Strategic Characterization of the Index of an Equilibrium

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Theorem

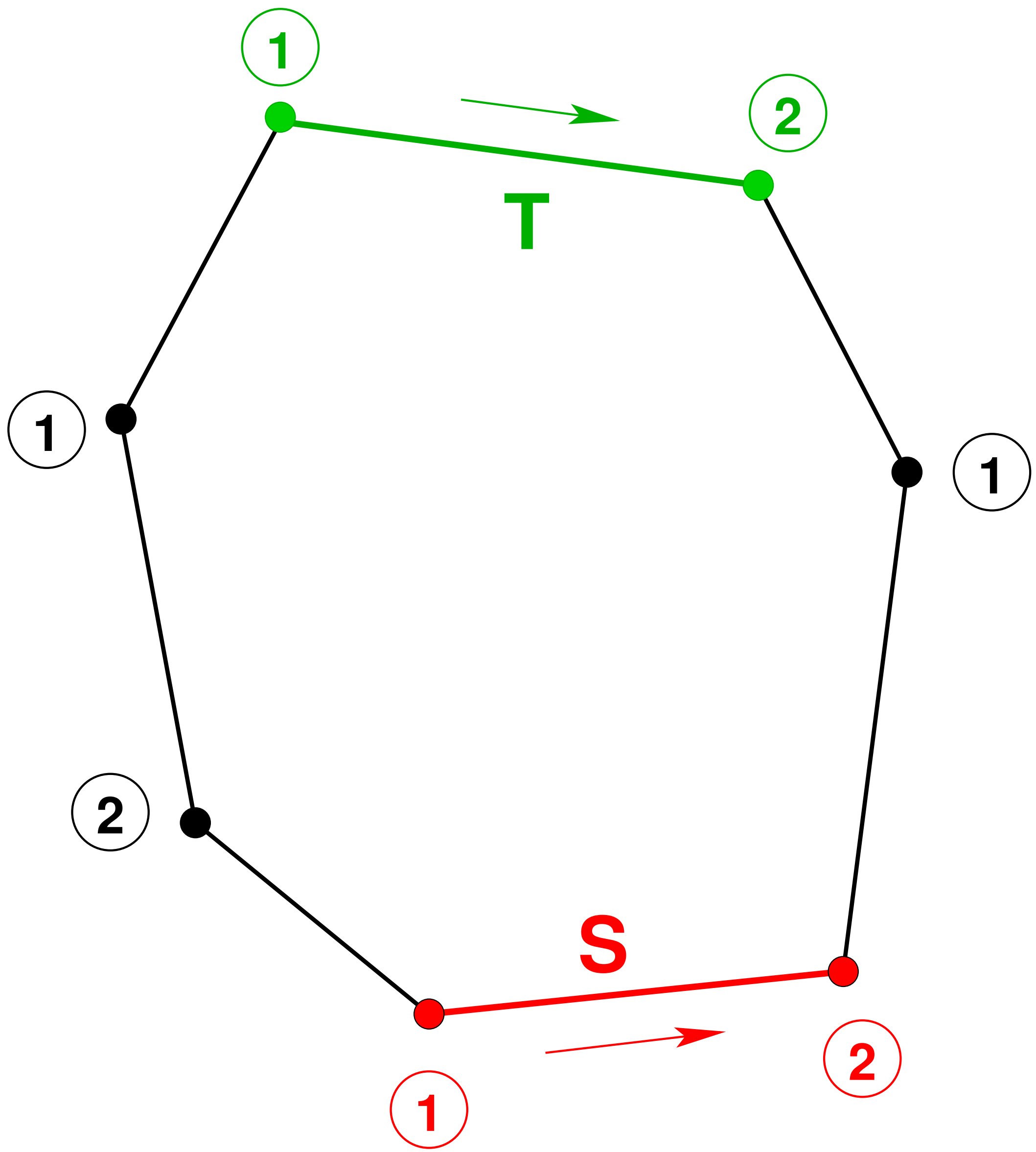
Given:

- simplicial d-polytope **P**
- each vertex has a label $\in$ { 1, 2, ..., d }
- two (disjoint) completely labelled facets **S**, **T** of opposite orientation

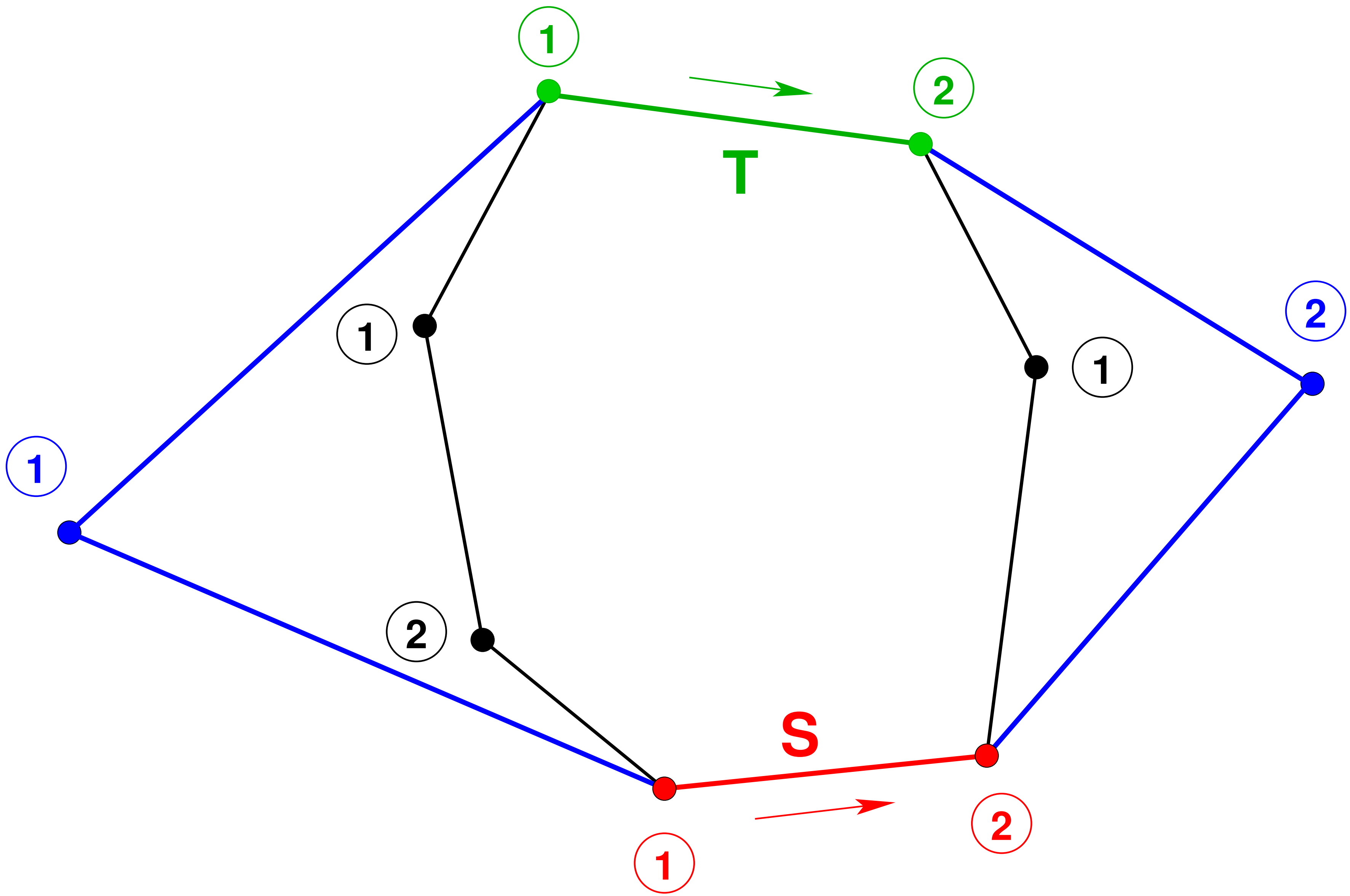
Then

- there are labelled points $c_1, \ldots, c_k$ so that **S**, **T** are the **only** completely labelled facets of the convex hull of **P** $\cup$ { $c_1, \ldots, c_k$ }

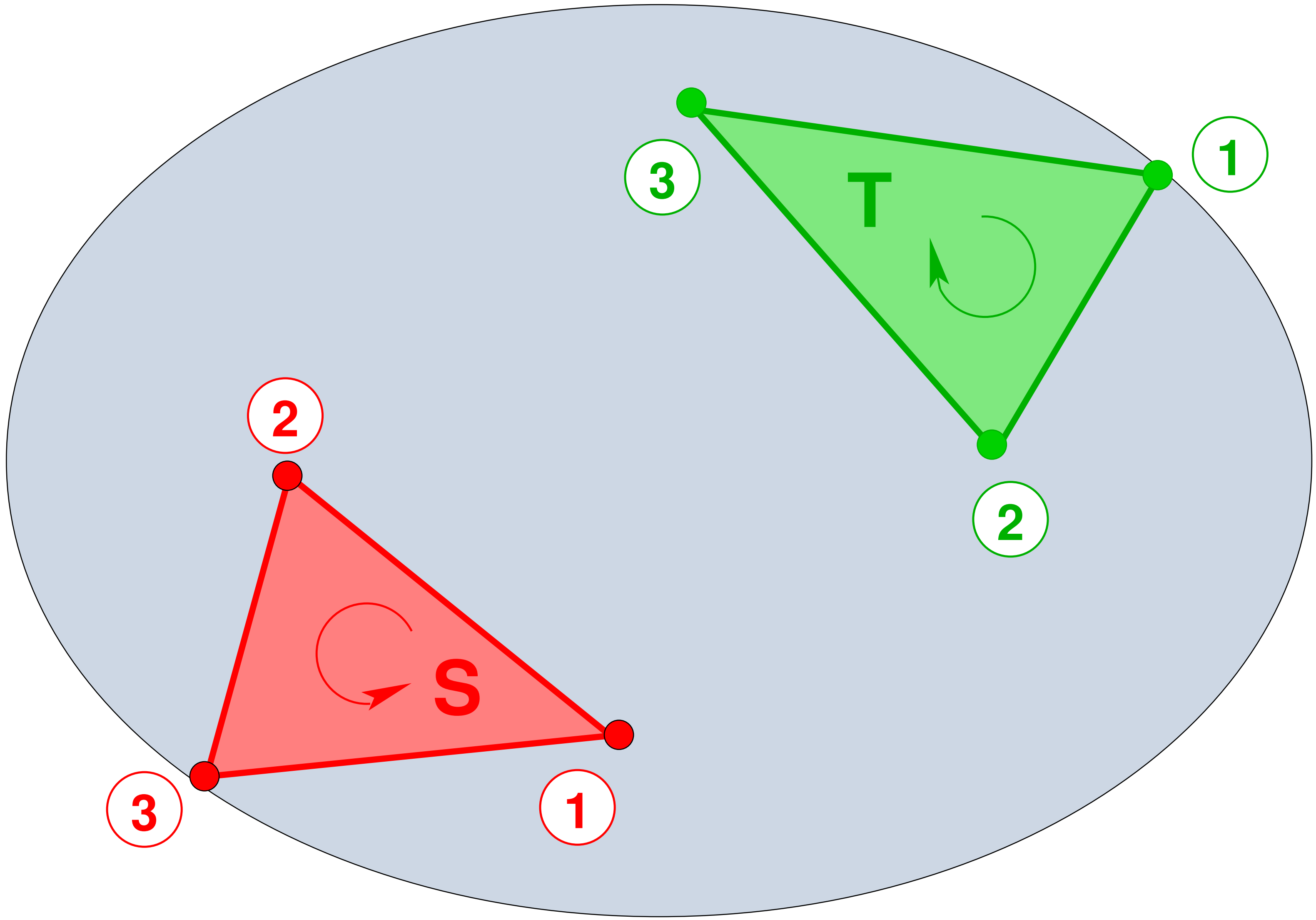
Example: d = 2



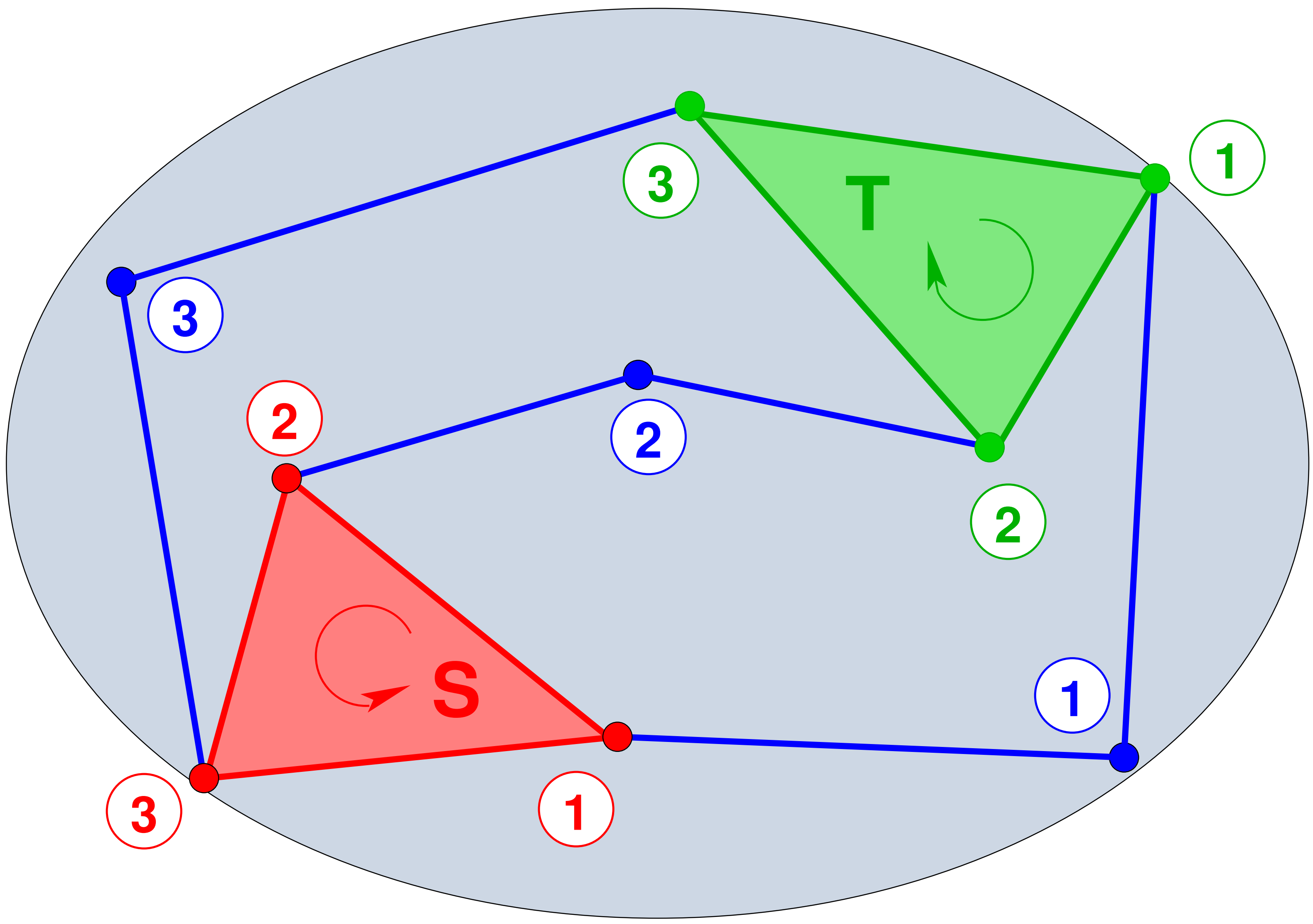
Example: d = 2



Topological proof for d > 2



Topological proof for d > 2



Nash equilibria of bimatrix games

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \begin{pmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{pmatrix}$$

$$x = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} \qquad x^T B = \begin{pmatrix} 5/3 & 5/3 \end{pmatrix}$$

$$Ay = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} \qquad y^T = \begin{pmatrix} 1/3 & 2/3 \end{pmatrix}$$

only **pure best responses** can have probability > 0

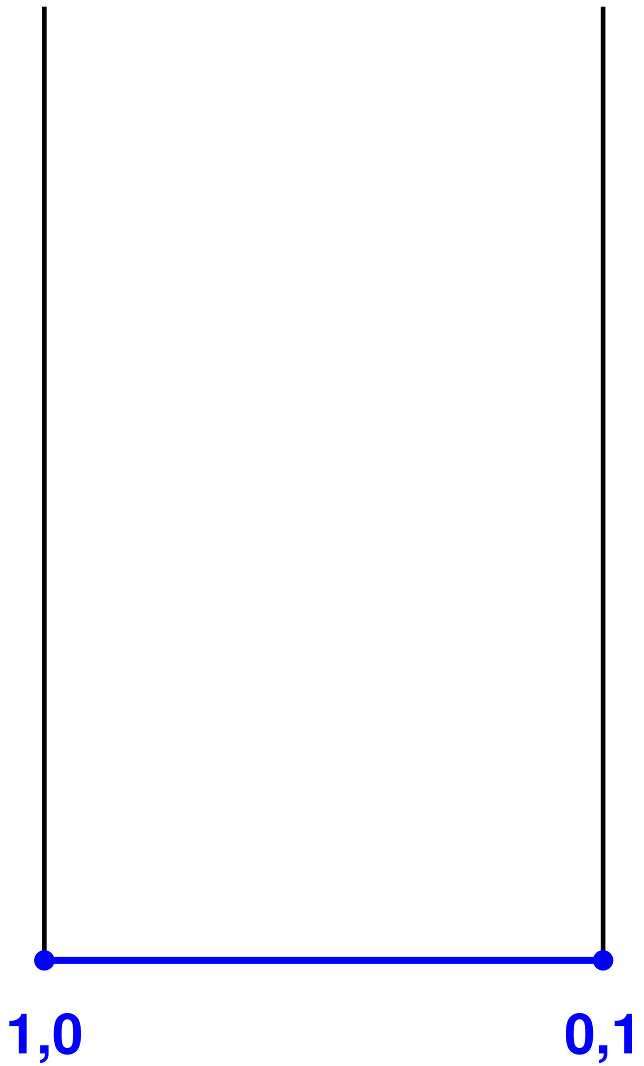
Best responses to mixed strategy of player 2

	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2



	4	5
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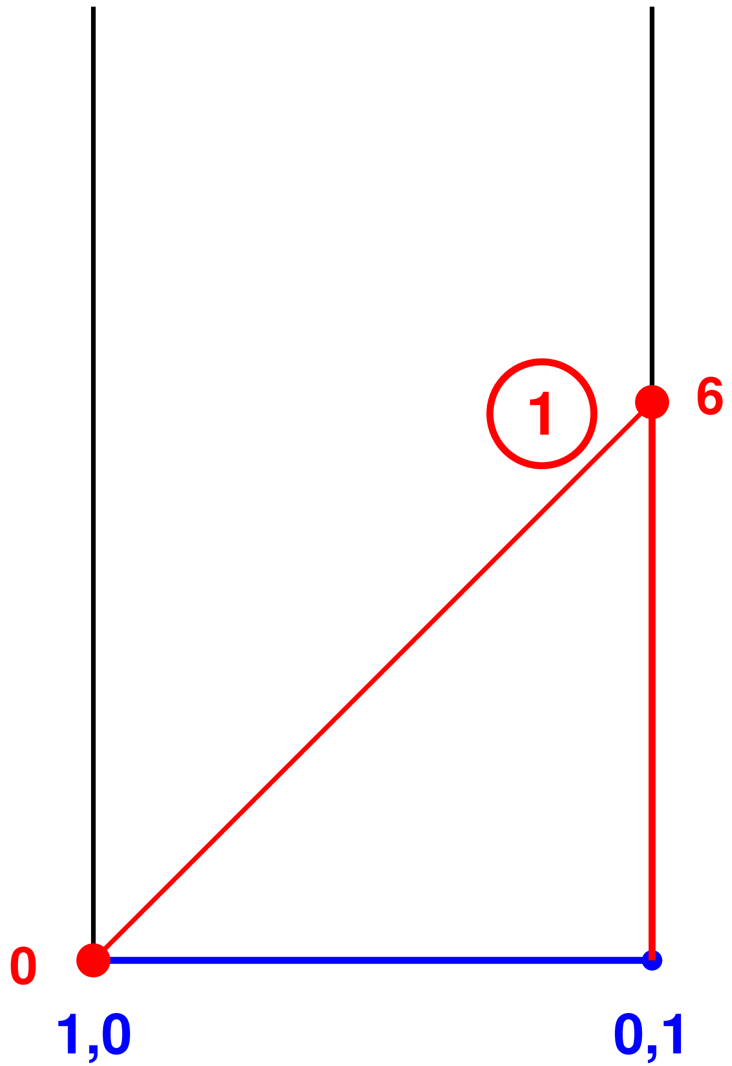
= A

payoffs to player I

Best responses to mixed strategy of player 2



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payoffs to
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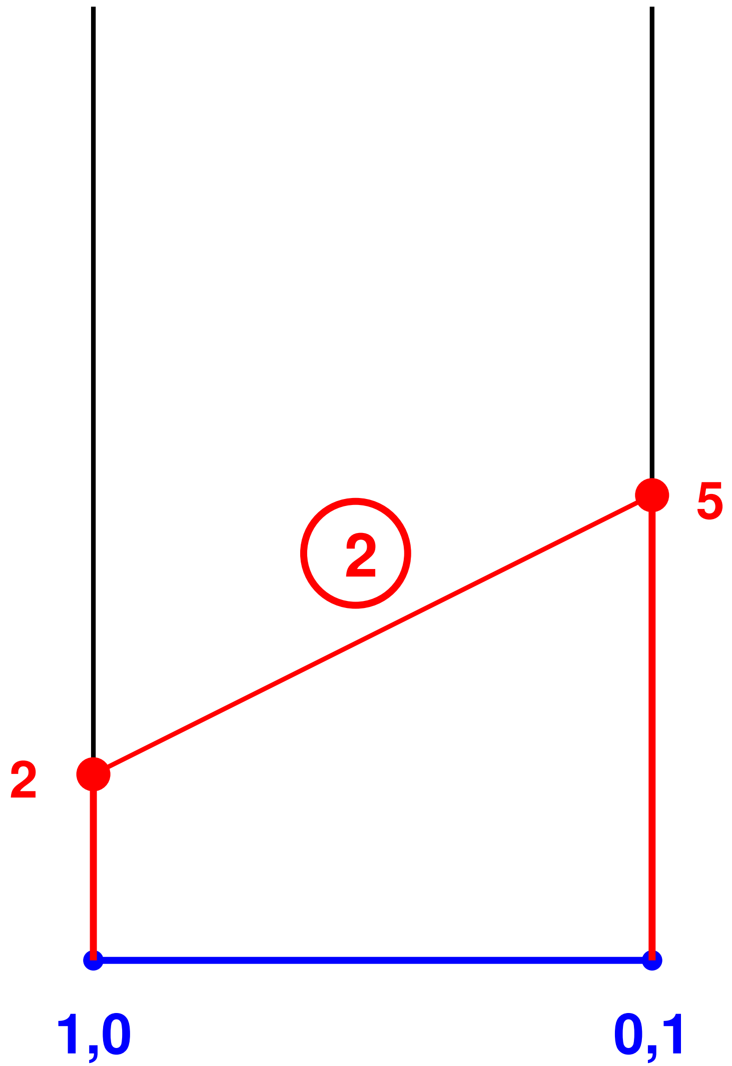

	4	5
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2	2	5
3	3	3

= A

payoffs to
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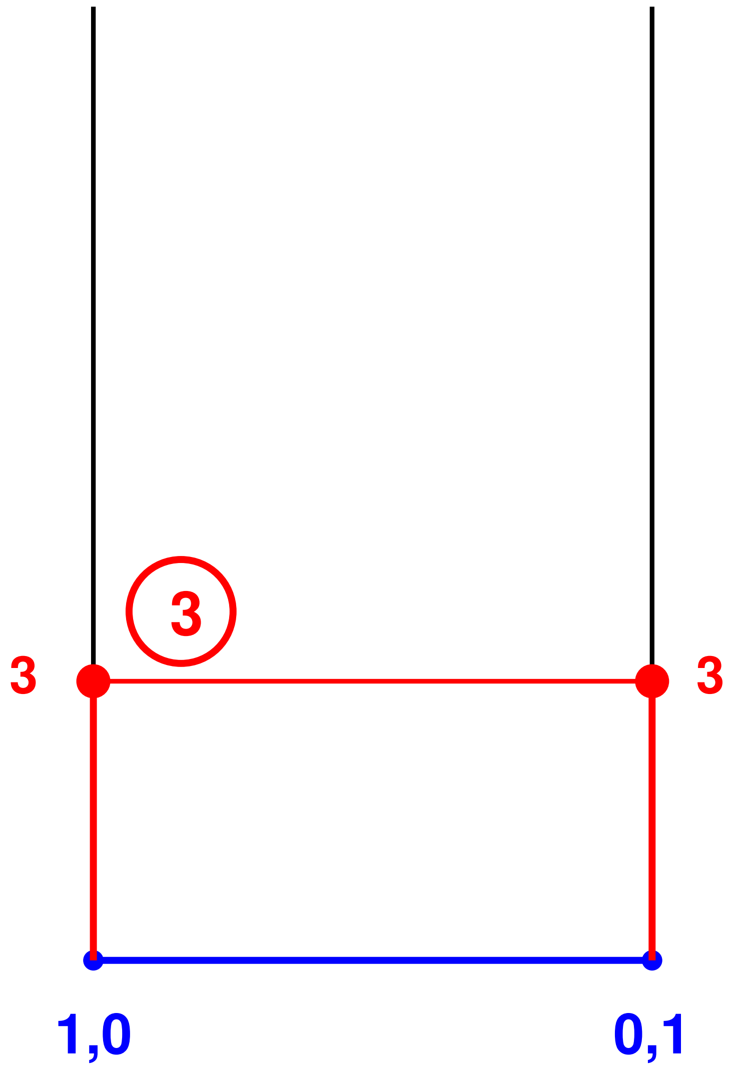
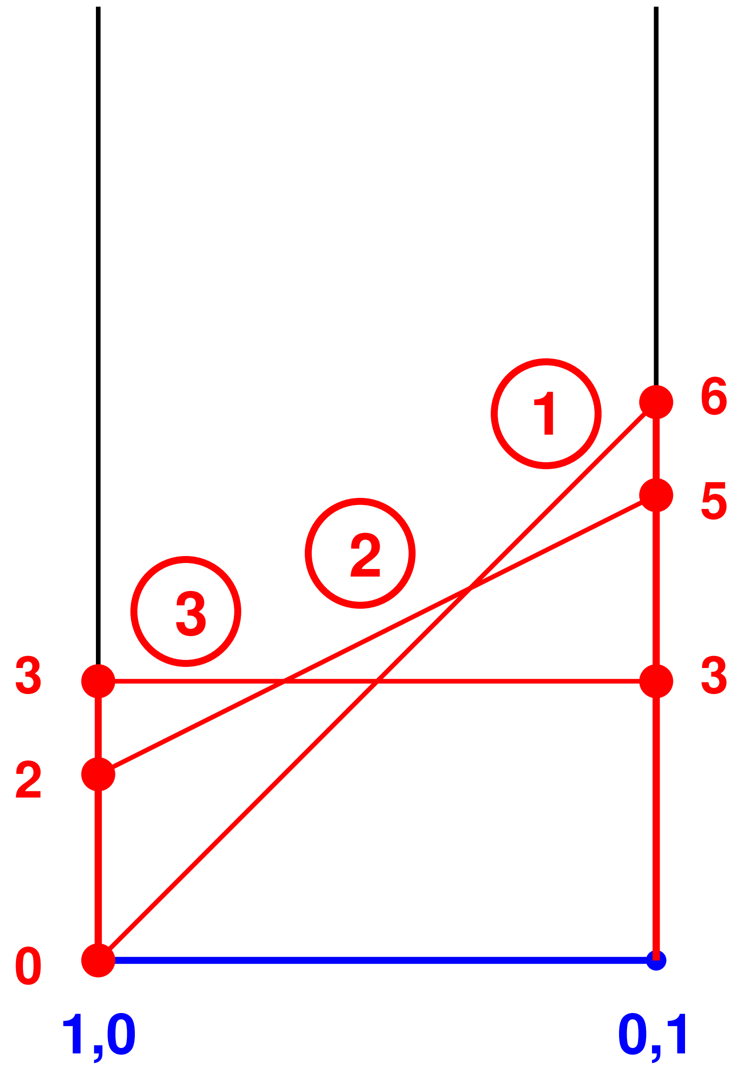


	4	5	
1	0	6	
2	2	5	= A
3	3	3	

payoffs to player I

Best responses to mixed strategy of player 2



	4	5
1	0	6
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= A

payoffs to
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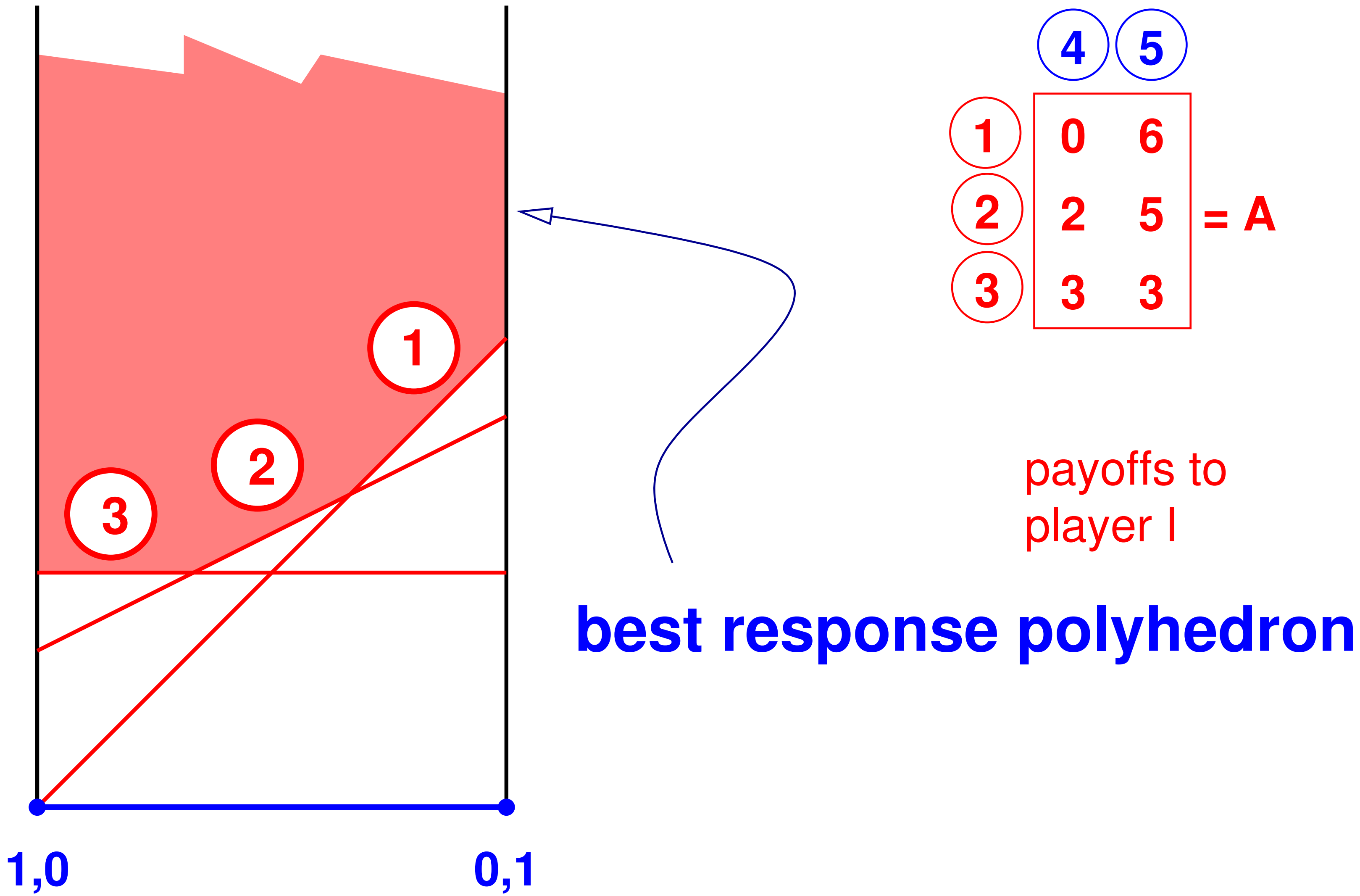
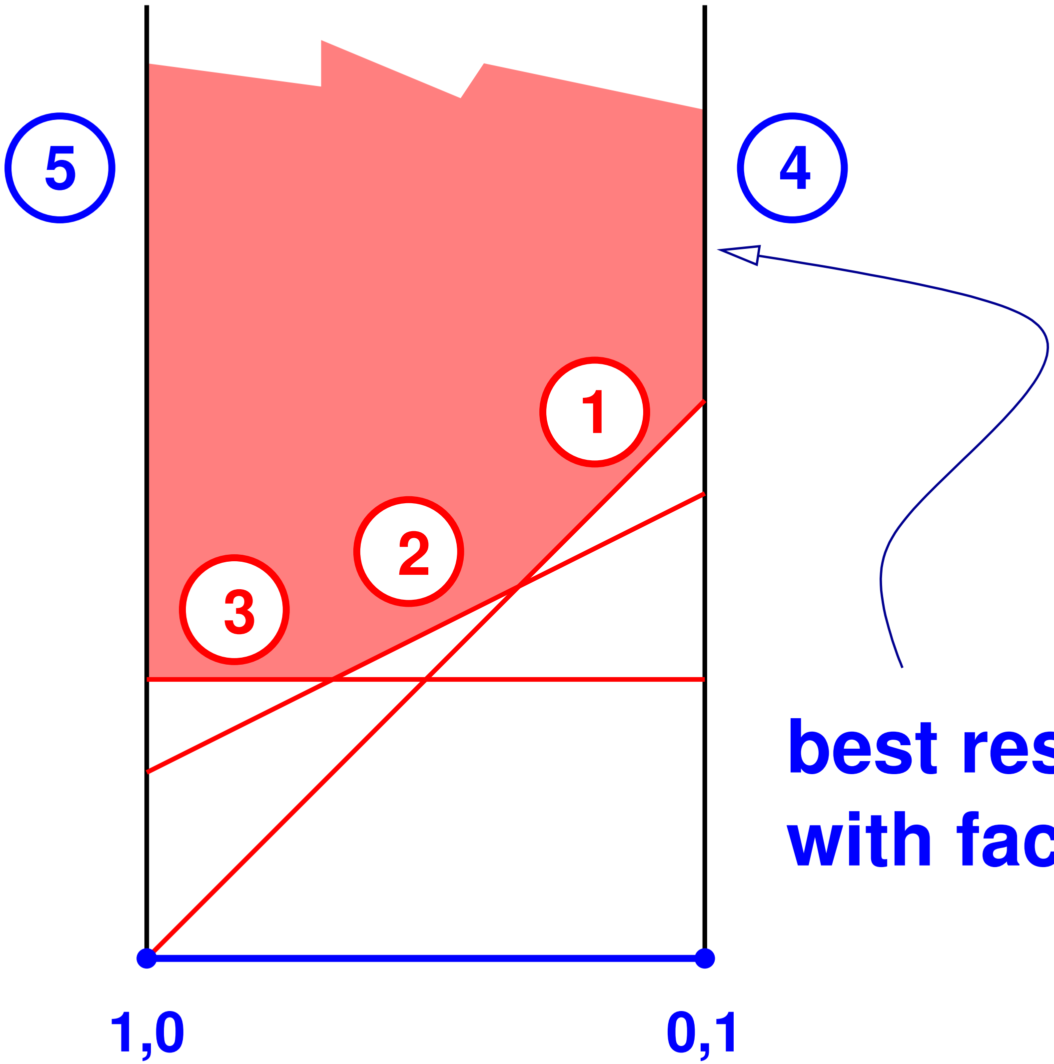
Best responses to mixed strategy of player 2



Best responses to mixed strategy of player 2



	4	5
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2	2	5
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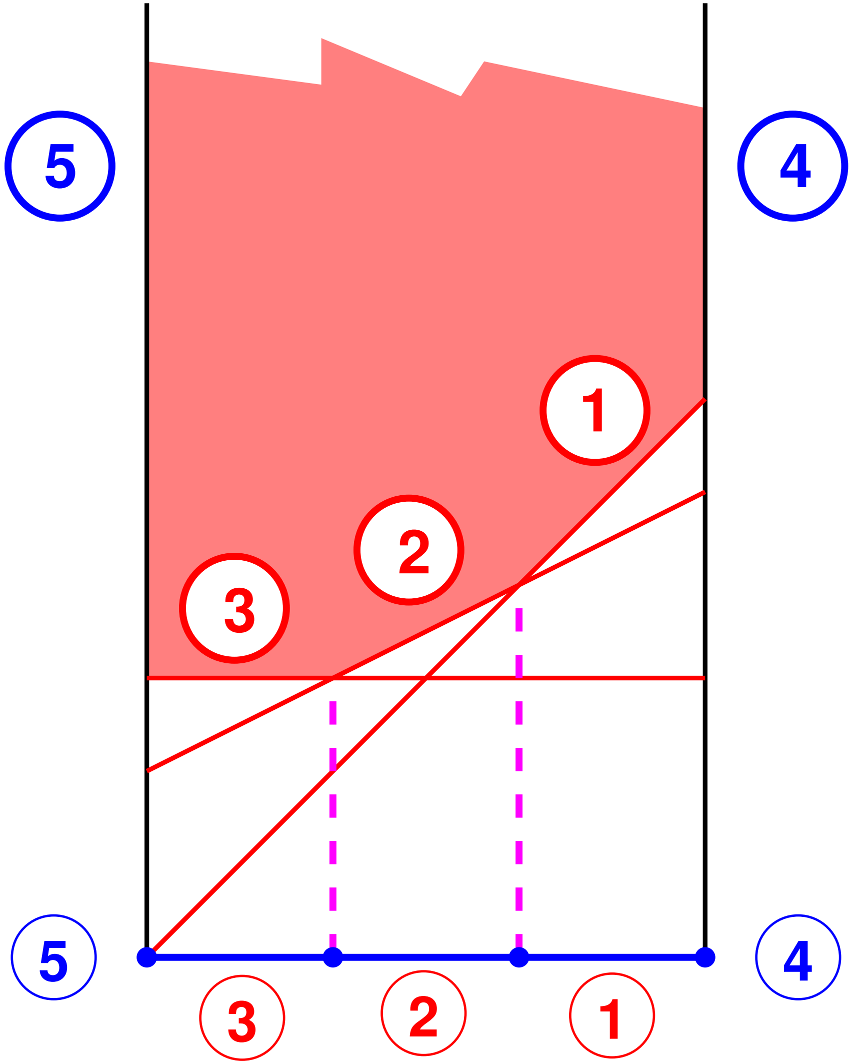
= A

payoffs to player I

best response polyhedron with facet labels

Best responses to mixed strategy of player 2



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payoffs to player I

Best responses to mixed strategy of player 2

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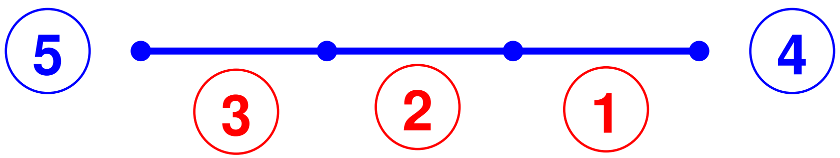
payoffs to
player I

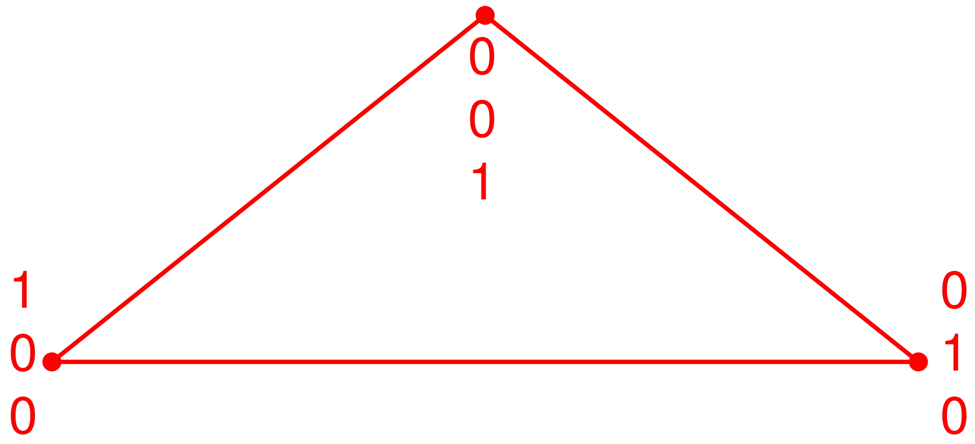


Best responses to mixed strategy of player 1

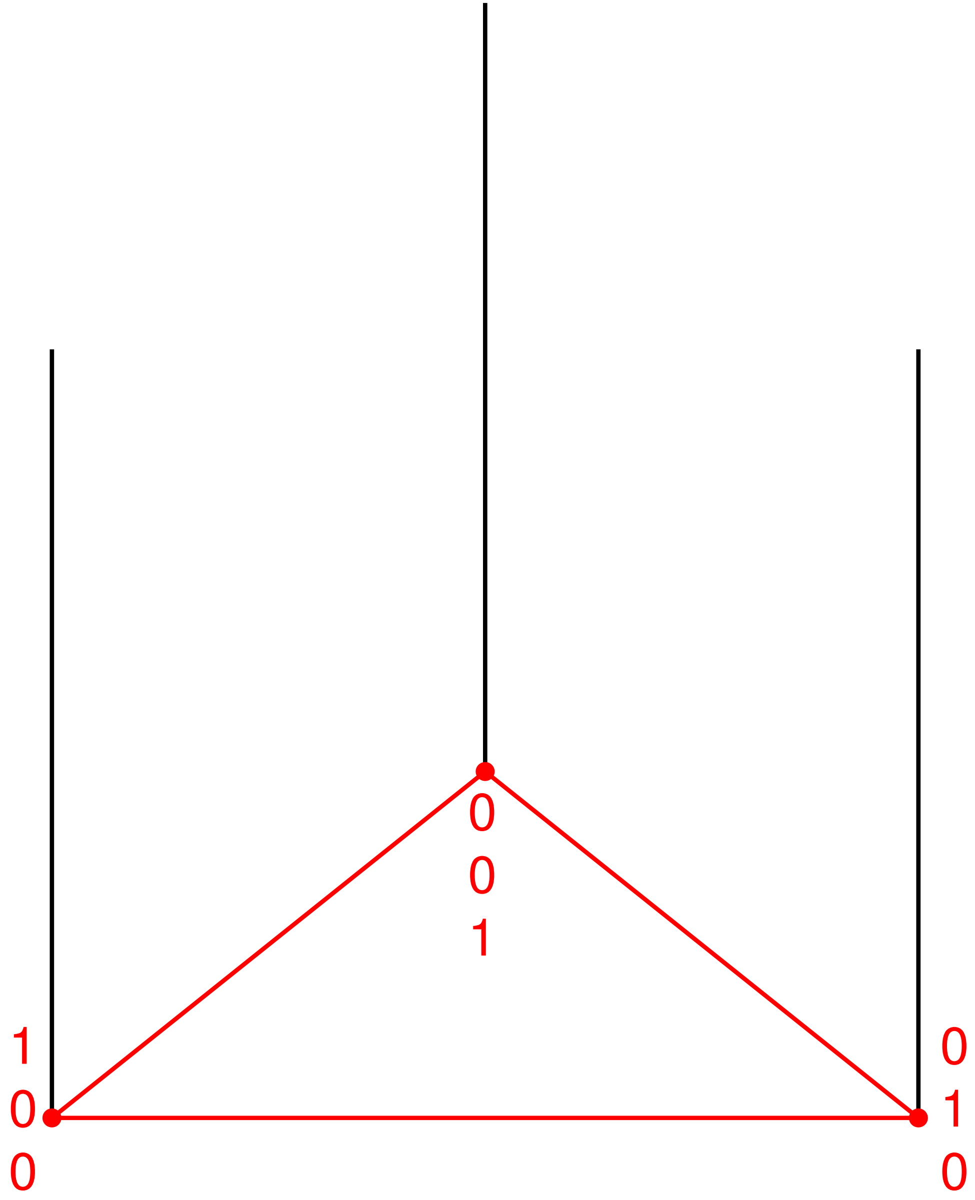
	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to player II



Best responses to mixed strategy of player 1



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= B

payoffs to
player II

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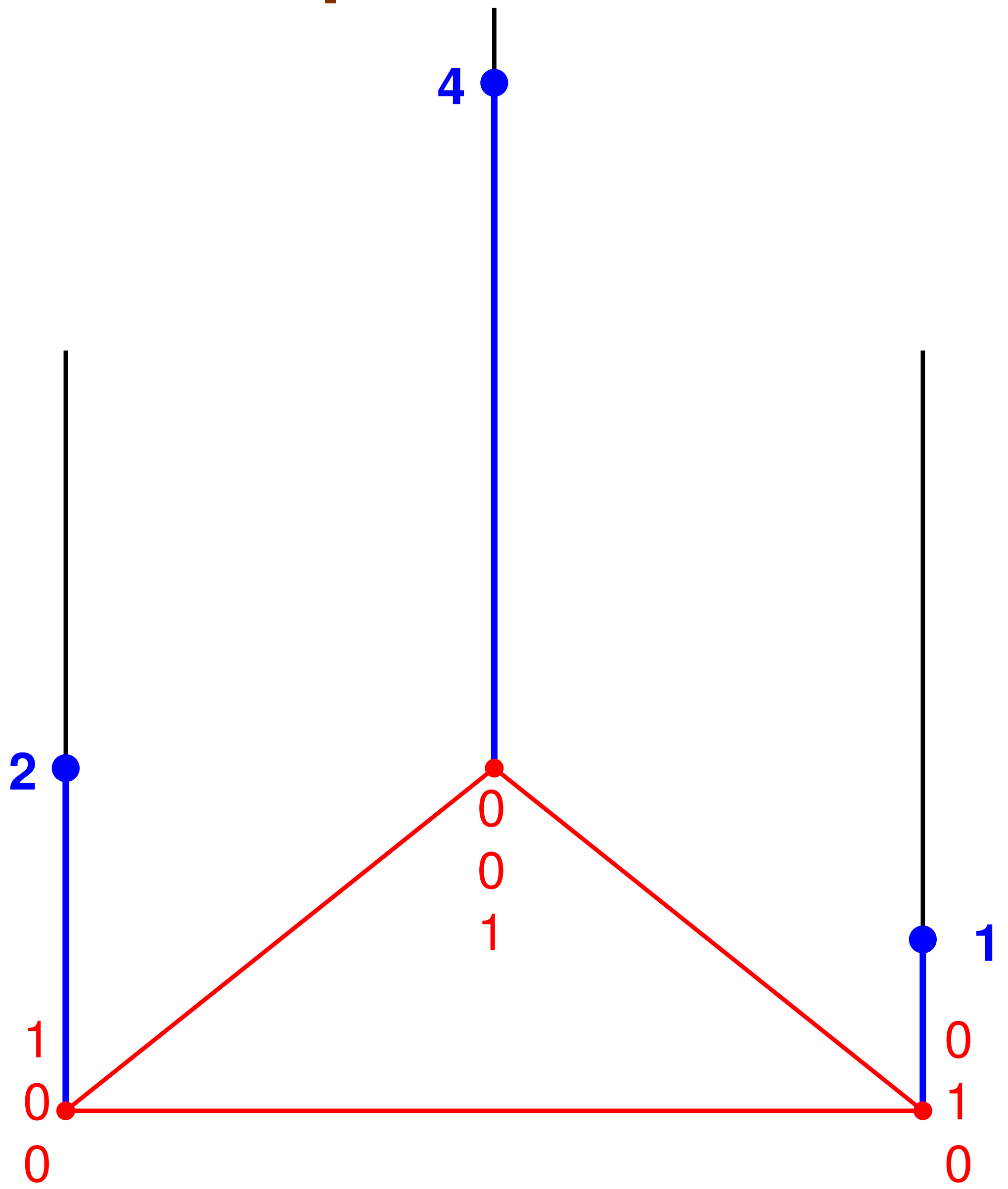

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payoffs to
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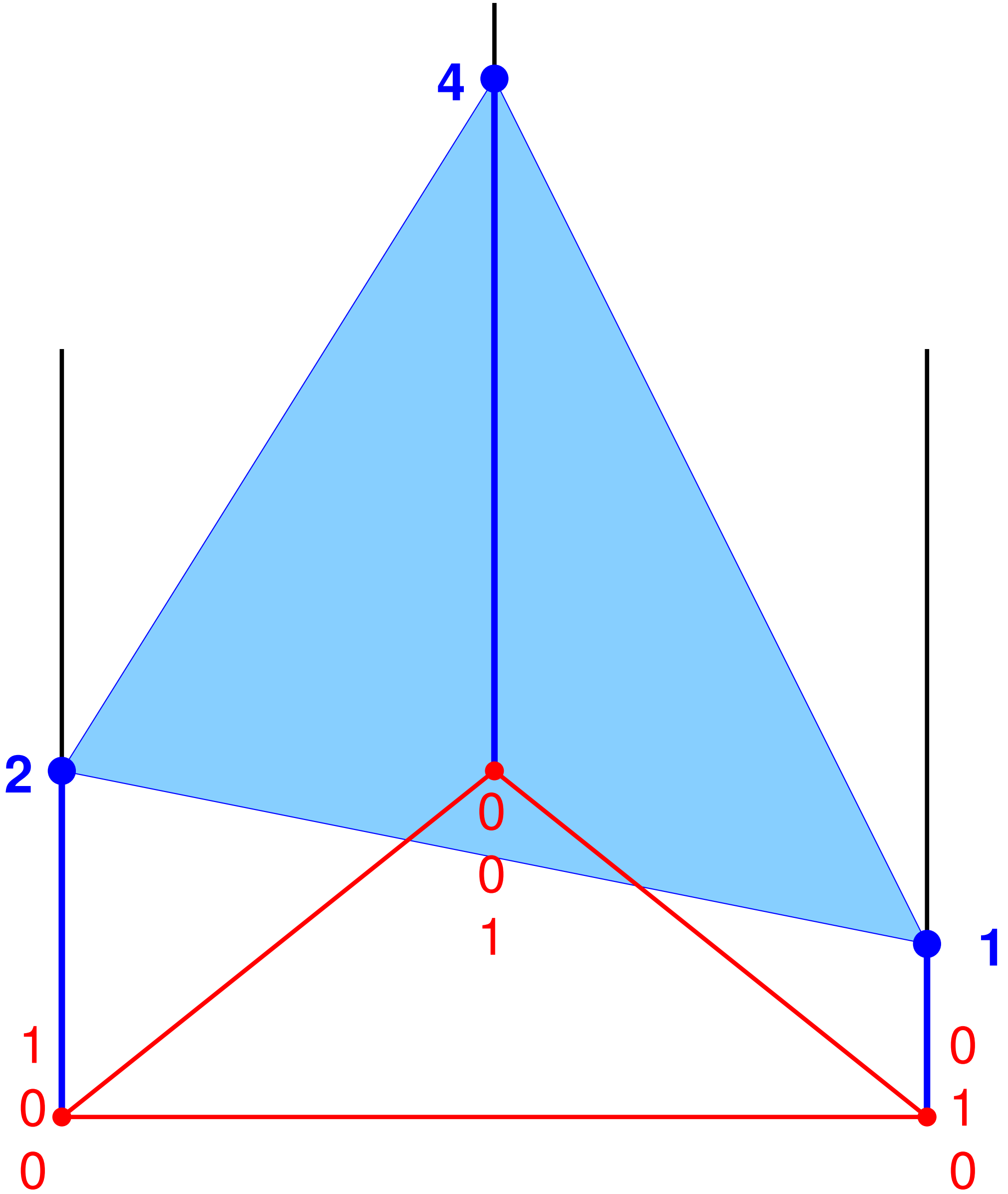

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payoffs to
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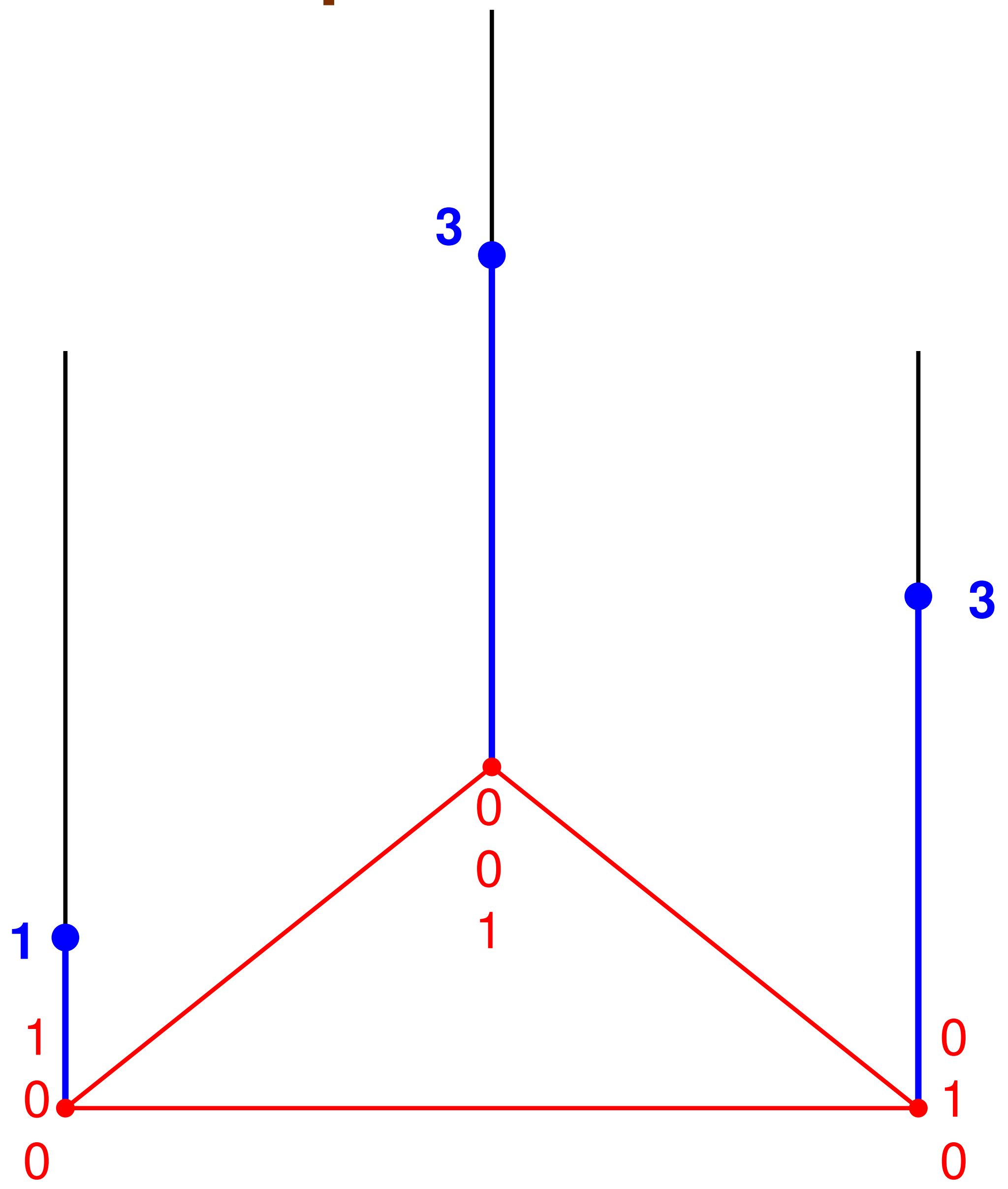

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payoffs to player II

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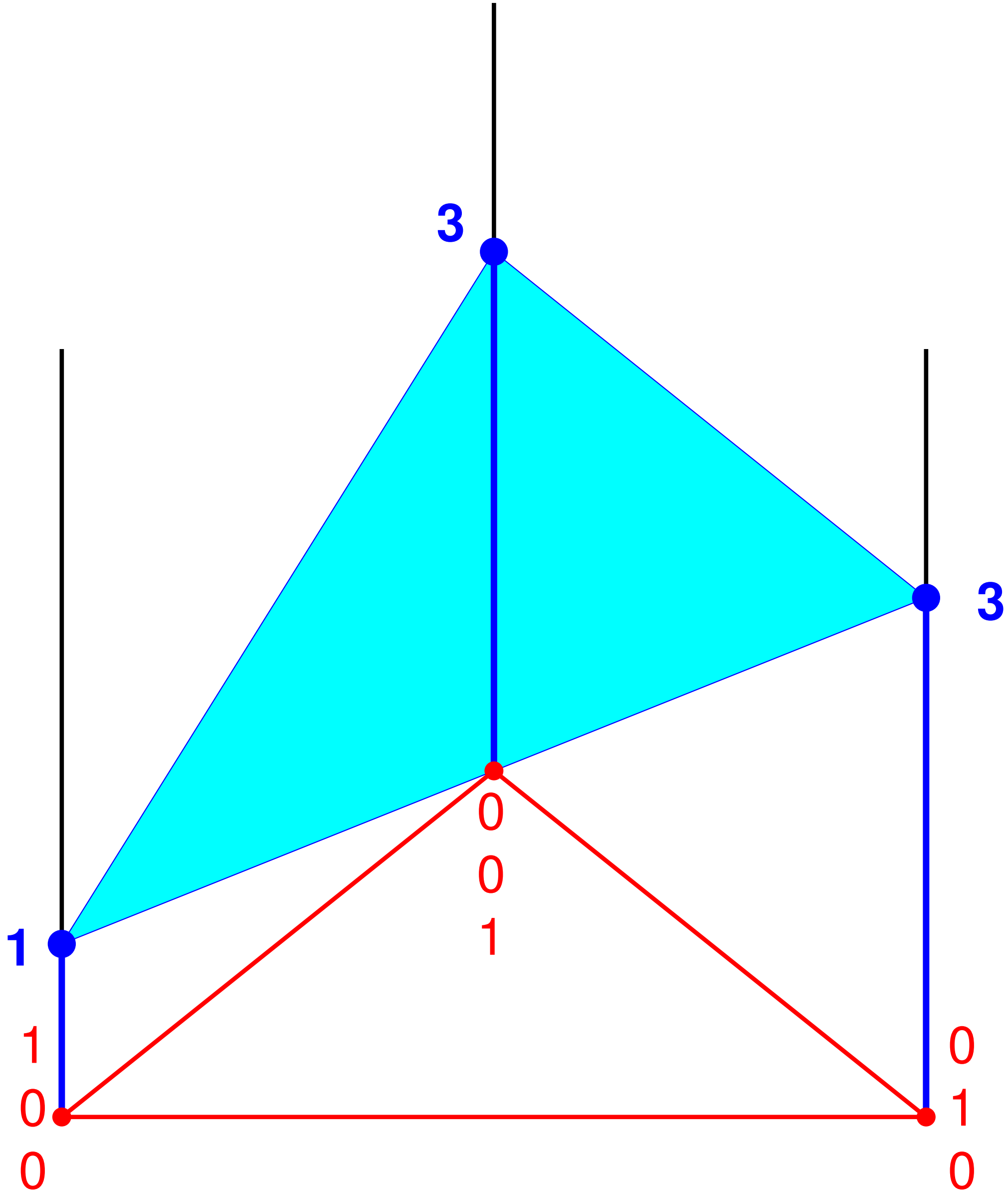

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= B

payoffs to
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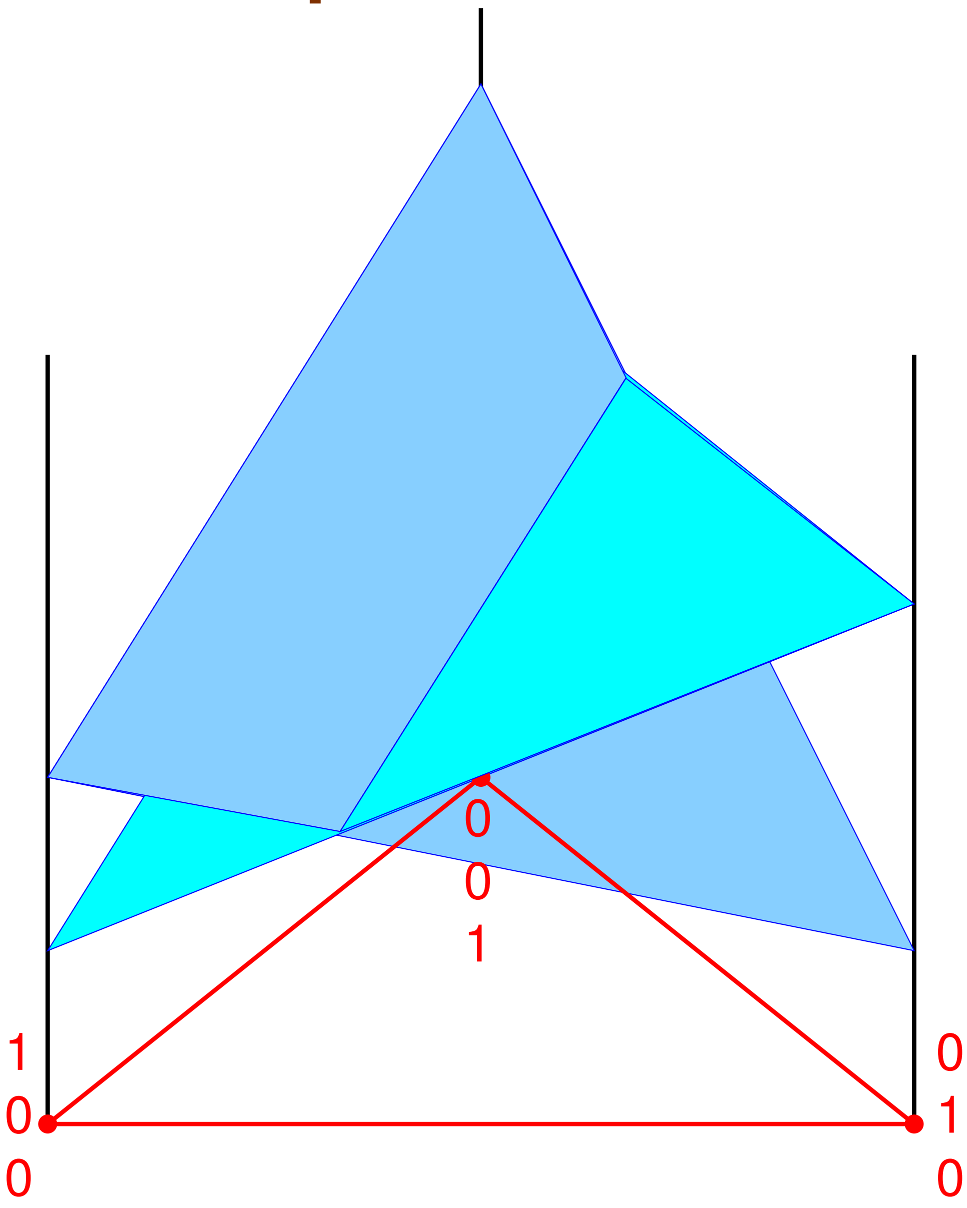

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payoffs to
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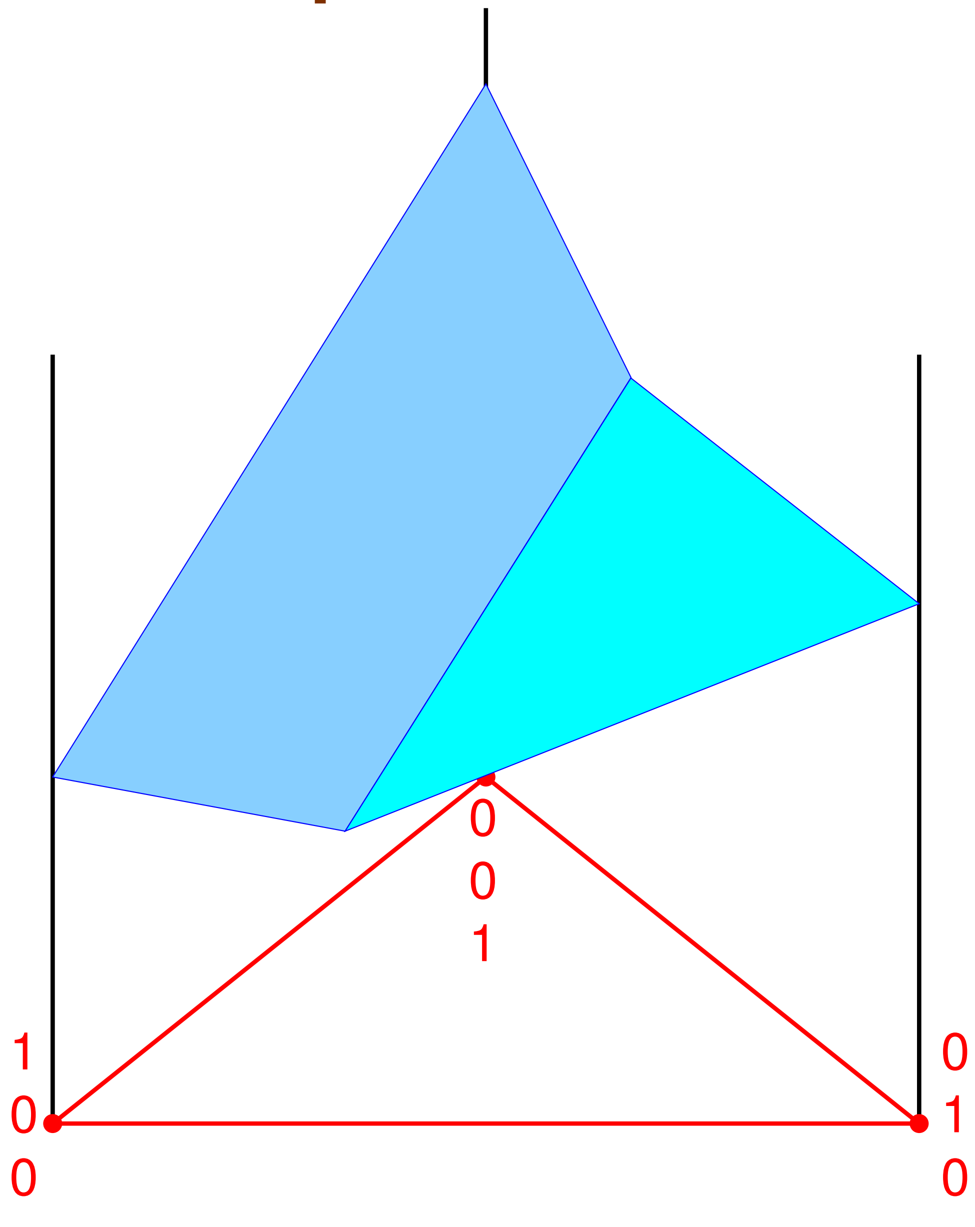
Best responses to mixed strategy of player 1



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payoffs to
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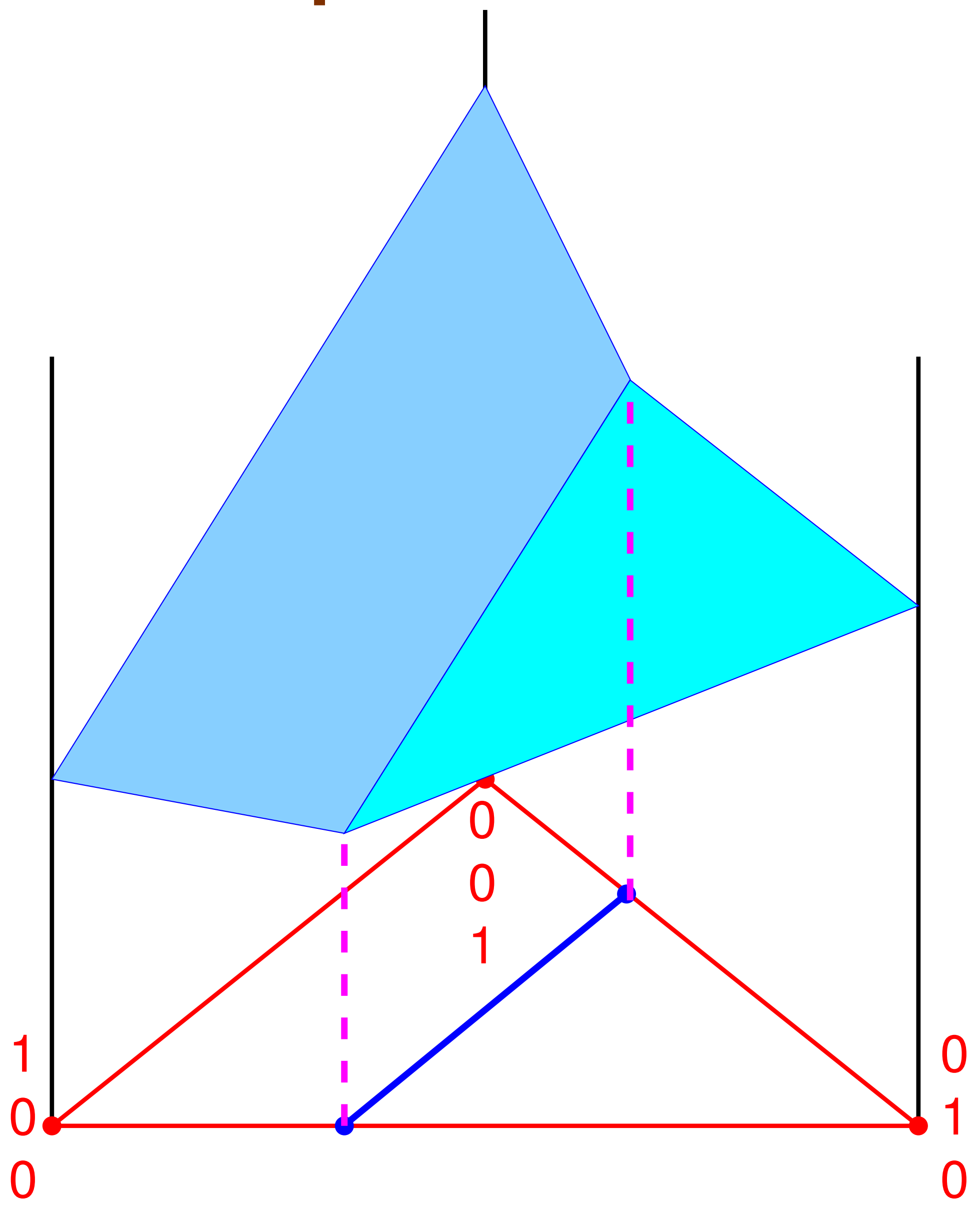

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payoffs to
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Best responses to mixed strategy of player 1

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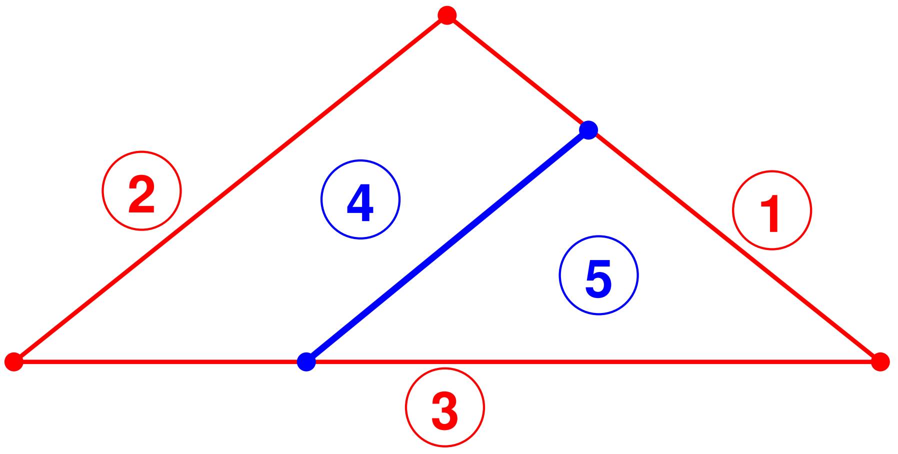
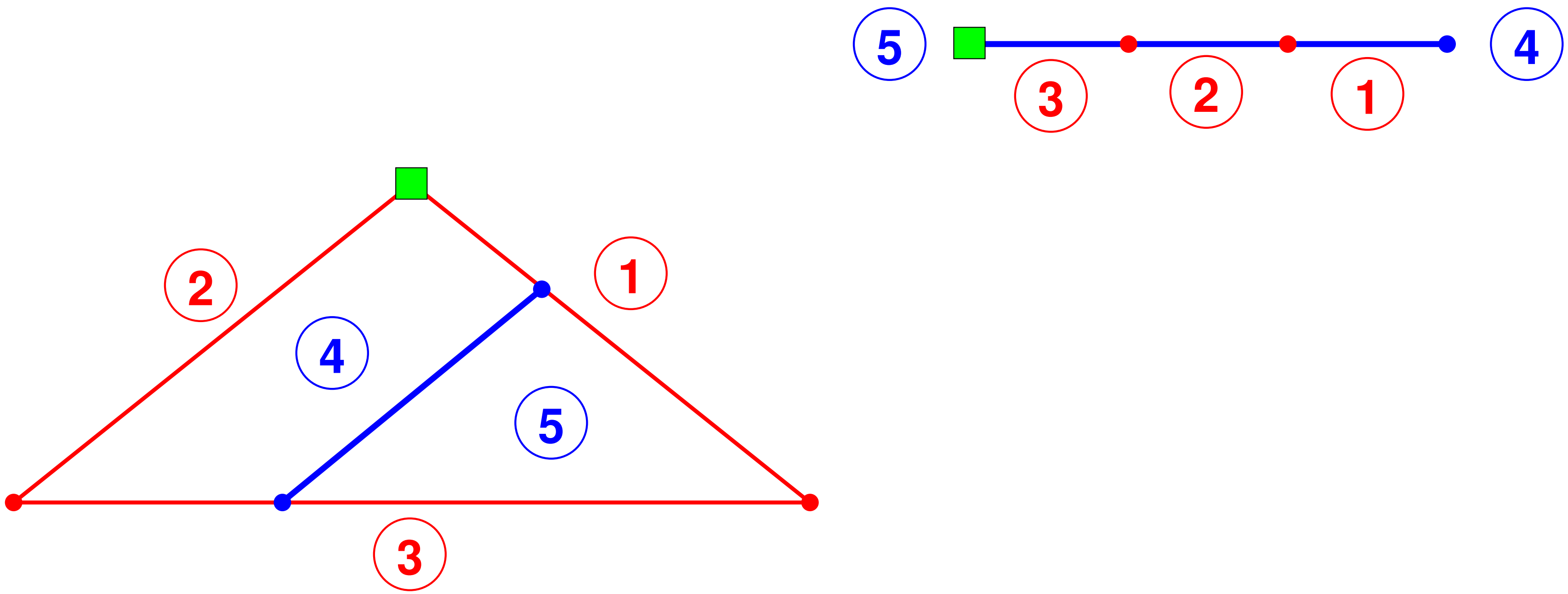
= B

payoffs to
player II



Equilibrium = completely labeled strategy pair



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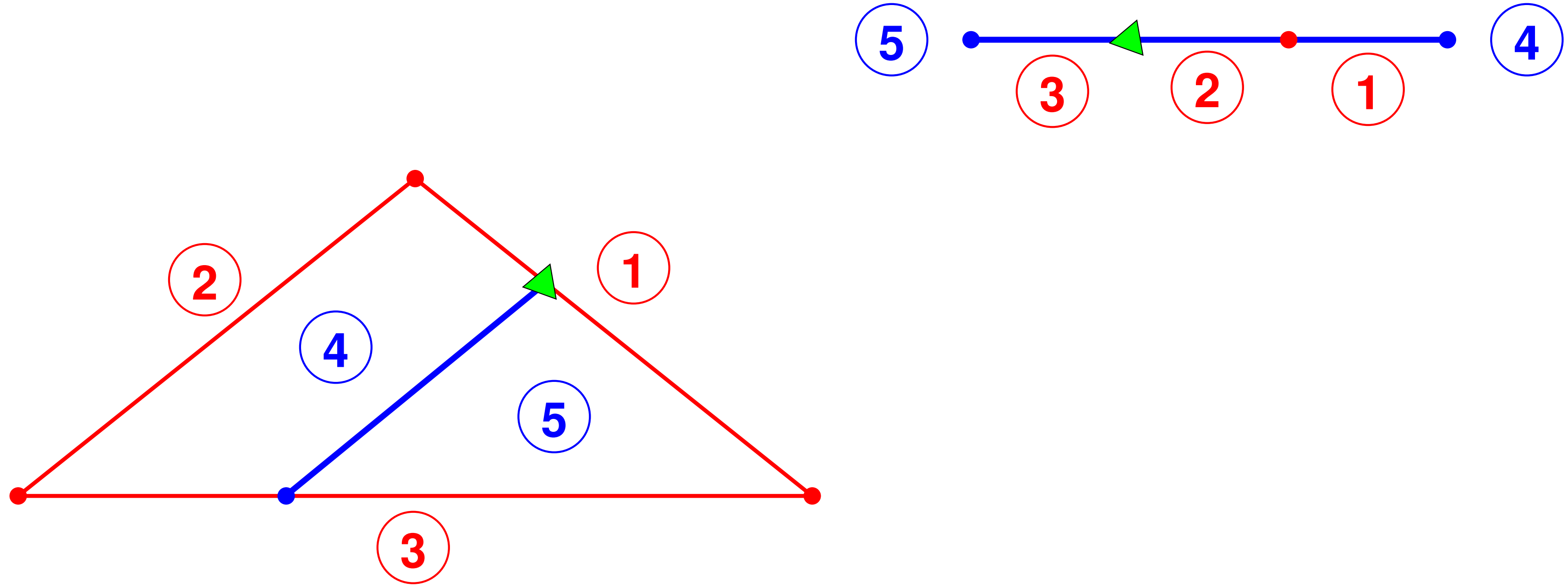

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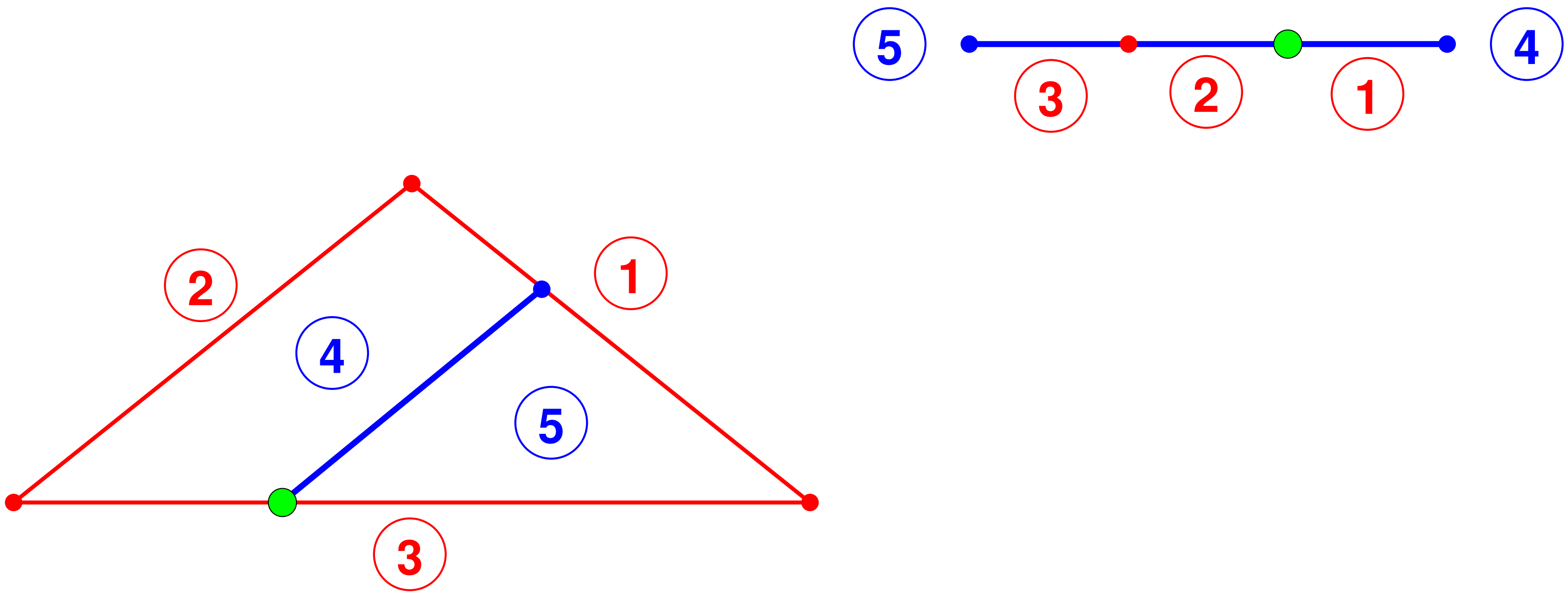


Nondegenerate bimatrix games

Given: $m \times n$ bimatrix game (A,B)

$X = \{ x \in \mathbf{R}^m \mid x \geq \mathbf{0},\ x_1 + \ldots + x_m = 1 \}$

$Y = \{ y \in \mathbf{R}^n \mid y \geq \mathbf{0},\ y_1 + \ldots + y_n = 1 \}$

$\mathrm{supp}(x) = \{ i \mid x_i > 0 \}$

$\mathrm{supp}(y) = \{ j \mid y_j > 0 \}$

(A,B) **nondegenerate** $\Leftrightarrow$ $\forall\, x \in X,\ y \in Y$:

$| \{ j \mid j$ best response to $x \} | \leq | \mathrm{supp}(x) |$,

$| \{ i \mid i$ best response to $y \} | \leq | \mathrm{supp}(y) |$.

Nondegeneracy via labels

$m \times n$ bimatrix game (A,B) **nondegenerate**

$\Leftrightarrow$ no $x \in X$ has more than m labels,
no $y \in Y$ has more than n labels.

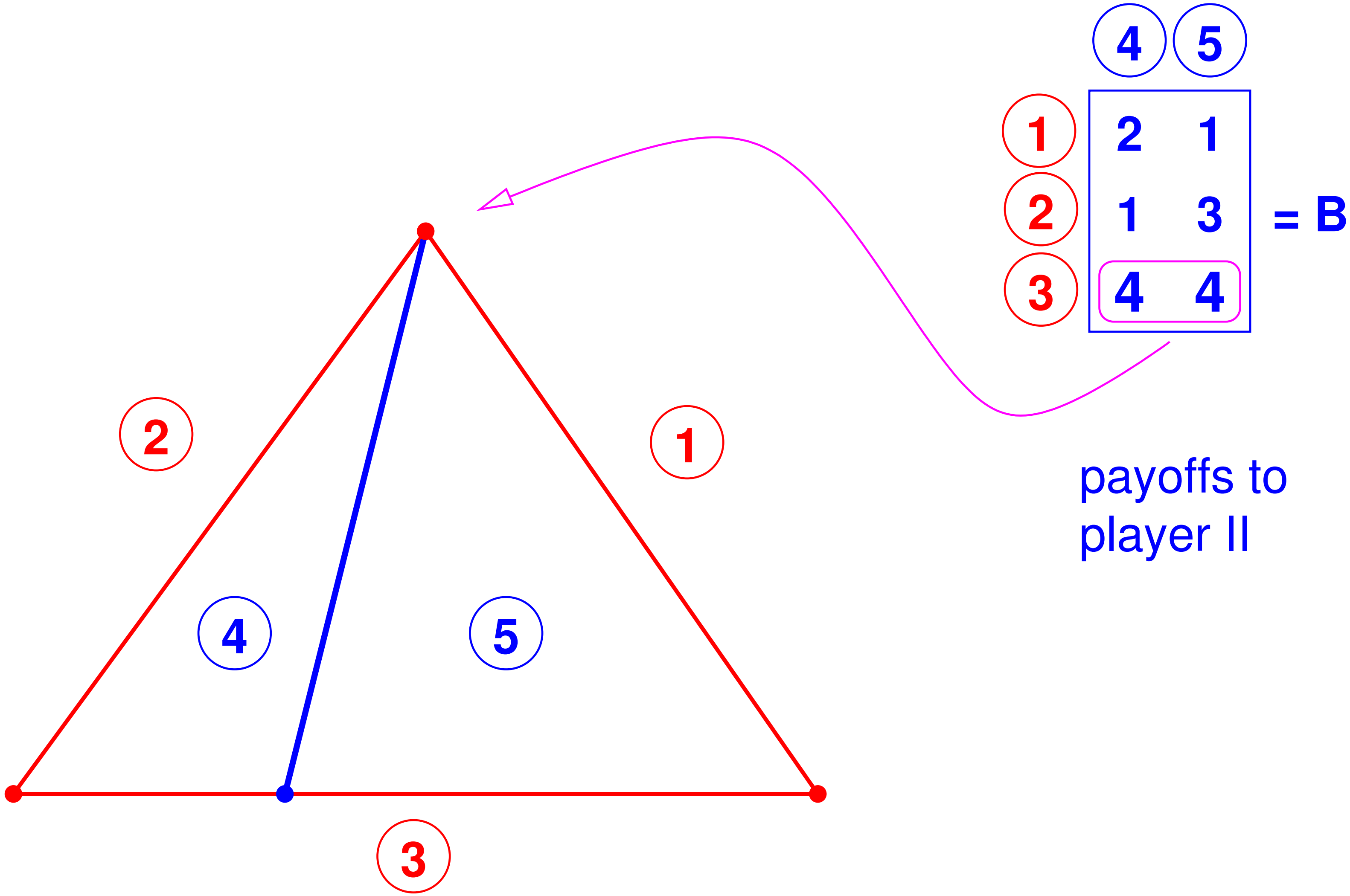
E.g. x with $> m$ labels,
s labels from $\{1, \ldots, m\}$,

$\Rightarrow$ $> m-s$ labels from $\{m+1, \ldots, m+n\}$

$\Leftrightarrow$ $> |\text{supp}(x)|$ best responses to x.

$\Rightarrow$ degenerate.

Example of a degenerate game



Making equilibria unique

Given:

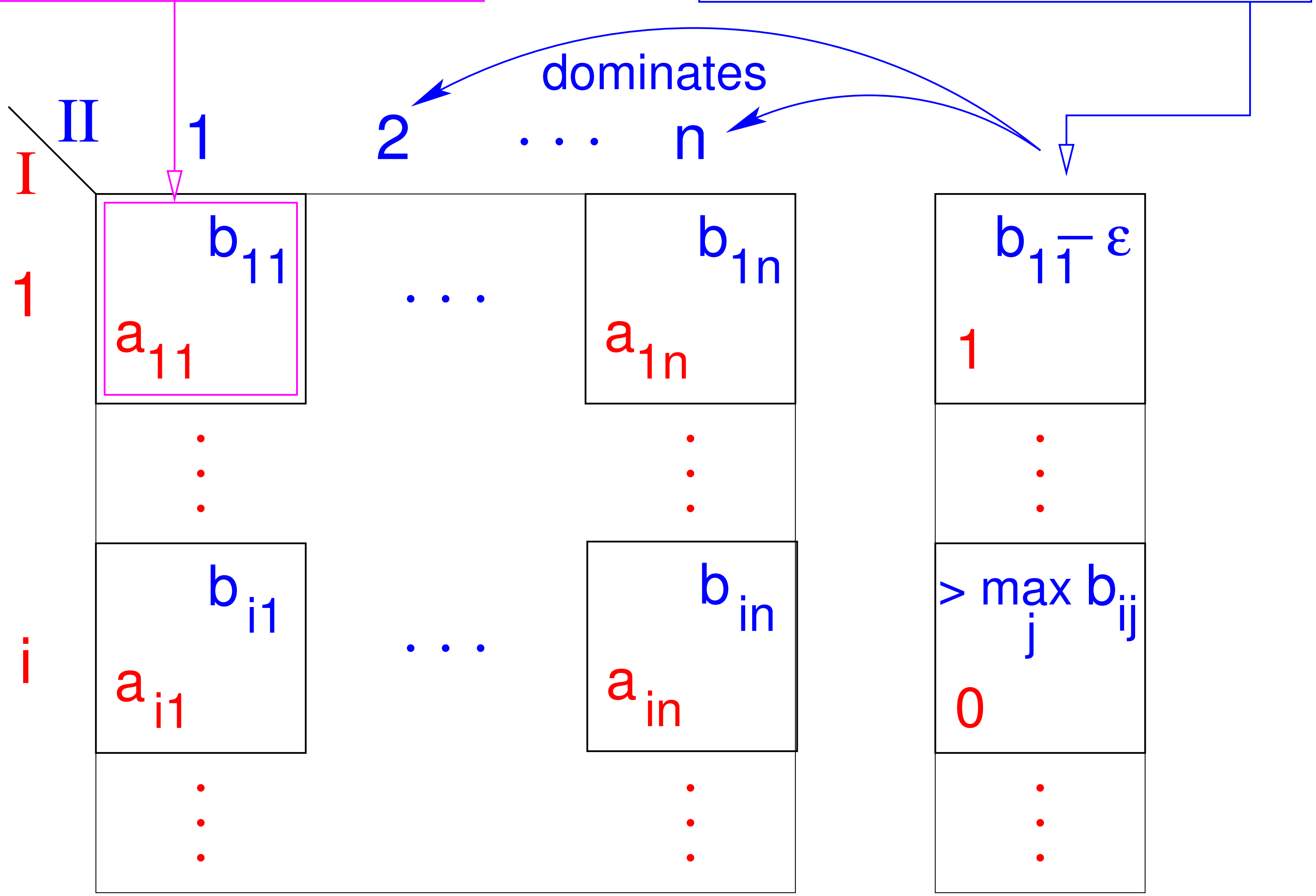
nondegenerate (A,B), Nash equilibrium (x,y).

Question:

∃ game G *extending* (A,B) by adding strategies so that (x,y) is the **unique** equilibrium of G?

e.g.: G obtained from (A,B) by adding *columns*, (x,y) becoming (x,[y, 0,0,...,0]) for G.

Pure equilibrium: need one extra column



Strategic characterization of the index

We will show a conjecture by Josef Hofbauer:

Theorem:

For nondegenerate (A,B), Nash equilibrium (x,y):

index $(x,y) = +1$

$\Leftrightarrow$ $\exists$ game G *extending* (A,B)
so that (x,y) is the **unique** equilibrium of G.

suffices: G obtained from (A,B) by adding *columns*,
(x,y) becoming $(x,[y, 0,0,\ldots,0])$ for G.

Sub-matrices of equilibrium supports

Given: nondegenerate (A,B), $A>0$, $B>0$,
Nash equilibrium (x,y).

$A = (a_{ij})$, $B = (b_{ij})$

$A_{xy} = (a_{ij})$ $i \in supp(x), j \in supp(y)$

$B_{xy} = (b_{ij})$ $i \in supp(x), j \in supp(y)$

A_{xy}, B_{xy} have **full rank** $|supp(x)|$,
nonzero determinants.

Index of an equilibrium (Shapley 1974)

Given: nondegenerate (A,B), $A>0$, $B>0$,
Nash equilibrium (x,y).

$$\text{Index}(x,y) = -\,\text{sign det} \begin{bmatrix} 0 & A_{xy} \\ B_{xy}^T & 0 \end{bmatrix}$$

$$= -\,\text{sign det}(A_{xy})\,\det(B_{xy})\,(-1)^{|\text{supp}(x)|}$$

$$\in \{+1, -1\}$$

Example: Matching Pennies

I \ II	*l*	*r*
T	2, 1	1, 2
B	1, 2	2, 1

$$-\ \text{sign}\ \det\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\ \det\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\ (-1)^2 = +1$$

Mixed equilibrium in Battle of Sexes

I \ II	*l*	*r*
T	2, 3	1, 1
B	1, 1	3, 2

$$-\ \text{sign}\ \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} (-1)^2 = -1$$

Properties of the index

- independent of
  - positive constant added to all payoffs
  - order of pure strategies
  - pure strategy payoffs outside equilibrium support

- pure-strategy equilibria have index $+1$

- sum of indices over all equilibria is $+1$

- the two endpoints of any *Lemke-Howson path* are equilibria of opposite index.

New "dual" construction

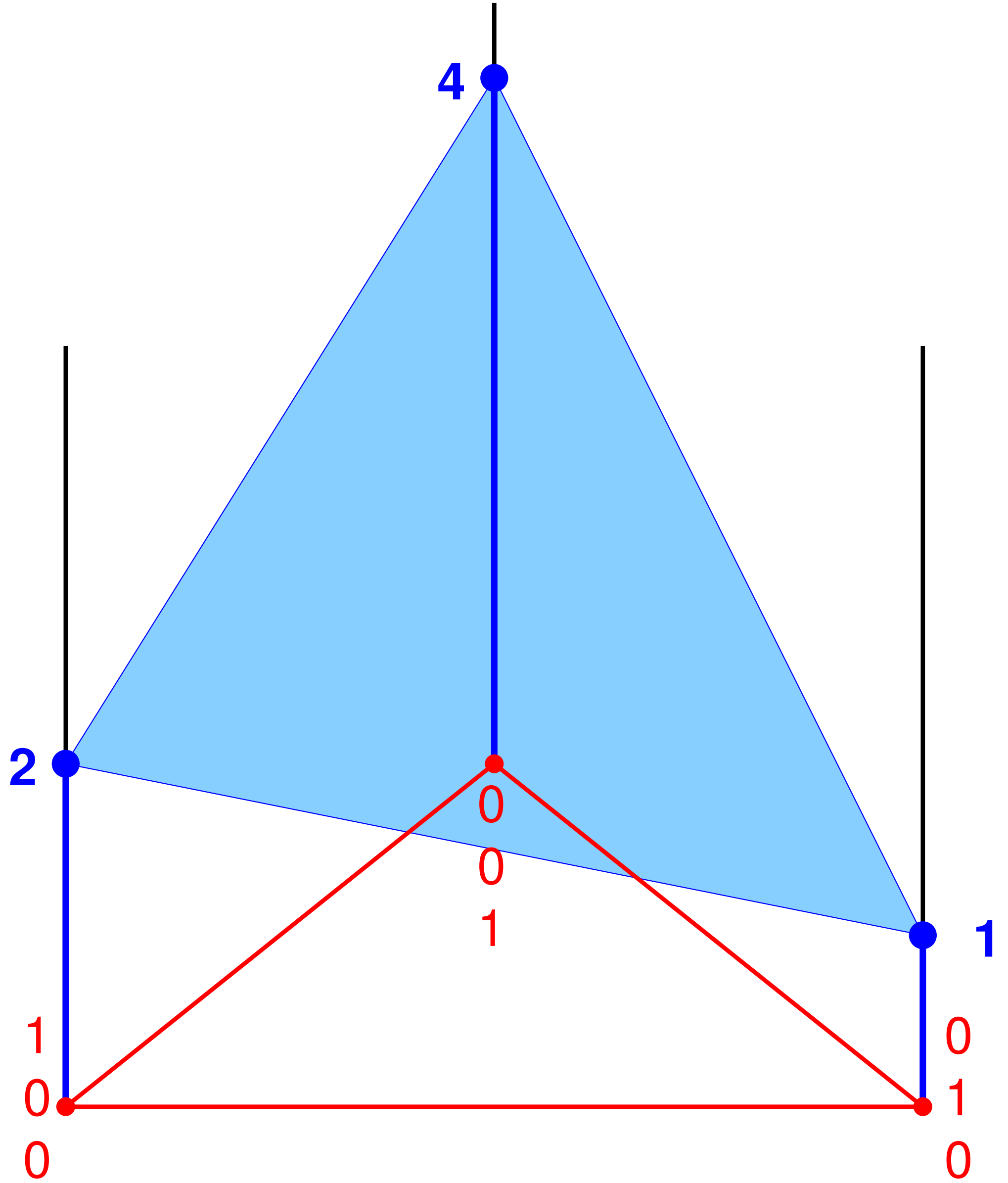
Given: nondegenerate $m \times n$ game (A,B), $m \leq n$.

- X subdivided into best response regions
- **dualize** X: best response **regions** for $\mathbf{j}$ $\rightarrow$ **points** $\mathbf{j}^{\Delta}$

  "unplayed strategy" facets of X $\rightarrow$ large unit vectors

  technical construction: "dual polytopes"
- **vertices** x of regions become **simplices** x^{Δ}

Best responses to mixed strategy of player 1

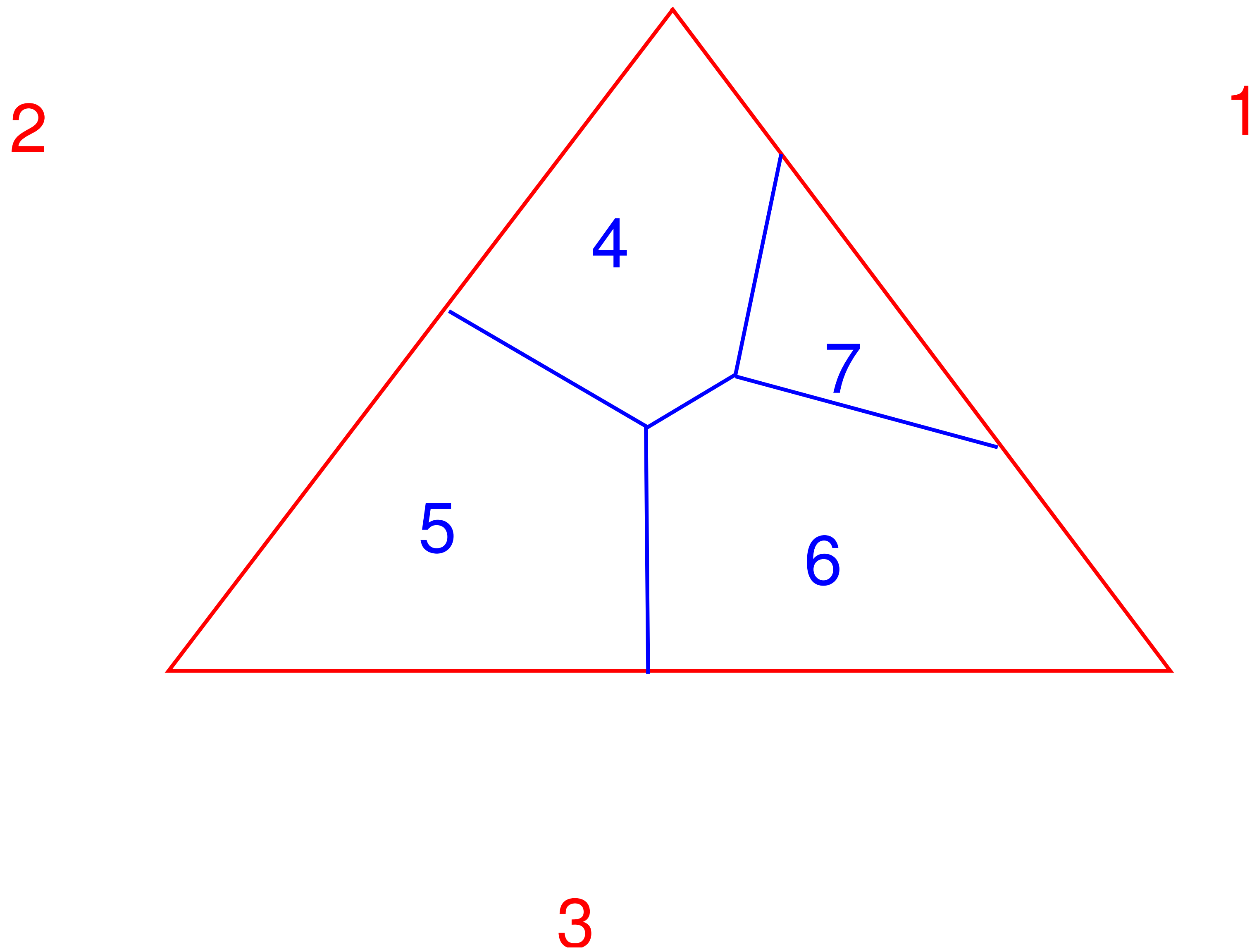


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Construction of X^{Δ}



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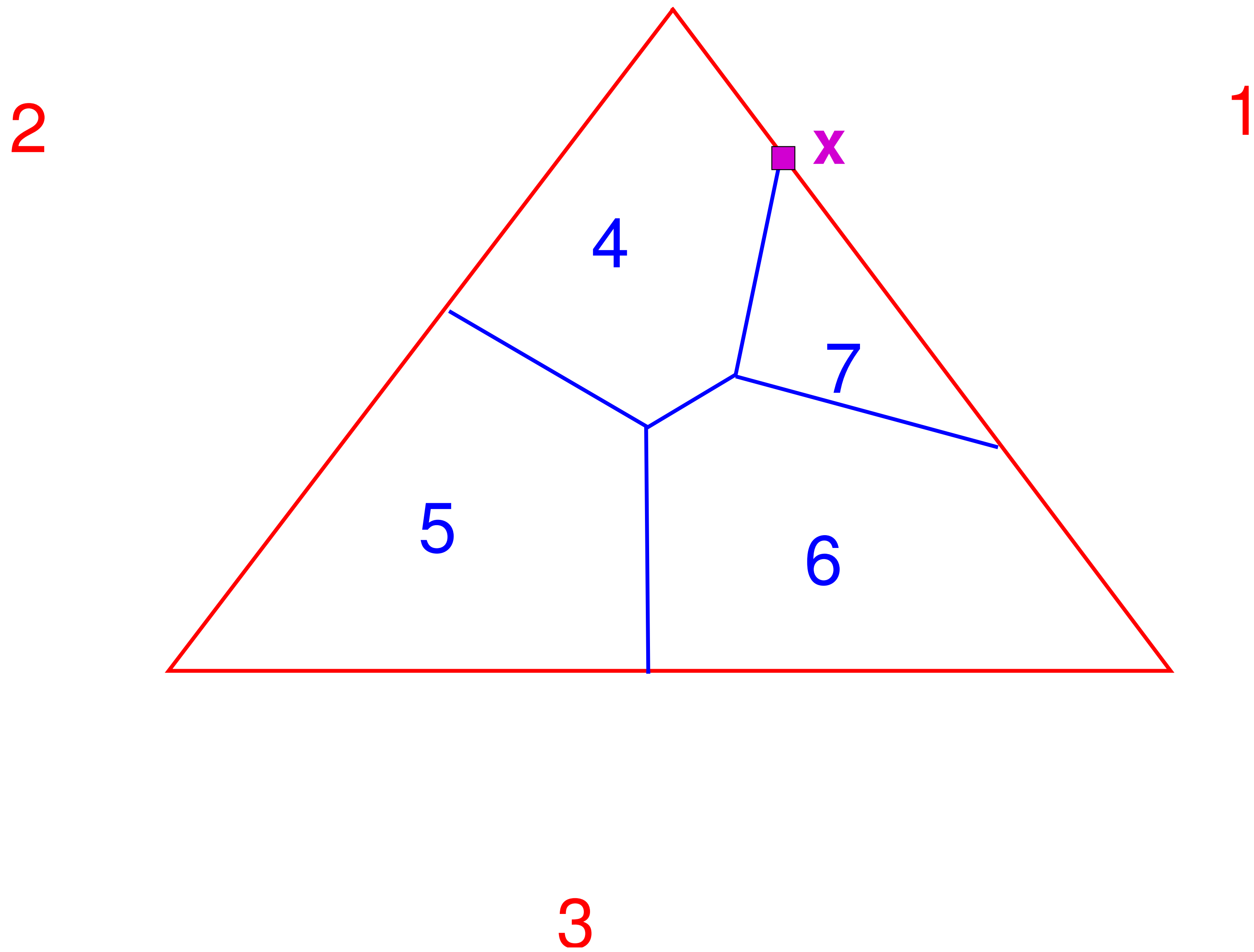

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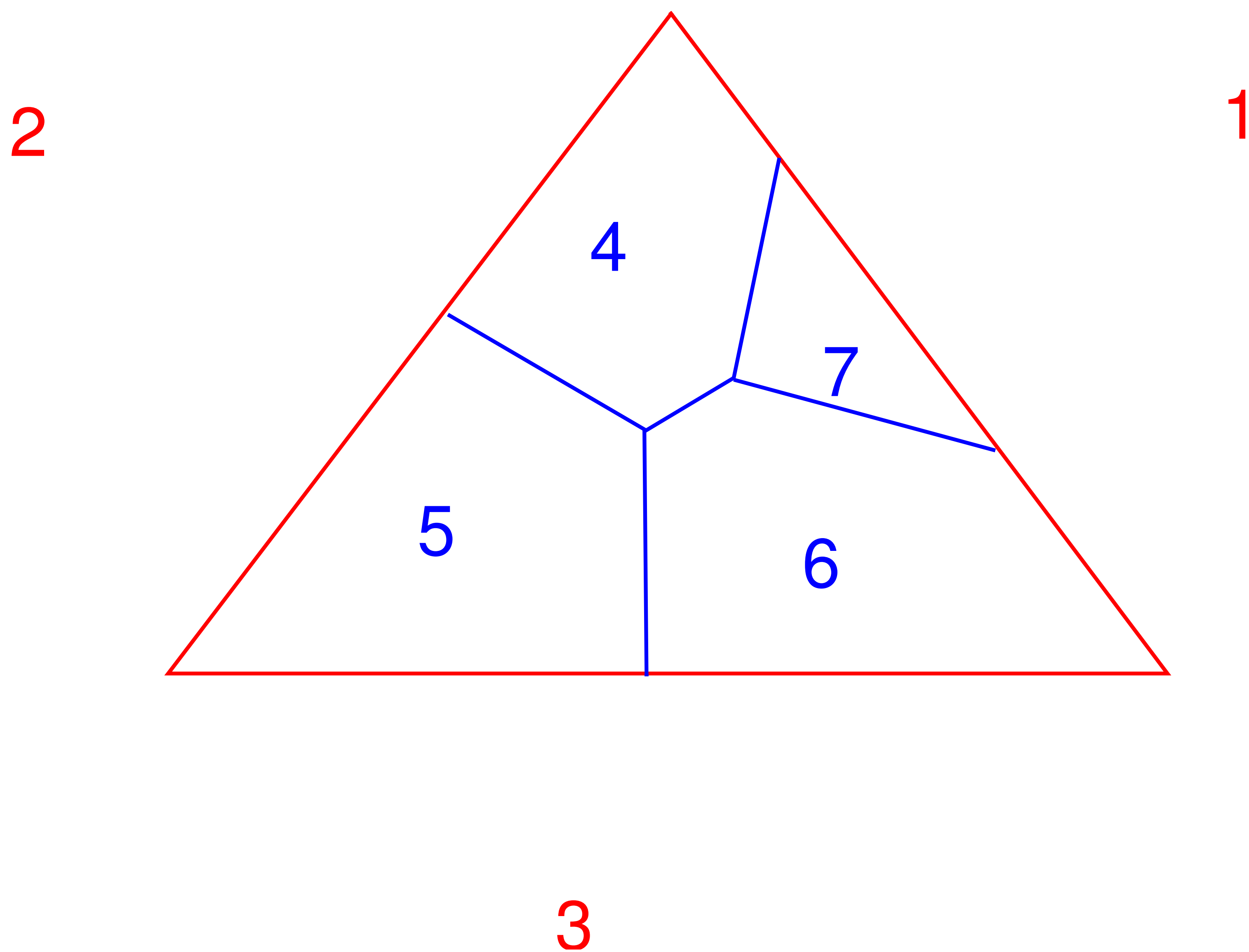
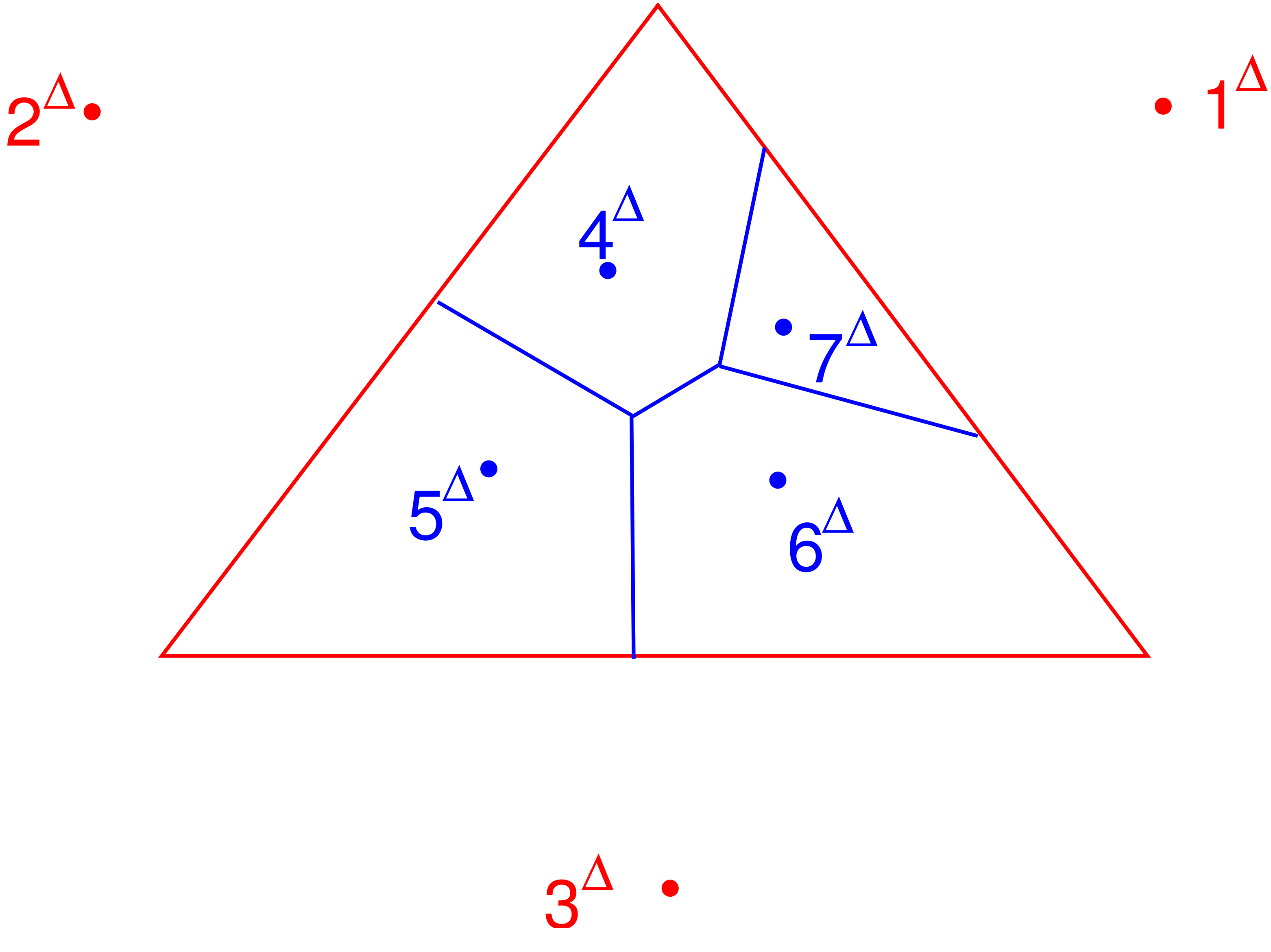
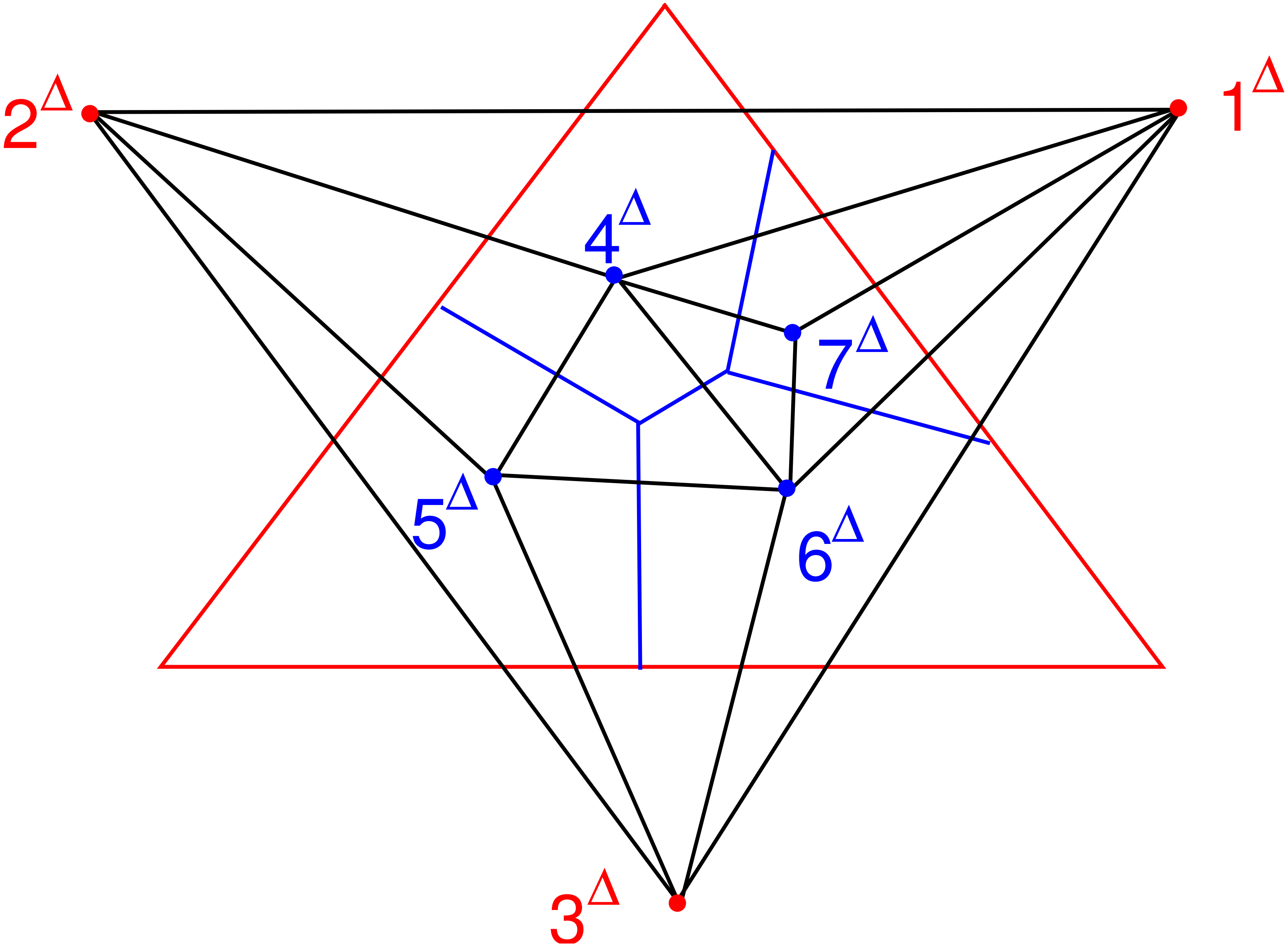


Construction of X^{Δ}



Construction of X^{Δ}



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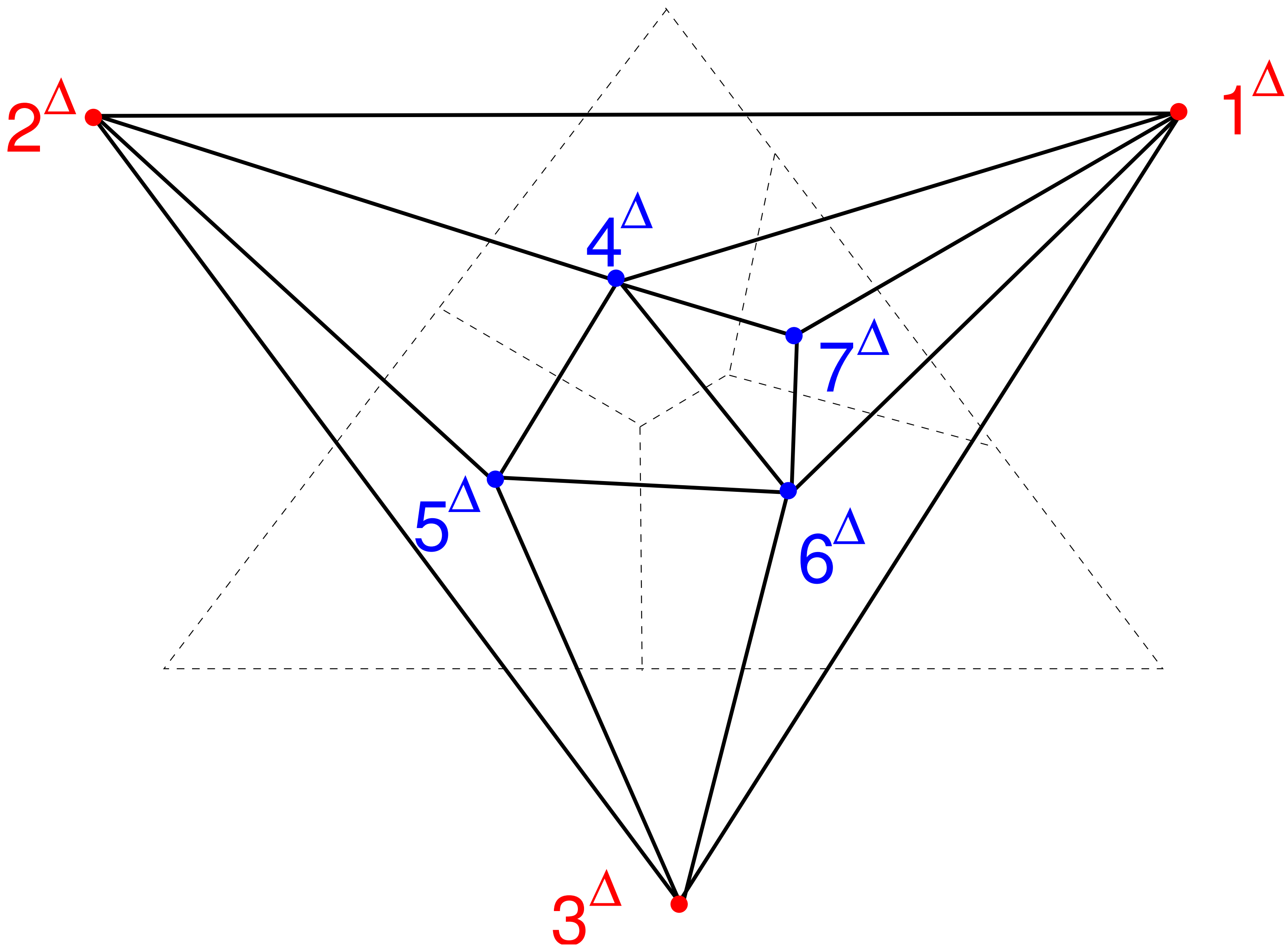
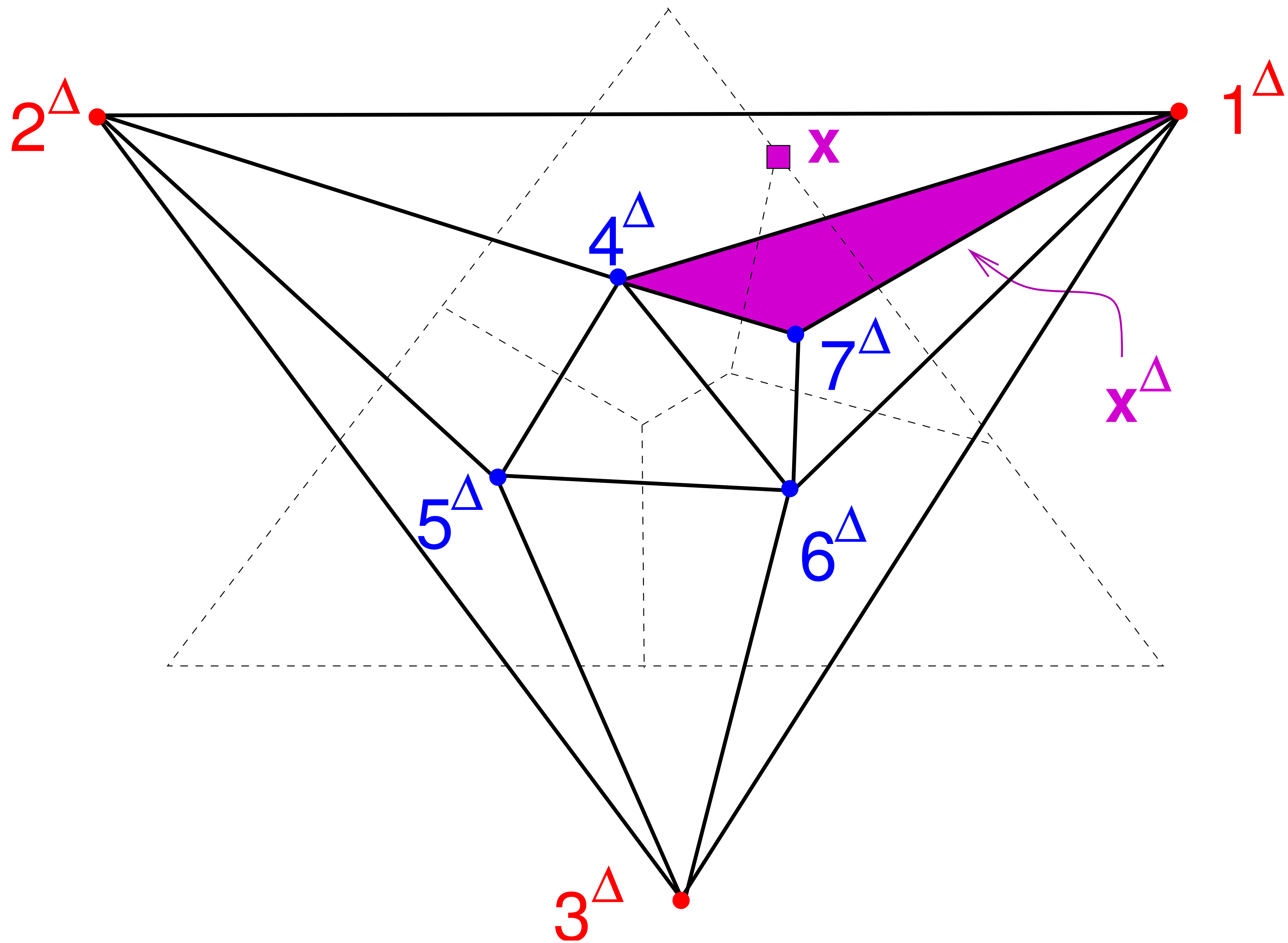


Construction of X^{Δ}



Incorporating the other player

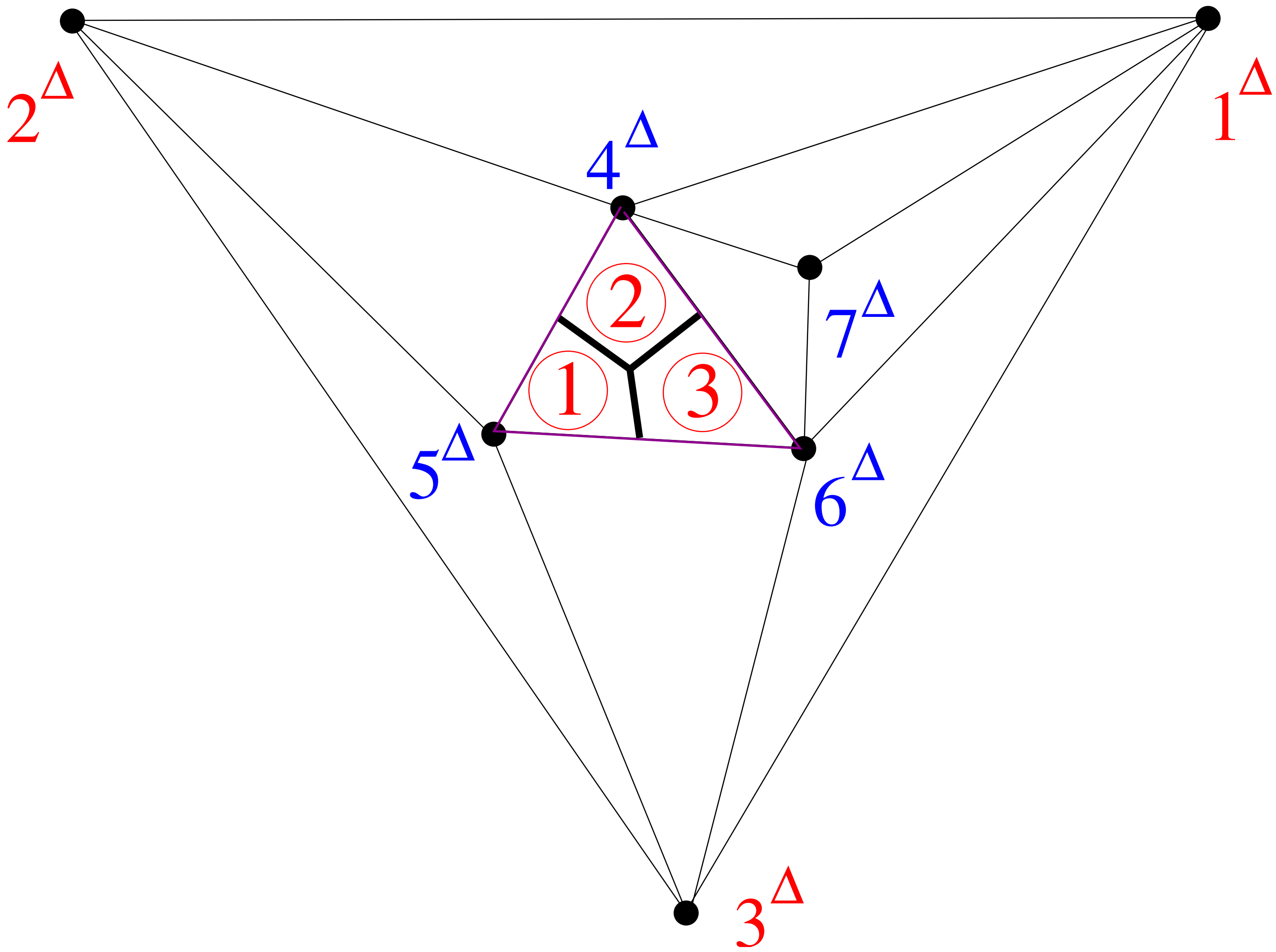
So far: X^Δ subdivided, dualized
according to player **II**'s payoffs B

Now: for each vertex x of a best response region,
with labels $k > m$: best response of player **II**
or $k \le m$: unplayed strategy of player **I**

- subdivide x^Δ into regions of player **I**'s best responses
  where – if $k > m$: use column k of A
  – if $k \le m$: player **I** "as if" playing k
  (artificial unit vector payoff),

⇒ picture X^Δ with labels 1...m *only*, equilibria: **all** labels.

subdivide x^{Δ} via player I's best responses

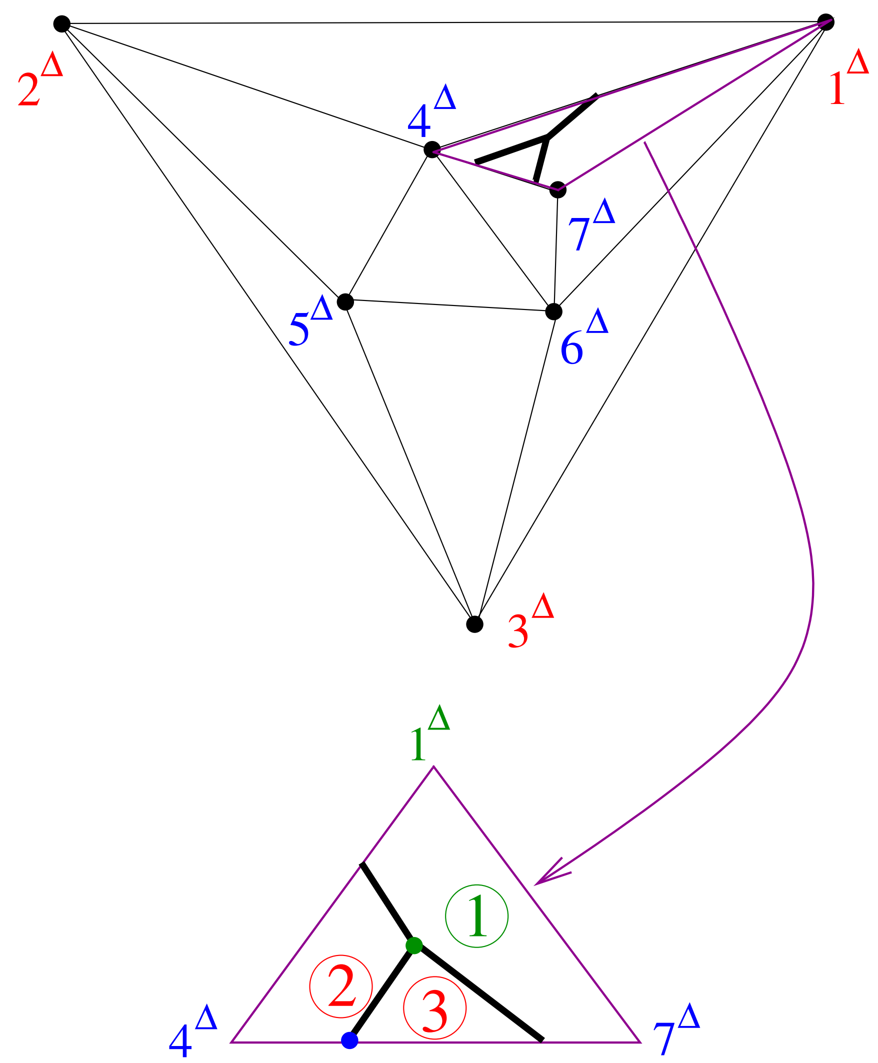


subdivide x^{Δ}

equilibrium iff
all labels 1...m

4/5	1/5	0

	④	⑦	
①	0	10	2
②	10	0	8
③	8	8	8



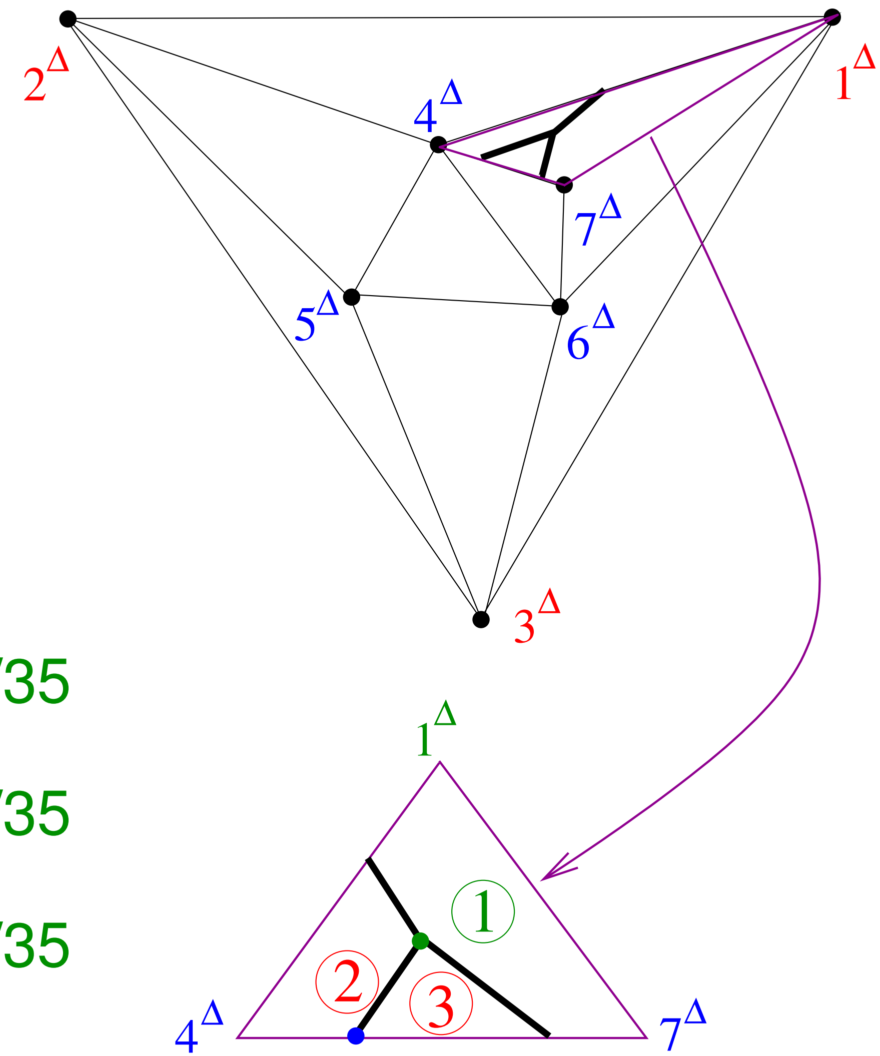
subdivide x^{Δ}

equilibrium iff

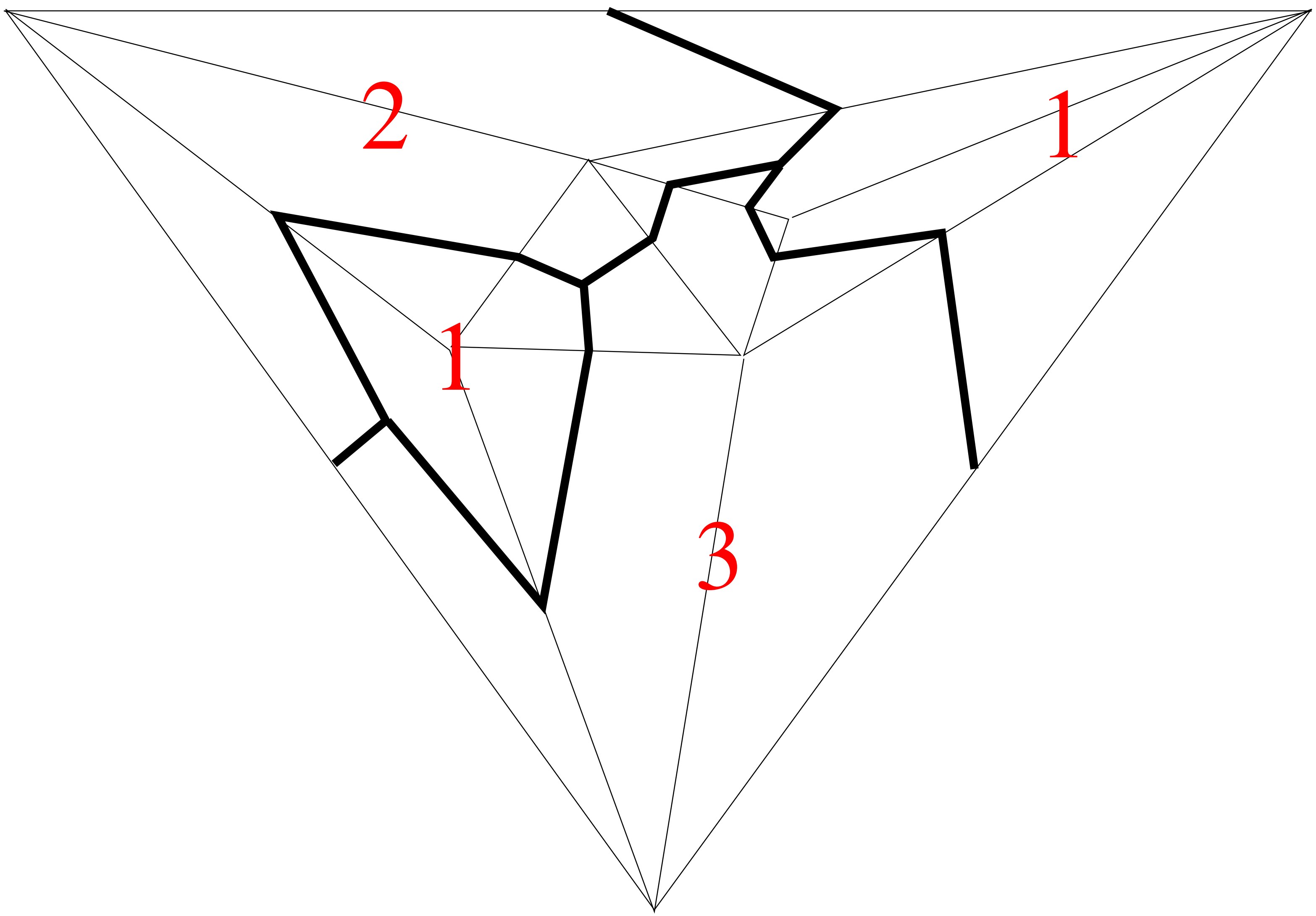
all labels 1...m

4/35	1/35	30/35
4/5	1/5	0

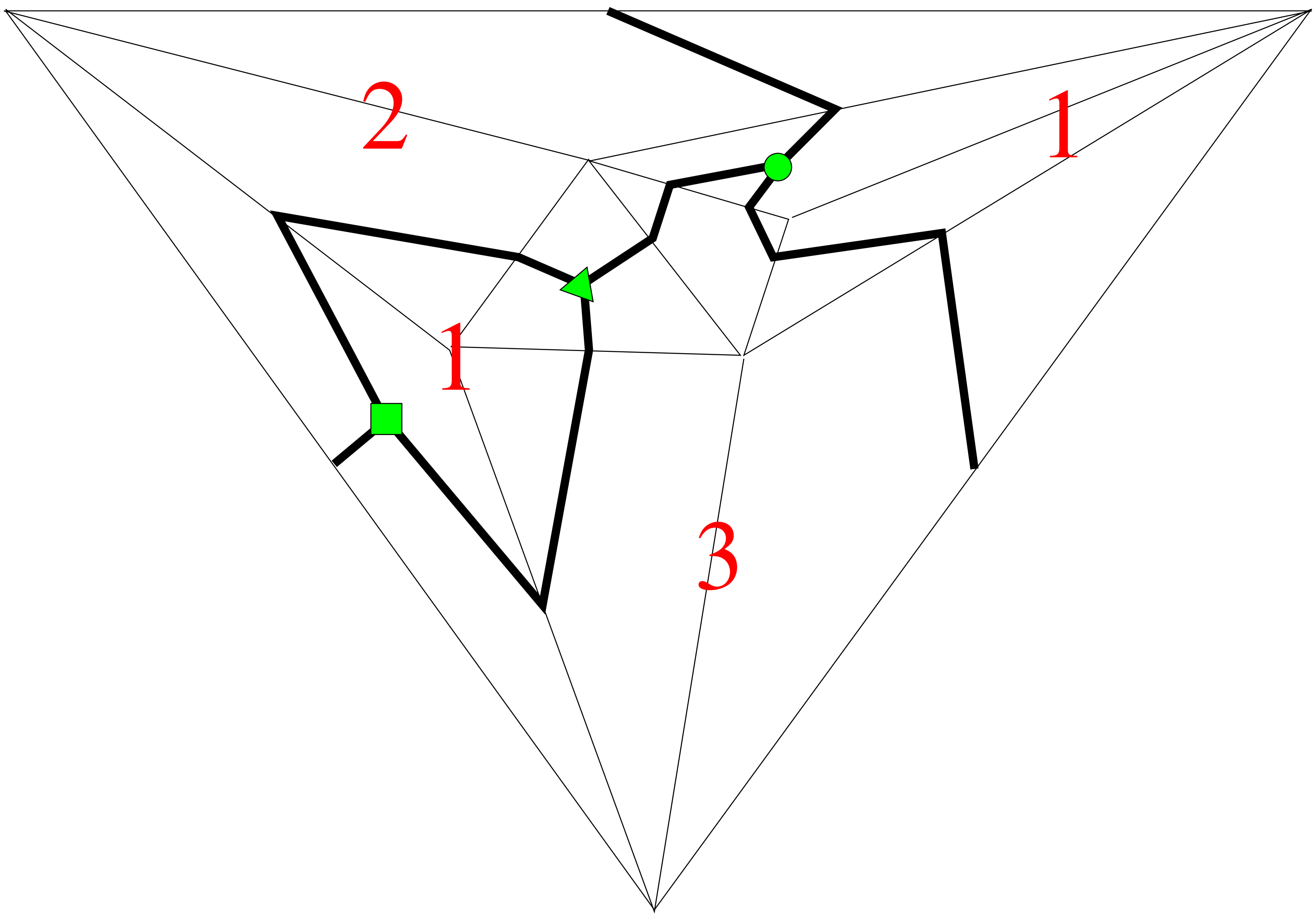
	(4)	(7)	(1)		
(1)	0	10	1	2	40/35
(2)	10	0	0	8	40/35
(3)	8	8	0	8	40/35



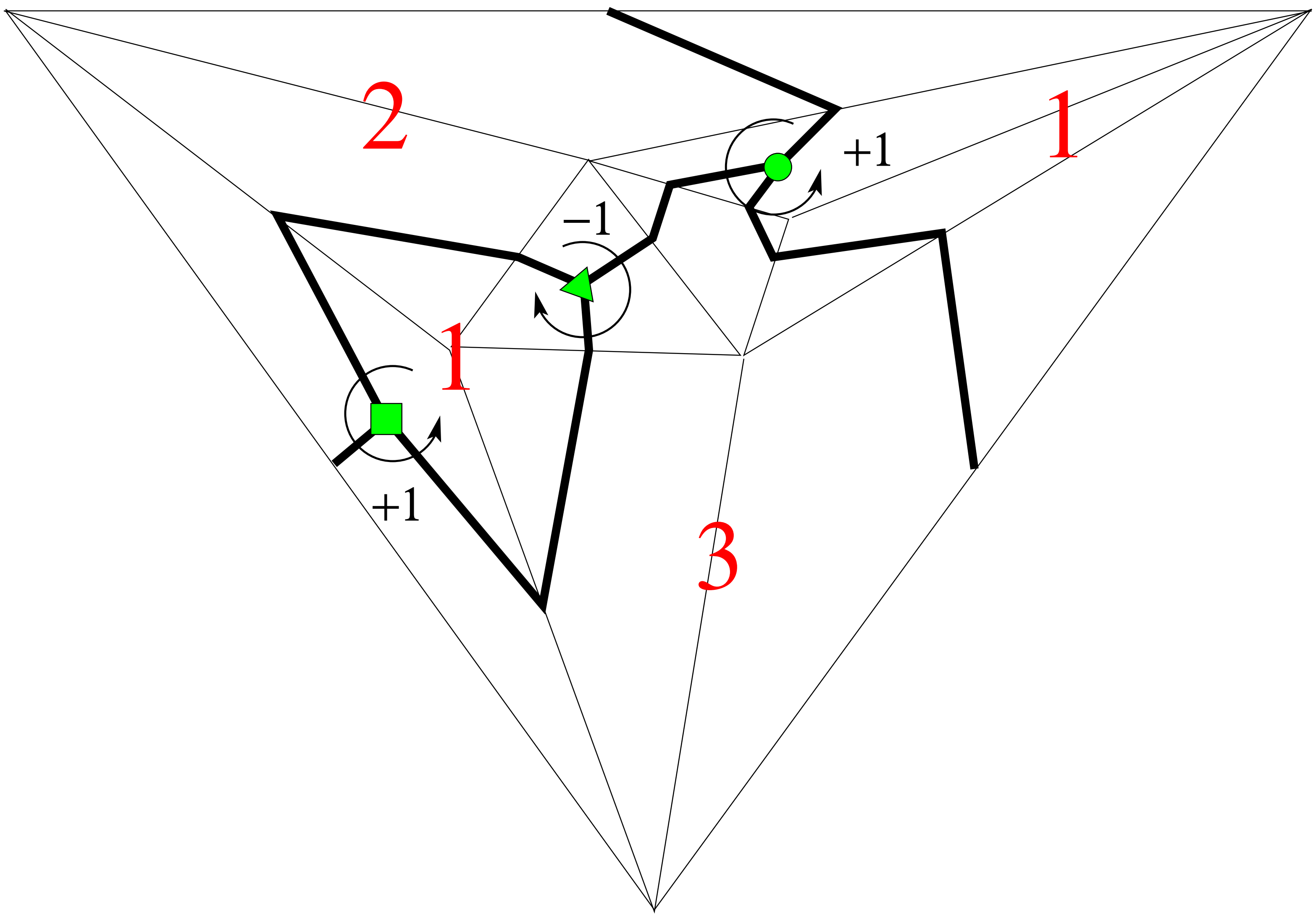
The full dual construction



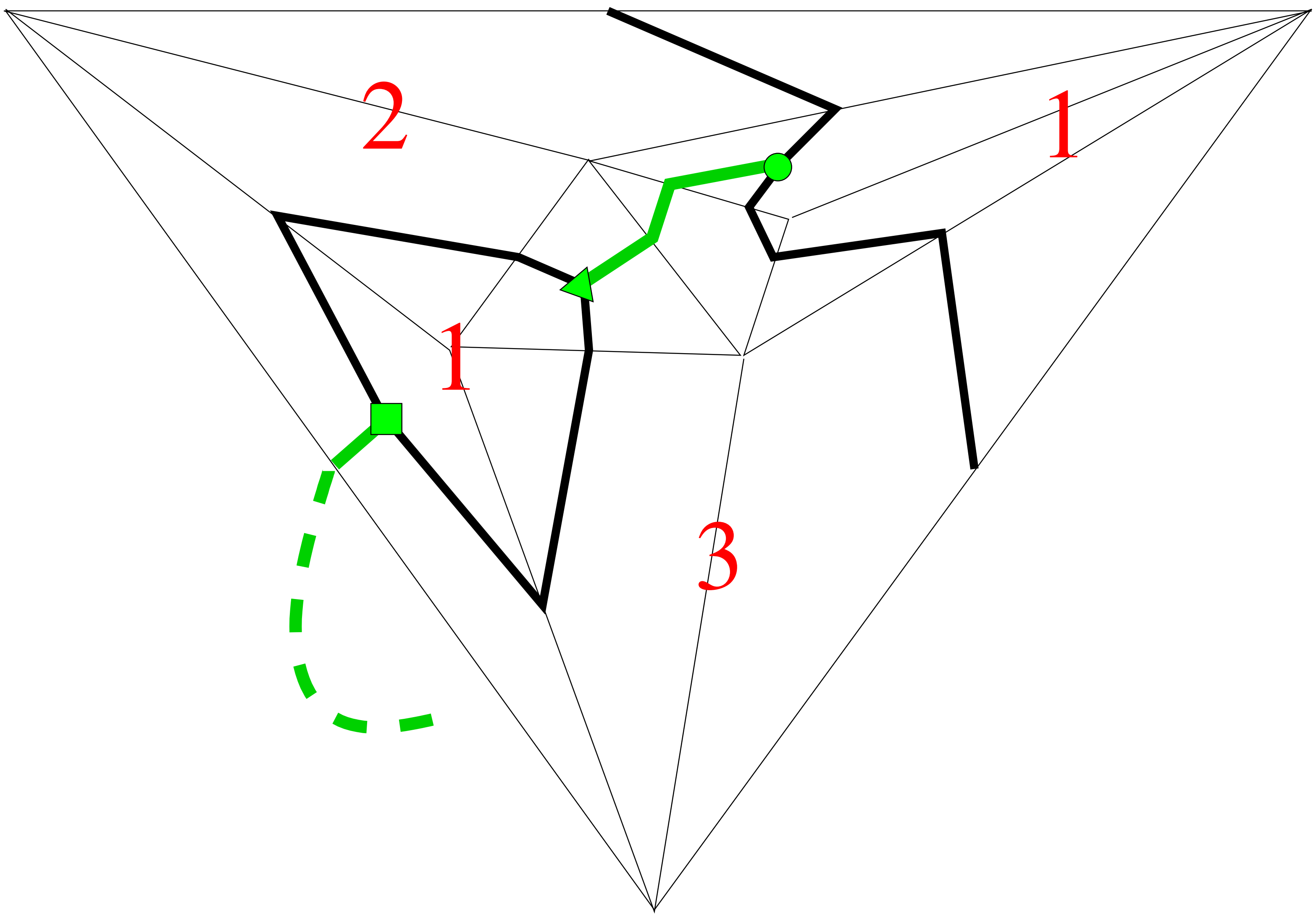
Equilibria have all m labels



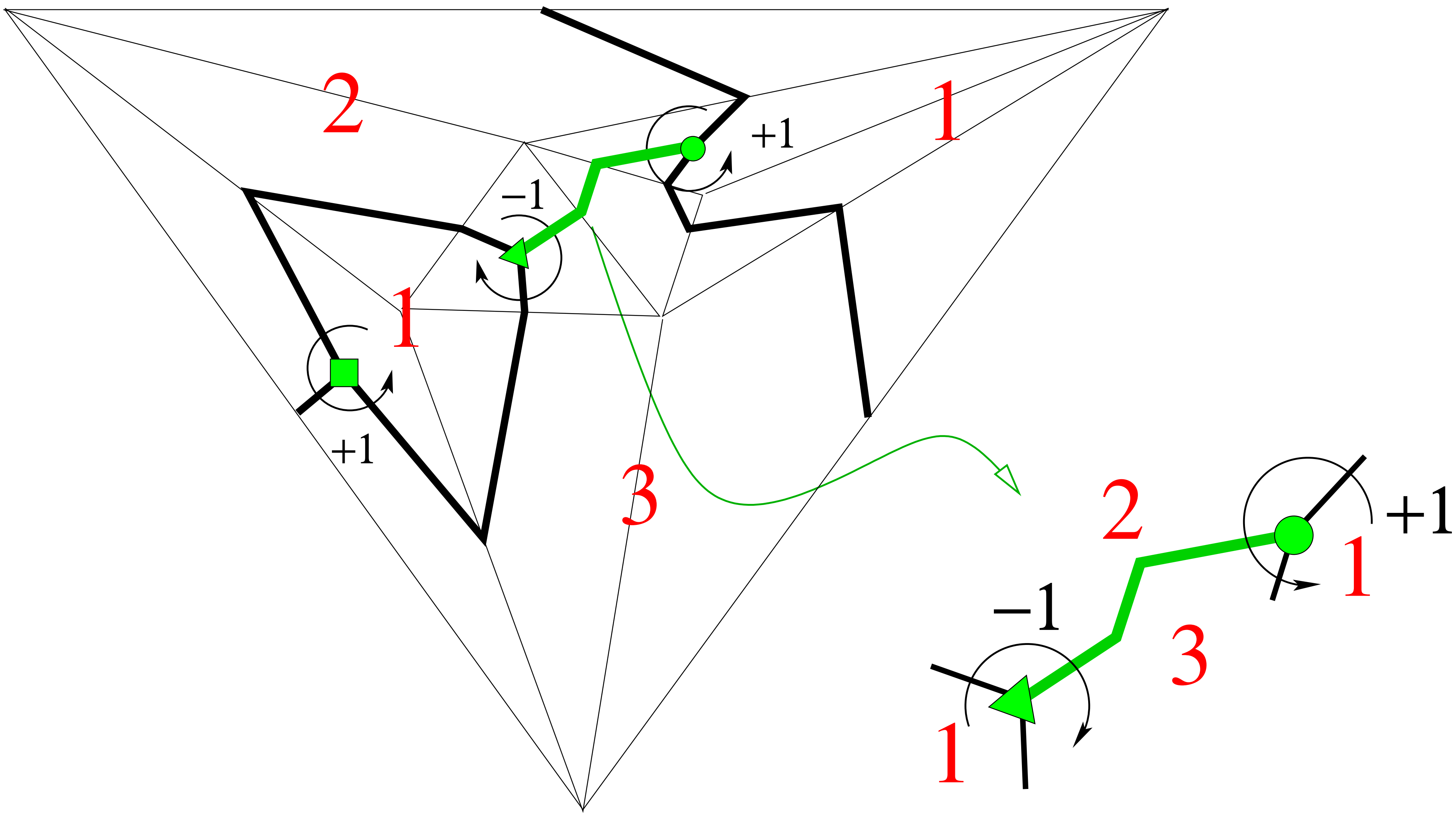
Index = orientation



Lemke–Howson paths



Opposite index of endpoints



Summary

New construction

- Triangulation reflects player **II**'s best replies
- Division player **I**'s best replies
- Intuitive definition of index
- Illustration of L-H algorithm and related index results
- Low dimension for easy visualization

Applications

- Strategic definition of index
- Components of equilibria / hyperstability
- Fixed point theory / Sperner's lemma