Leadership Games

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Part I:

Follower Payoffs in Symmetric Duopoly Games

Cournot vs. Stackelberg

Quantity competition - **Cournot**

payoff **I**:	x(1 – y – x)	**I** chooses x
payoff **II**:	y(1 – x – y)	**II** chooses y

Cournot (= Nash) x, y : 1/3, 1/3, payoffs 1/9, 1/9

Best response of **II**: y(x) = (1 – x) / 2

Stackelberg: commitment to x with response y(x)

Leader **I**, follower **II**: 1/2, 1/4, payoffs 1/8, 1/16

Symmetric Duopoly Games

player **I**: strategy $x \geq 0$, payoff $a(x,y)$

player **II**: strategy $y \geq 0$, payoff $b(x,y) = a(y,x)$

Assume: - **unique best response** $r(y)$ to y:

$a(r(y), y) > a(x, y)$ all $x \neq r(y)$

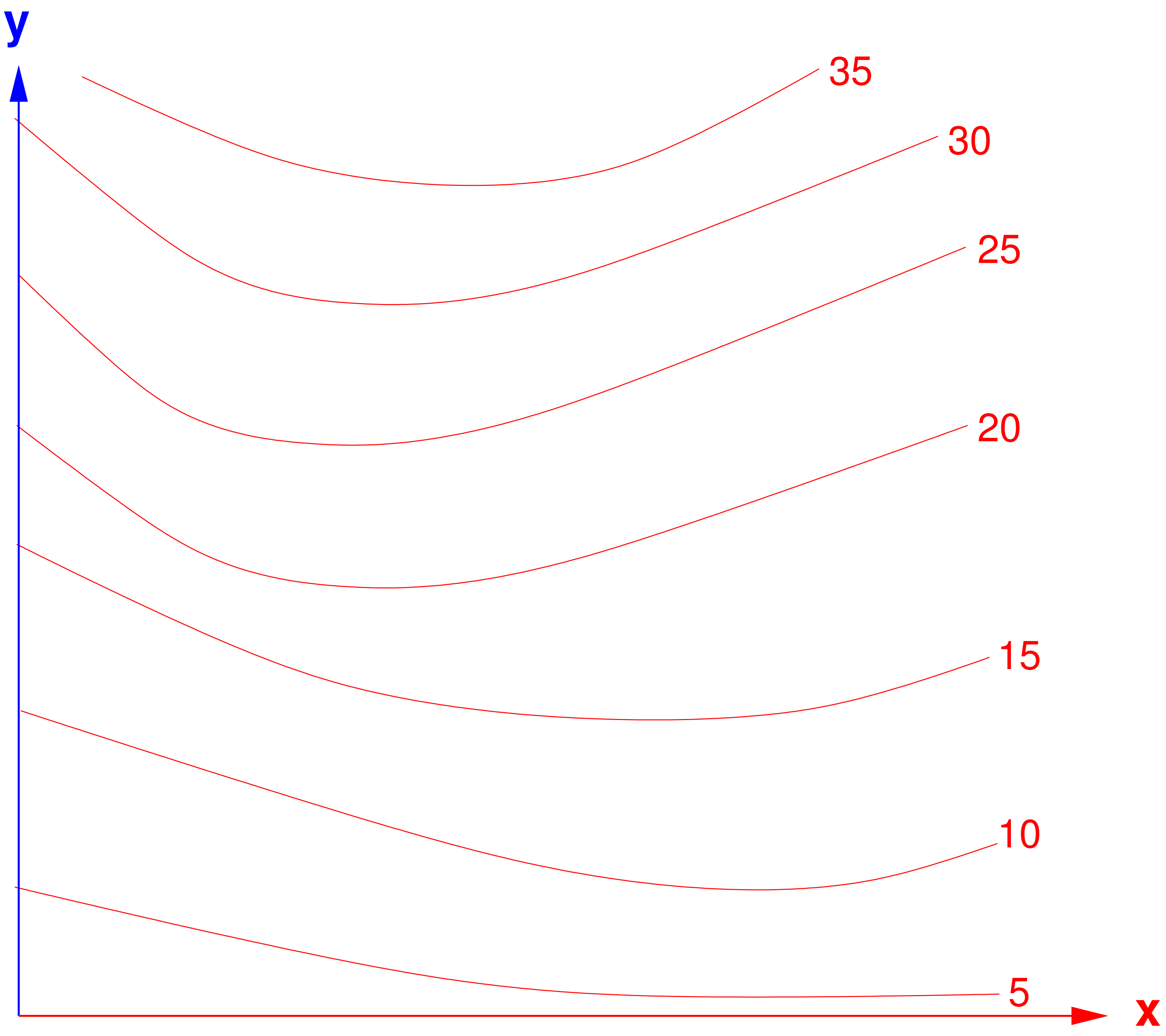
- and further assumptions

Leadership game: maximize $a(x, r(x))$ for $x = L$

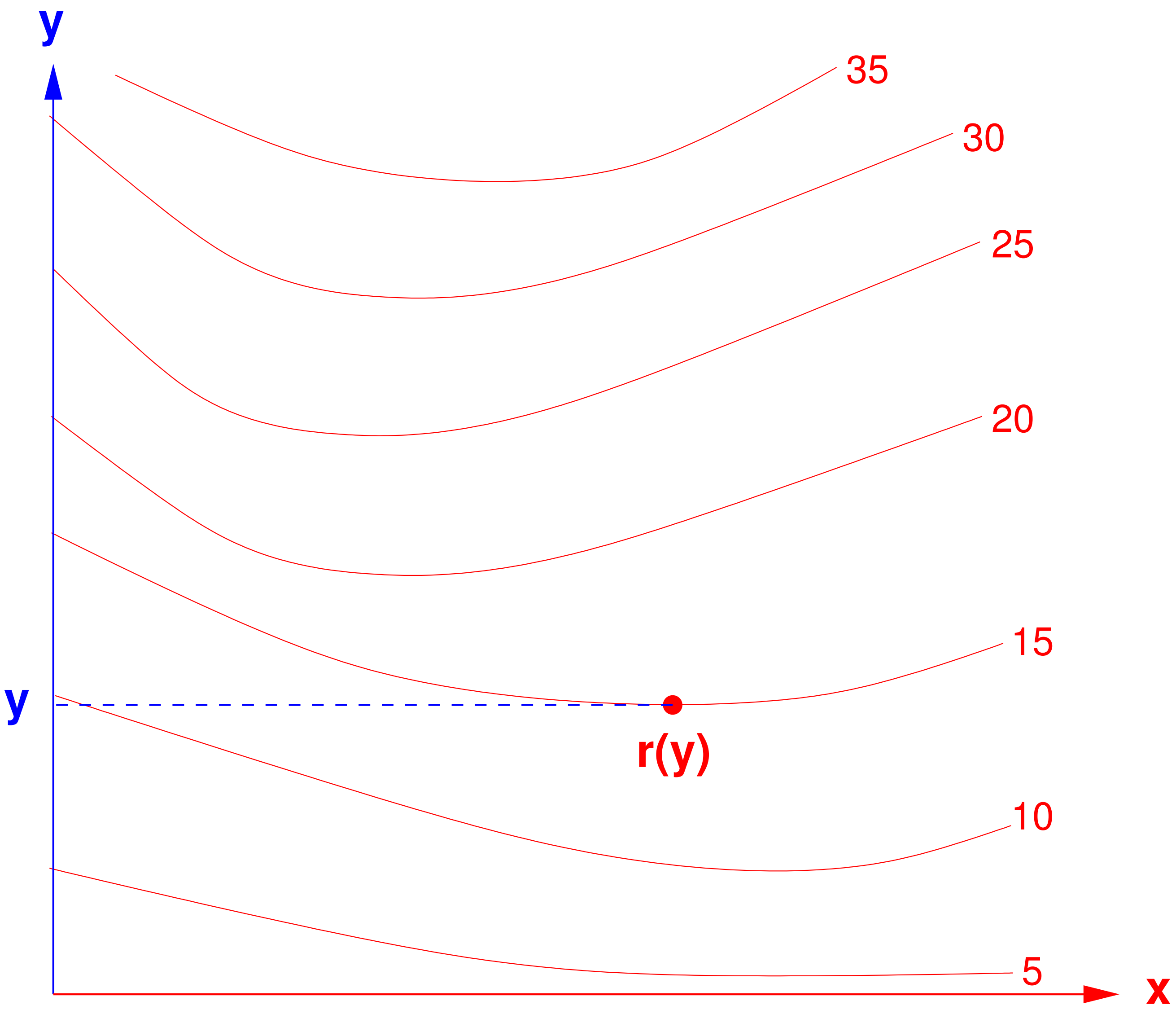
compare:

Leader payoff	$a(L, r(L))$
Nash payoff	$a(N, N)$
Follower payoff	$a(r(L), L)$

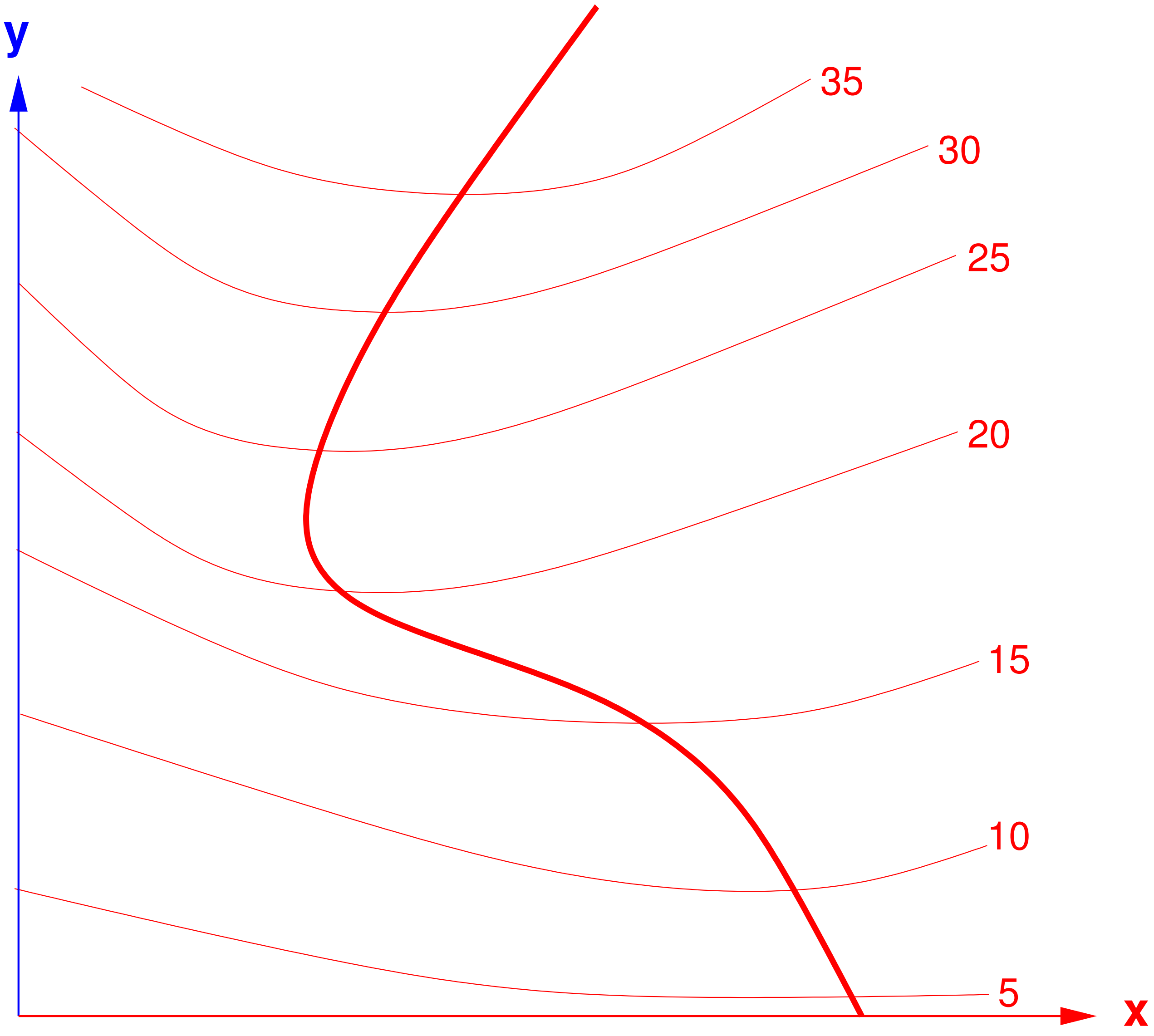
Contour lines of a(x,y)



unique best responses r(y)

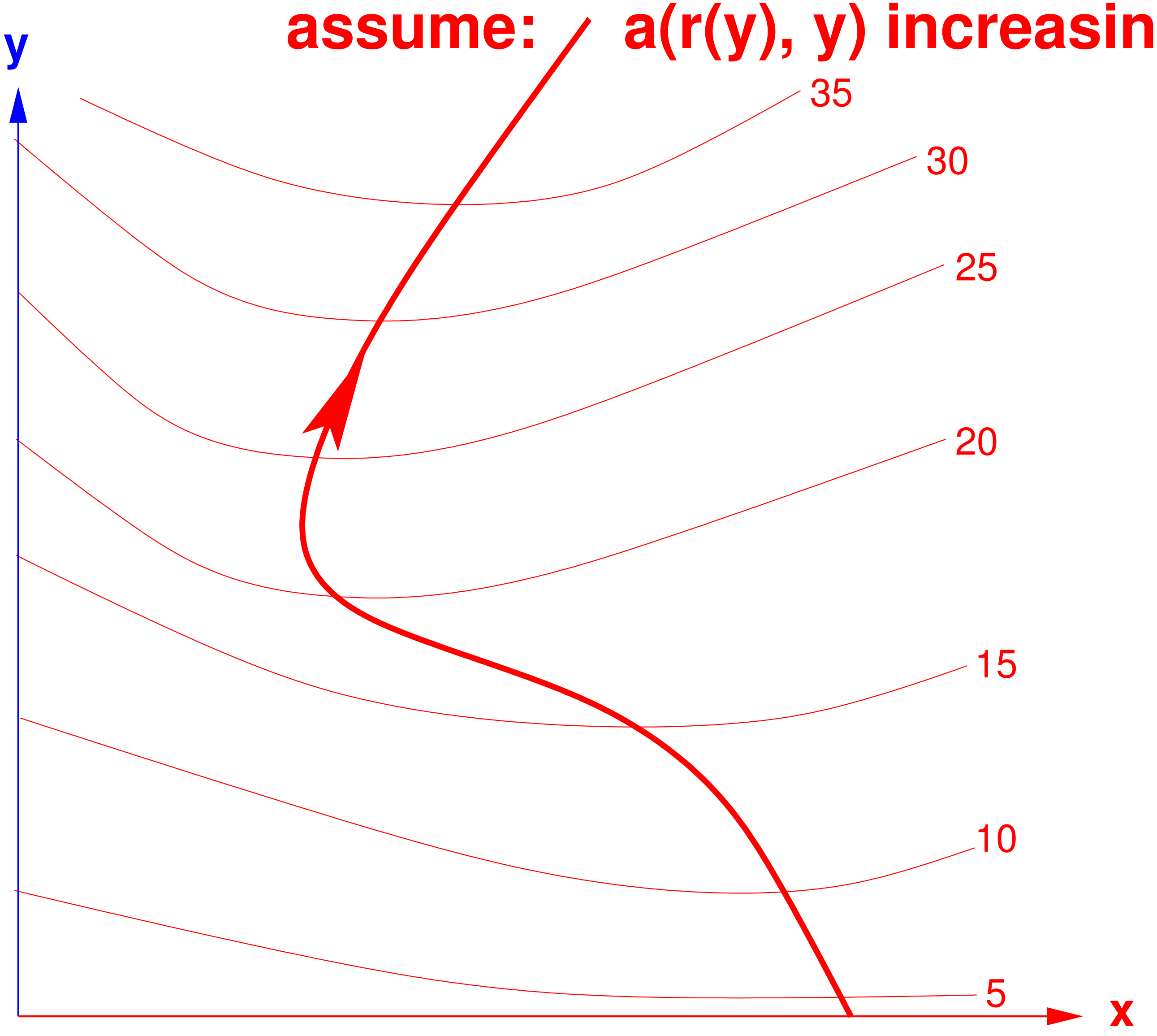


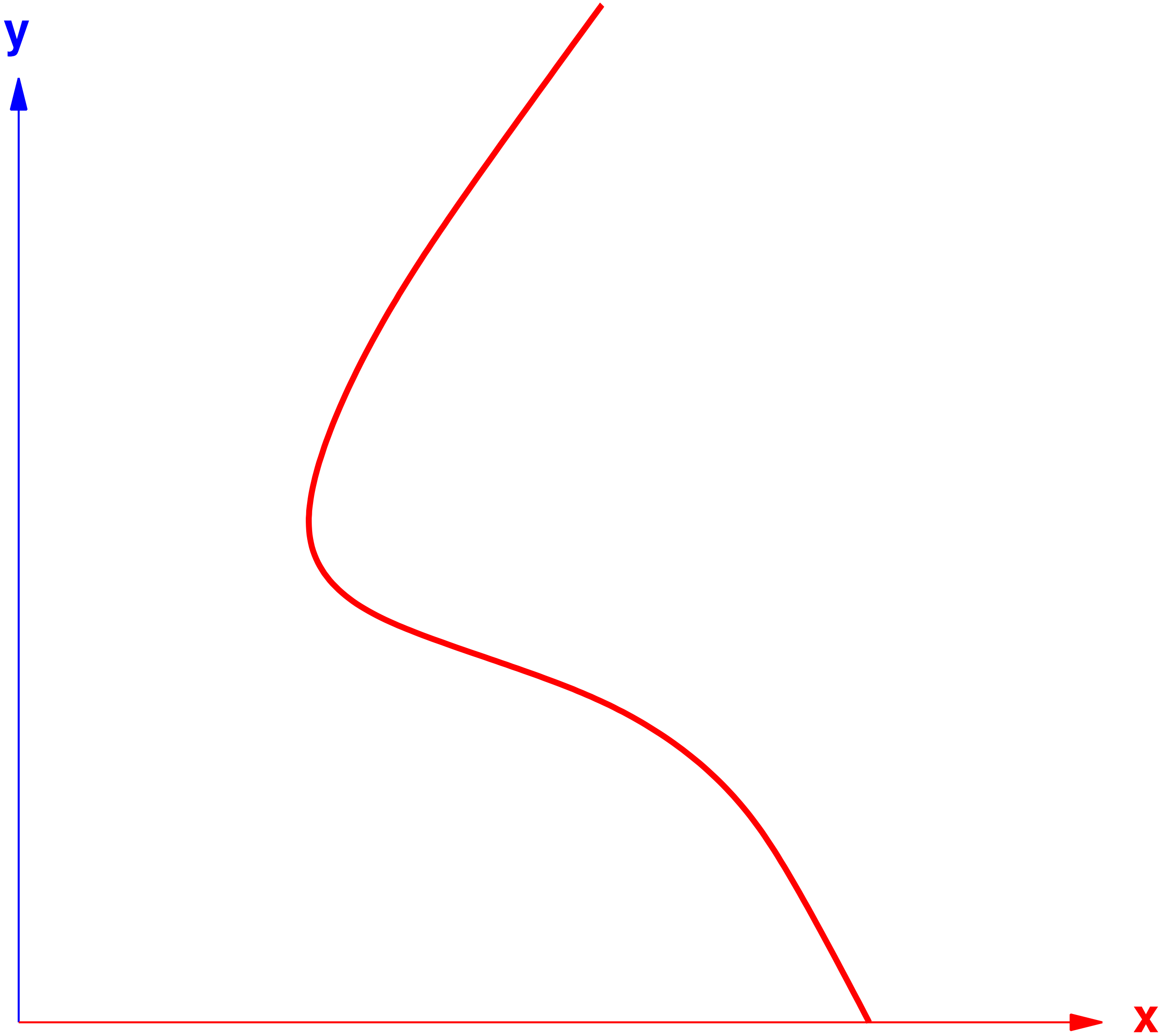
best response function r(y)



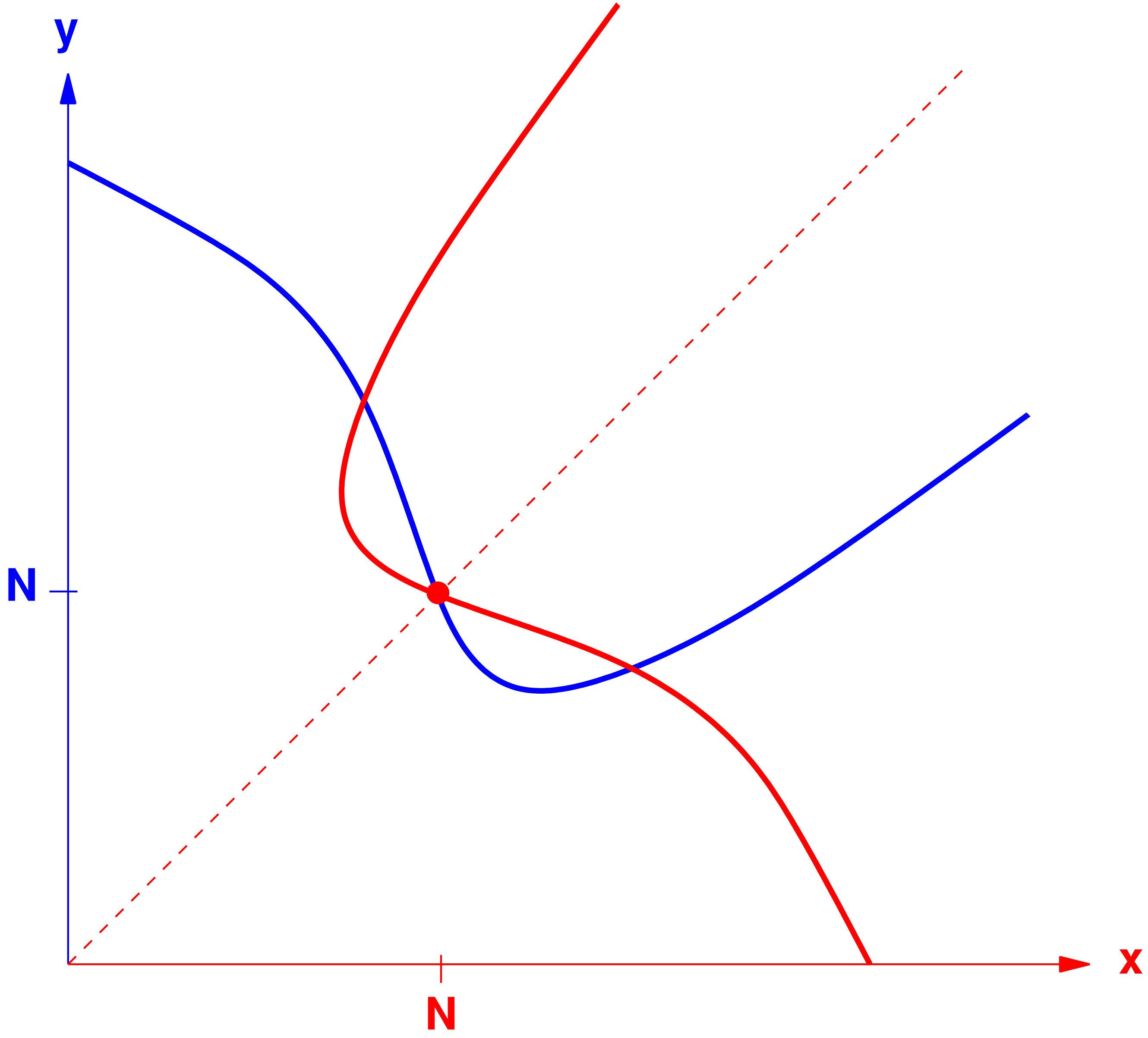
best response function r(y)

assume: a(r(y), y) increasing in y

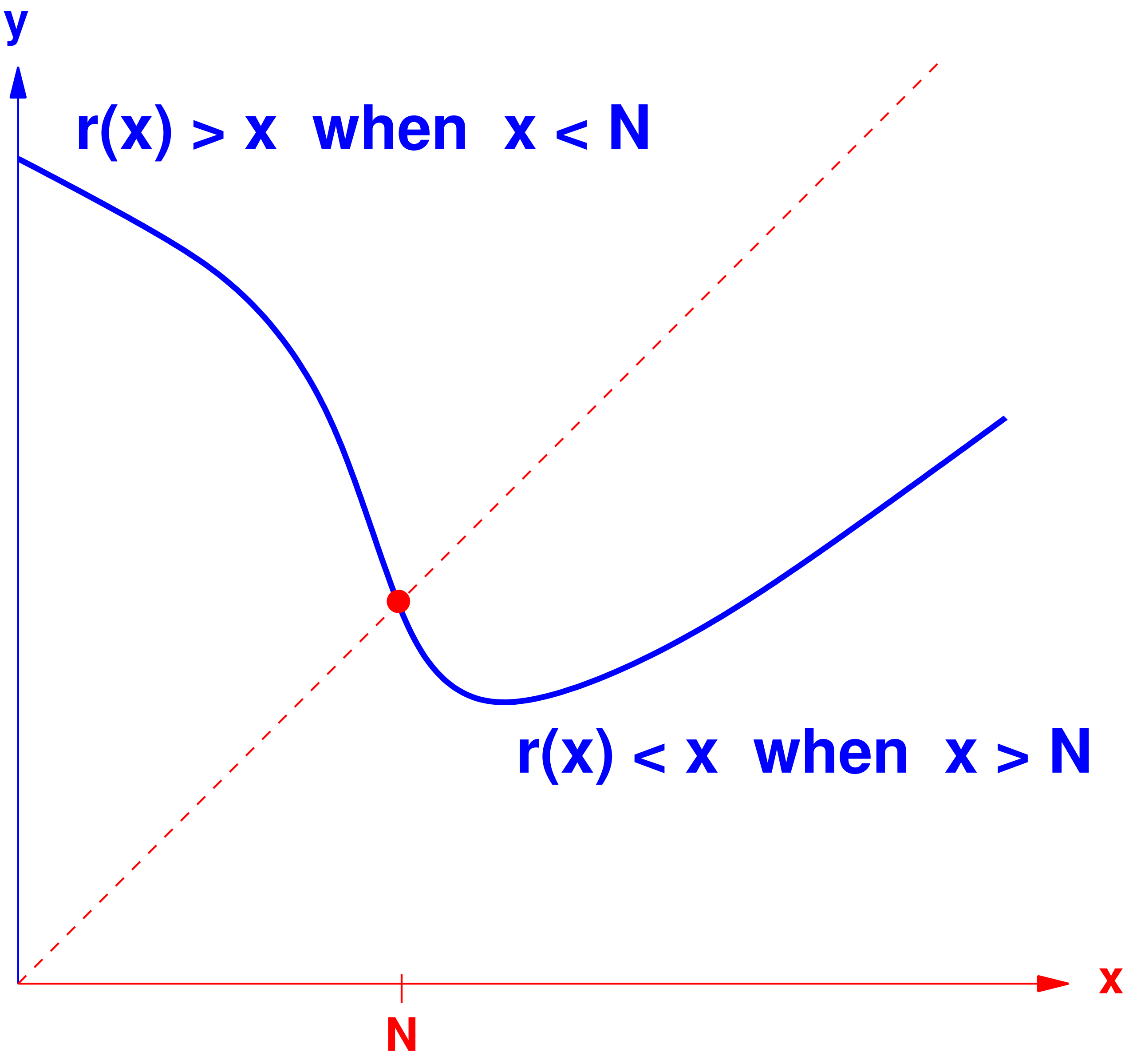



y
x

Only one symmetric equilibrium (N,N)

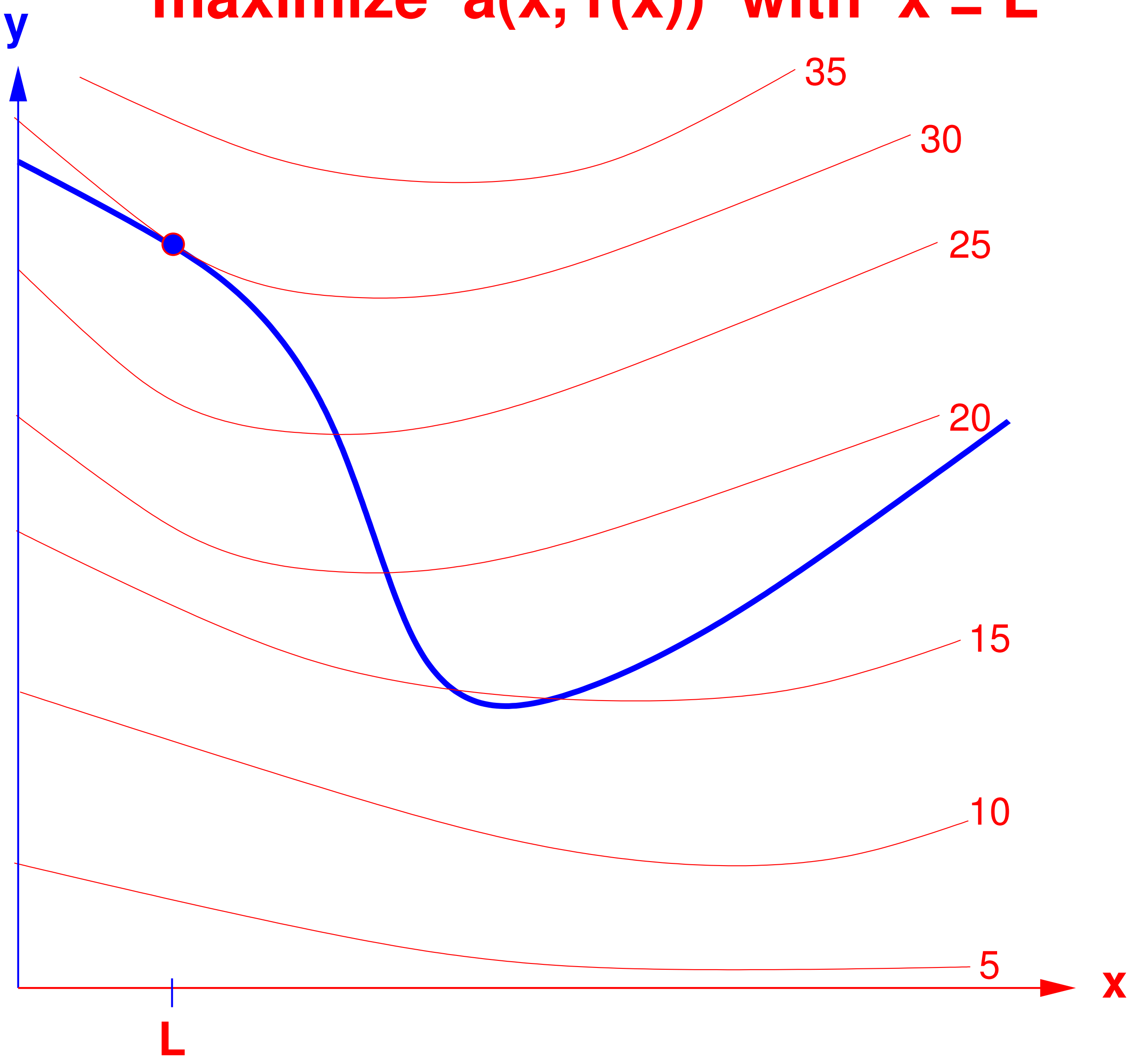


$r(x) = x$ only when $x = N$

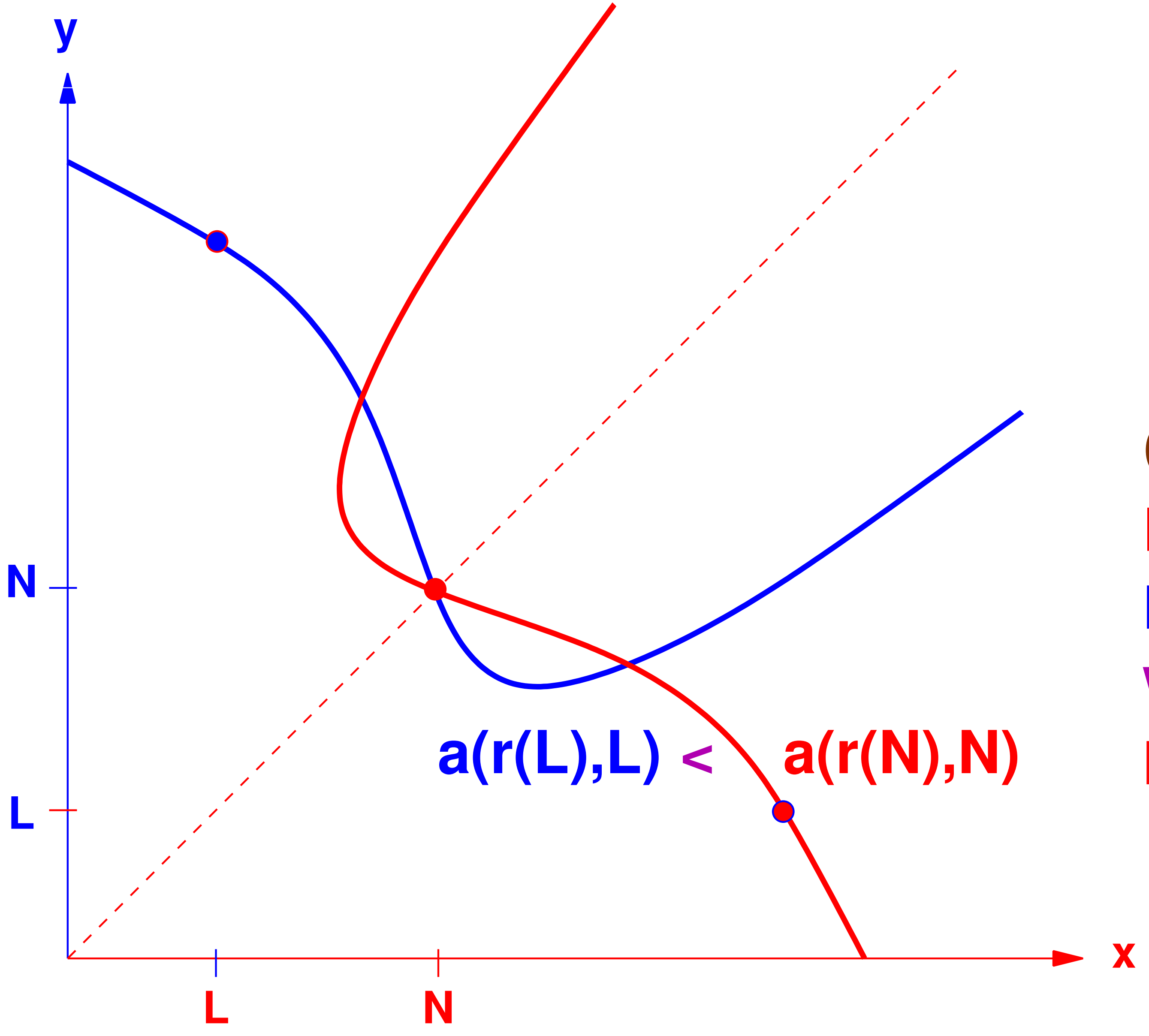


Leadership payoff a(L, r(L))

maximize a(x, r(x)) with x = L



Follower payoff a(r(L), L)



$a(r(L),L) < a(r(N),N)$

Case 1:

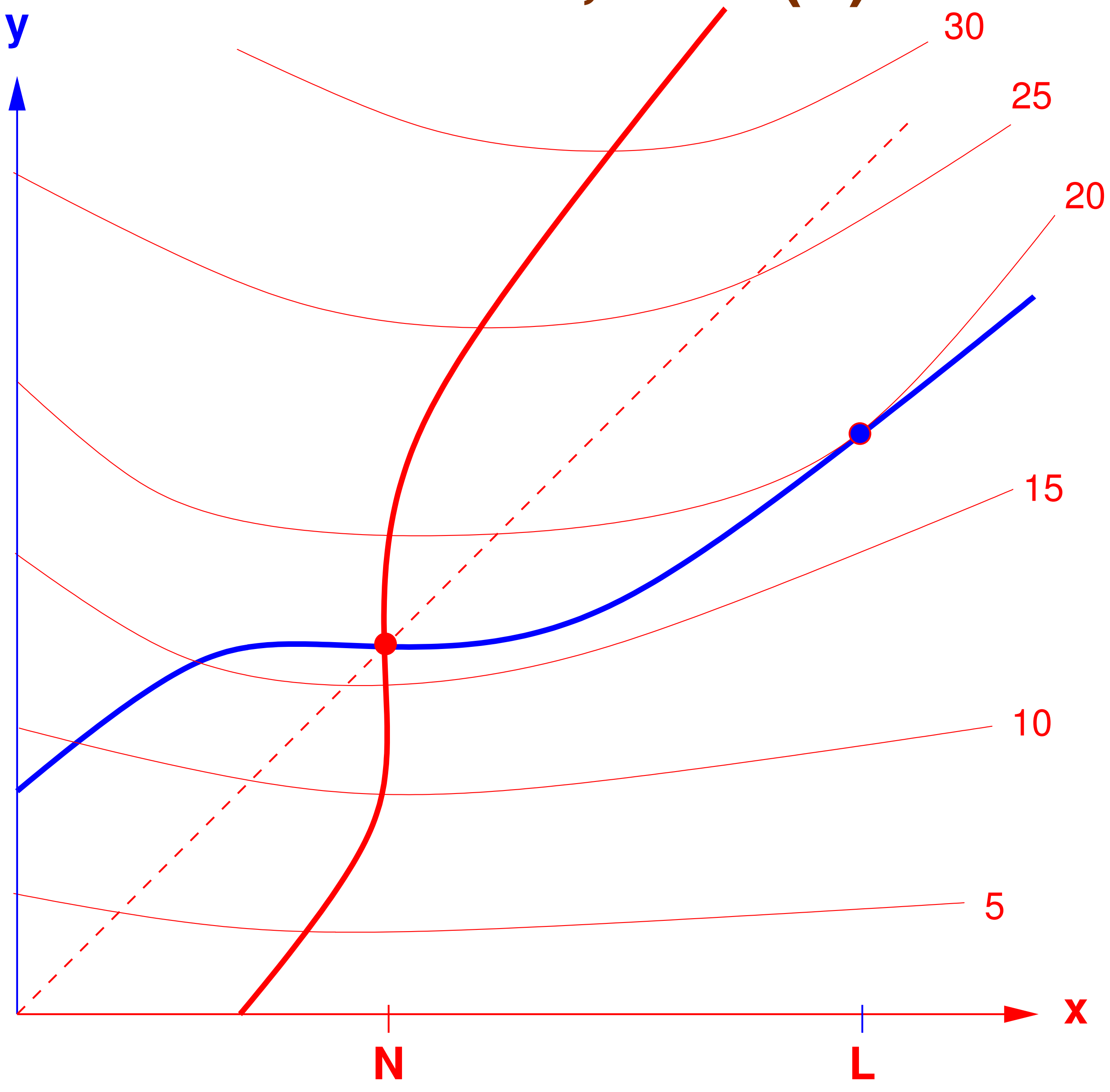
L < N :

Follower

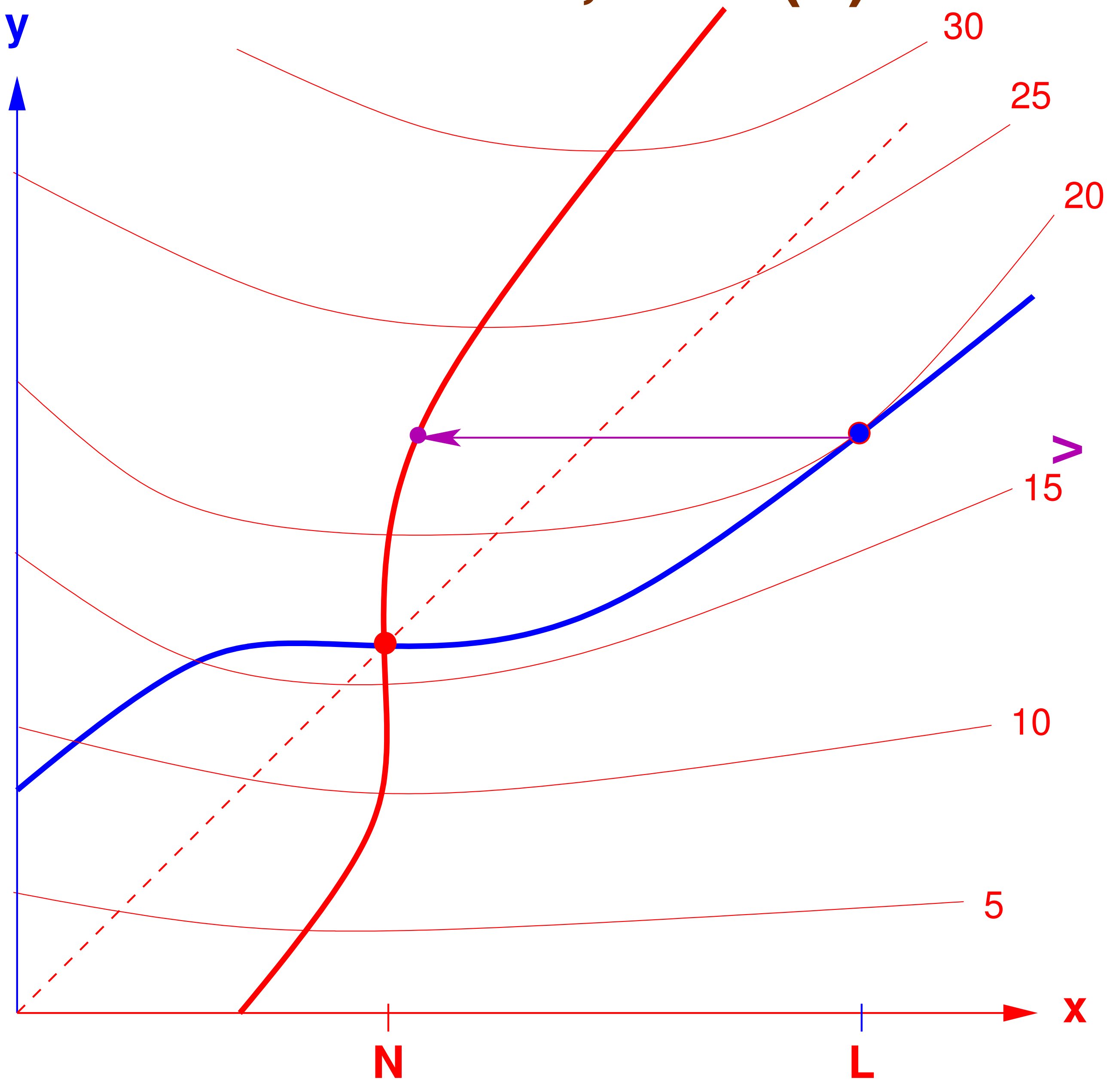
worse than

Nash!

Case 2: L > N, so r(L) < L



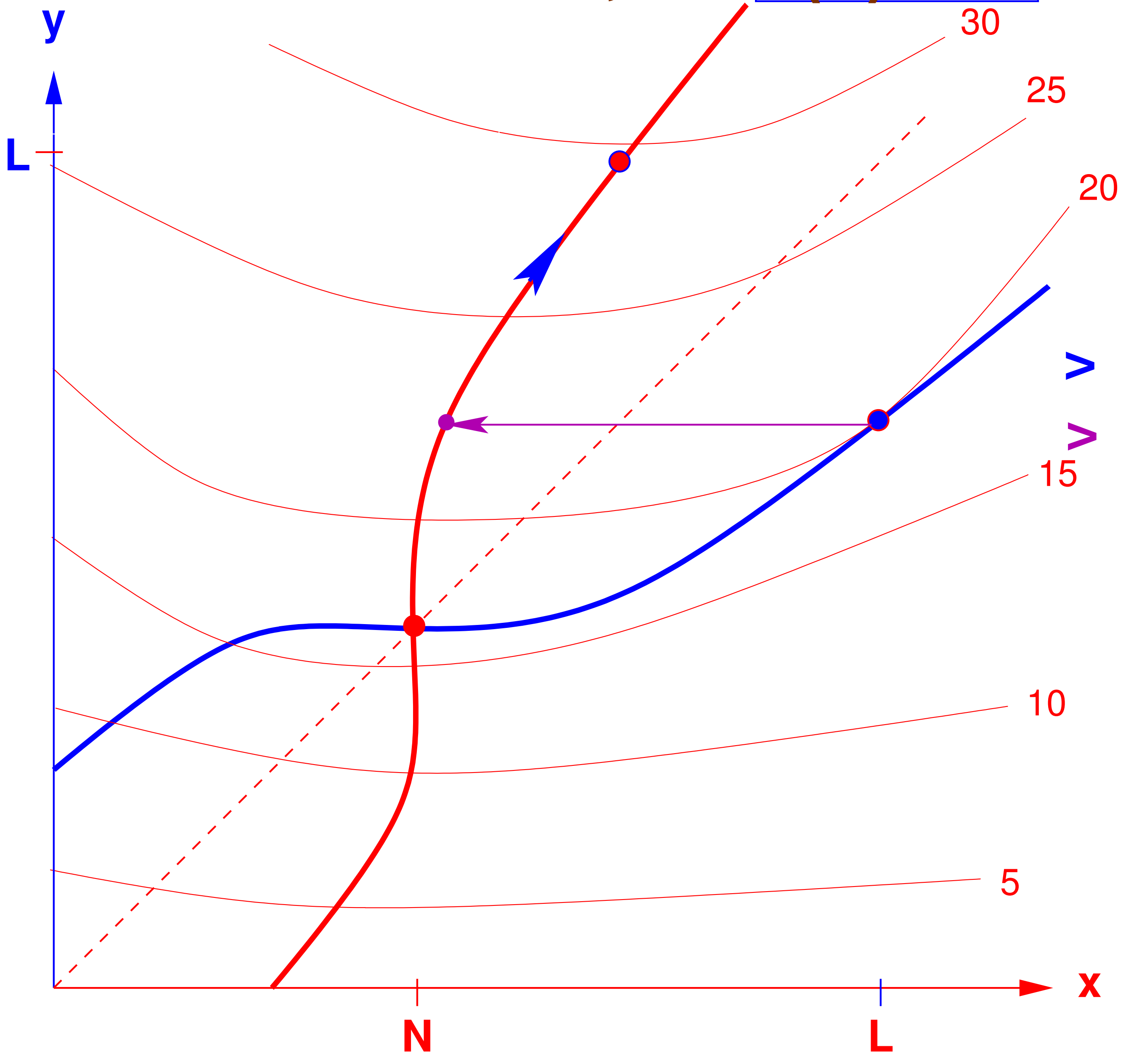
Case 2: L > N, so r(L) < L



$a(r(r(L)), r(L))$

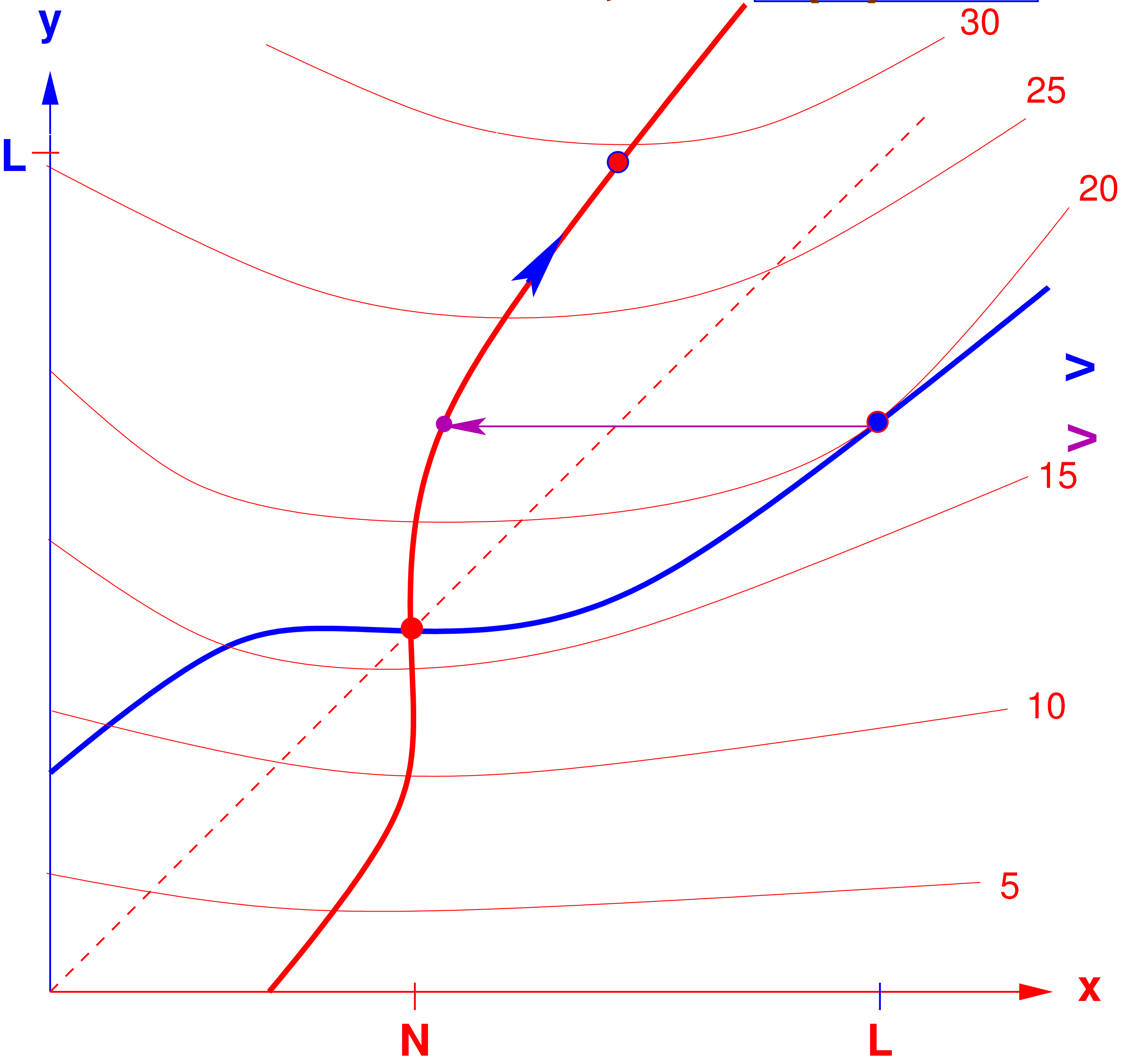
$> \; a(L, r(L))$

Case 2: L > N, so $\boxed{r(L) < L}$



$$a(r(L), L)$$
$$> \ a(r(r(L)), r(L))$$
$$> \ a(L, r(L))$$

Case 2: L > N, so $\boxed{r(L) < L}$



a(r(L), L)

> a(r(r(L)), r(L))

> a(L, r(L))

Follower
better than
Leader!

Theorem

Given: Symmetric duopoly game with

- **continuous** payoffs a(x,y), a(y,x), for x,y in intervals
- **unique** best responses r(y)
- payoff a(r(y), y) **monotonic** in y
- **unique symmetric Nash** equilibrium (N,N), r(N) = N.

Then the **follower** payoff a(r(L), L) is **either**

- **worse** than the **Nash** payoff a(N,N) **or**
- **strictly better** than the **Leader** payoff a(L, r(L))

but does not belong to the interval

(a(N,N), a(L,r(L))] **!**

Interpretation

Endogenizing leadership ...

see: Hamilton, J. and S. Slutsky (1990),
Endogenous timing in duopoly games: Stackelberg or Cournot equilibria. *Games Econ. Behav*. **2**, 29-46.

... in symmetric duopoly games is difficult!

either - both want to go **first** as follower is hurt

or - both want to go **second** as follower profits.

⇒ back to **simultaneous** game (or "Stackelberg war"), respectively equilibrium **selection** problem.

Part II:

Leadership with Commitment to Mixed Strategies

joint work with **Shmuel Zamir**

CNRS, Paris, and
The Hebrew University, Jerusalem

Simultaneous vs. Leadership Game, Commit to Mixed Strategies

- 2 players, player **I** vs. player **II**, finite game

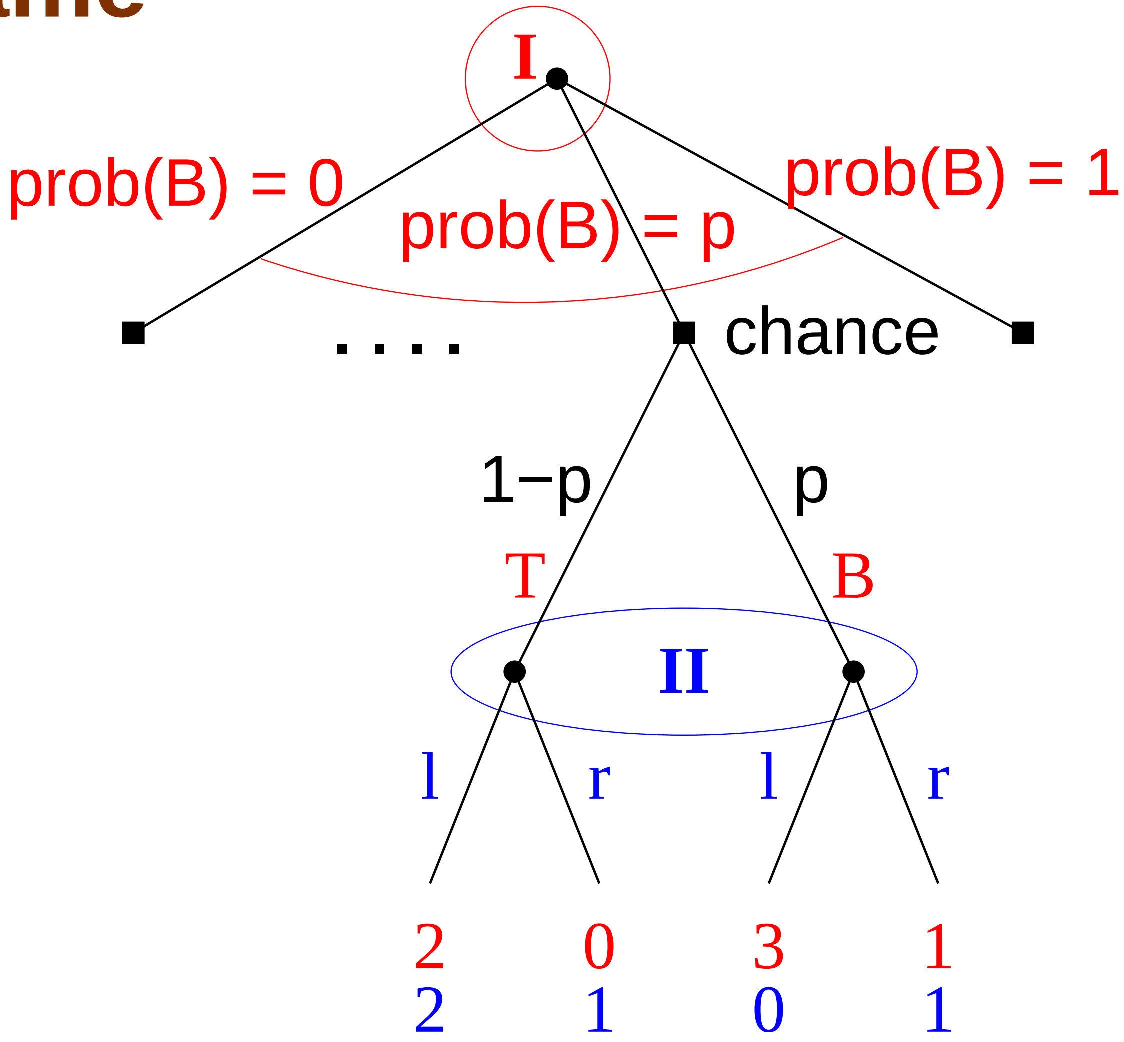
- **simultaneous** game,
  Nash equilibria,

compared with

- **leadership** game:
  player **I** commits to a **mixed** strategy
  player **II** always chooses best response
  (subgame perfect equilibria)

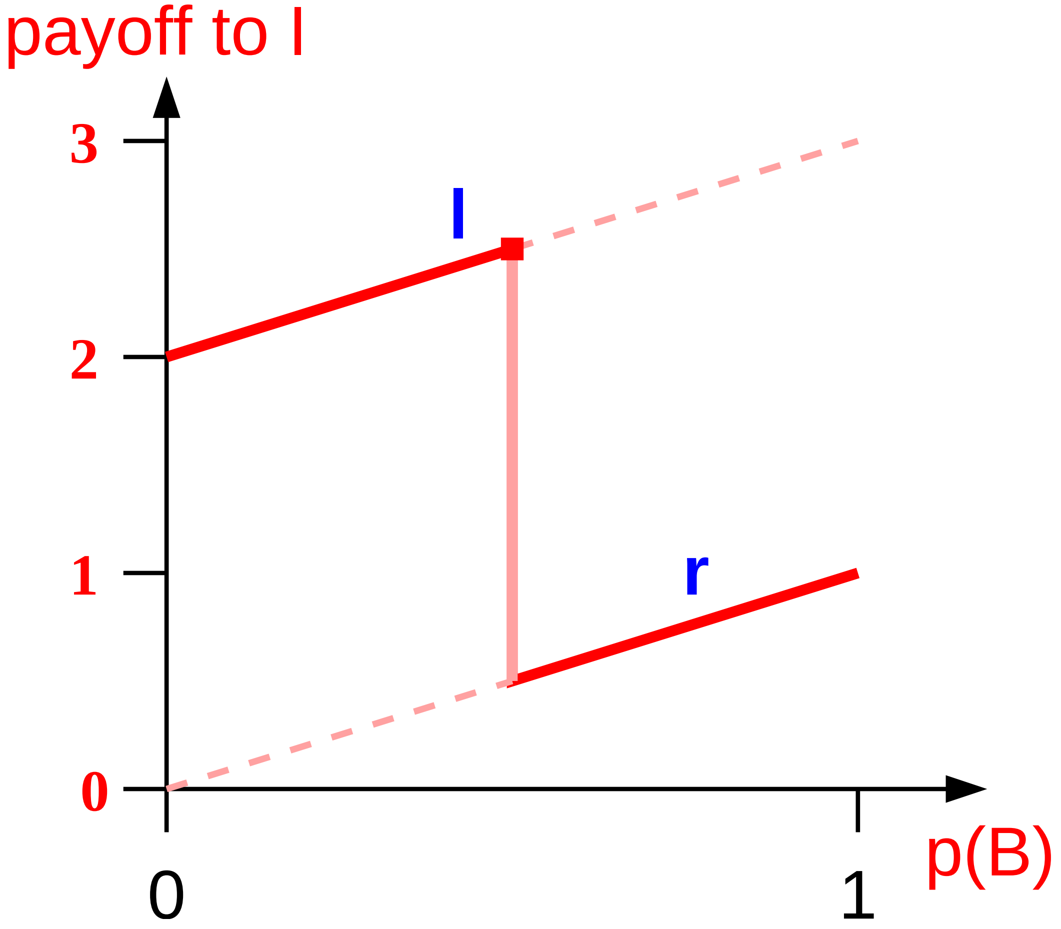
Leadership Game

II I	l	r
T	2 2	1 0
B	0 3	1 1



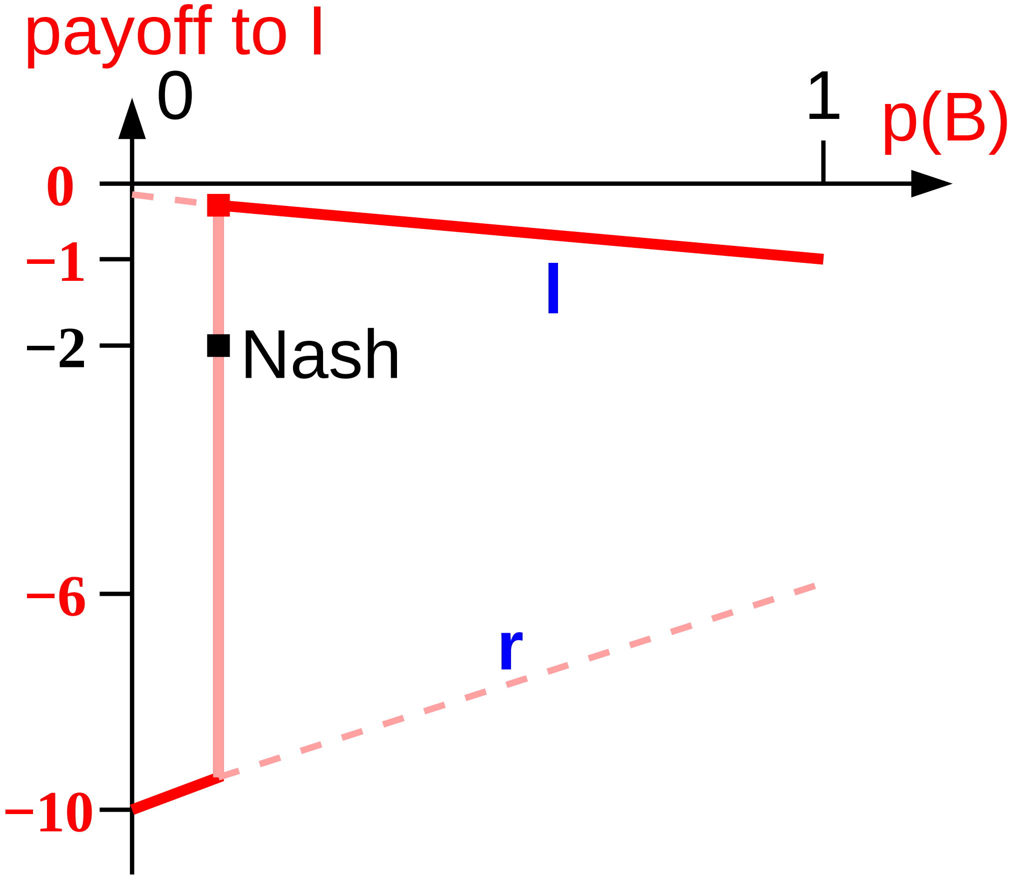
The Quality Game

I \ II	l	r
T	2 2	1 0
B	0 3	1 1



Inspection game

I \ II	l	r
T	0 0	1 −10
B	0 −1	−9 −6



Issues Not Considered Here

- **verifiability** ("mixing" credible?)
- **observability** [Bagwell; Hurkens / van Damme]
- **robustness**: induce unique best response by changing commitment by ε [Maschler]
- Nash equilibria that **survive** commitment [Rosenthal] - very restrictive
- "**endogenous**" commitment?
  Here: study **consequences** if
  commitment power **given**
  (natural for e.g. inspection games)

Our Results

- **commitment always helps**

- even in **nongeneric** games
  (will give more examples, **Theorem**)

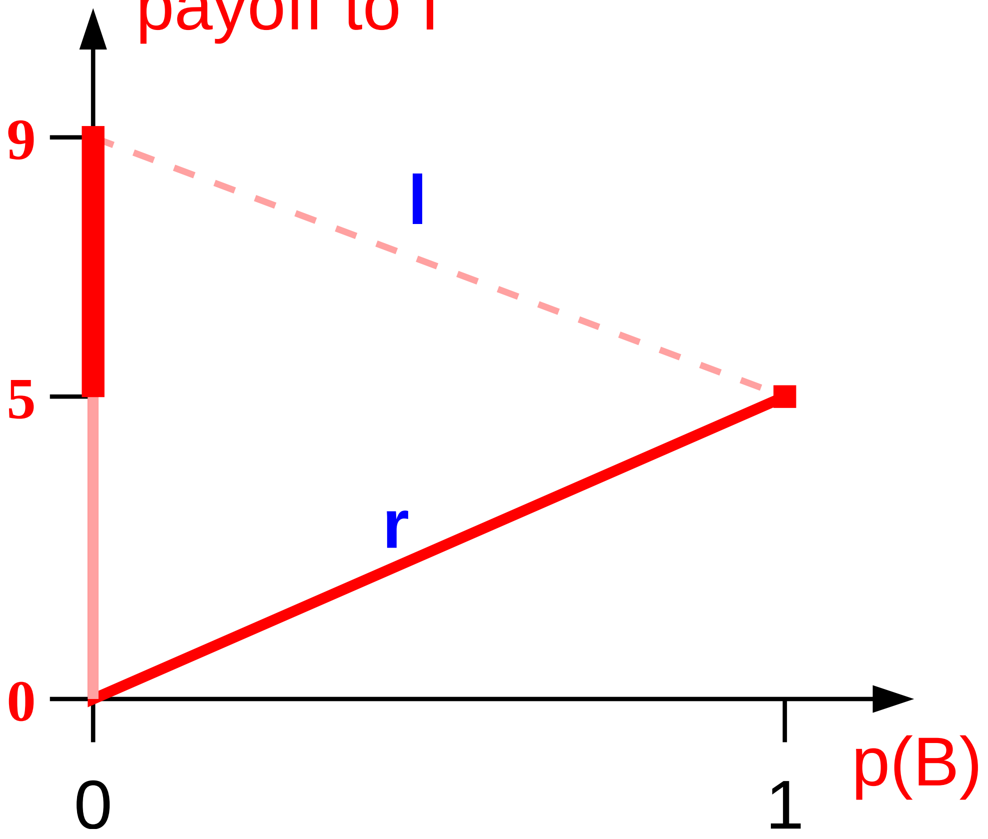
- commitment as **coordination** device:
  even **correlated equilibria** are not better

- **3** or more **players**: commitment may **hurt**

"Best" remote from "safe" commitment

Interval [5, 9] of leader payoffs

II I	l	r
T	9 9	9 0
B	6 5	7 5



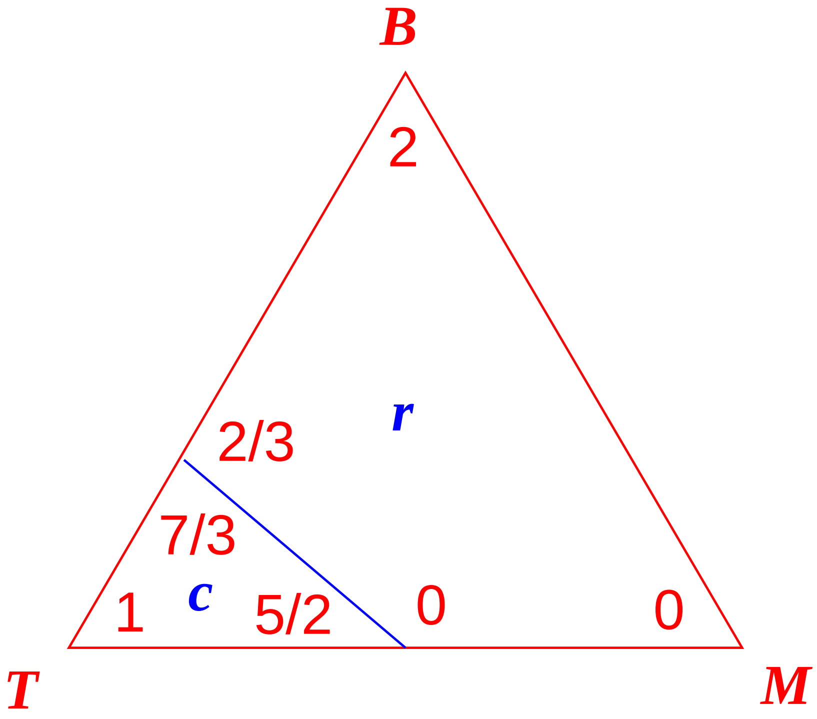
Symmetric game, 3 strategies

I \ II	*l*	*c*	*r*
T	**0** **0**	**2** **1**	**1** **0**
M	**1** **2**	**4** **4**	**5** **0**
B	**0** **1**	**0** **5**	**2** **2**

$c > r \quad \Longleftrightarrow$

$2T + 4M > T + 5M + 2B \quad \Longleftrightarrow$

$T > M + 2B$



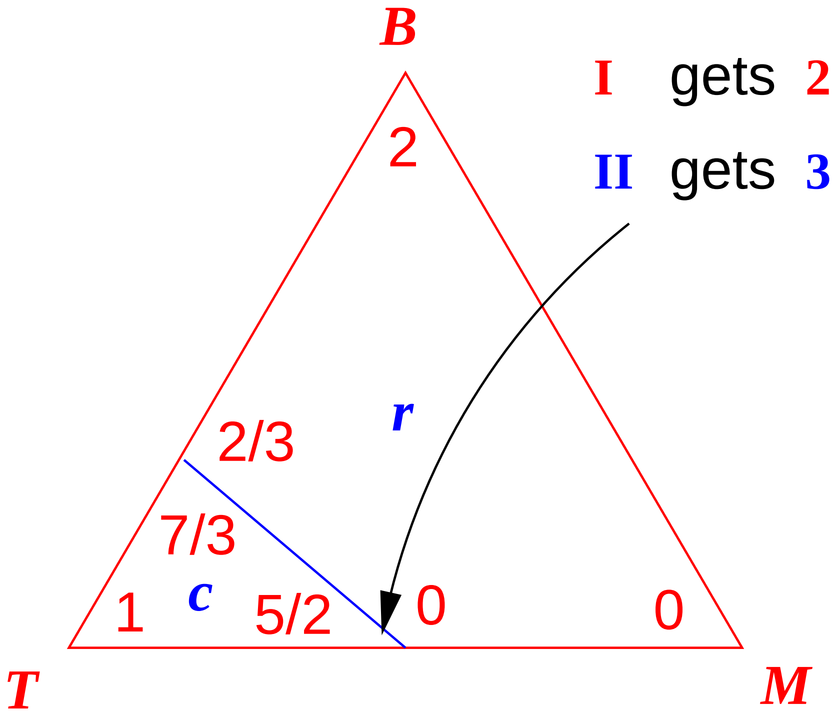
Symmetric game, 3 strategies

II I	*l*	*c*	*r*
T	0 0	2 1	1 0
M	1 2	4 4	5 0
B	0 1	0 5	2 2

$c > r$ <=>

$2T + 4M > T + 5M + 2B$ <=>

$T > M + 2B$



I gets **2.5**

II gets **3**

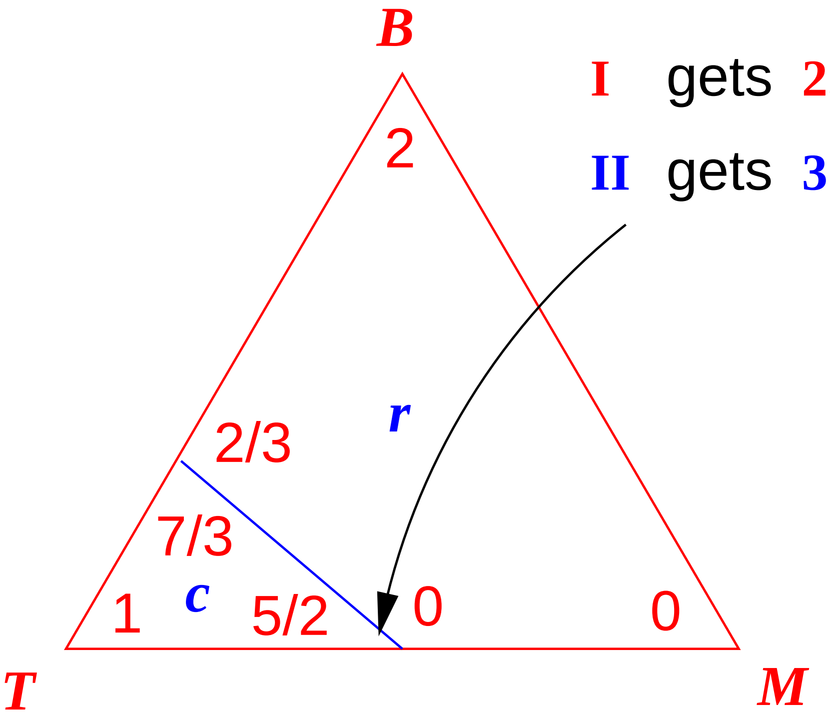
Arbitrary follower–payoffs

II I	*l*	*c*	*r*
T	d d	2+d 1	1+d 0
M	1 2+d	4 4	5 0
B	0 1+d	0 5	2 2

$c > r \iff$

$2T + 4M > T + 5M + 2B \iff$

$T > M + 2B$



I gets **2.5**

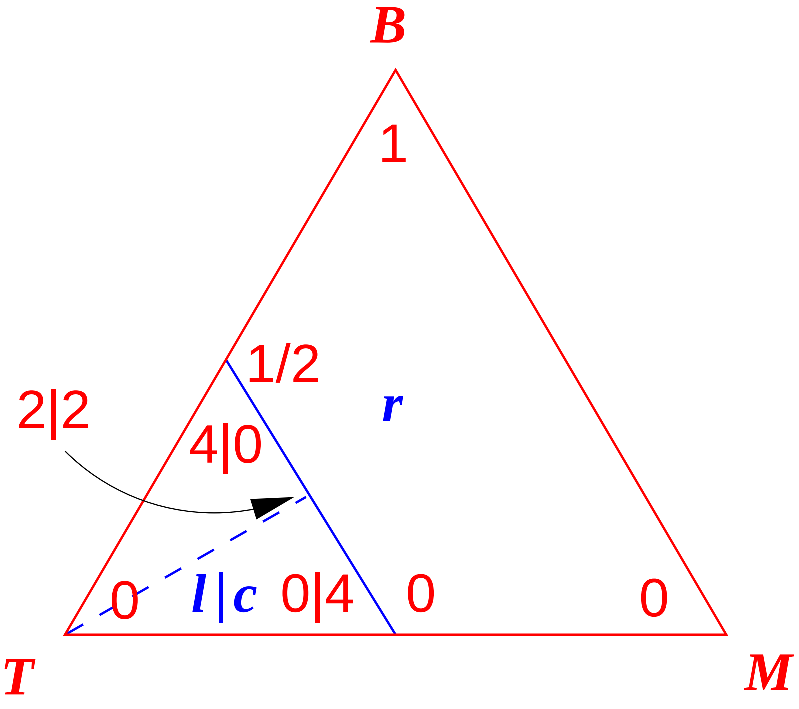
II gets **3+d/2**

Identical Follower columns

II I	*l*	*c*	*r*
T	1 0	1 0	0 0
M	0 0	0 8	1 0
B	0 8	0 0	1 1

$l\,|\,c > r$ <=>

$T > M + B$



Theorem

m x n payoff matrices: $A = [A_1 \ldots A_n]$, $B = [B_1 \ldots B_n]$

$X = \{ x \geq 0 \mid x_1 + \ldots + x_m = 1\}$

$X(j) = \{ x \in X \mid j$ best response to $x \}$ $(1 \leq j \leq n)$

$F = \{ j \mid X(j)$ **full-dimensional** $\}$ (any **unique** b.r. in F)

Theorem

$m \times n$ payoff matrices: $A = [A_1 \ . \ . \ . \ A_n]$, $B = [B_1 \ . \ . \ . \ B_n]$

$X = \{ x \geq 0 \mid x_1 + \ldots + x_m = 1 \}$

$X(j) = \{ x \in X \mid j$ best response to $x \}$ $(1 \leq j \leq n)$

$F = \{ j \mid X(j)$ **full-dimensional** $\}$ (any **unique** b.r. in F)

Then in any subgame perfect equilibrium of the leadership game, the set of **leader** payoffs is $[L, H]$, where

$$L = \max_{j \in F} \ \max_{x \in X(j)} \ \min_{k:\, B_k = B_j} x A_k \,,$$ $\geq$ **some** Nash payoff,

$$H = \max_{1 \leq j \leq n} \ \max_{x \in X(j)} x A_j \,,$$ $\geq$ **all** Nash payoffs to **I** .

Generic games

If the game is generic, then

in any subgame perfect equilibrium of the leadership game, the **leader** payoff is = L = H,

$$H = \max_{1 \leq j \leq n} \quad \max_{x \in X(j)} \quad x A_j ,$$

where **any Nash** equilibrium payoff to player **I** is $\leq$ H .

"Pessimistic" Leader, Many Followers

- player **I** commits to mixed strategy x

- n followers play Nash equilibrium y
  of resulting n-player game (subgame perfection)
  from set N(x) [n = 1: N(x) = best responses to x.]

- player **I** gets payoff a(x, y)

⇒ then the **lowest** leadership payoff is

$$L = \sup_{x} \min_{y \in N(x)} a(x, y)$$

... but in the subgame perfect equilibrium, the
followers typically **don't** choose the worst response.

Commitment and correlated equilibria

Theorem:

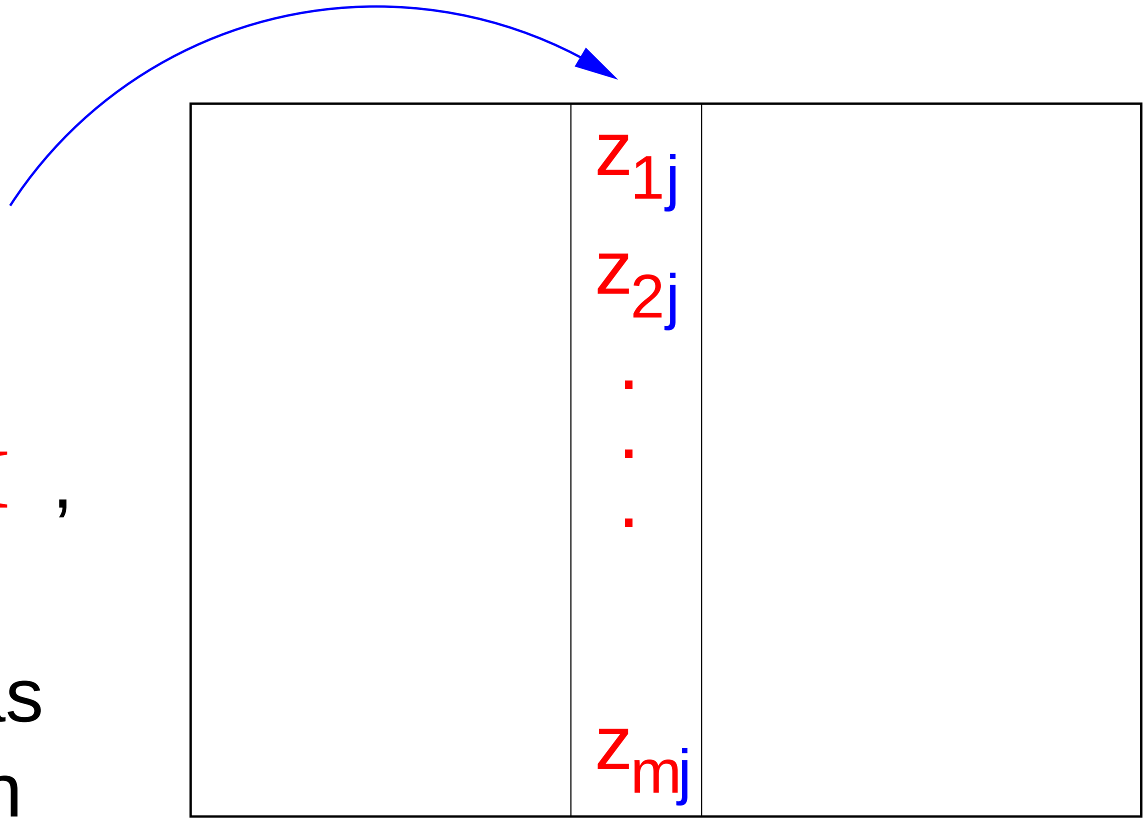
In any subgame perfect equilibrium of the leadership game, the set of **leader** payoffs is [L, H], where any **correlated equilibrium payoff** to player **I** is $\leq$ H .

Reminder: Correlated equilibrium = joint distribution **z**$_{ij}$ on strategies i,j of player **I**, **II** fulfilling **incentive** constraints (for player **I**): for all i, k

$$\sum_j z_{ij} a_{ij} \geq \sum_j z_{ij} a_{kj}$$

and analogously for player **II**.

Proof:



In given CE, pick column j with largest conditional payoff C to **I** ,

Commit to that column as distribution on rows, response j subgame perfect by incentive constraints.

Then CE−payoff ≤ C ≤ H .

Weakly correlated equilibrium

[Moulin & Vial, 1978]

- as in CE: correlation device with joint prob's $\mathbf{z_{ij}}$

- now players can **either**
  commit to using the recommended action
  or choose their own strategy, knowing only
  marginal probabilities of device.

- Equilibrium: prefer **device**, for player **I**,

$$\sum_{i,j} z_{ij}\, a_{ij} \ \geq\ \sum_j \Big(\sum_i \ z_{ij} \Big)\, a_{kj} \qquad \text{all } k = 1,\ldots, m,$$

analogously player **II**.
(„all i“ instead of $\sum_i$ = incentive constraints of CE)

Example: Rock–Scissors–Paper

WCE–payoff = 0

I \ II	*r*	*s*	*p*
R	0 0	–2 1	1 –2
S	1 –2	0 0	–2 1
P	–2 1	1 –2	0 0

1/3	**0**	**0**
0	**1/3**	**0**
0	**0**	**1/3**

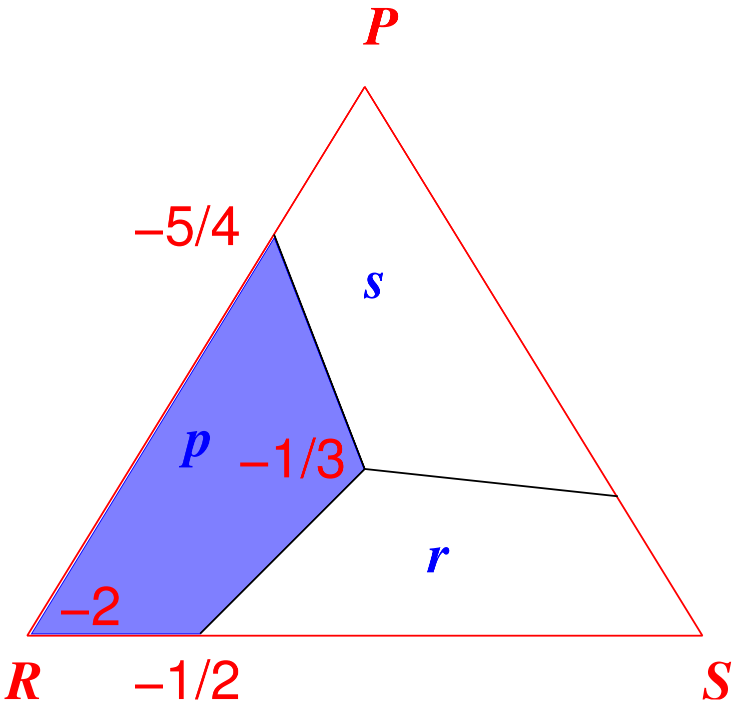
Example: Rock–Scissors–Paper

II I	*r*	*s*	*p*
R	0 0	−2 1	1 −2
S	1 −2	0 0	−2 1
P	−2 1	1 −2	0 0

WCE–payoff = 0

Nash = CE = maxmin =

leader–payoff = −1/3 < 0 !



3 (or more) players

- player **I** commits to mixed strategy
- **II**, **III** play equilibrium of resulting 2-player game (subgame perfection)

⇒ commitment may hurt player **I** !

Example:

II and **III** team with identical, zero-sum payoffs against **I**.

Then commitment by **I** helps **II**, **III** to co-ordinate, usually worse for **I** .

Example: Leader vs. 2−player team

II \ III	p	q
P	−1 1 1	0 0 0
Q	0 0 0	−4 4 4

I: L

II \ III	p	q
P	−4 4 4	0 0 0
Q	0 0 0	−1 1 1

R