Finding all Nash equilibria of a bimatrix game

David Avis
McGill University

Gabriel Rosenberg
Yale University

Rahul Savani
University of Warwick

Bernhard von Stengel
London School of Economics

Nash equilibria of bimatrix games

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix} \qquad y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} \qquad x^T B = \begin{bmatrix} 8/3 & 8/3 \end{bmatrix}$$

only **pure best responses** can have probability > 0

Best response polyhedron H_2 for player 2

$$A = \begin{array}{c|cc} & \overline{y}_4 & \overline{y}_5 \\ \hline ① & 3 & 3 \\ ② & 2 & 5 \\ ③ & 0 & 6 \end{array}$$

$$H_2 = \{ (\overline{y}_4, \overline{y}_5, u) \mid$$

① : $3\overline{y}_4 + 3\overline{y}_5 \le u$

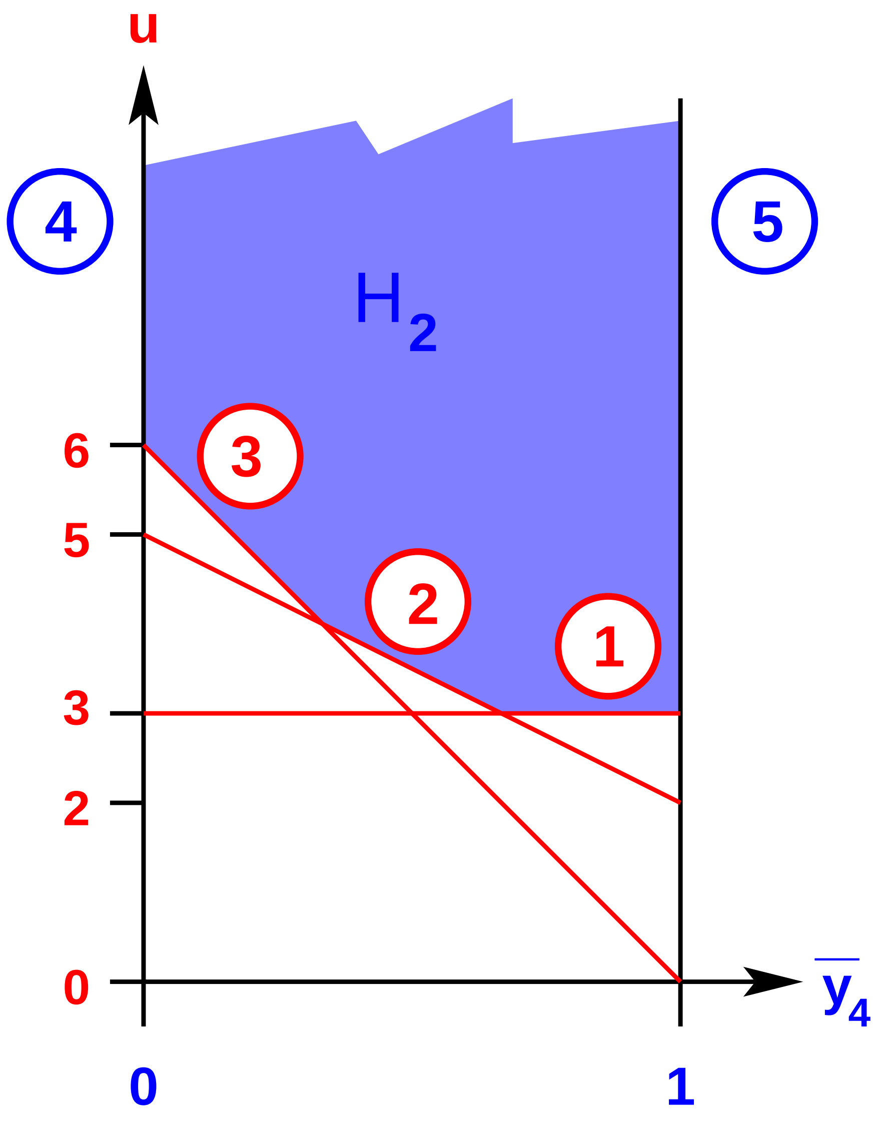
② : $2\overline{y}_4 + 5\overline{y}_5 \le u$

③ : $6\overline{y}_5 \le u$

$\overline{y}_4 + \overline{y}_5 = 1$

④ : $\overline{y}_4 \ge 0$

⑤ : $\overline{y}_5 \ge 0 \}$

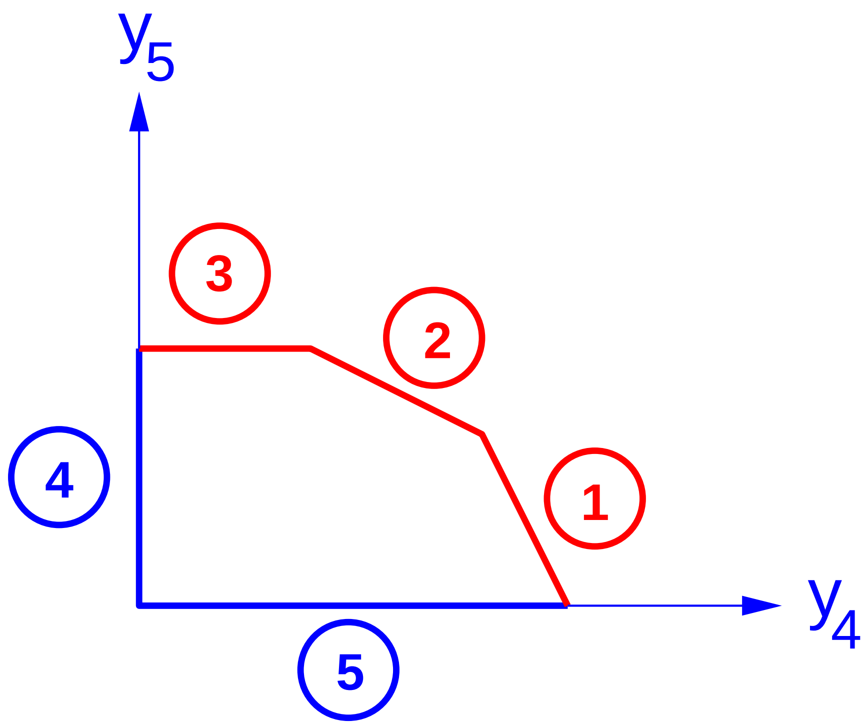


Best response polytope Q for player 2

	y_4	y_5
①	3	3
②	2	5
③	0	6

$= A$

$Q = \{\, y \mid Ay \leq 1,\ y \geq 0 \,\}$



$Q = \{\ (y_4, y_5)\ \mid$

① : $3y_4 + 3y_5 \leq 1$

② : $2y_4 + 5y_5 \leq 1$

③ : $6y_5 \leq 1$

④ : $y_4 \geq 0$

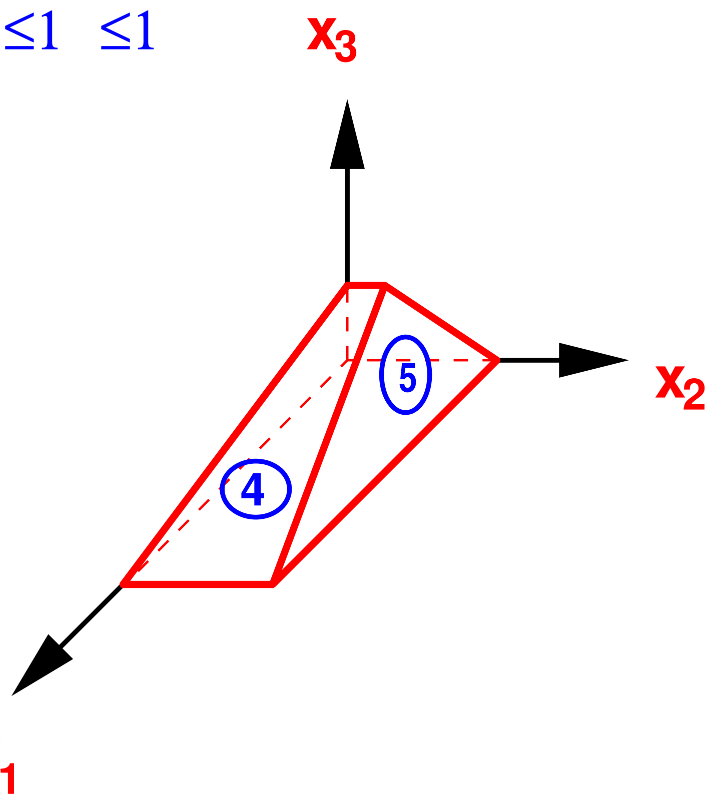
⑤ : $y_5 \geq 0\ \}$

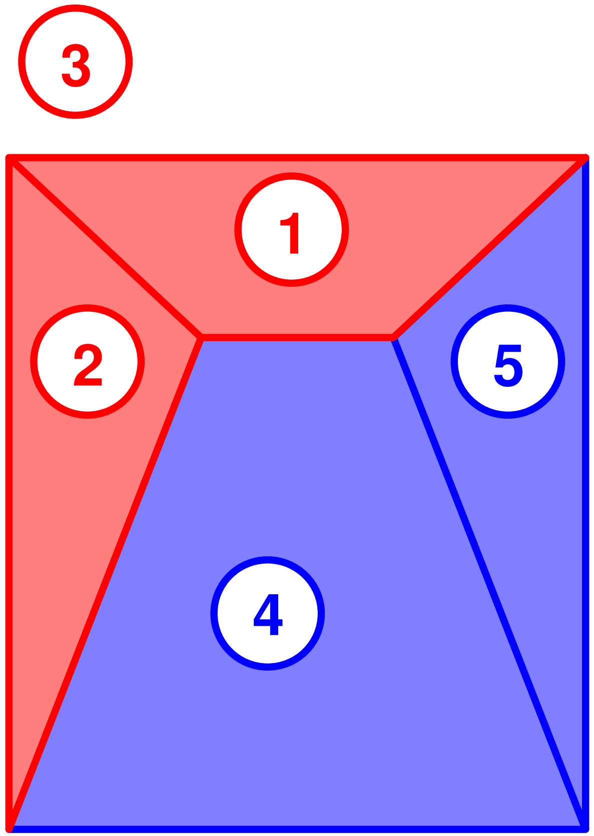
Best response polytope P for player 1

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix} = B$$

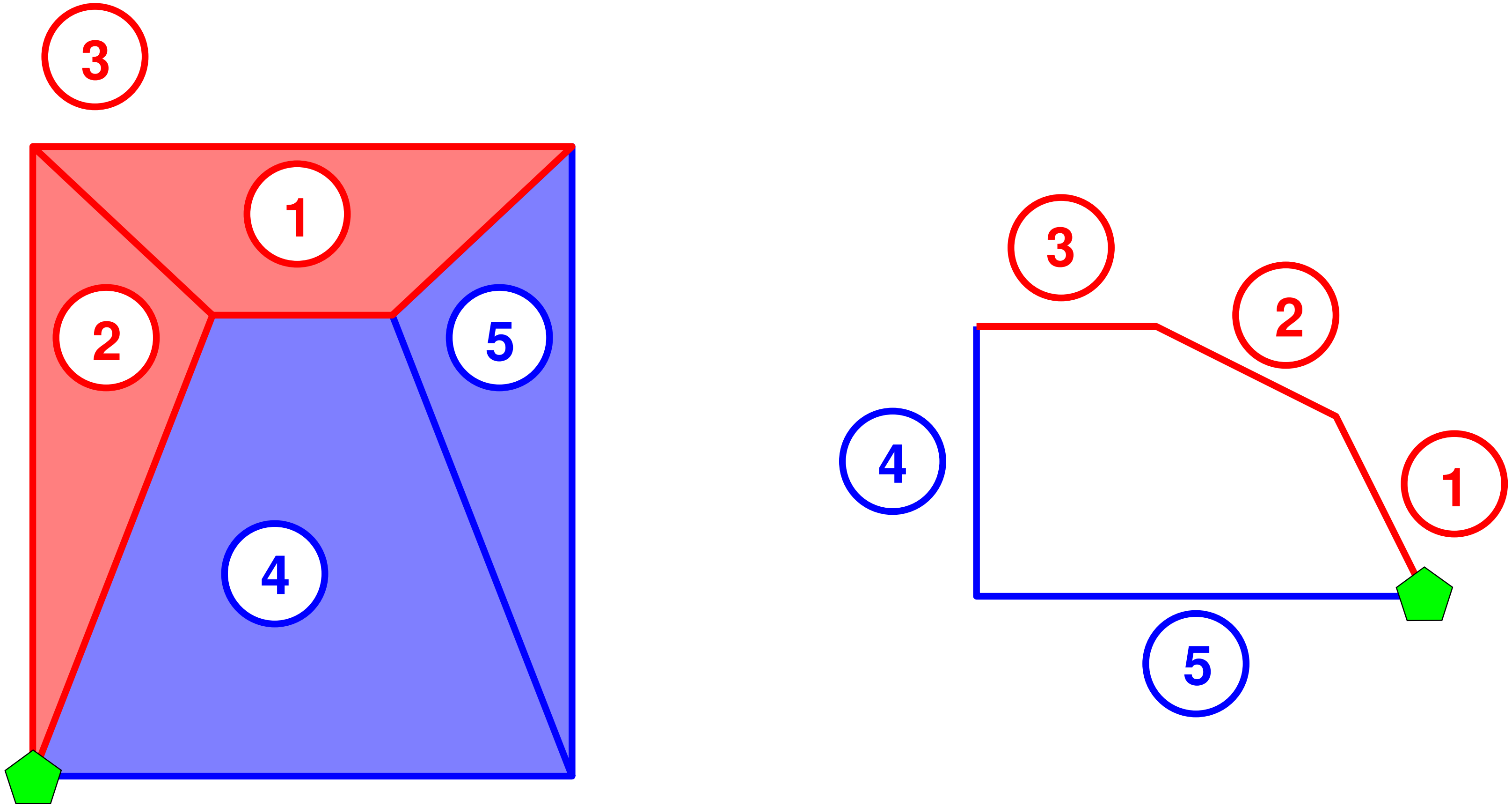
$$\begin{array}{cc} \le 1 & \le 1 \end{array}$$

$$P = \{ x \mid x \geq 0, \; x^{\top} B \leq 1 \}$$



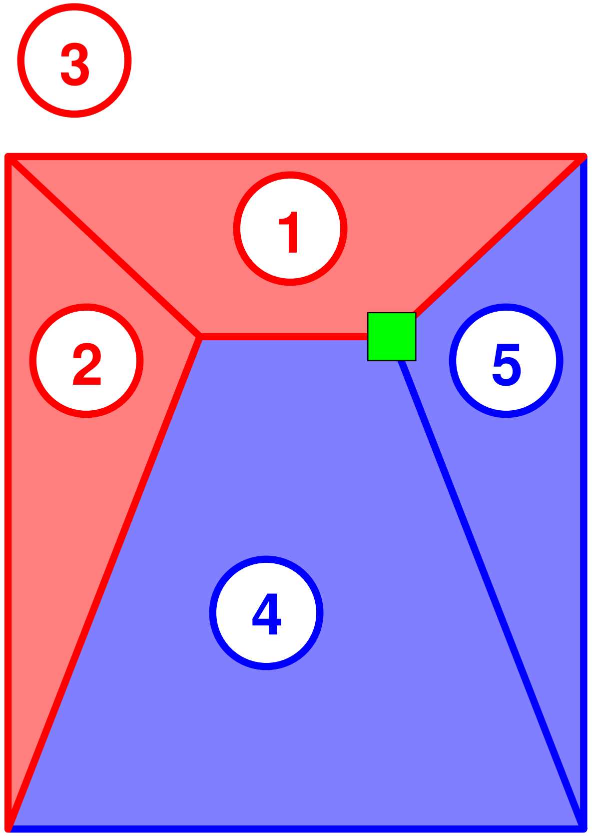


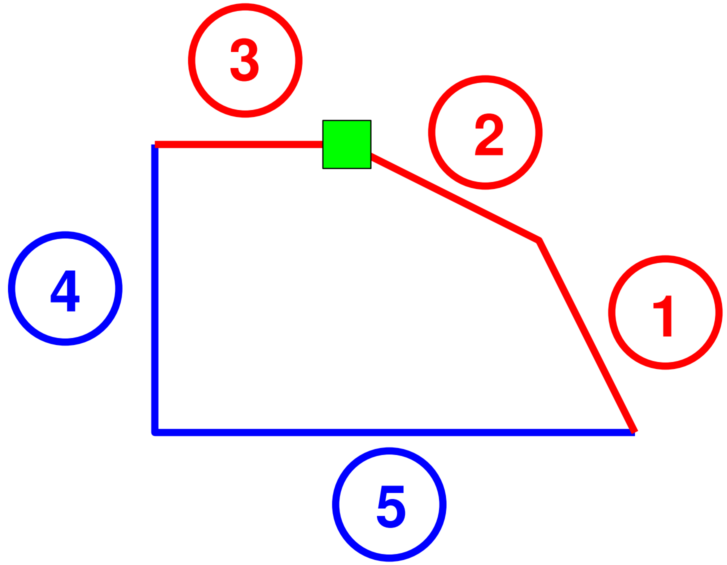
Equilibrium = completely labeled pair



pure equilibrium

Equilibrium = completely labeled pair





mixed equilibrium

Convex equilibrium components

[Winkels 1979 / Jansen 1980]

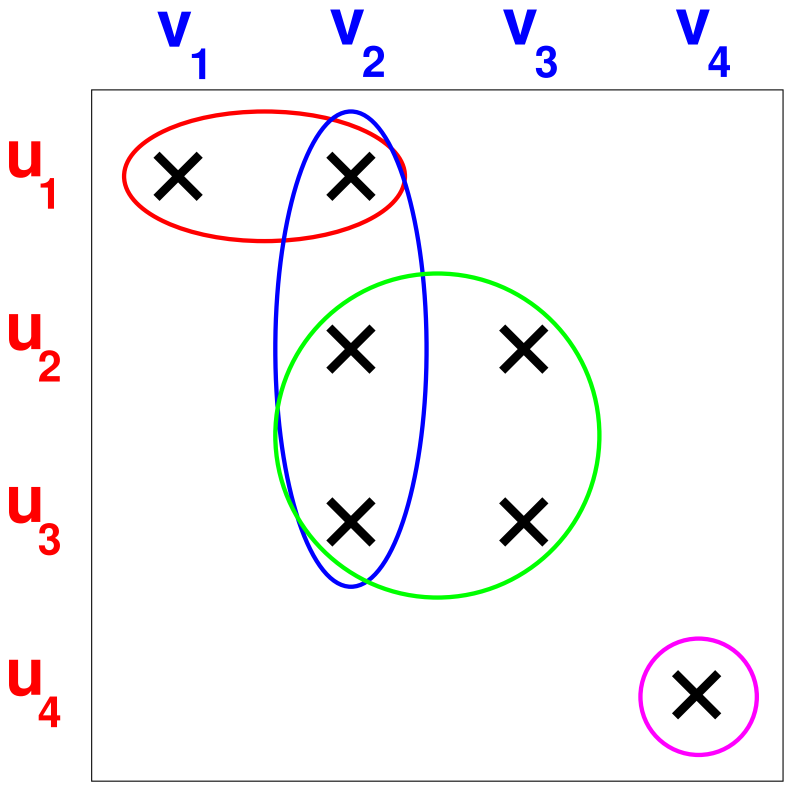
(**x**,**y**) is an equilibrium of (**A**,**B**) $\Leftrightarrow$

(**x**,**y**) is in the convex hull of **U** x **V** , where all (**u**, **v**) $\in$ **U** x **V** are completely labelled vertex pairs of **P** x **Q – (0,0)**

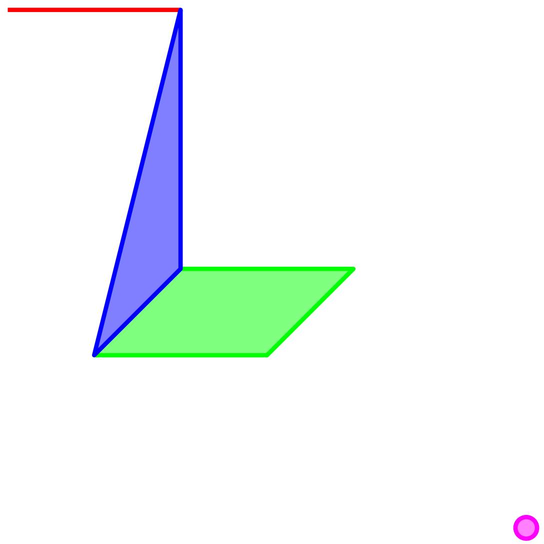
Convex equilibrium components U x **V**

= maximal cliques of bipartite graph

Convex equilibrium components



Geometry:



Clique enumeration

[Bron & Kerbosch 1973]

Recursive bottom-up generation of maximal cliques by elegant backtracking and branch and bound.

Adapted to bipartite graphs, outputs 2000 cliques / second independent of graph size.

So far

Problem:

Given: bimatrix game (**A**,**B**). What are its Nash equilibria?

Overview:

Any equilibrium is a convex combination of extreme equilibria = certain vertices of polytopes derived from **A**, **B**.

Enumerate extreme equilibria (finitely many).

Output convex equilibrium components.

No efficient equilibrium enumeration

[Gilboa & Zemel 1989]
Uniqueness of Nash equilibrium is NP -hard.

$\Longrightarrow$ No output efficient enumeration of equilibria possible
(unless P = NP).

The EEE algorithm

Audet, C., P. Hansen, B. Jaumard, and G. Savard (2001), Enumeration of all Extreme Equilibria of bimatrix games. *SIAM Journal on Scientific Computing* **23**, 323-338.

- depth-first exploration of search tree for both polyhedra via parameterized LPs

Our improvements:

- stand-alone instead of using CPLEX
- exact arithmetic instead of floating-point
- much faster degeneracy handling

Equilibria via LP duality

x best response to y

⇔ solves **primal LP**:

$$\max\ x^T(Ay)$$
$$\text{s.t. } \mathbf{1}^T x = 1$$
$$x \geq \mathbf{0}$$

dual LP: $\min\ u$

$$\text{s.t. } \mathbf{1}u \geq Ay$$

optimal ⇔ $u = x^T Ay$

complementary slackness:

⇔ $x \geq \mathbf{0} \perp \mathbf{1}u \geq Ay$

i.e. $x^T(\mathbf{1}u - Ay) = 0$

Equilibria via LP duality

x best response to y	y best response to x
$\Leftrightarrow$ solves **primal LP**:	
$\max\ x^T(Ay)$	$\max\ (x^TB)y$
s.t. $\mathbf{1}^T x = 1$	s.t. $\mathbf{1}^T y = 1$
$x \geq \mathbf{0}$	$y \geq \mathbf{0}$
dual LP: $\min\ u$	$\min\ v$
s.t. $\mathbf{1}u \geq Ay$	s.t. $\mathbf{1}v \geq B^T x$
optimal $\Leftrightarrow$ $u = x^TAy$	$v = x^TBy$
complementary slackness:	
$\Leftrightarrow$ $x \geq \mathbf{0} \perp \mathbf{1}u \geq Ay$	$y \geq \mathbf{0} \perp \mathbf{1}v \geq B^T x$
i.e. $x^T(\mathbf{1}u - Ay) = 0$	$y^T(\mathbf{1}v - B^T x) = 0$

Equilibria via Linear Complementarity

x best response to y	y best response to x
$\Leftrightarrow$ solves **primal LP**:	
$\max\ x^T(Ay)$	$\max\ (x^TB)y$
s.t. $\mathbf{1}^T x = 1$	s.t. $\mathbf{1}^T y = 1$
$x \geq \mathbf{0}$	$y \geq \mathbf{0}$
dual LP: $\min\ u$	$\min\ v$
s.t. $\mathbf{1}u \geq Ay$	s.t. $\mathbf{1}v \geq B^T x$
optimal $\Leftrightarrow$ $u = x^T A y$	$v = x^T B y$

complementary slackness:

$\Leftrightarrow$ $x \geq \mathbf{0} \perp \mathbf{1}u \geq Ay$ $\quad$ $y \geq \mathbf{0} \perp \mathbf{1}v \geq B^T x$

i.e. $x^T(\mathbf{1}u - Ay) = 0$ $\quad$ $y^T(\mathbf{1}v - B^T x) = 0$

Best response polyhedra

$$\mathbf{P} = \{\, (x, v) \in \mathbf{R}^{m+1} \;\mid\; x \geq \mathbf{0},\; B^{\mathsf{T}} x \leq \mathbf{1} v,\; \mathbf{1}^{\mathsf{T}} x = 1 \,\}$$

$$\perp$$

$$\perp$$

$$\mathbf{Q} = \{\, (y, u) \in \mathbf{R}^{n+1} \;\mid\; A y \leq \mathbf{1} u,\; y \geq \mathbf{0},\; \mathbf{1}^{\mathsf{T}} y = 1 \,\}$$

Best response polyhedra

$$\mathbf{P} = \{\ (x, v) \in \mathbf{R}^{m+1} \ \mid \ x \geq \mathbf{0},\quad B^T x \leq \mathbf{1} v,\quad \mathbf{1}^T x = 1\ \}$$

$$\mathbf{Q} = \{\ (y, u) \in \mathbf{R}^{n+1} \ \mid \ A y \leq \mathbf{1} u,\quad y \geq \mathbf{0},\quad \mathbf{1}^T y = 1\ \}$$

$$\perp \qquad \perp$$

with slack variables r, s:

$$x, s \geq \mathbf{0} \qquad B^T x + s = \mathbf{1} v \qquad \mathbf{1}^T x = 1$$

$$x \perp r \qquad s \perp y$$

$$r + A y = \mathbf{1} u \qquad r, y \geq \mathbf{0} \qquad \mathbf{1}^T y = 1$$

Faces of P, Q via label sets K

Let K, L $\subseteq$ M $\cup$ N = set of labels (pure stategies)

$P(K) = \{ (x, v) \in P \mid x_i = 0, \ (B^T x)_j = v \ \text{ for } i, j \in K \}$

$[\Leftrightarrow \ s_j = 0 \ ]$

$Q(L) = \{ (y, u) \in Q \mid (Ay)_i = u, \ y_j = 0 \ \text{ for } i, j \in L \}$

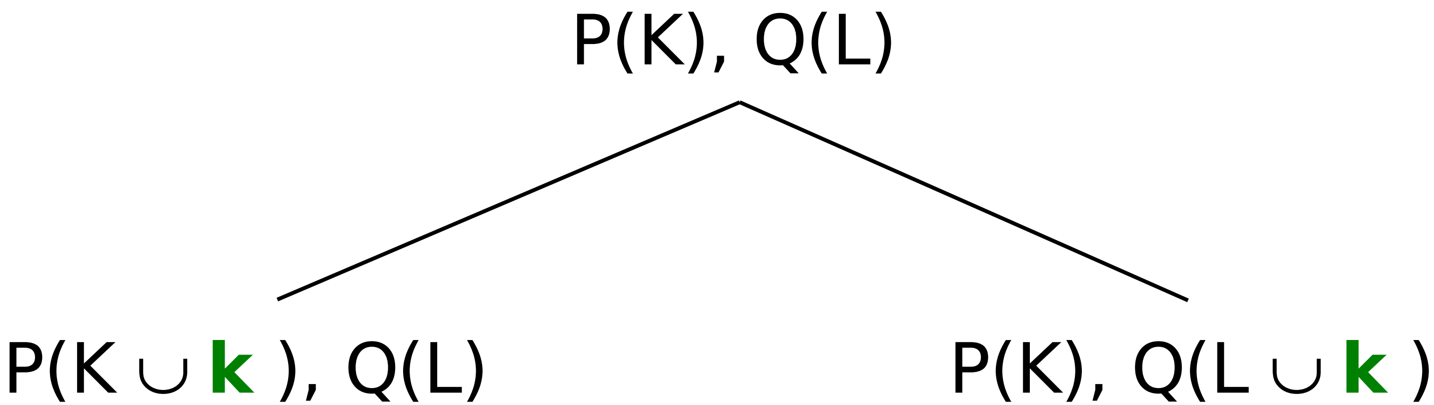
$[\Leftrightarrow \ r_i = 0 \ ]$

EEE - binary search tree

root: $P, Q \quad = \quad P(\varnothing), Q(\varnothing)$

nodes: $P(K), Q(L)$ with disjoint K, L

binary branching: new label **k** added to K or L



Face representatives

face P(K) of P represented by $x \in P(K)$,

face Q(L) of L represented by $y \in Q(L)$.

$x \in P(K)$, $y \in Q(L)$

$P(K \cup \mathbf{k})$, $Q(L)$
new x, same y

$P(K)$, $Q(L \cup \mathbf{k})$
same x, new y

Face representatives

face P(K) of P represented by $x \in P(K)$,

face Q(L) of L represented by $y \in Q(L)$.

$x \in P(K)$, $y \in Q(L)$

k=**i**: set $x_i=0$ set $r_i=0$

P(K ∪ **i**), Q(L)
new x, same y

P(K), Q(L ∪ **i**)
same x, new y

Face representatives

face P(K) of P represented by $x \in P(K)$,

face Q(L) of L represented by $y \in Q(L)$.

$x \in P(K)$, $y \in Q(L)$

k=**j**: set $s_j=0$ set $y_j=0$

$P(K \cup \mathbf{j})$, Q(L)
new x, same y

P(K), $Q(L \cup \mathbf{j})$
same x, new y

To smaller-dimensional face via LP

$x \in P(K)$, $y \in Q(L)$

set $x_i = 0$ . . .

$P(K \cup \mathbf{i})$, $Q(L)$

new x as solution to LP

$$\max \; x^T(Ay) - v$$

$$\text{s.t.} \quad (x, v) \in P(K), \quad x_i = 0$$

To smaller-dimensional face via LP

$x \in P(K)$, $y \in Q(L)$

set $x_i = 0$ . . .

$P(K \cup \mathbf{i})$, $Q(L)$

new x as solution to LP

$$\max \quad x^T(Ay) - v$$

$$\text{s.t.} \quad (x, v) \in P(K), \quad x_i = 0$$

if infeasible: terminate search,

backtrack (depth-first search)

To smaller-dimensional face via LP

$x \in P(K)$, $y \in Q(L)$

set $x_i = 0$ . . .

$P(K \cup \mathbf{i})$, $Q(L)$

new x as solution to LP

$$\max \; x^T(Ay) - v$$

$$\text{s.t.} \quad (x, v) \in P(K), \quad x_i = 0$$

in LP tableau: drive x_i out of basis (use tableau row to minimize x_i), then **delete** cobasic column for x_i

Choice of label to branch on

given node with $x \in P(K)$, $y \in Q(L)$,

branch on **k** that violates complementarity most, i.e.

$$\mathbf{k} = \arg\max \{ x_i r_i , s_j y_j \mid \mathbf{i} , \mathbf{j} \notin K \cup L \}$$

$\Rightarrow$ hope to discover infeasibility quickly
to prune search tree.

Choice of label to branch on

given node with $x \in P(K)$, $y \in Q(L)$,

branch on **k** that violates complementarity most, i.e.

$$\mathbf{k} = \arg\max \{ x_i r_i , s_j y_j \mid i, j \notin K \cup L \}$$

possible: $\max \{ x_i r_i , s_j y_j \} = 0$

$\Leftrightarrow$ (x,y) equilibrium,

but: is **ignored** unless $|K| = m$, $|L| = n$

(Nash equilibrium at leaf of search tree, node at depth $m+n$).

Choice of objective function

given $y \in Q(L)$, $x \in P(K \cup \mathbf{k})$ is solution to LP

$$\max \; x^T(Ay) - v$$

$$\text{s.t.} \quad (x,v) \in P(K \cup \mathbf{k})$$

given $x \in P(K)$, $y \in Q(L \cup \mathbf{k})$ is solution to LP

$$\max \; (x^T B)y - u$$

$$\text{s.t.} \quad (y,u) \in Q(L \cup \mathbf{k})$$

Choice of objective function

given $y \in Q(L)$, $x \in P(K \cup \mathbf{k})$ is solution to LP

$$\max \; x^T(Ay) - v$$

$$\text{s.t.} \quad (x,v) \in P(K \cup \mathbf{k})$$

given $x \in P(K)$, $y \in Q(L \cup \mathbf{k})$ is solution to LP

$$\max \; (x^T B)y - u$$

$$\text{s.t.} \quad (y,u) \in Q(L \cup \mathbf{k})$$

Why these objective functions? Their sum

$$x^T(Ay) + (x^T B)y - u - v \qquad [\, \leq 0 \,]$$

is negative of duality gap, try to make that zero!

Other objective functions

ignoring $y \in Q(L)$, $x \in P(K \cup \mathbf{k})$ is solution to LP

$$\max \boxed{-v}$$

s.t. $(x,v) \in P(K \cup \mathbf{k})$

ignoring $x \in P(K)$, $y \in Q(L \cup \mathbf{k})$ is solution to LP

$$\max \boxed{-u}$$

s.t. $(y,u) \in Q(L \cup \mathbf{k})$

find | lowest point on upper envelope |

Other objective functions

$x \in P(K \cup \mathbf{k})$ is solution to LP

max $\boxed{0}$

s.t. $(x, v) \in P(K \cup \mathbf{k})$

$y \in Q(L \cup \mathbf{k})$ is solution to LP

max $\boxed{0}$

s.t. $(y, u) \in Q(L \cup \mathbf{k})$

find $\boxed{\text{feasible vertex}}$

$\Rightarrow$ still on upper envelope

Run time comparisons [sec]

Objective function	Random (average) 17 x 17 57 NE	Guessing Game 22 x 22 3 NE	Dollar Game 10 x 10 91 NE
P: $\max x^T A y - v$ Q: $\max x^T B y - u$	186	44	126
P: $\max -v$ Q: $\max -u$	408	185	119
P: $\max 0$ Q: $\max 0$	397	169	117
P: $\max x^T (A+B) y - v$ Q: $\max x^T (A+B) y - u$	146	33	122

Extra multiple branching needed

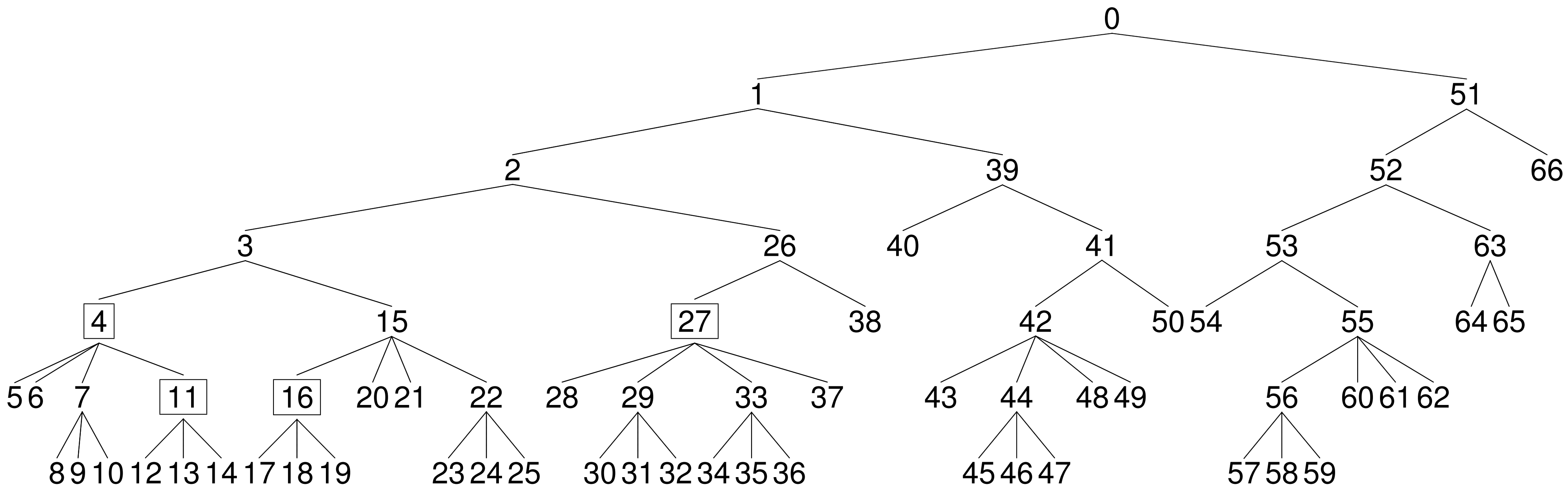
In degenerate games, **not all** extreme equilibria appear at level $m+n$ in the binary search tree.

Audet et al. add **multiple** branches at level $m+n$: each positive variable x_i , s_j , r_i , y_j at that level is attempted to be set to zero (typically infeasible, $m+n$ extra pivots), continuing search.

One observes a **very high number** of extra nodes for degenerate games.

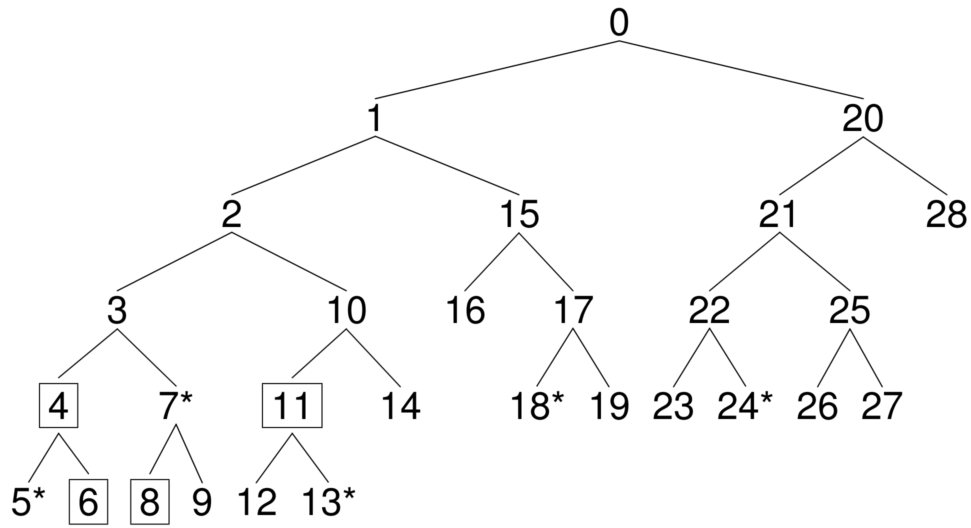
Example of original EEE search tree

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



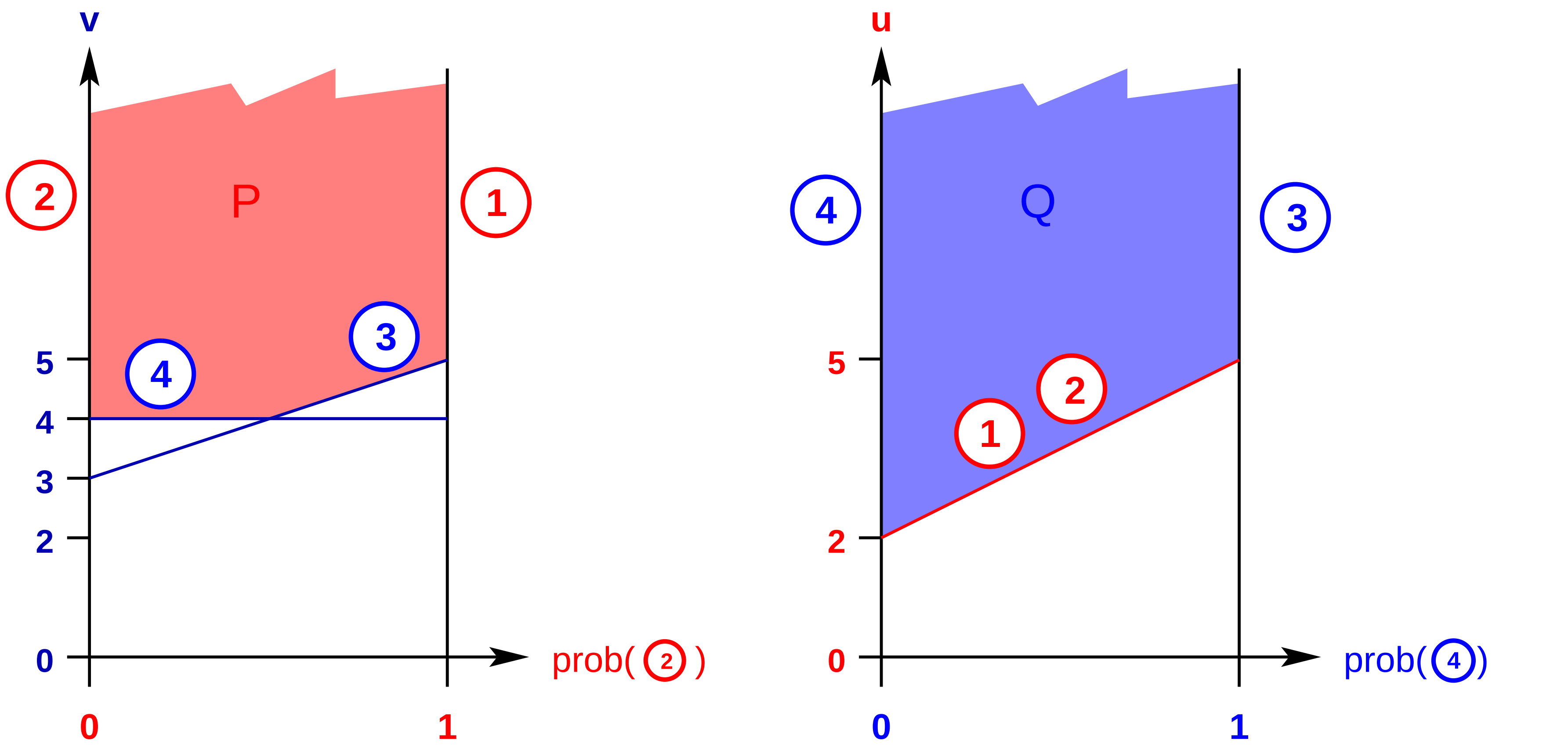
Improved EEE search tree

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



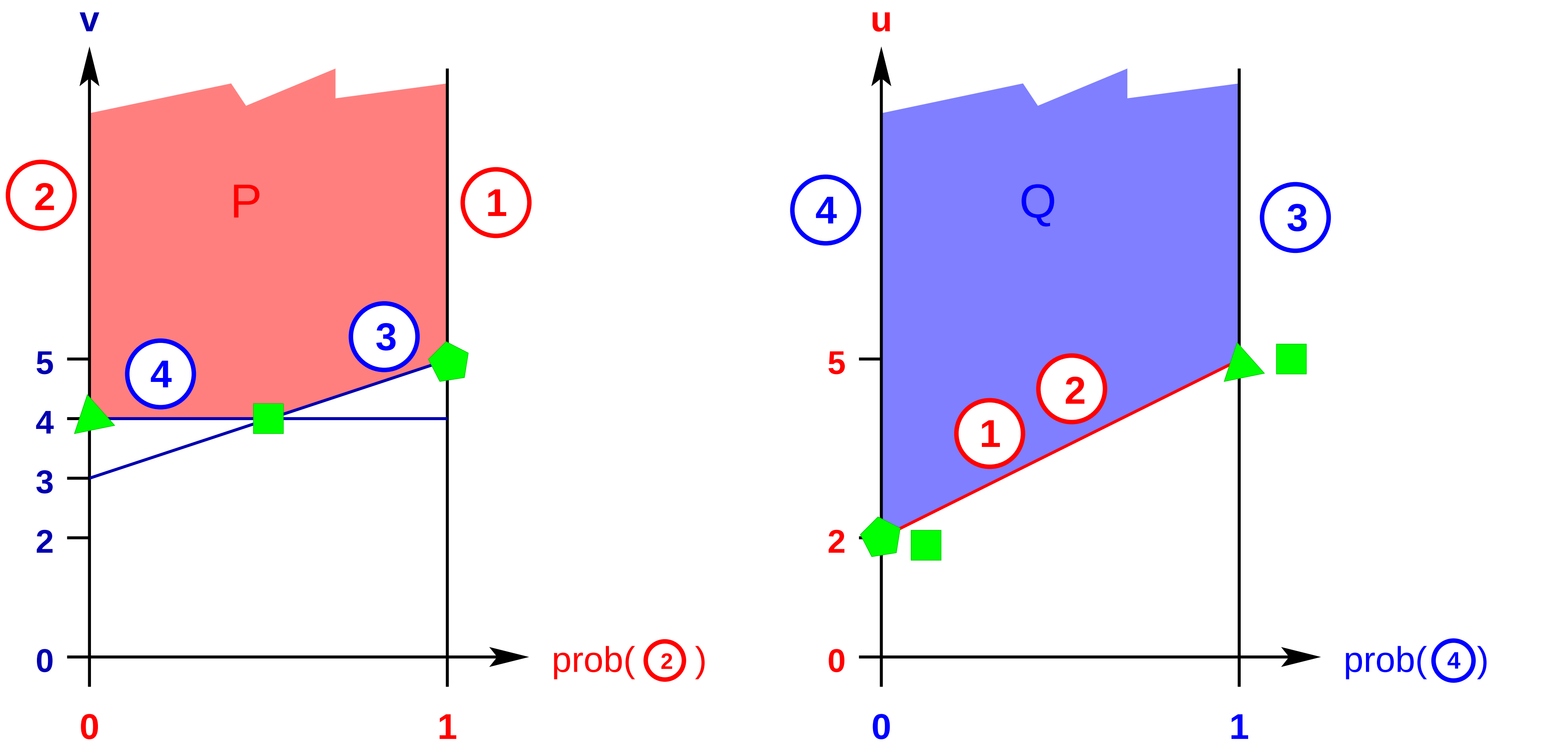
Polyhedra for example

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



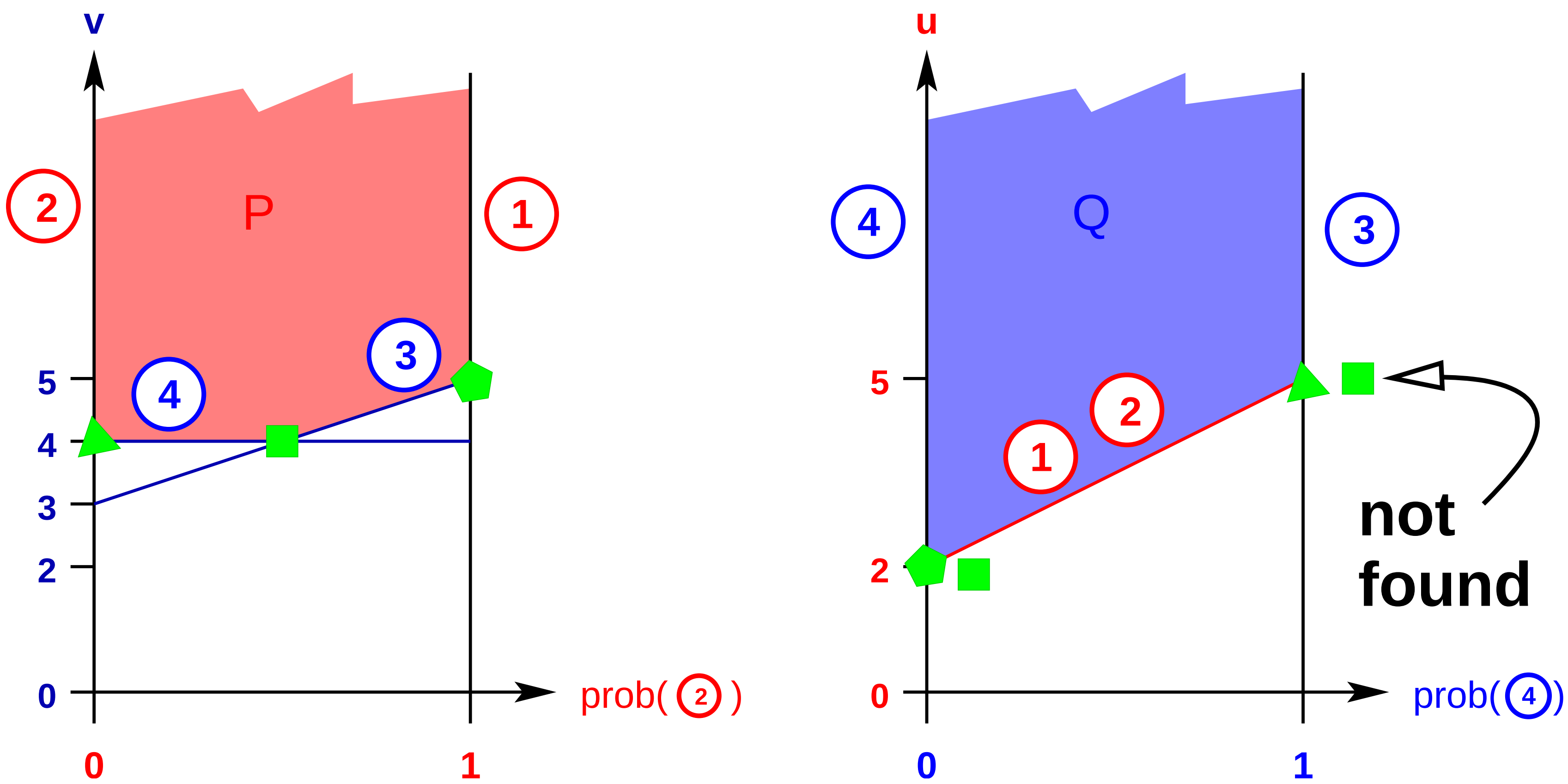
Extreme equilibria

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



One equilibrium missed by objective function

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



Improved degeneracy treatment

Vertices are missed due to **redundant equalities** (degeneracy in **description** of polyhedron!).

These appear as **zero rows** in tableaux
and **cannot** be made **cobasic**.

$\Rightarrow$ **delete** zero row, keep extra cobasic variable;

only these extra cobasic variables are searched (eliminated) beyond search level $m+n$.

Run time comparisons

Degeneracy check	Guessing Game 5 NE	Dollar Game 73 NE
Original EEE time nodes pivots	0.36 427 530	14806.06 279020949 20326989
Improved EEE time nodes pivots	0.37 337 530	25.25 141531 43987