Sperner's Lemma

- simplex S with r vertices, triangulation T
- each vertex of T has color in {1, ..., r}
- color of a vertex of S **not** found on opposite facet (Sperner condition).

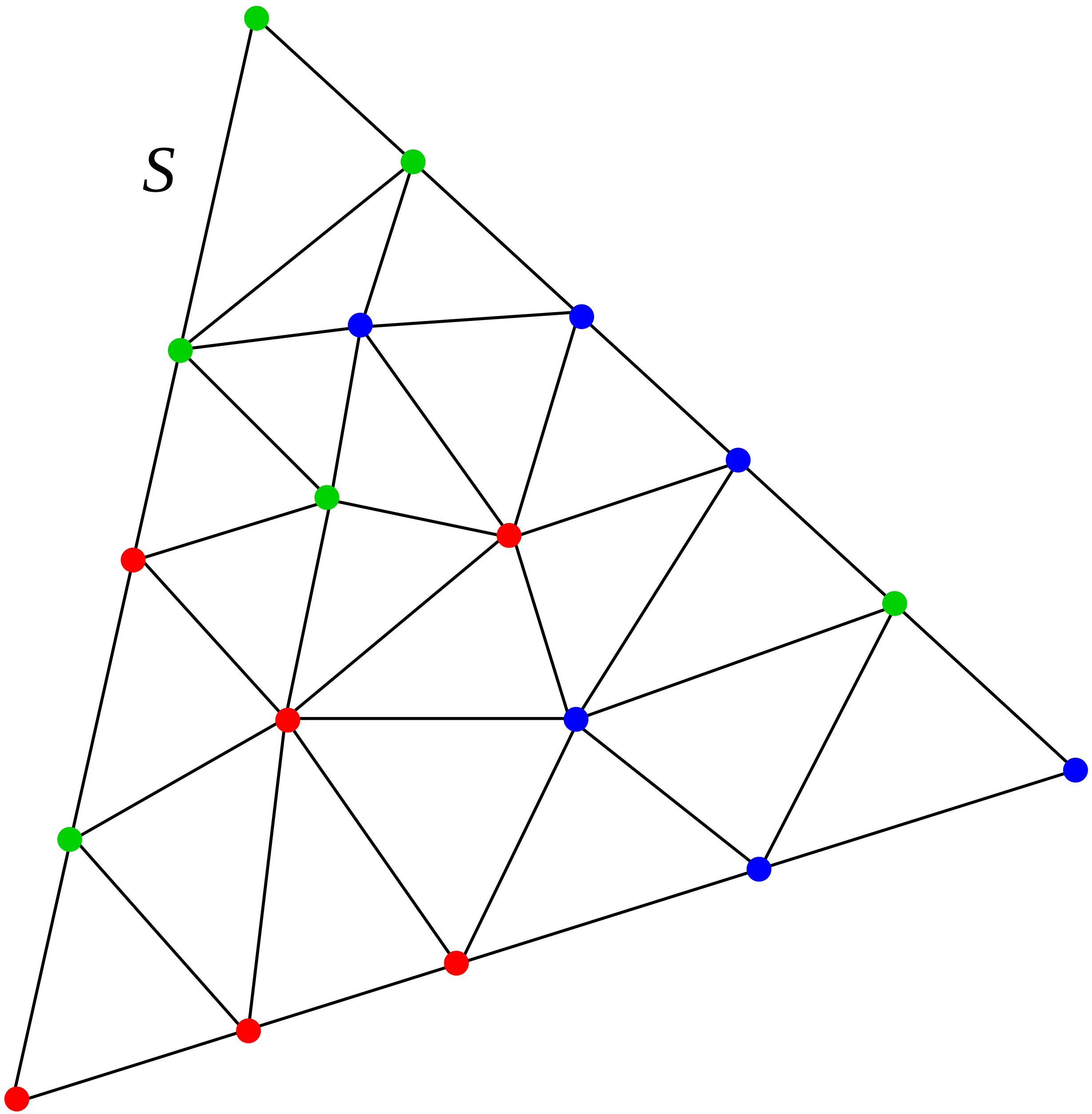
Theorem:

T has an **odd** number of panchromatic simplices.

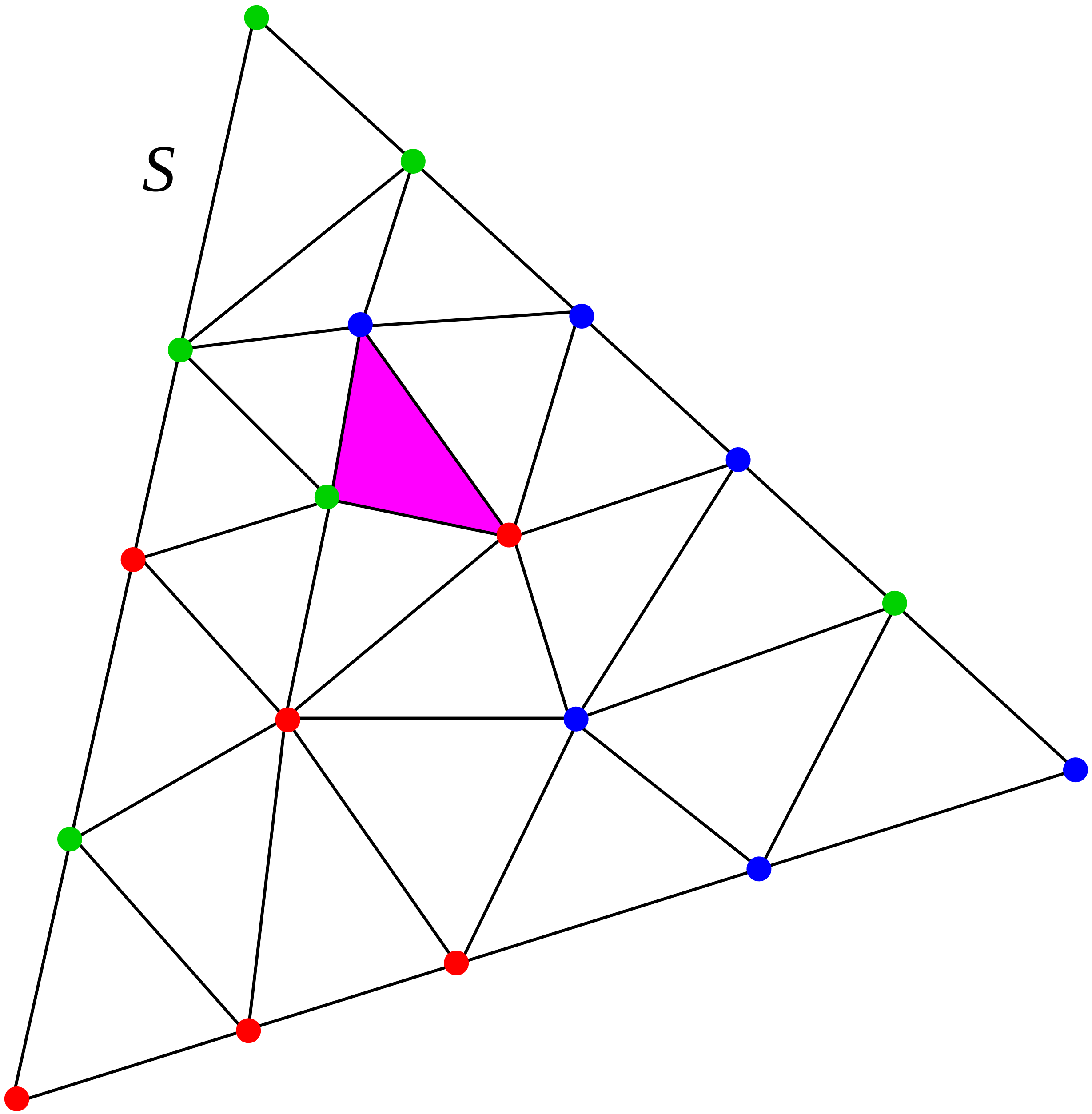
Proof:

Inductive hypothesis (lower dimension): each facet has an odd number of panchromatic simplices on that facet.

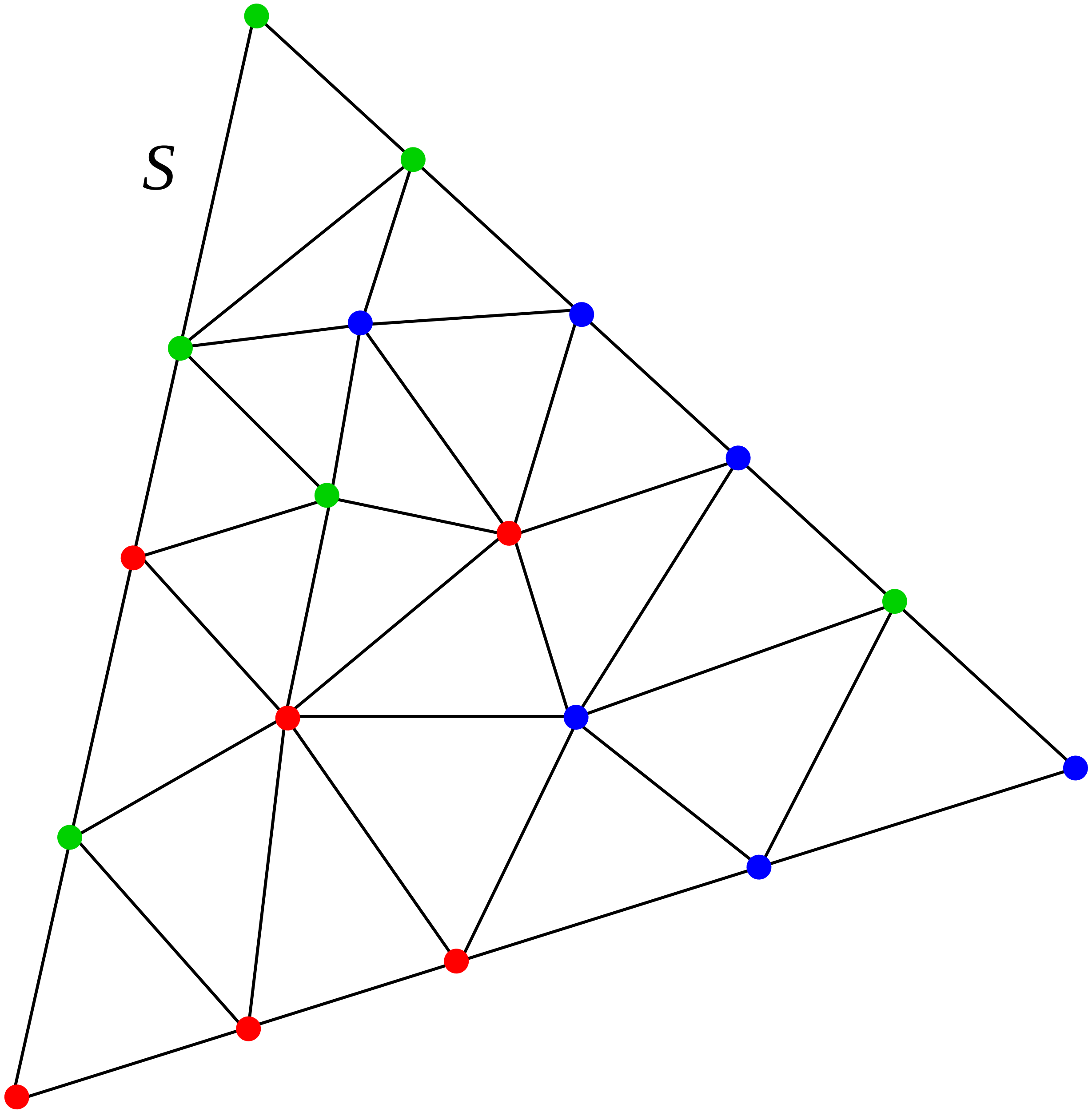
Simplex S with triangulation T and colors



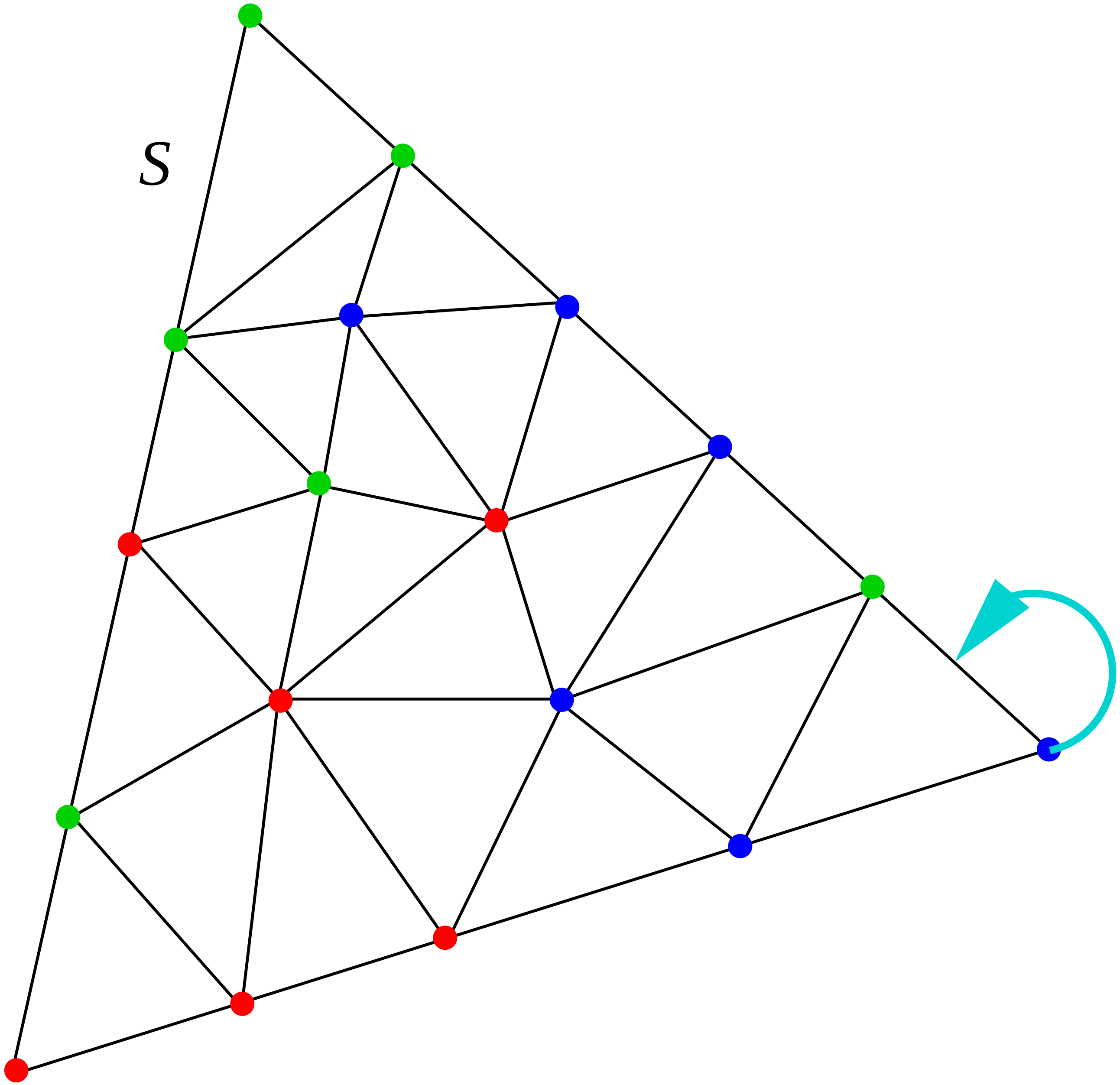
1 panchromatic simplex



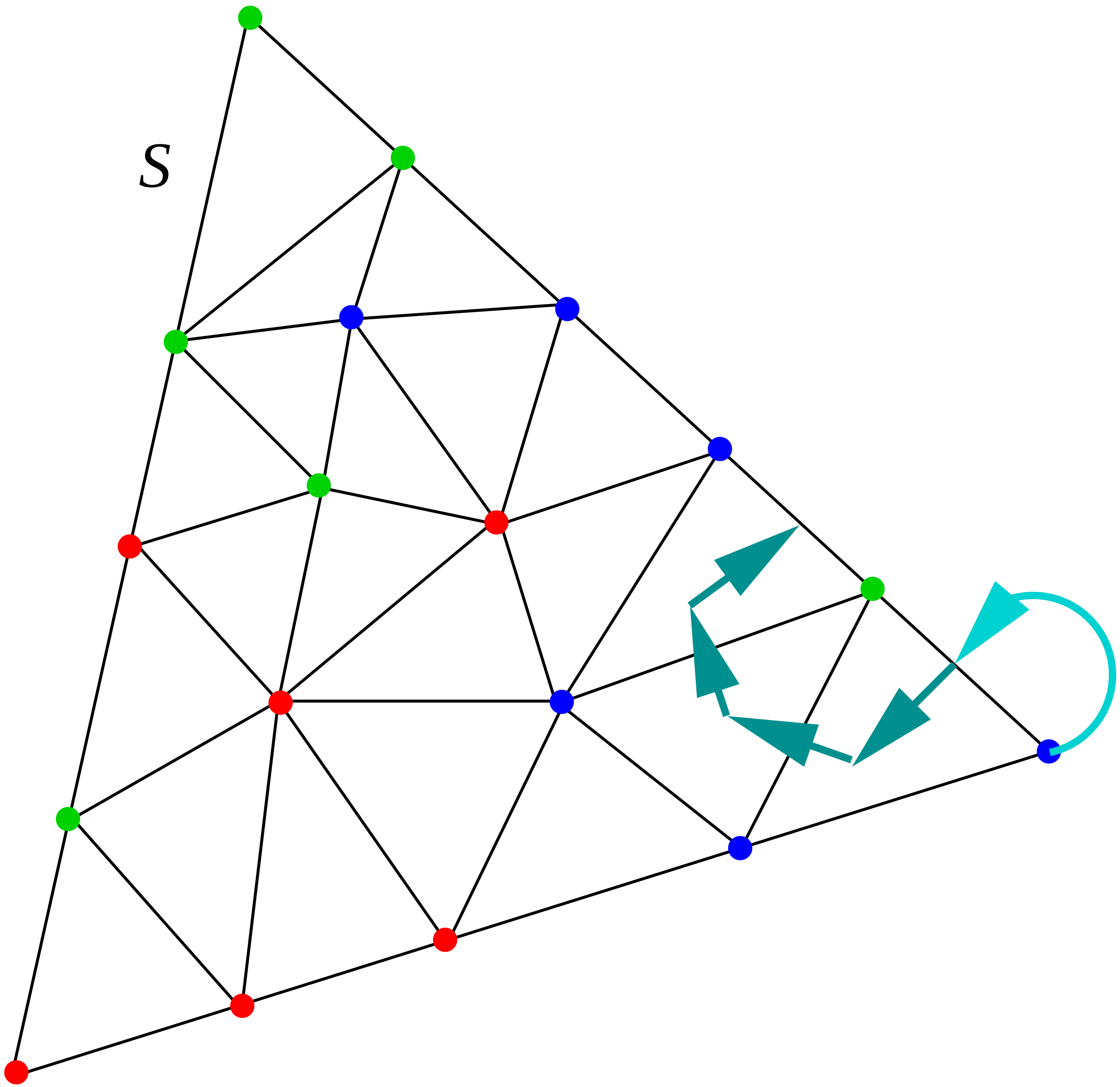
start at blue vertex, look for green



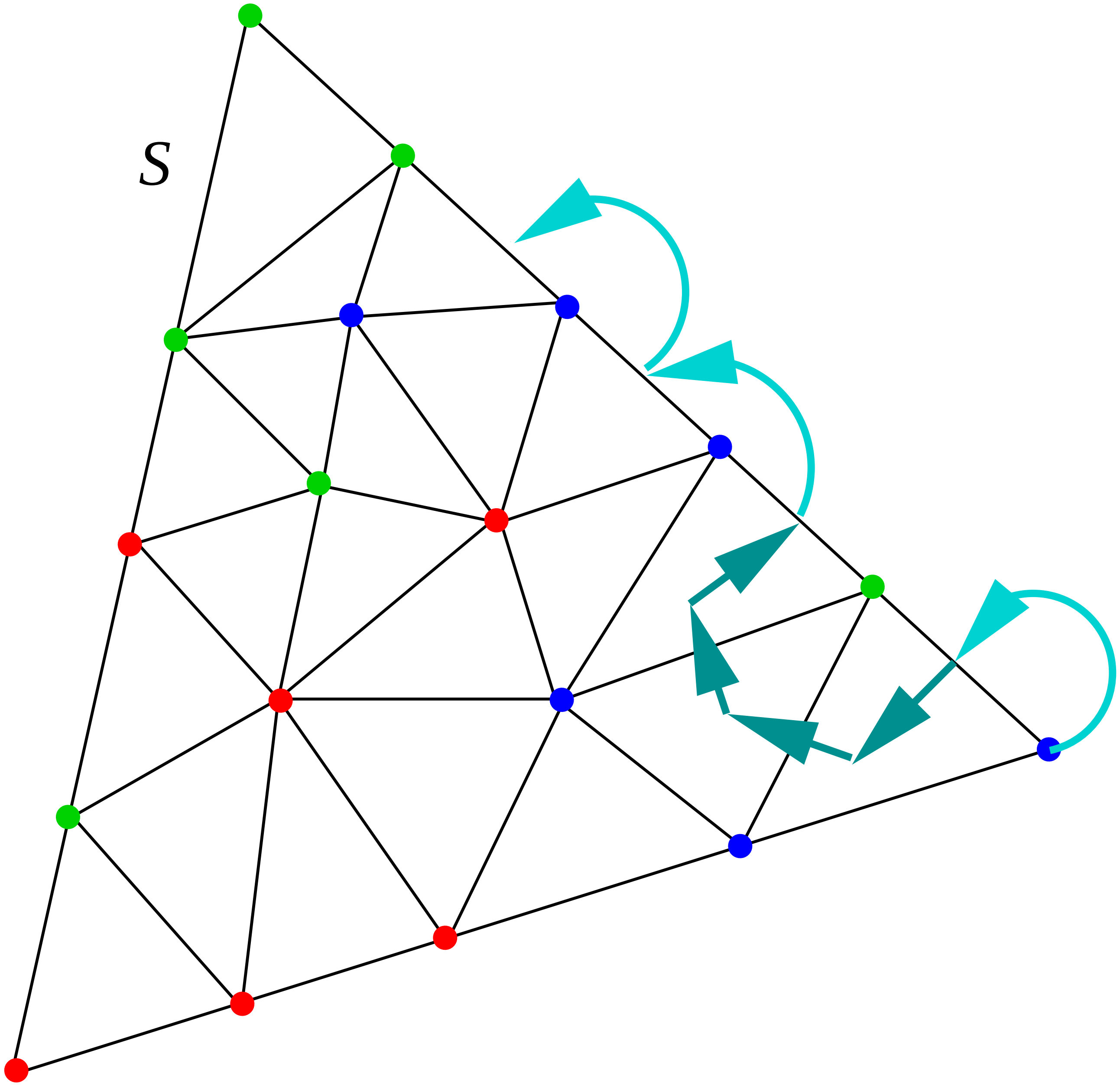
found green, look for red



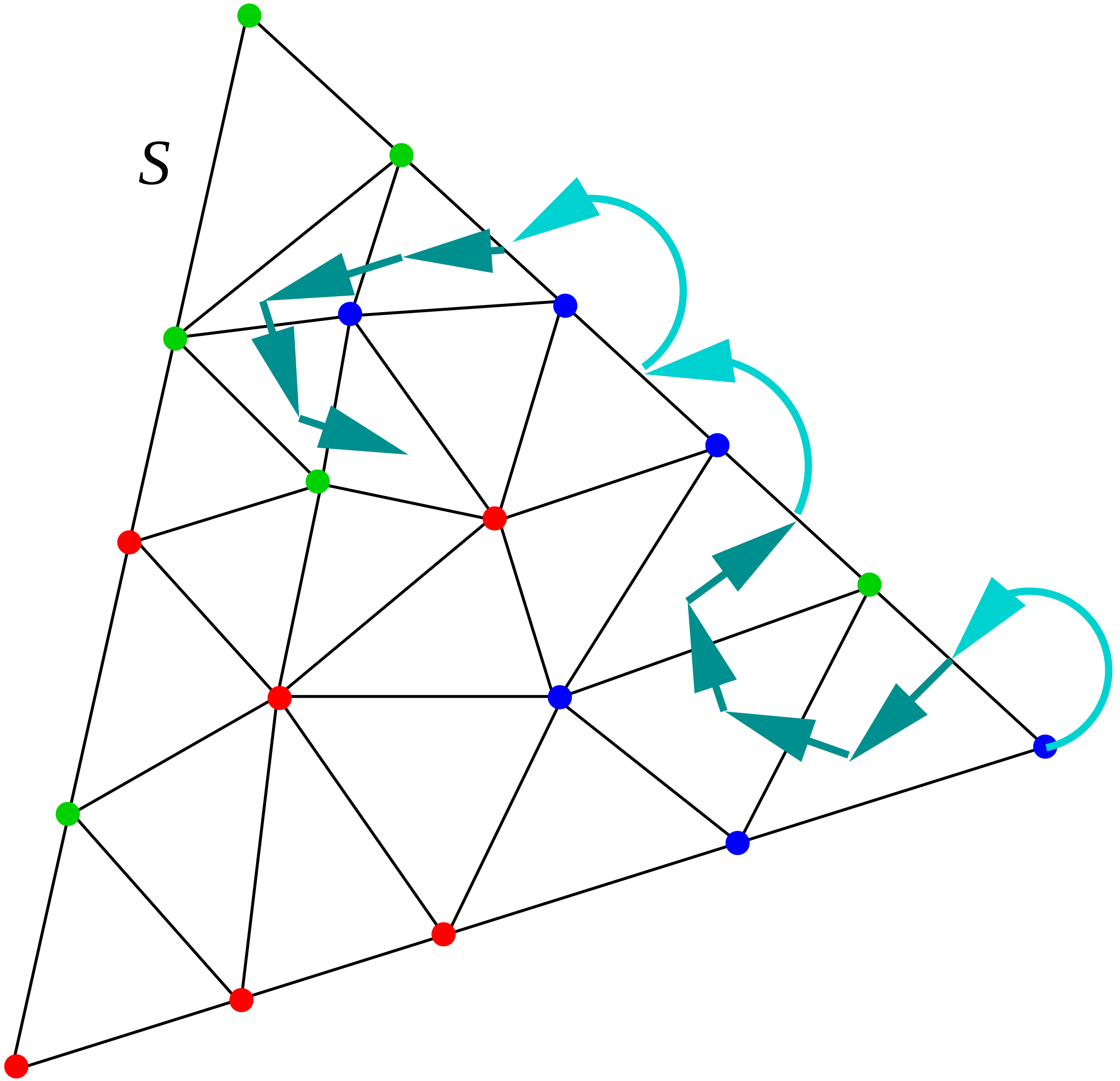
got back to lower dimension



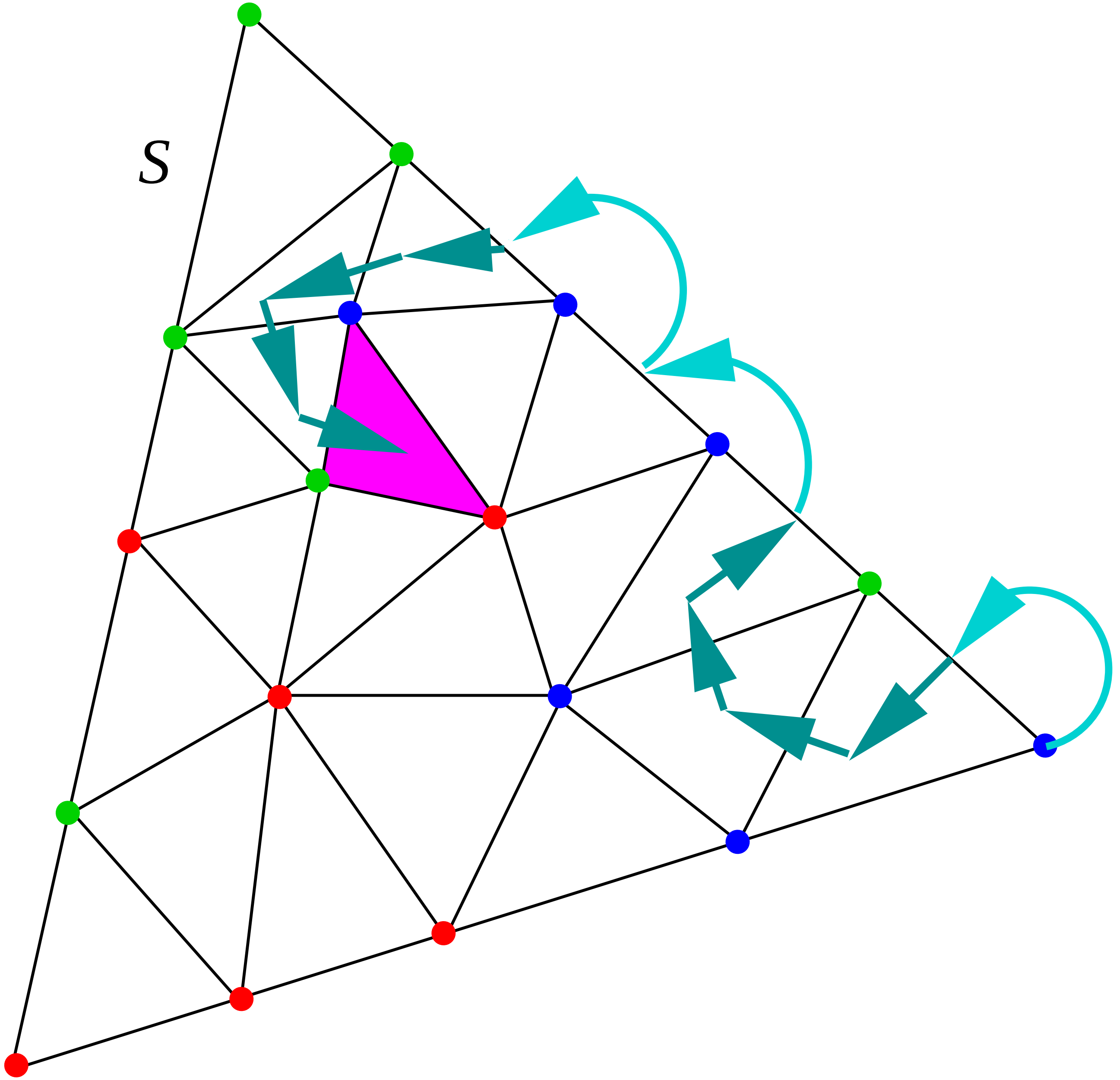
look for green



found green, look for red



found red, panchromatic simplex found



Brouwer Fixed Points via Sperner

Brouwer's FPT: Unit simplex S, continuous f: S $\rightarrow$ S.
Then f has a **fixed point**: f(**s**) = **s** for some **s** $\in$ S.

Proof:
$x = (x_1, \ldots, x_r)$, $f(x) = (f_1(x), \ldots, f_r(x))$.

Triangulate S, each vertex x gets some color
$i \in \{1, \ldots, r\}$ if $x_i > 0$ and $x_i \geq f_i(x)$.

Panchromatic simplex via Sperner:
vertices are "approximately fixed" points.
Take finer and finer triangulations and convergent subsequence of (e.g., blue) vertices of panchromatic simplices: Limit is fixed point.

Combinatorial manifolds

Given: **rank** r

collection M of r-element sets called **rooms**
(also: abstract simplicial complex M, rooms = simplices)

set of **vertices** $V = \bigcup M$

wall = room without a vertex v (wall "opposite" v)

any wall belongs to exactly 2 rooms
(i.e. any $(r-1)$-set of vertices belongs to 0 or 2 rooms)

Call M a **manifold**.

Abstract Sperner

- Manifold M of rank r
- each vertex v has label (color) in {1, . . ., r}
- call a set of vertices **panchromatic** if no two of its vertices have the same color.

Theorem:

M has an **even** number of panchromatic rooms.

Abstract Sperner

- Manifold M of rank r
- each vertex v has label (color) in {1, . . ., r}
- call a set of vertices **panchromatic** if no two of its vertices have the same color.

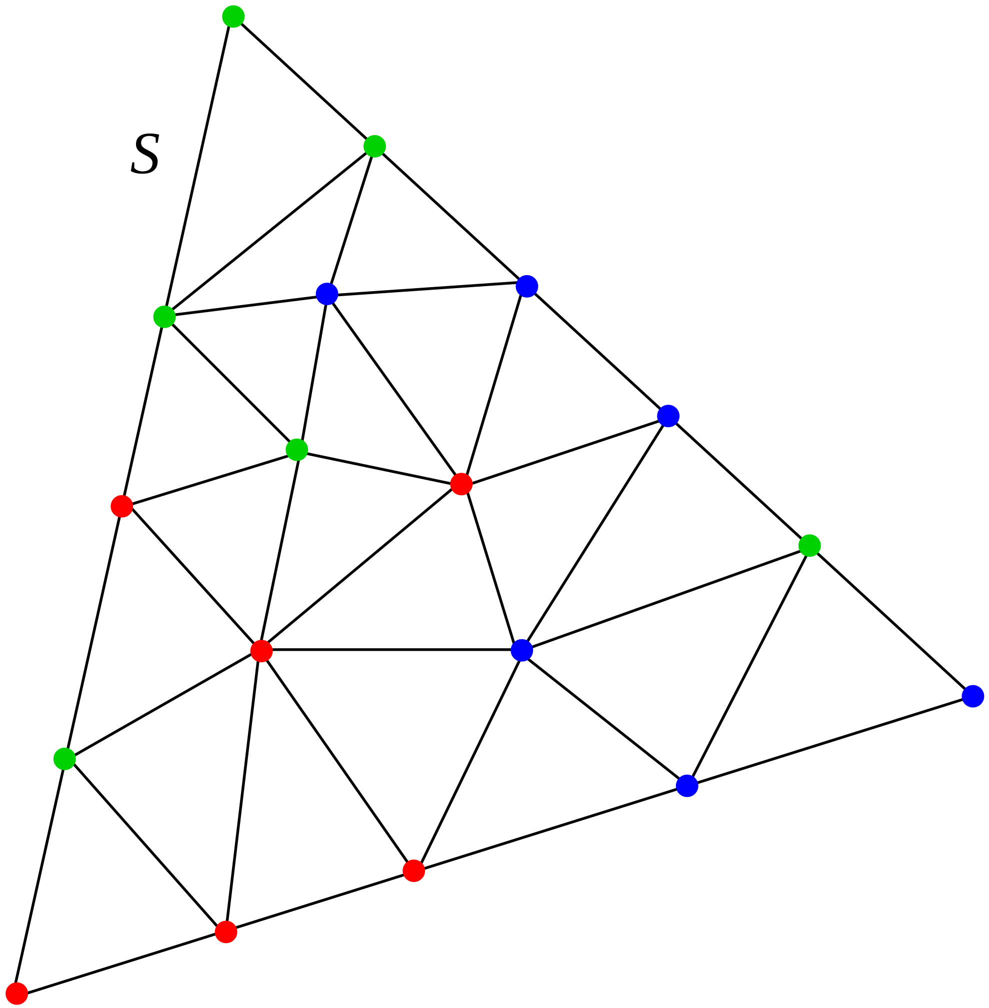
Theorem:

M has an **even** number of panchromatic rooms.

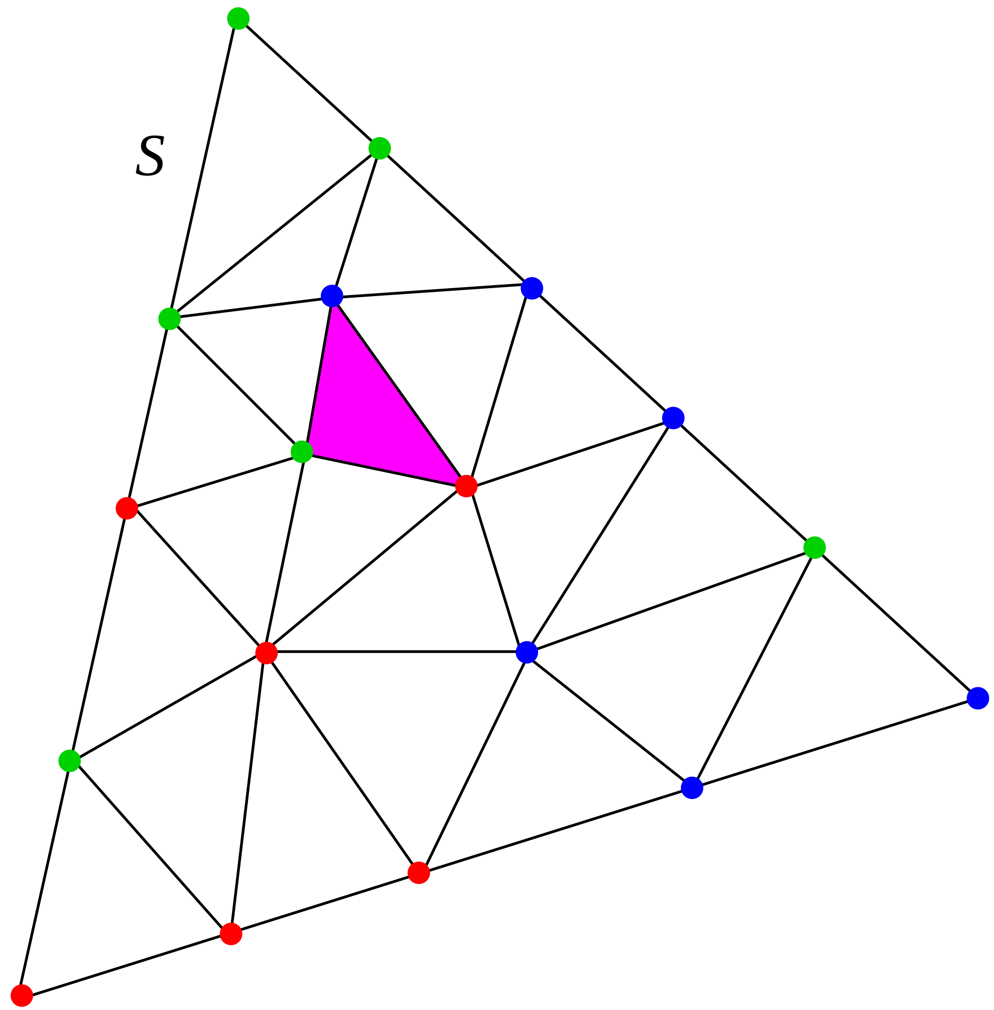
Corollary:

Sperner's Lemma.

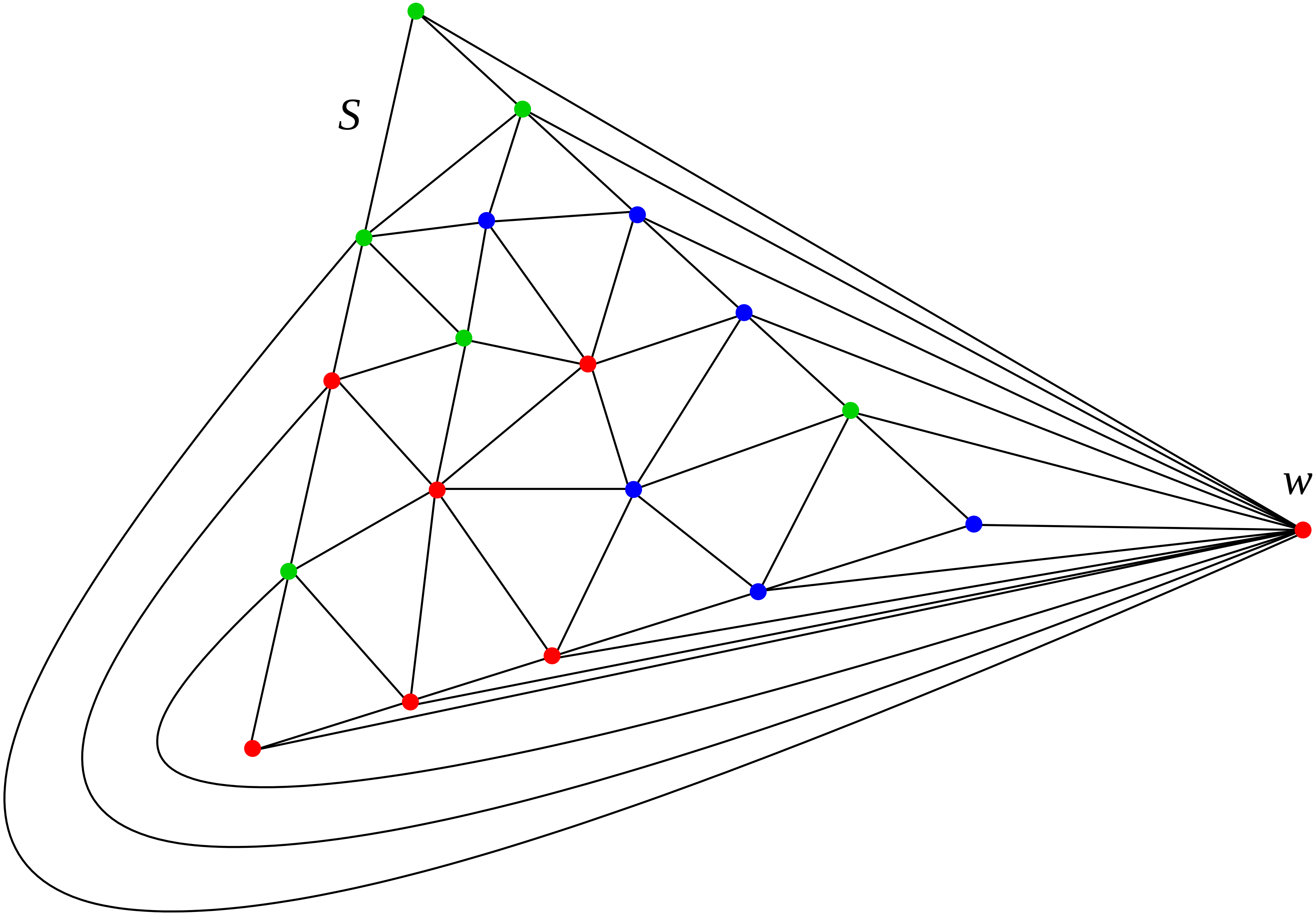
Sperner's lemma via Abstract Sperner



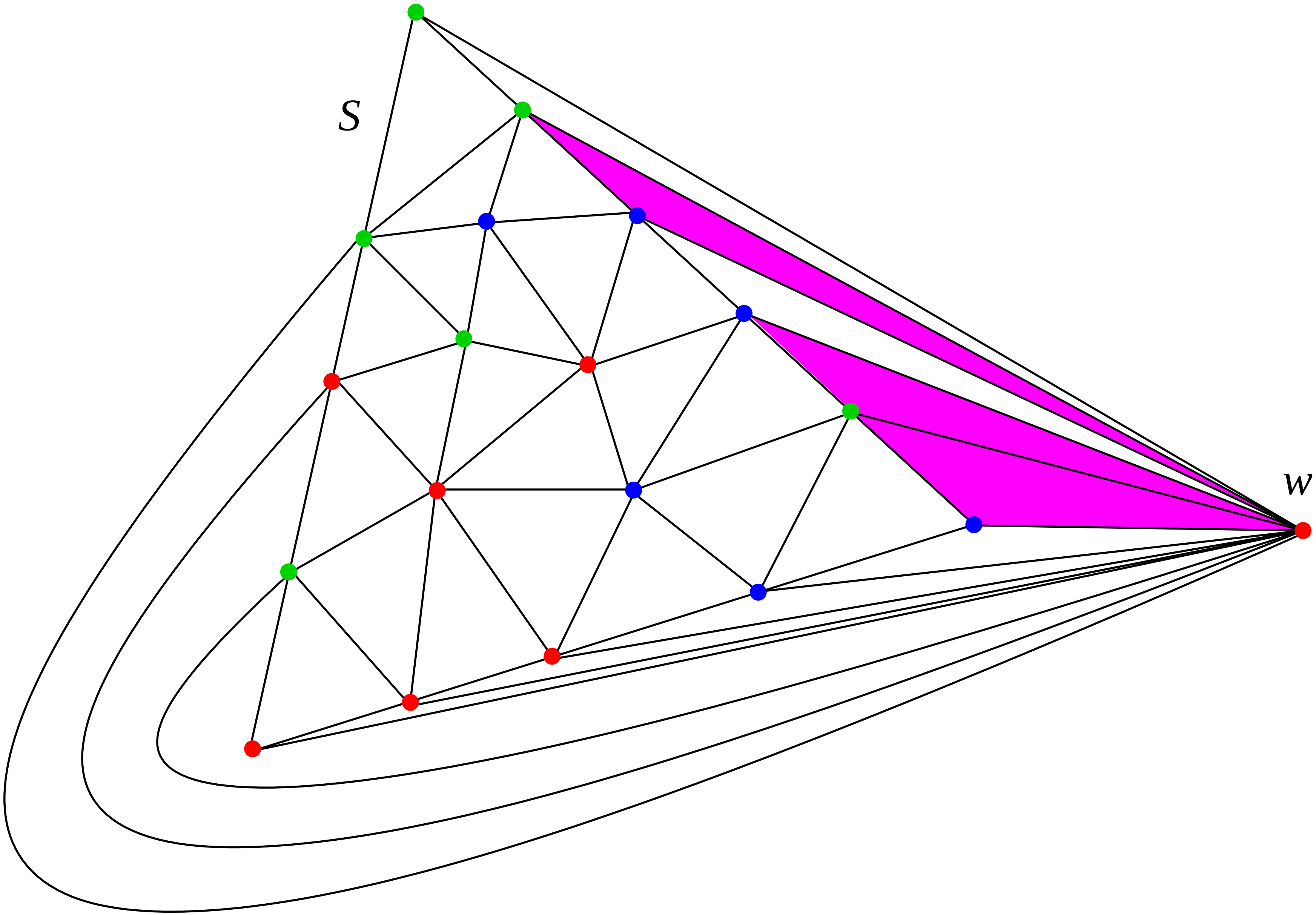
Want to show:
Odd number of panchromatic simplices



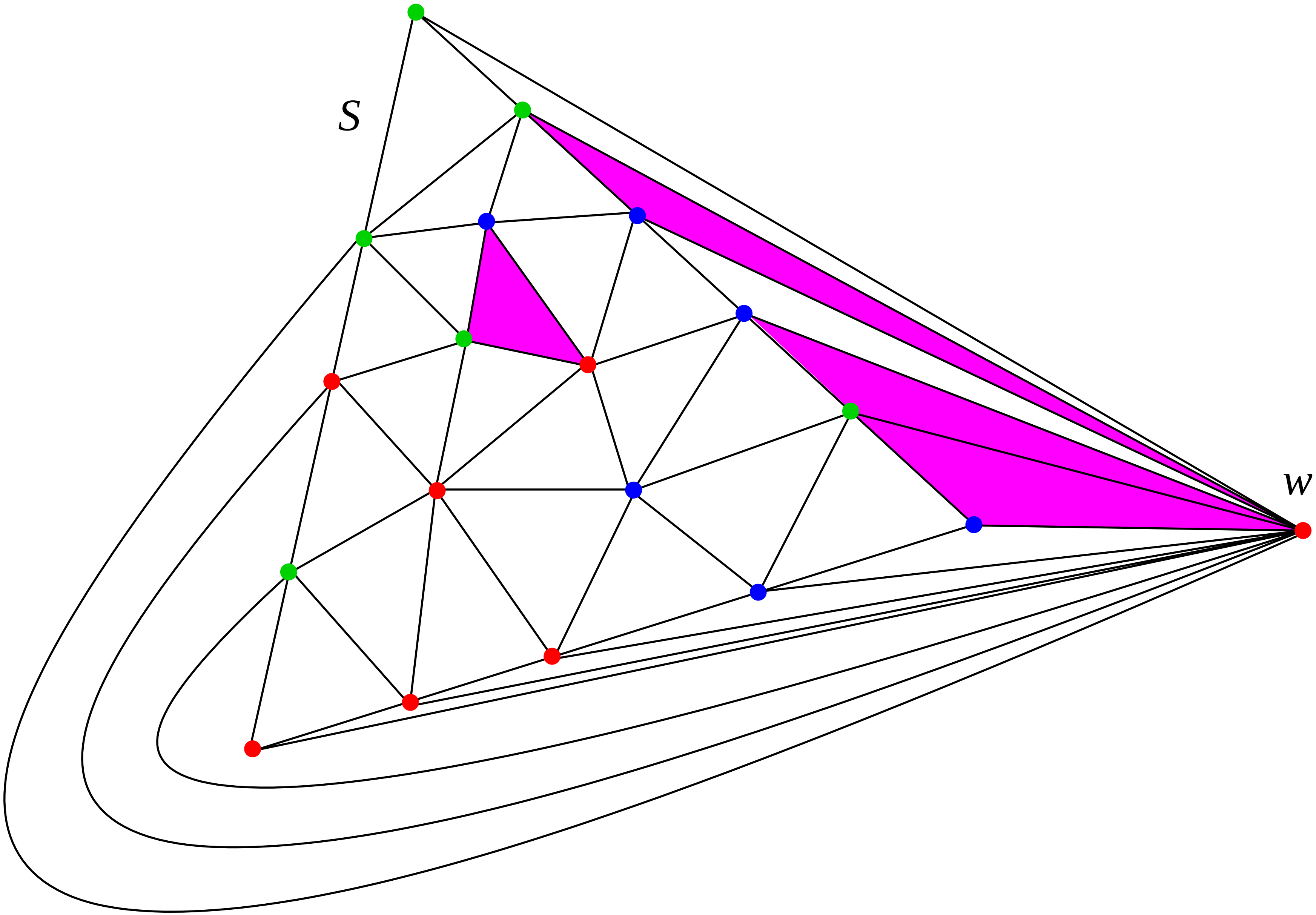
Add new vertex **w**, any color, connect to outside vertices to get manifold



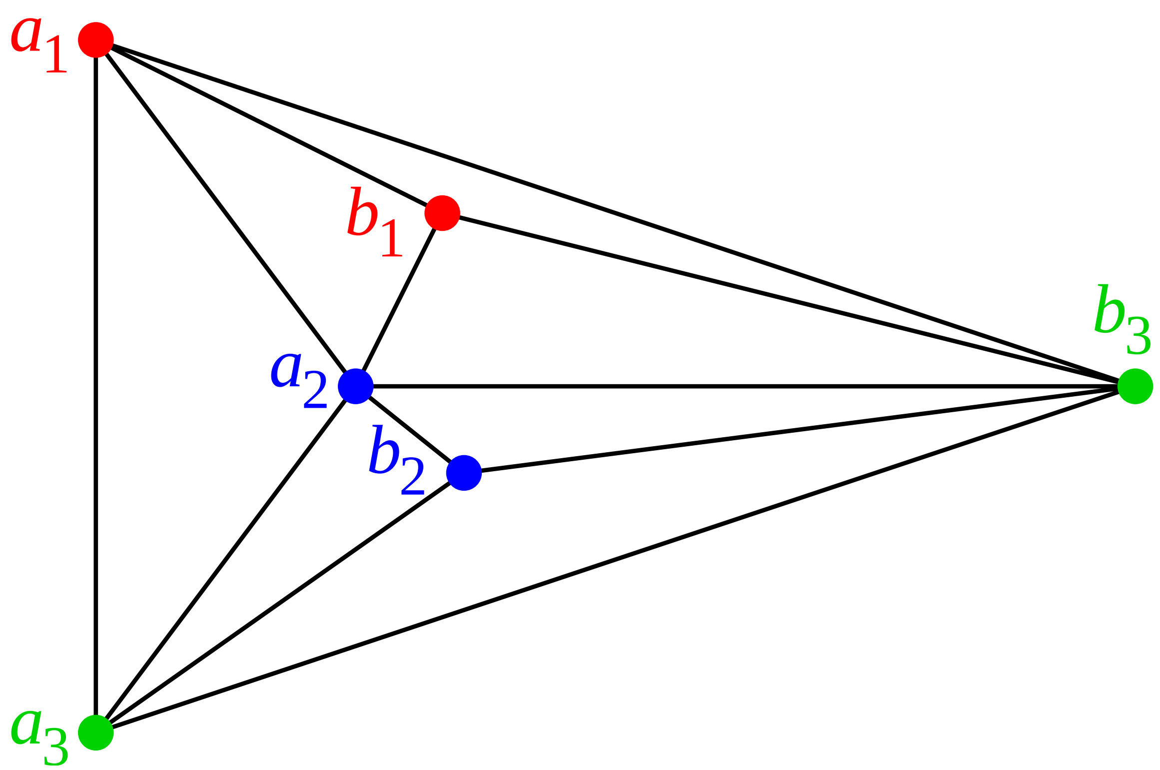
By induction: odd # of panchromatic rooms that contain w (outside rooms)



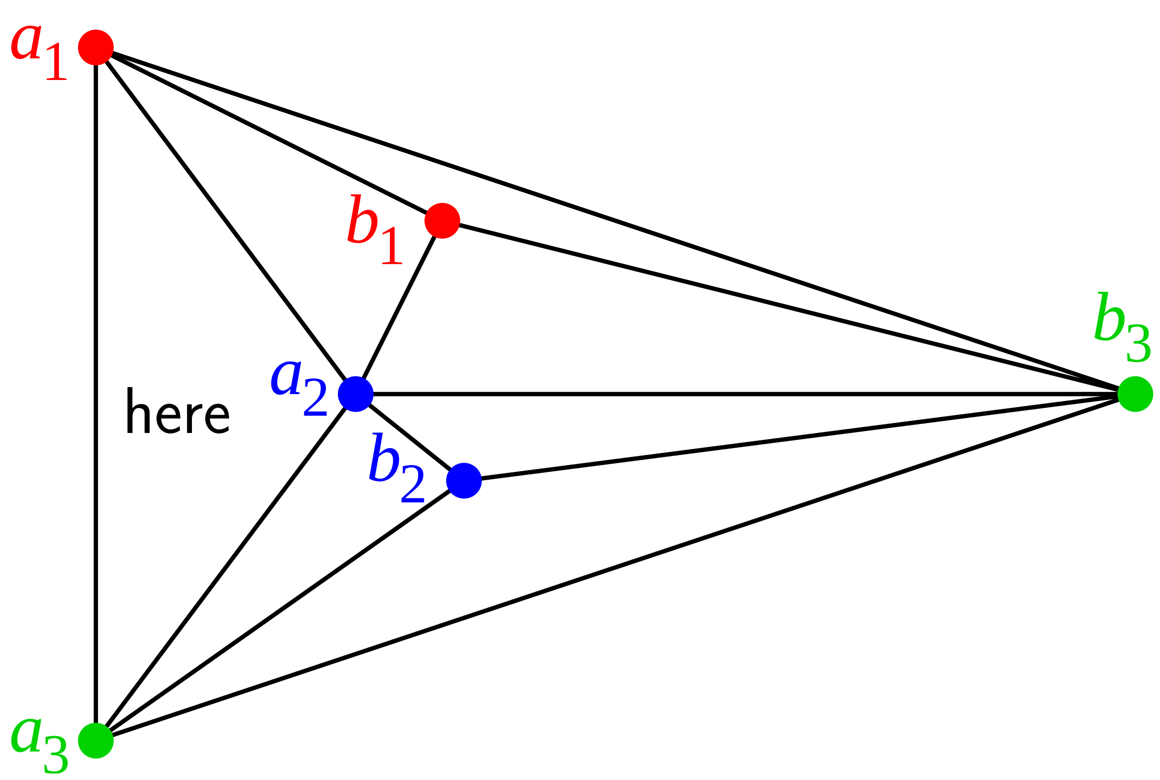
By Abstract Sperner (even total #):
odd # of inside panchromatic rooms



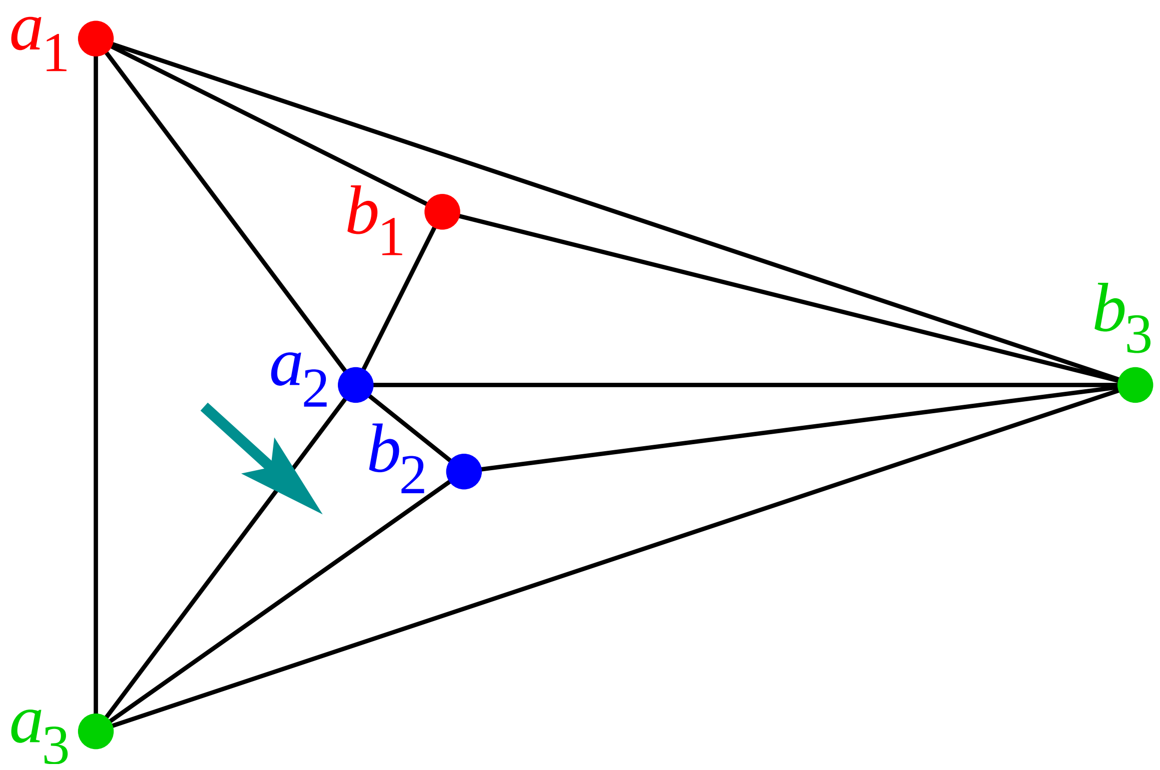
Abstract Sperner: Proof



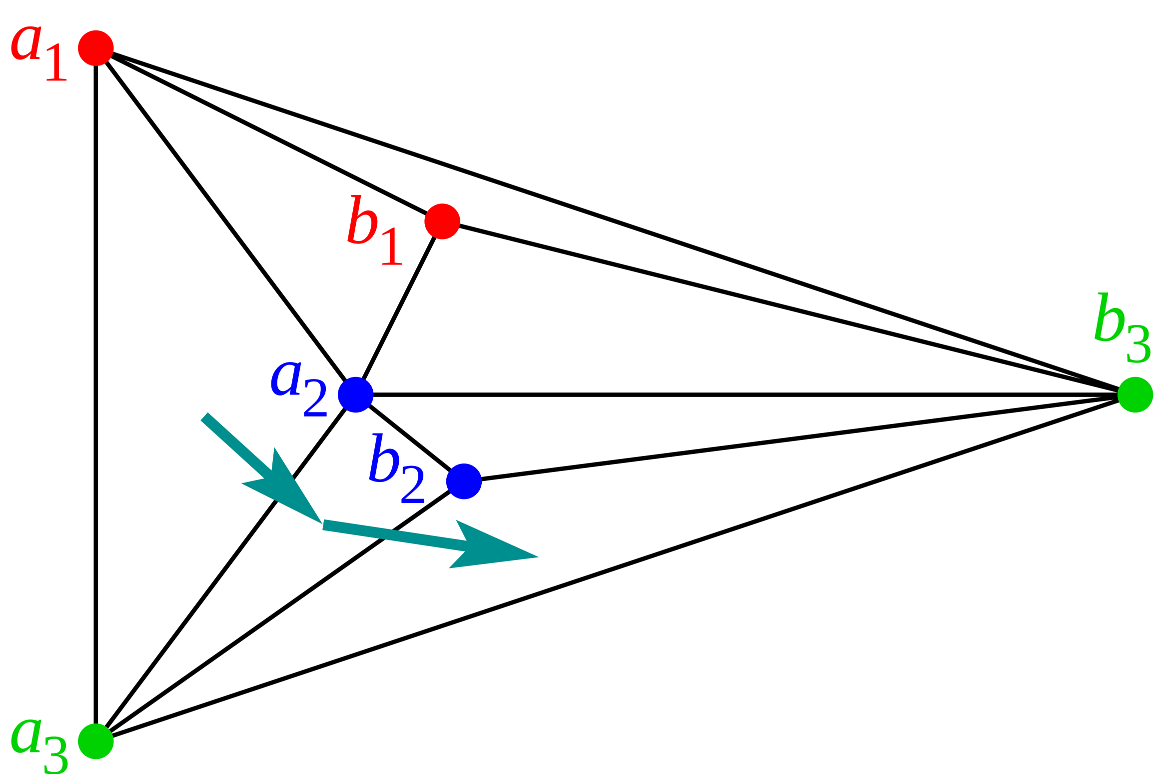
start at panchromatic room



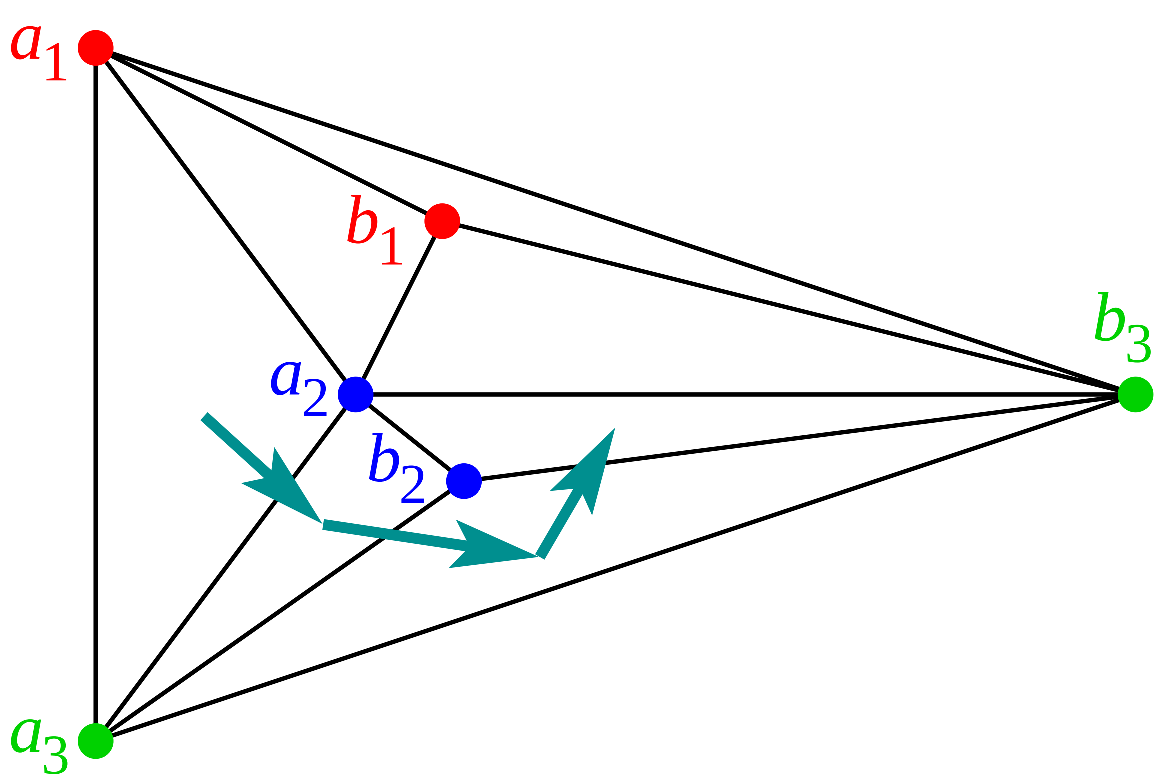
allow missing color 1



allow missing color 1



allow missing color 1



find color 1, done!