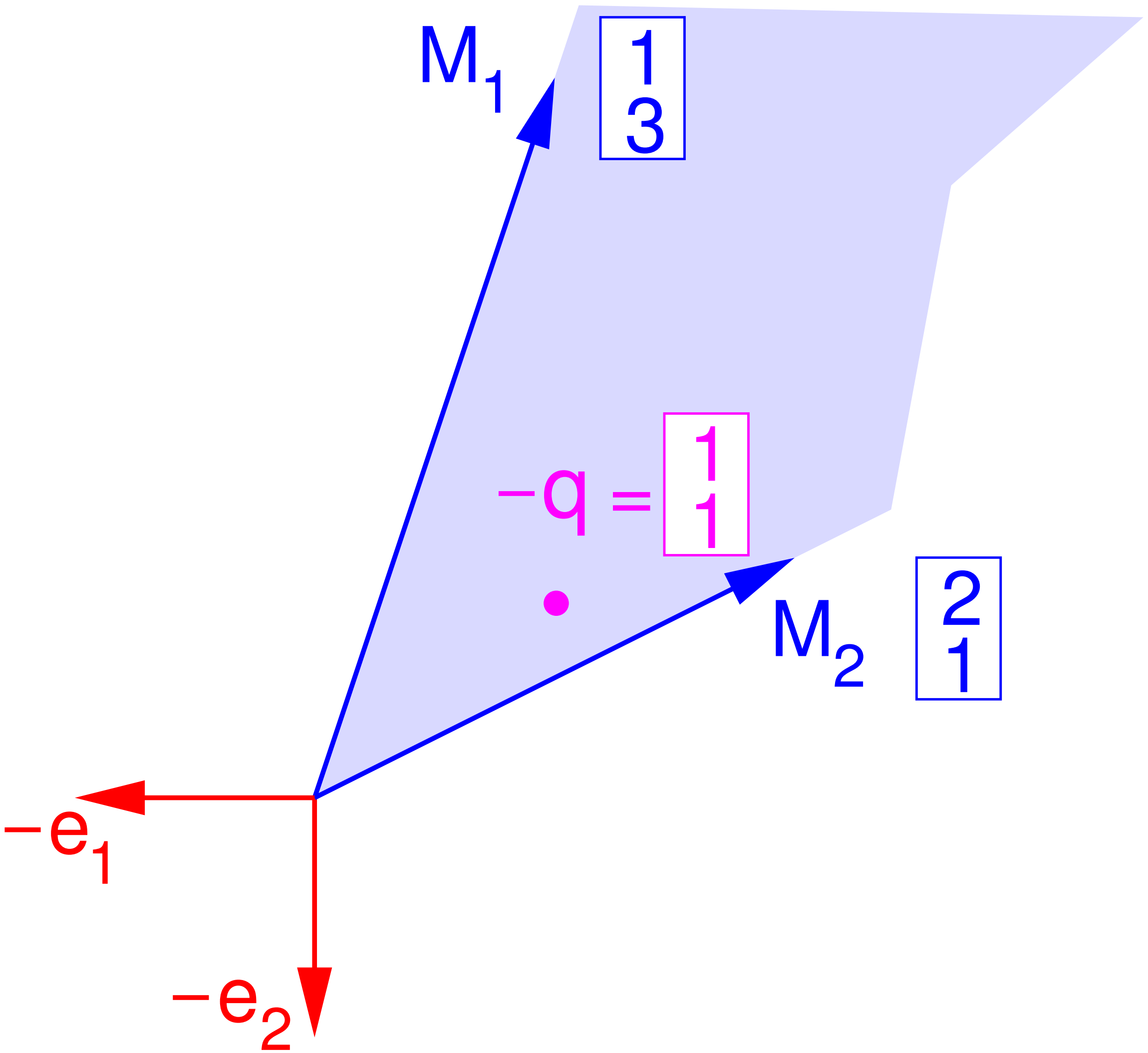
Complementary cone C({1})


M_1
$\begin{bmatrix}1\\3\end{bmatrix}$
$-q = \begin{bmatrix}1\\1\end{bmatrix}$
M_2
$\begin{bmatrix}2\\1\end{bmatrix}$
$-e_1$
$-e_2$

Complementary cone C({2})


M_1 $\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_2 $\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$

Complementary cone C({1,2})


M_1 $\boxed{\begin{matrix}1\\3\end{matrix}}$
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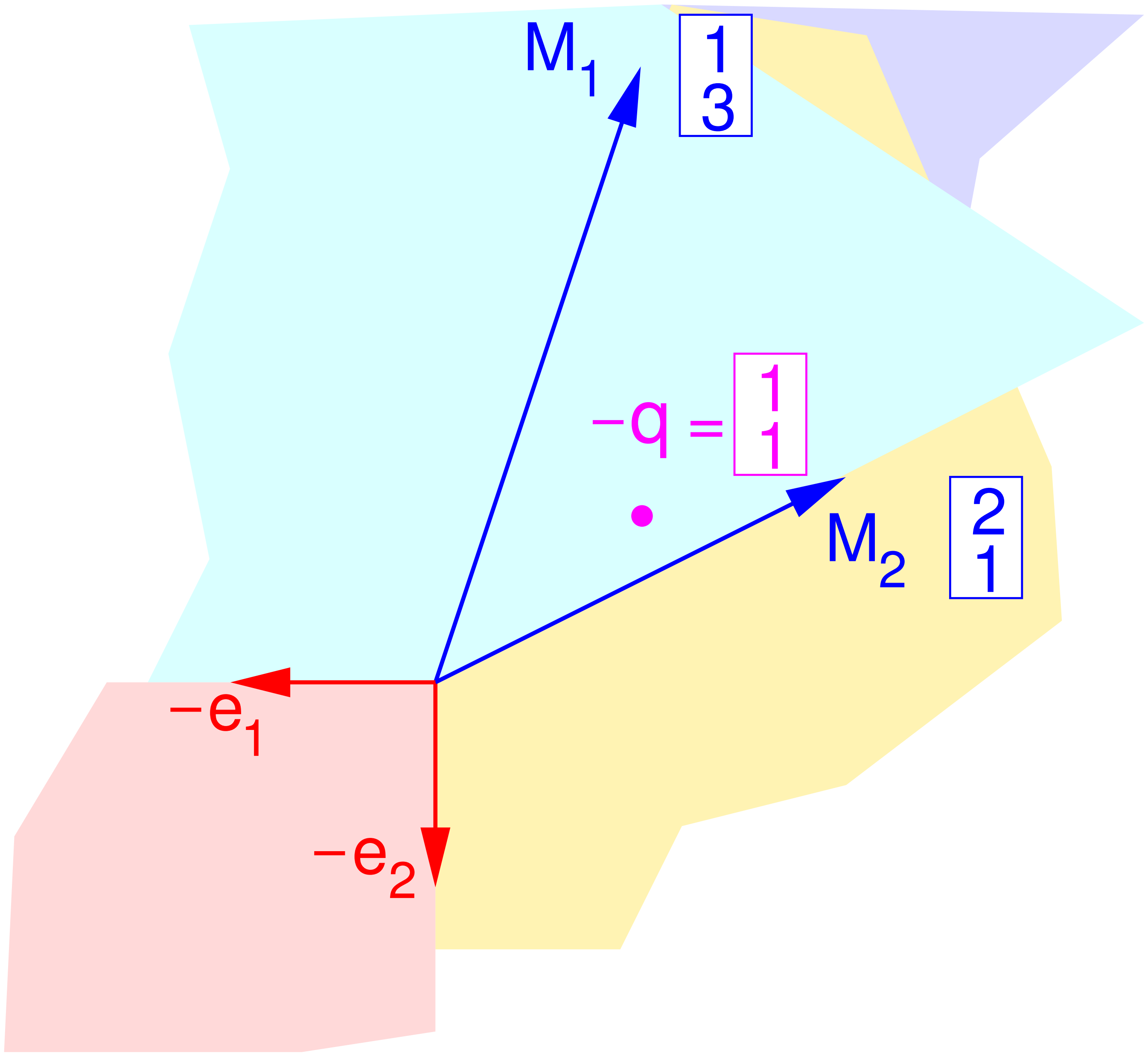
M_2 $\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$

Non-injective LCP map F


M_1 $\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_2 $\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$

F is surjective for M > 0

Given: $p \in \mathbf{R}^n$.

Claim: $$\boxed{\exists\, x : \ F(x) = Mx^+ + x^- = p}$$

Proof (solving $F(x) = p$)

Let $p \in \mathbf{R}^n$, $\alpha = \{ i \mid p_i > 0 \}$.

Step 1. Consider only rows $i \in \alpha$. Solution x^+ to

$$\forall i \in \alpha \qquad x_i \quad \perp \quad \sum_{j \in \alpha} m_{ij} x_j \geq p_i$$

Proof (solving $F(x) = p$)

Let $p \in \mathbf{R}^n$, $\alpha = \{ i \mid p_i > 0 \}$.

Step 1. Consider only rows $i \in \alpha$. Solution x^+ to

$$\forall\, i \in \alpha \qquad x_i \quad \perp \quad \sum_{j \in \alpha} (m_{ij} / p_i)\, x_j \geq 1$$

exists as Nash equilibrium (game matrix m_{ij} / p_i).

Proof (solving $F(x) = p$)

Let $p \in \mathbf{R}^n$, $\alpha = \{ i \mid p_i > 0 \}$.

Step 1. Consider only rows $i \in \alpha$. Solution x^+ to

$$\forall i \in \alpha \qquad x_i \quad \perp \quad \sum_{j \in \alpha} (m_{ij} / p_i)\, x_j \geq 1$$

exists as Nash equilibrium (game matrix m_{ij} / p_i).

Step 2. $\forall k \notin \alpha$ choose $-x_k^- = w_k \geq 0$ so that

$$\sum_{j \in \alpha} m_{kj}\, x_j^+ - w_k = p_k \ (\leq 0).$$

Gives $\boxed{F(x) = p.}$

Lemke via complementary cones

Invert the piecewise linear map $F(x)$ along the line segment $[\,-d, -q\,]$:

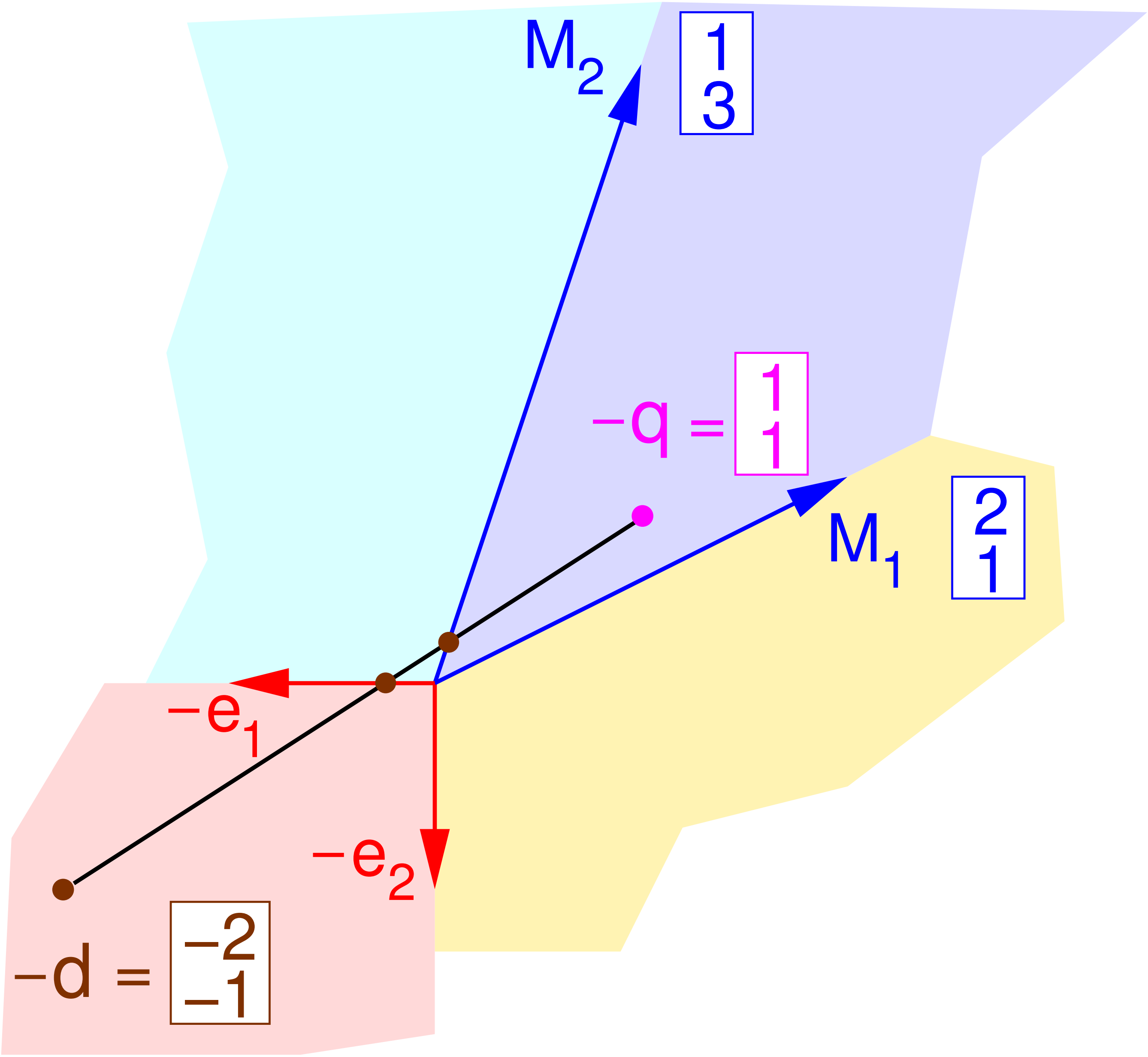
$$\boxed{F(x) = Mx^{+} + x^{-} = (-d)(1-\mathbf{t}) + (-q)\mathbf{t}} \qquad (0 \le \mathbf{t} \le 1)$$

$\mathbf{t} > 0$:

$$\Leftrightarrow \qquad Mx^{+}(1/\mathbf{t}) + x^{-}(1/\mathbf{t}) = (-d)(1-\mathbf{t}\,)/\mathbf{t} + (-q)$$

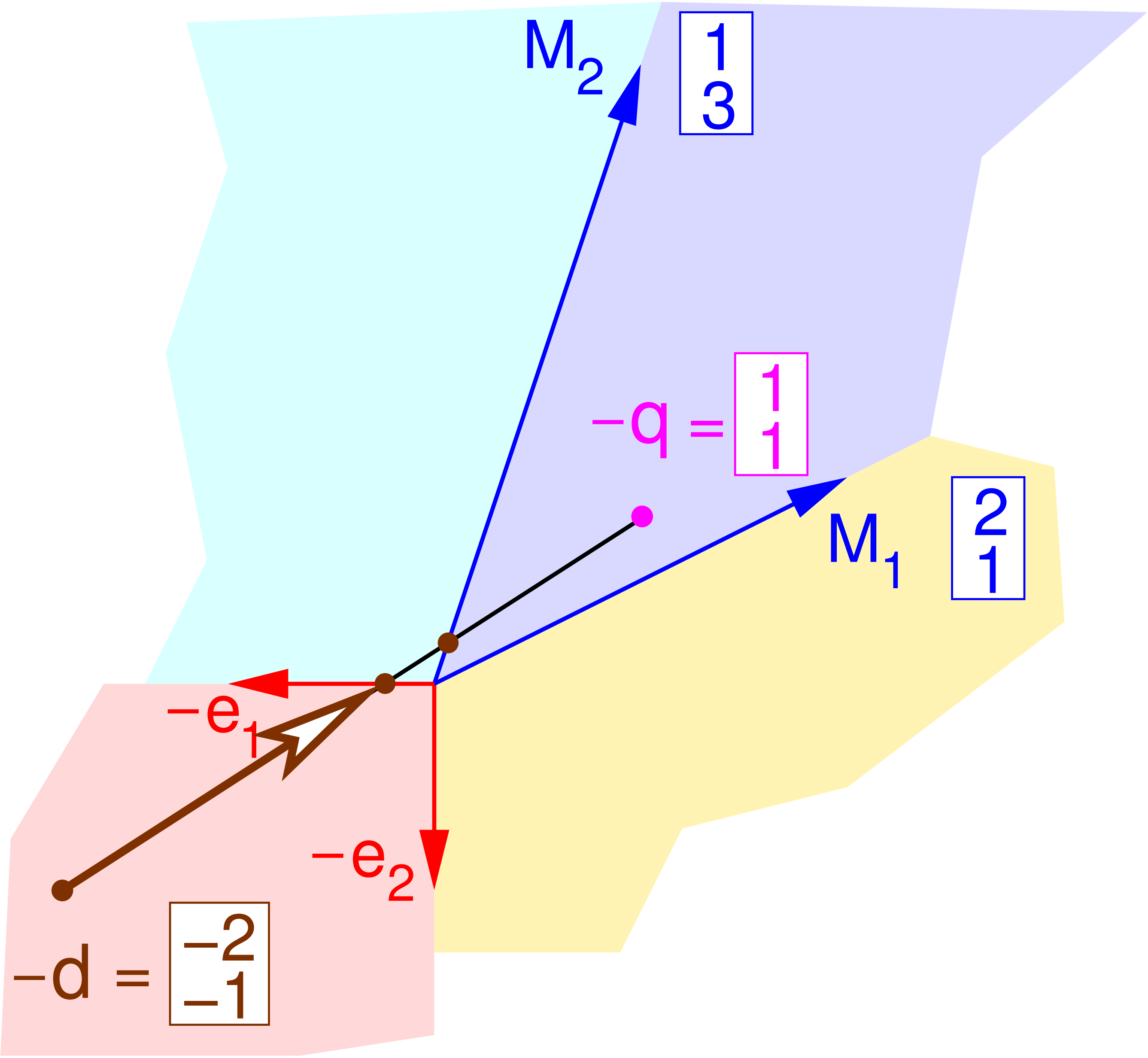
$$\Leftrightarrow \qquad \boxed{Mz - w = (-d)z_{\mathbf{0}} + (-q)}\,, \qquad z \ge 0 \perp w \ge 0\,.$$

Inverting the LCP map F

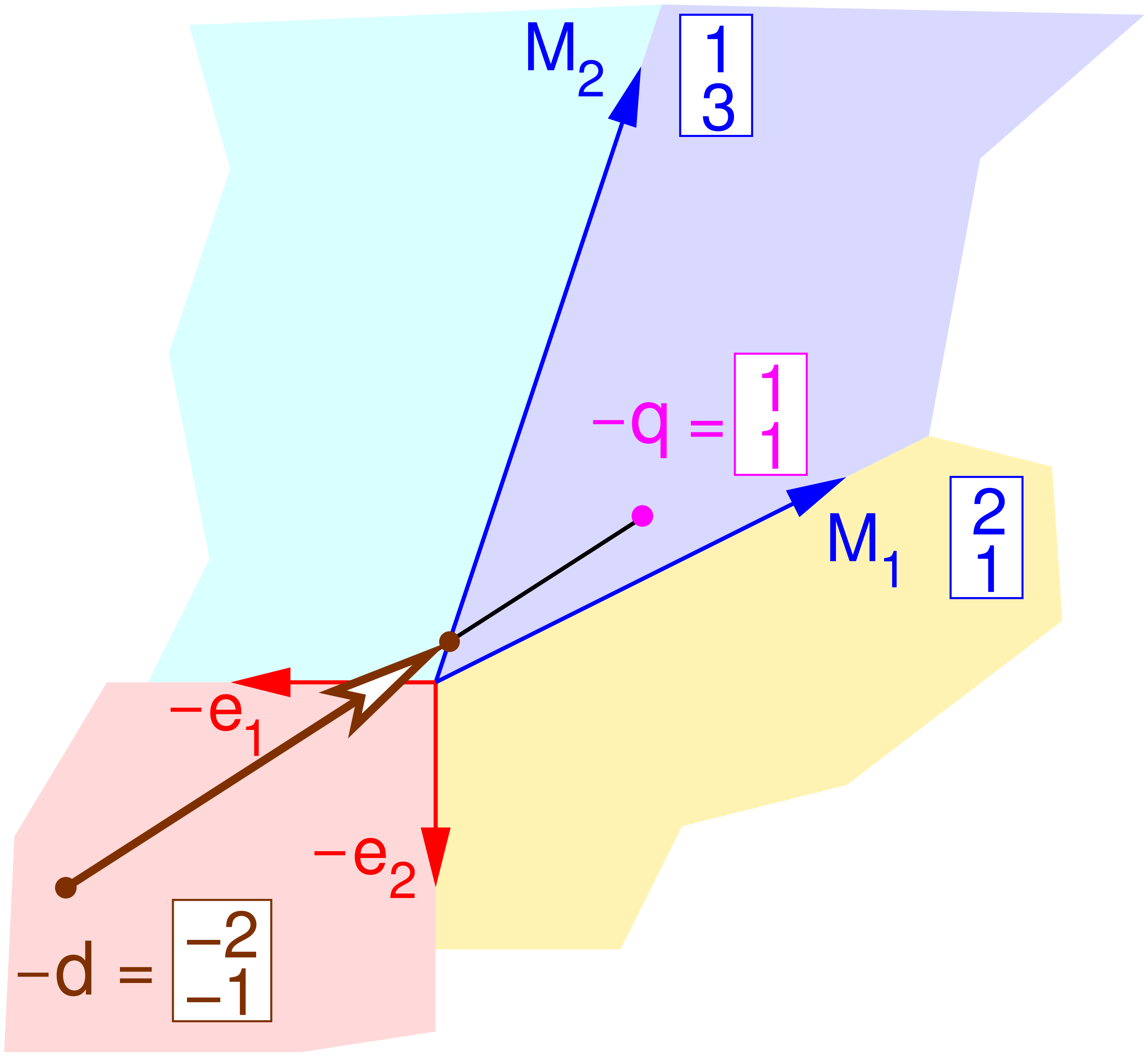


M_2 $\begin{bmatrix}1\\3\end{bmatrix}$
$-q = \begin{bmatrix}1\\1\end{bmatrix}$
M_1 $\begin{bmatrix}2\\1\end{bmatrix}$
$-e_1$
$-e_2$
$-d = \begin{bmatrix}-2\\-1\end{bmatrix}$

Inverting the LCP map F


M_2 $\begin{bmatrix}1\\3\end{bmatrix}$
$-q = \begin{bmatrix}1\\1\end{bmatrix}$
M_1 $\begin{bmatrix}2\\1\end{bmatrix}$
$-e_1$
$-e_2$
$-d = \begin{bmatrix}-2\\-1\end{bmatrix}$

Inverting the LCP map F


M_2
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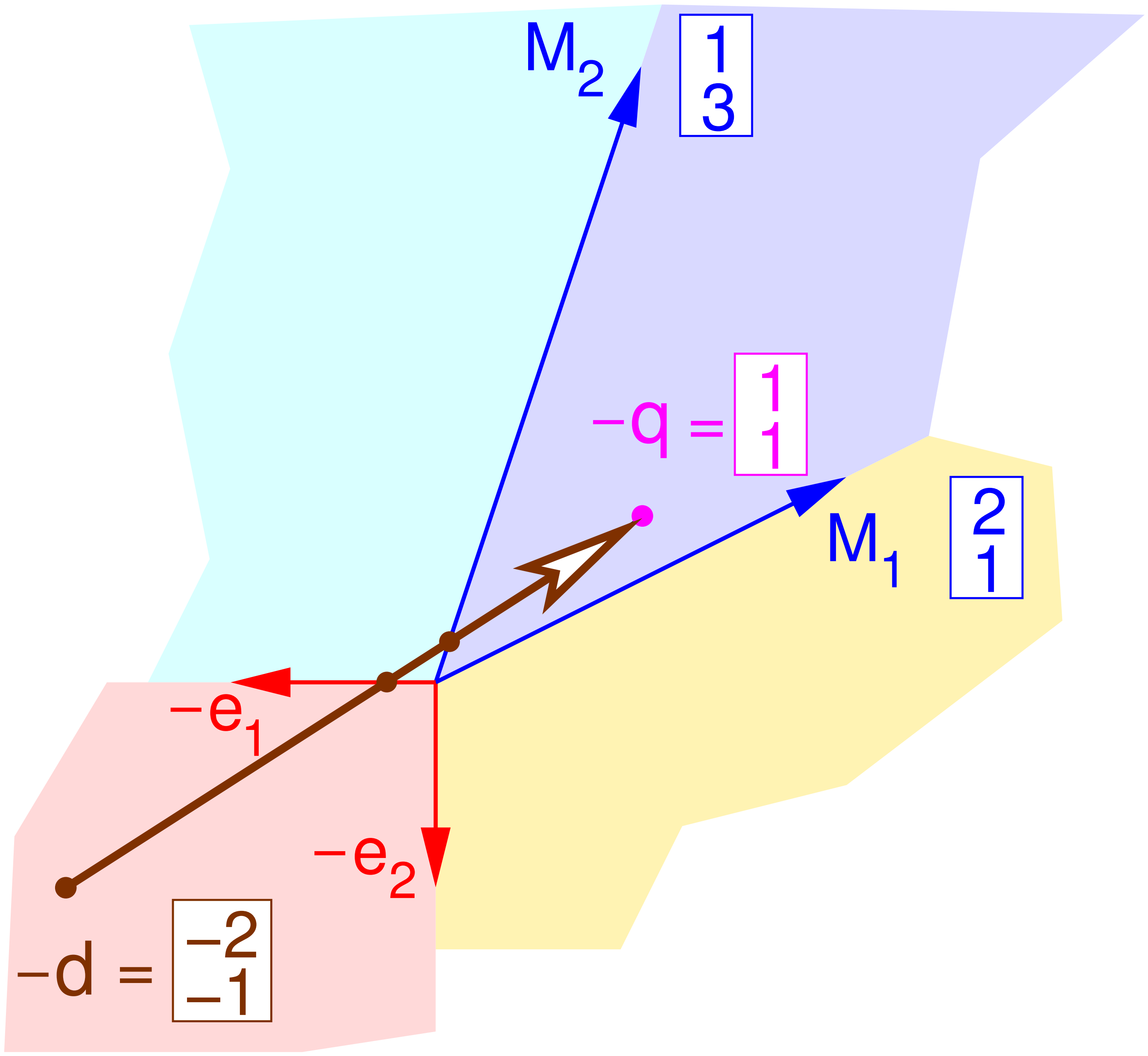
M_1
$\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$
$-d = \boxed{\begin{matrix}-2\\-1\end{matrix}}$

Inverting the LCP map F



M_2
$\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_1
$\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-e_2$
$-d = \boxed{\begin{matrix}-2\\-1\end{matrix}}$

Lemke-Howson: –d = unit vector

Theorem:

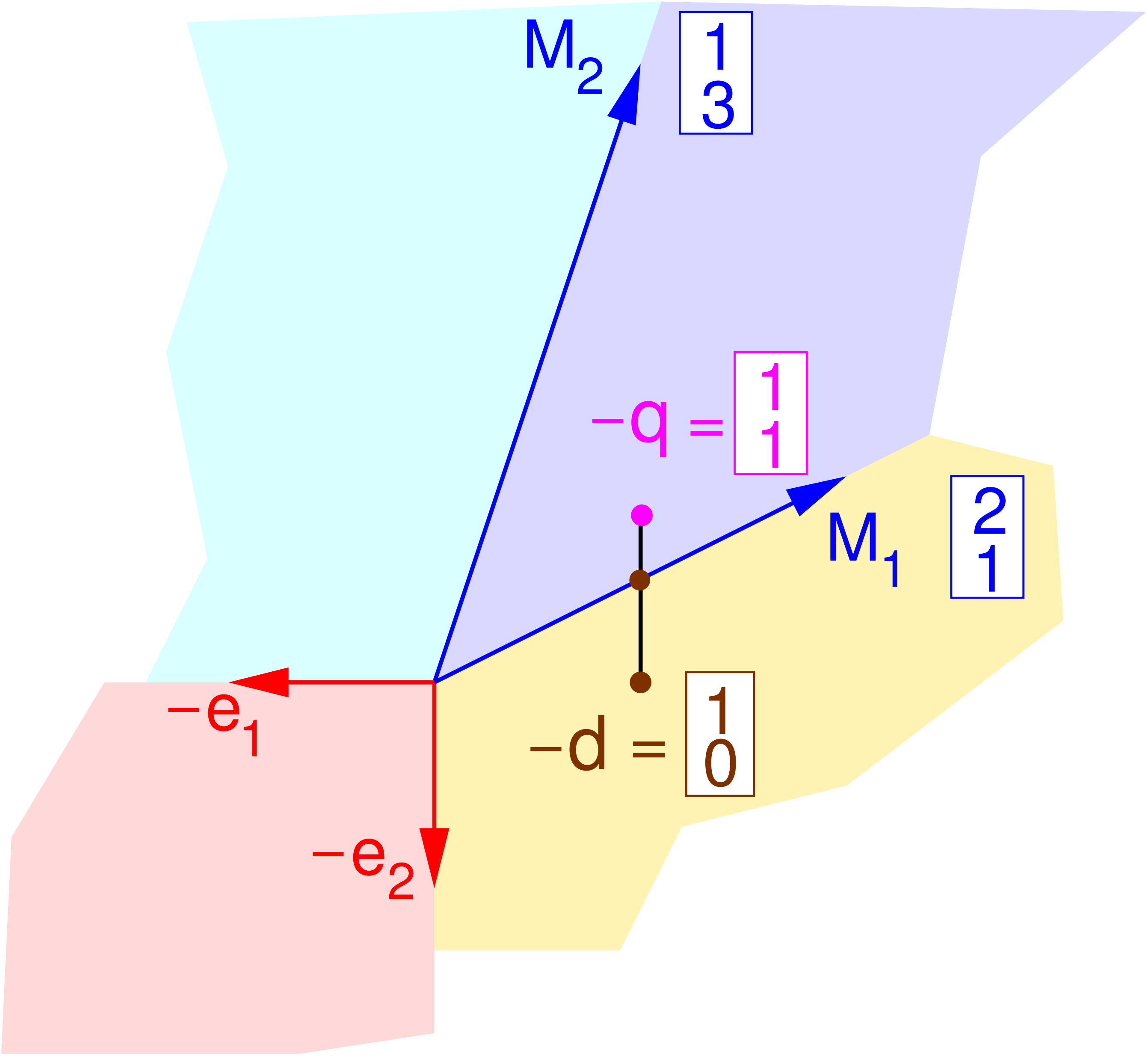
Symmetric Lemke-Howson with missing label k

= Lemke started at $-d = e_k$ in cone $\mathbf{C}(\{k\})$

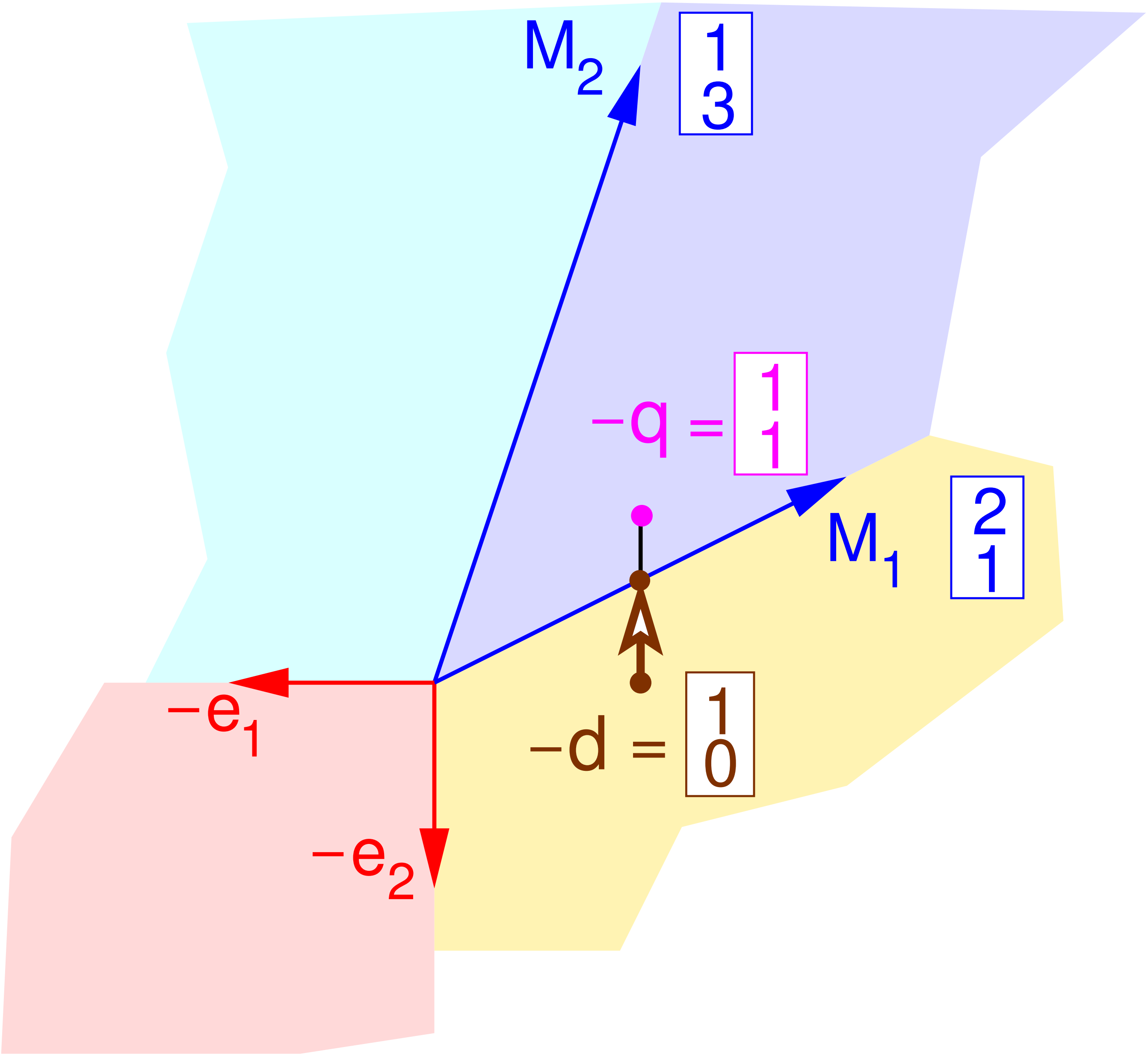
Proof:

- initialize by pivoting z_0 in, w_k out (still infeasible!), w_k stays in negative unit column
- pivot z_k in (note $M_k > \mathbf{0}$), gives start in cone $\mathbf{C}(\{k\})$

Start at unit vector


M_2
$\boxed{\begin{matrix}1\\3\end{matrix}}$
$-q = \boxed{\begin{matrix}1\\1\end{matrix}}$
M_1
$\boxed{\begin{matrix}2\\1\end{matrix}}$
$-e_1$
$-d = \boxed{\begin{matrix}1\\0\end{matrix}}$
$-e_2$

Start at unit vector



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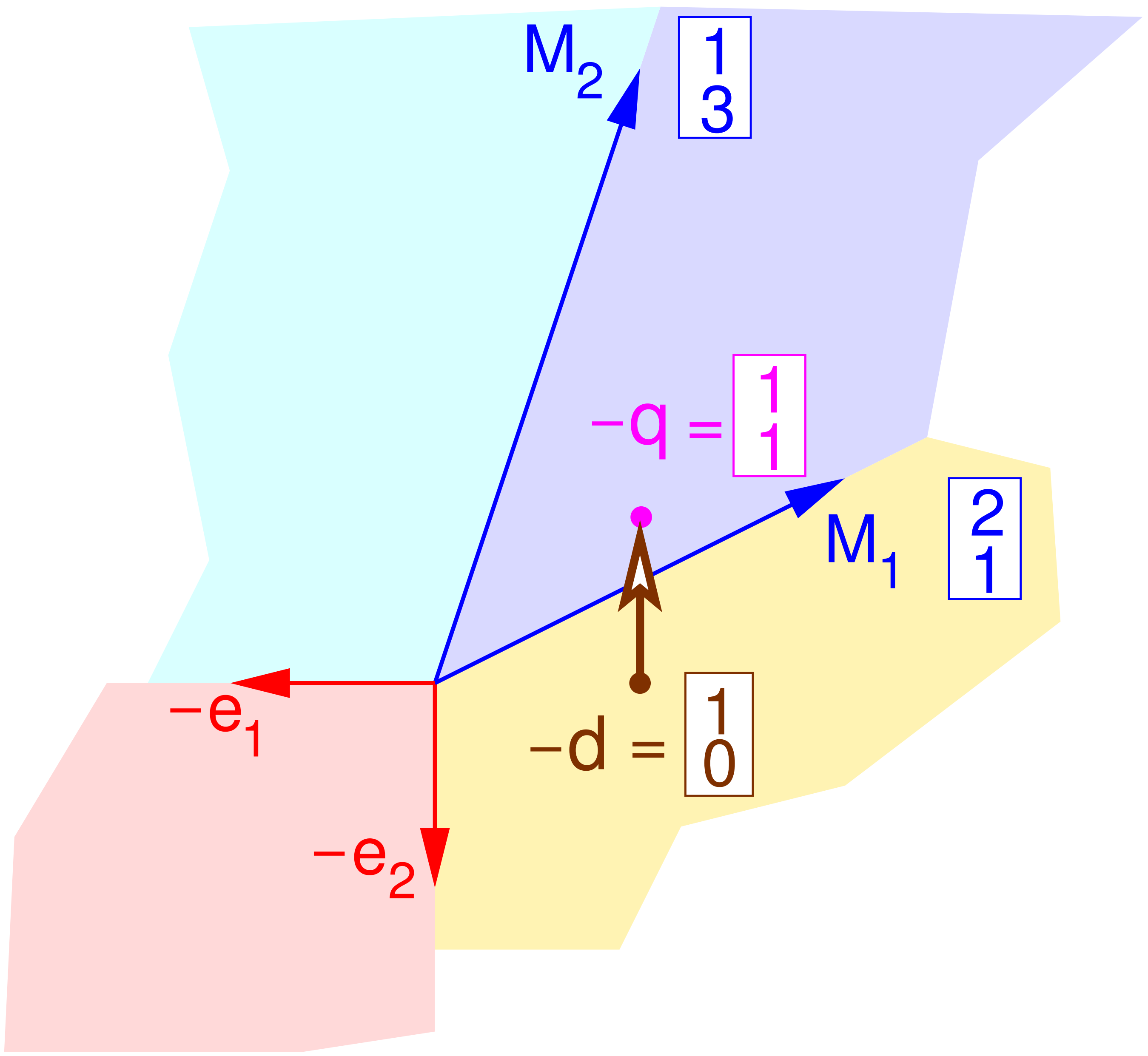
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$-d = \begin{bmatrix}1\\0\end{bmatrix}$
$-e_2$

Complexity

A result of **Morris** implies that the symmetric LH can be best-case **exponential** (i.e., for **any** missing label).

Savani & von Stengel showed that for bimatrix games LH can be best-case **exponential** (i.e., for **any** missing label).

Murty and **Goldfarb** (independently):
Lemke's algorithm derived from an **LP** can be **exponential** for the specific covering vectors $(0,...,0,1,...,1)^T$ resp. $(1,...,1,0,...,0)^T$.

Megiddo: Lemke for **random** M (not > **0**) has **expected**

- **exponential** running time when **d** = $(1,1,...,1)^T$
- **quadratic** running time when **d** = $(\varepsilon, \varepsilon^2,..., \varepsilon^n)^T$.

Harsanyi-Selten tracing procedure

Given: bimatrix game $(\boldsymbol{A}, \boldsymbol{B})$, **prior** strategy pair $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$.

Then with $\boldsymbol{M} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} \\ \boldsymbol{B}^\top & \mathbf{0} \end{bmatrix}$, $\boldsymbol{d} = \begin{bmatrix} \boldsymbol{A}\overline{\boldsymbol{y}} \\ \boldsymbol{B}^\top\overline{\boldsymbol{x}} \end{bmatrix}$,

Lemke's algorithm mimicks the **Harsanyi-Selten** procedure of **tracing** equilibria

$$(1 - z_0)(x, y) + z_0(\overline{x}, \overline{y})$$

with $z_0 \in [0, 1]$, starting with $z_0 = 1$ and ending with $z_0 = 0$ ⇒ Nash equilibrium (x, y).

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Then with $M = \begin{bmatrix} 0 & A \\ B^\top & 0 \end{bmatrix}$, $d = \begin{bmatrix} A\overline{y} \\ B^\top\overline{x} \end{bmatrix}$,

Lemke's algorithm mimicks the **Harsanyi-Selten** procedure of **tracing** equilibria

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[Goldberg / Papadimitriou / Savani]:
Finding this Nash equilibrium is PSPACE-complete.

Summary

- Lemke’s algorithm = complementary pivoting
- polyhedral and complementary-cones geometric views
- stable implementation exists
- includes Lemke-Howson as a special case
- open question: “average” running time?