NCNC: Nash Codes for Noisy Channels

Penélope Hernández and Bernhard von Stengel

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Tenerife Airport, Canary Islands, 27 March 1977. Fog.

Two jumbo jets, from KLM and PanAm.

PanAm 1736 is taxiing back on the **runway**.

1705:44 KLM 4805:	The KLM 4805 is now ready for takeoff and we are waiting for your ATC clearance.
1705:53 Tower:	KLM 8705 **you are cleared** to the Papa Beacon, climb to maintain flight level [...]
1706:09 KLM 4805:	Ah–roger sir, we are cleared to the Papa beacon [...]

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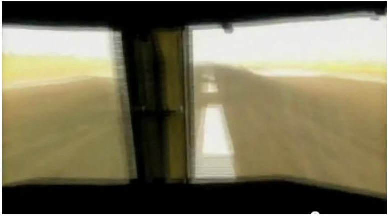

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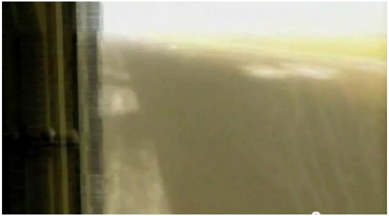

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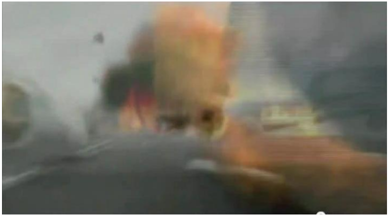
1706:50 **collision**	

Survivors from the KLM flight: 0, from the PanAm flight: 70.

583 lives lost, the deadliest aviation accident in history.

Motivation

- Communication is basic to interaction
- Coordination may be hindered by communication **errors**, including imprecisely worded or misunderstood messages
- Dealing with errors deserves game-theoretic analysis
- Model: noisy channel, requires codebook
- Using the codebook should define a Nash equilibrium
- We will describe *some* equilibrium codes.

Sender-receiver games in economics

- **Sender**, fully informed about state of nature
  sends message to **receiver**, who chooses action

- Crawford and Sobel (*Econometrica* 1982):
  state and message from $[0, 1]$,
  single-peaked but non-identical preference for action

  Equilibrium: finite partition of $[0, 1]$, sender only tells partition
  $\Rightarrow$ noise introduced strategically, endogenous from model

- Our model of communication: consider

  – given finitely many states and possible messages,
  – coinciding interests of sender and receiver,
  – noise exogenously given by channel

Sender-receiver game and noisy channel

- Two players: Sender and Receiver
- Nature chooses a *state* i from a set $\Omega = \{0, 1, \ldots, M-1\}$ with positive *prior* probability q_i
- Channel:
  - Input set X, output set Y.
  - Transition probabilities $p(y|x)$ for each $x \in X$, $y \in Y$.
  - The channel is used n times independently without feedback.
  - An input $x = (x_1, \ldots, x_n)$ is transmitted through the channel. It is altered to an output $y = (y_1, \ldots, y_n)$ according to the probability $p(y|x)$ given by

$$p(y|x) = \prod_{j=1}^{n} p(y_j|x_j).$$

Strategies

- Sender strategy: A codebook $(x^0, x^1, \ldots, x^{M-1})$ where x^i is the codeword for state i:

$$\begin{aligned} \Omega &\rightarrow X^n \\ i &\mapsto x^i \end{aligned}$$

- Receiver strategy: The receiver uses a probabilistic decoding function

$$d : Y^n \times \Omega \rightarrow \mathbb{R},$$

where $d(y, i)$ is the probability that y is decoded as i.

Payoff / Nash equilibrium

- Sender and receiver have common interest: If state i is decoded correctly, they get positive payoff U_i and V_i, respectively, otherwise both get payoff zero.

- A codebook $(x^0, x^1, \ldots, x^{M-1})$ defines a Nash equilibrium if:

  - Receiver Condition: $d(y, i) > 0$ only if

$$q_i V_i\, p(y|x^i) \geq q_k V_k p(y|x^k) \ \ \forall k \in \Omega$$

  - Sender Condition: At state i the sender *uses the codebook*, i.e. transmits codeword x^i which fulfills for any other possible channel input x in X^n

$$U_i \sum_{y \in Y^n} p(y|x^i)\, d(y, i) \geq U_i \sum_{y \in Y^n} p(y|x)\, d(y, i).$$

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From best response partition to best response codebook

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with priors

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From best response partition to best response codebook

Receiver Condition:

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with priors and utilities

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From best response partition to best response codebook

Receiver Condition:

$d(y, i) > 0$ only if $y \in Y_i$

with decoding partition with priors and utilities

$$Y_i = \{ y \in Y^n \mid \ q_i V_i \ p(y|x^i) \geq q_k V_k \ p(y|x^k) \ \ \forall k \in \Omega \}$$

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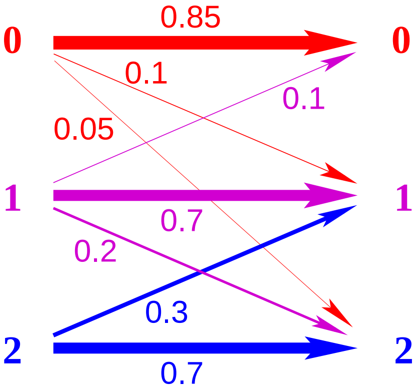
Sender Condition:

$\sum_{y \in Y_i} p(y|x) \, d(y, i)$ maximized for $x = x^i$.

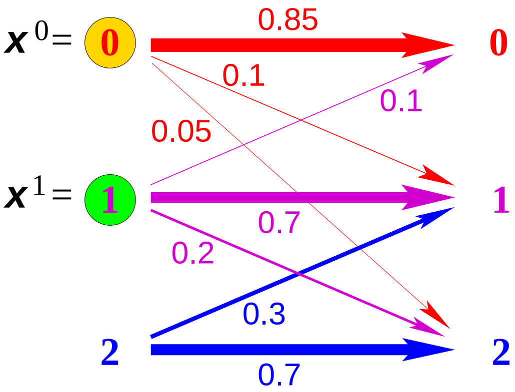
Questions

1. **Is every code a Nash code?**

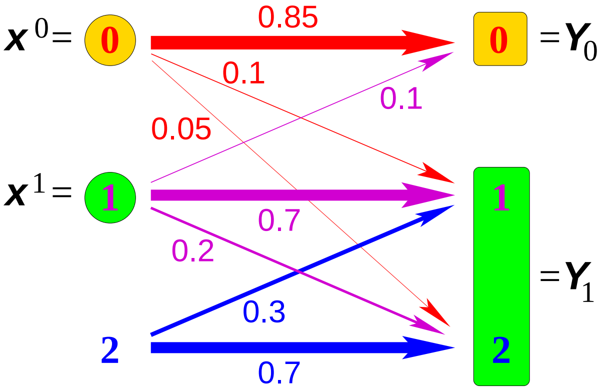
Noisy channel: Example



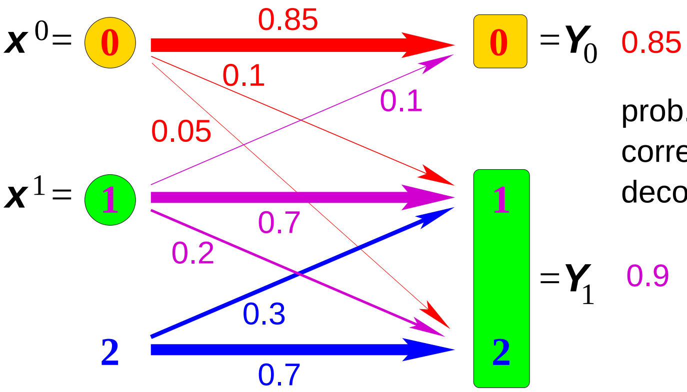
Encoding two states 0 and 1



Best-response decoding: partition of $\boldsymbol{Y}$

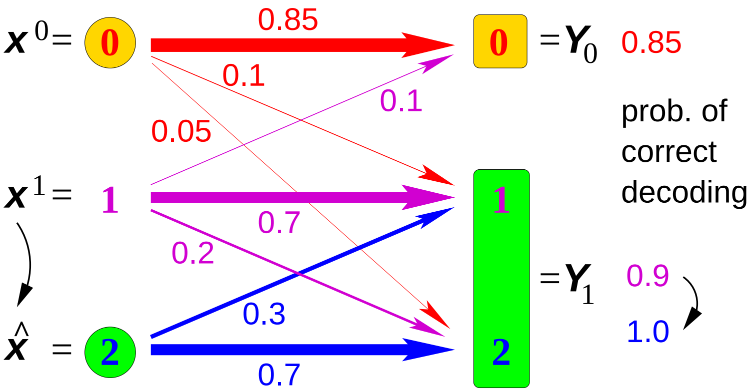


Sender payoff for this code



prob. of correct decoding

Sender deviation: not a Nash code!



Questions

1. Is every code a Nash code? – no

2. **Is some code a Nash code?**

Sufficient condition for Nash codes

Definition:

A receiver-optimal code is a codebook $\boldsymbol{c} = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^{M-1})$ that maximizes the receiver payoff

$$V(\boldsymbol{c}, \boldsymbol{d}) = \sum_{i \in \Omega} q_i V_i \sum_{y \in Y_i} p(y|\boldsymbol{x}^i)\, d(y, i)$$

for best-response decoding $\boldsymbol{d}$.

Theorem: Every receiver-optimal code is a Nash code.

Proof:

Let $\boldsymbol{c} = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^{M-1})$ be a receiver-optimal codebook with best-response decoding $\boldsymbol{d}$.

Profitable sender deviation from $\boldsymbol{x}^i$ to $\hat{\boldsymbol{x}}$ means

$$\sum_{y \in Y_i} p(y|\hat{\boldsymbol{x}})\, d(y,i) > \sum_{y \in Y_i} p(y|\boldsymbol{x}^i)\, d(y,i)$$

$\Rightarrow$ sender **and** receiver improve

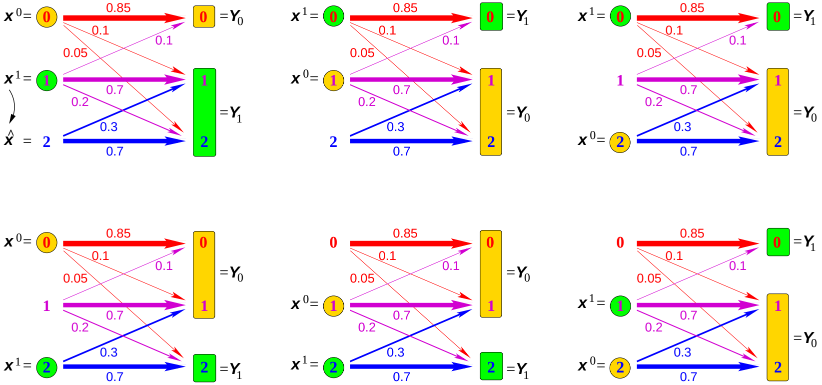
$\Rightarrow$ for codebook $\boldsymbol{c}' = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^{i-1}, \hat{\boldsymbol{x}}, \boldsymbol{x}^{i+1}, \ldots, \boldsymbol{x}^{M-1})$ and best-response decoding $\boldsymbol{d}'$: Receiver payoffs fulfill

$$V(\boldsymbol{c}, \boldsymbol{d}) < V(\boldsymbol{c}', \boldsymbol{d}) \leq V(\boldsymbol{c}', \boldsymbol{d}')$$

$\Rightarrow$ $\boldsymbol{c}'$ is better code for receiver than $\boldsymbol{c}$, contradiction.

Sender-optimal code not necessarily Nash code!

utilities for states 0, 1 for sender: $U_0 = 1.0$, $U_1 = 9.0$
receiver: $V_0 = 7.6$, $V_1 = 2.4$



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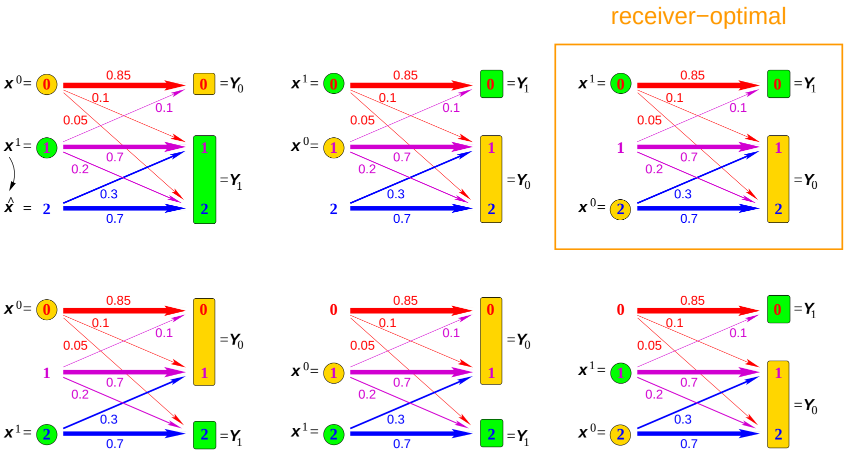

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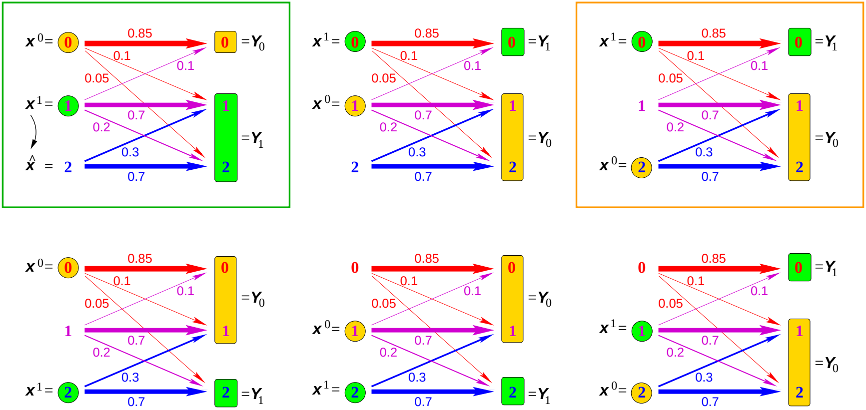


Questions

1. Is every code a Nash code? – no

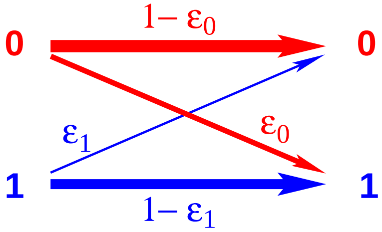
2. Is some code a Nash code? – yes, receiver-optimal code is Nash

3. **Do small alphabets allow for more Nash codes?**

Binary channel

$X = Y = \{0, 1\}$

Transmission errors:



Can assume $\varepsilon_0 + \varepsilon_1 < 1$.

Symmetric channel: $\varepsilon_0 = \varepsilon_1 = \varepsilon$

Use n times independently.

Conditions needed for binary code to be Nash

- ? receiver-optimal code
- ? symmetric channel errors, $\varepsilon_0 = \varepsilon_1$
- ? uniform priors q_i
- ? unit payoffs, $V_i = 1$
- ? equal payoffs for sender and receiver, $U_i = V_i$
- ? consistent tie breaking if $q_i V_i \, p(y|x^i) = q_k V_k \, p(y|x^k)$

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Examples: uniform (or any fixed) probability among tied states i, k; fixed-order tie breaking (always i before k).

For generic priors q_i there are no ties.

Conditions needed for binary code to be Nash

no receiver-optimal code

no symmetric channel errors, $\varepsilon_0 = \varepsilon_1$

no uniform priors q_i

no unit payoffs, $V_i = 1$

no equal payoffs for sender and receiver, $U_i = V_i$

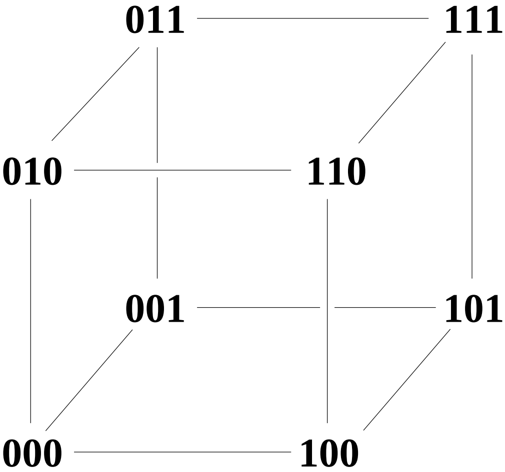
yes consistent tie breaking if $q_i V_i \, p(y|x^i) = q_k V_k \, p(y|x^k)$

Examples: uniform (or any fixed) probability among tied states i, k; fixed-order tie breaking (always i before k).

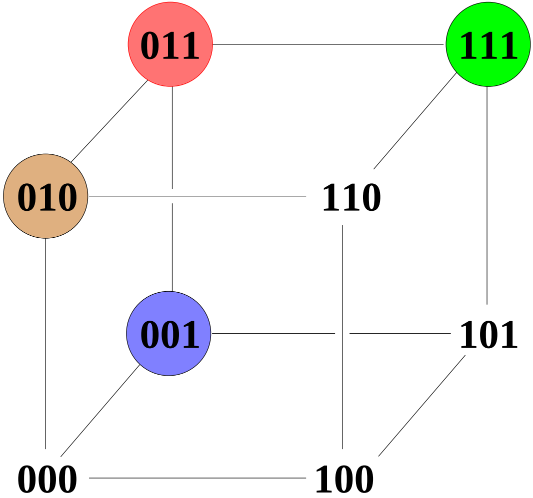
For generic priors q_i there are no ties.

Theorem: Every consistently decoded binary code is a Nash code.

Binary channel, codewords length 3

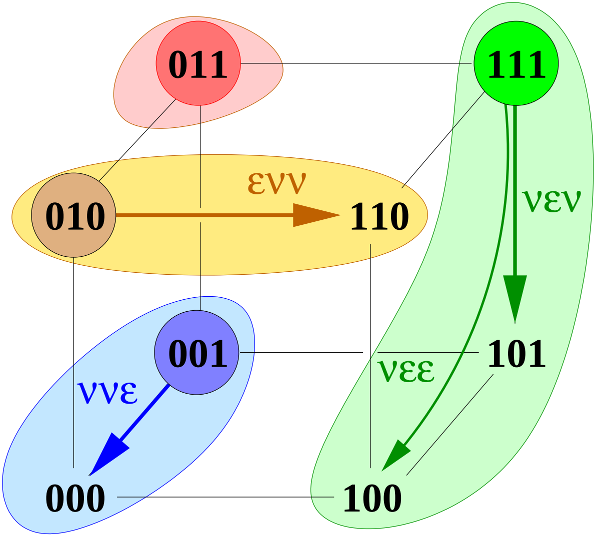


Four codewords



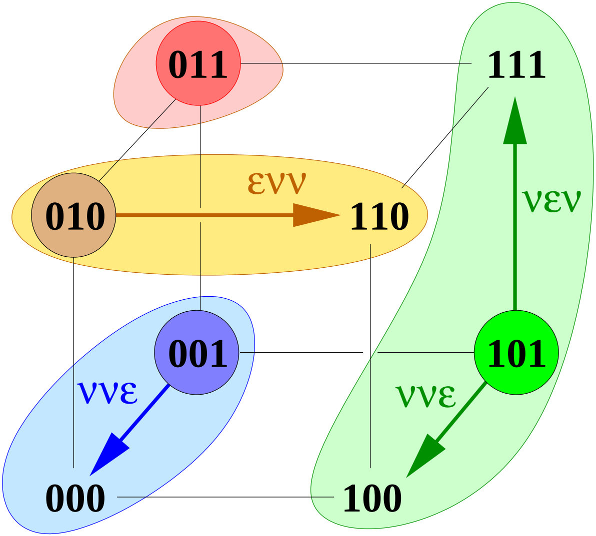
A decoding partition and its errors

$\nu = 1 - \varepsilon > \varepsilon$



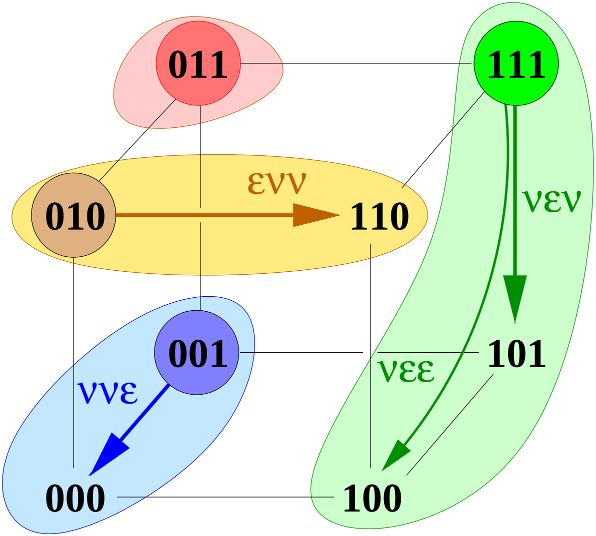
Deviate and get smaller errors: not Nash!

$\nu = 1 - \varepsilon > \varepsilon$

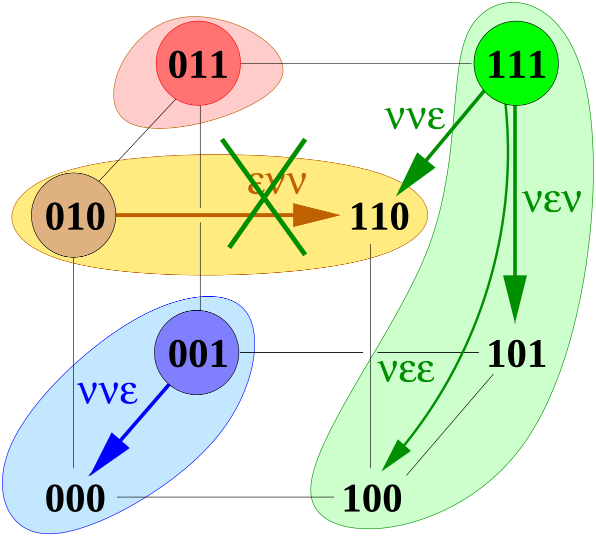


Decoding inconsistent: 110 should decode as 111

$$\nu = 1 - \varepsilon > \varepsilon$$



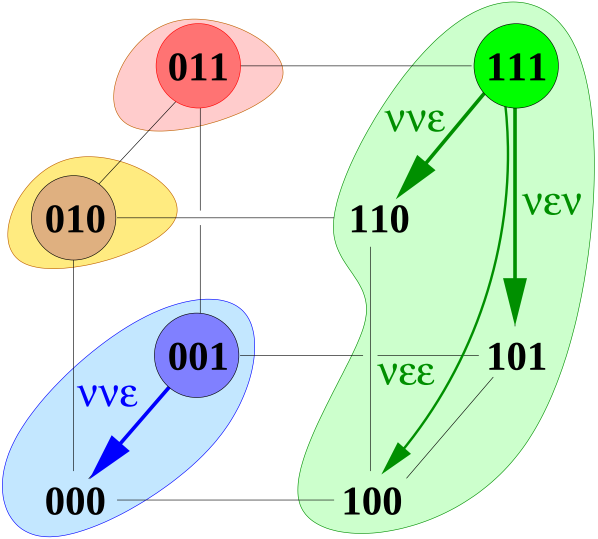
... because 100 decodes as 111



$\nu = 1 - \varepsilon > \varepsilon$

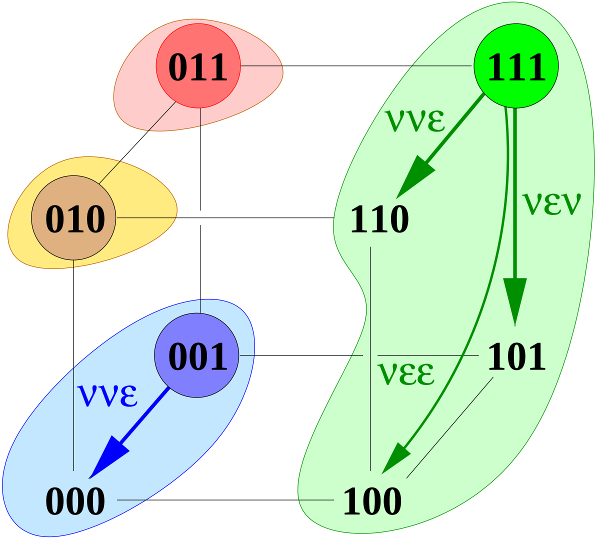
This decoding is consistent

$\nu = 1 - \varepsilon > \varepsilon$



This decoding is consistent ... and Nash

$\nu = 1 - \varepsilon > \varepsilon$



Main feature of consistent decoding

Decoding is **monotone**:

if y decoded as i and y' agrees with codeword x^i more than y, then y' also decoded as i.

Example: $x^i = \mathbf{111}$, $y = \mathbf{100}$, $y' = \mathbf{110}$

Main feature of consistent decoding

Decoding is **monotone**:

if y decoded as i and y' agrees with codeword x^i more than y, then y' also decoded as i.

Example: $x^i = 111$, $y = 100$, $y' = 110$

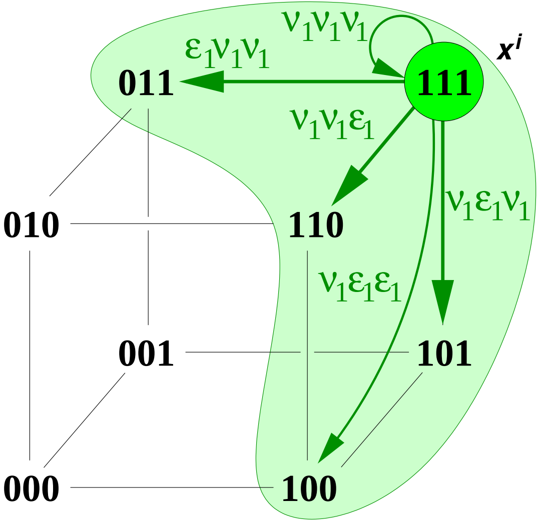
Definition of “consistent decoding” states a related monotonicity for decoding probabilities and sets of tied states:

$$i \in \{k \in \Omega \mid y' \in Y_k\} \subseteq \{k \in \Omega \mid y \in Y_k\}$$

$$\Rightarrow \quad d(y', i) \geq d(y, i)$$

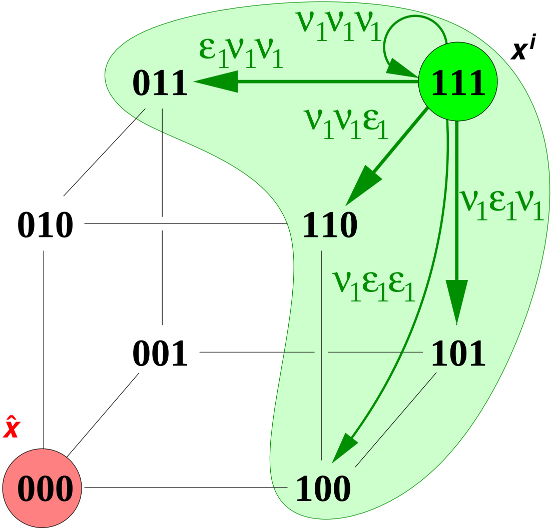
Monotone decoding

$$p(y \in Y_i \mid x^i) = \varepsilon_1\nu_1\nu_1 + \nu_1$$



Monotone decoding, consider deviation from x^i to $\hat{x}$

$p(y \in Y_i \mid x^i) = \varepsilon_1\nu_1\nu_1 + \nu_1$ $\quad$ $p(y \in Y_i \mid \hat{x}) = \nu_0\varepsilon_0\varepsilon_0 + \varepsilon_0$



(it suffices to consider only bits where x^i and $\hat{x}$ differ.)

Proof of Nash property

Want
$$\begin{aligned} p(y \in Y_i \mid x^i) &\geq p(y \in Y_i \mid \hat{x}) \\ \varepsilon_1\nu_1\nu_1 + \nu_1 &\geq \nu_0\varepsilon_0\varepsilon_0 + \varepsilon_0 \end{aligned}$$

Proof of Nash property

Want $\quad p(y \in Y_i \mid x^i) \geq p(y \in Y_i \mid \hat{x})$

$$\varepsilon_1 \nu_1 \nu_1 + \nu_1 \geq \nu_0 \varepsilon_0 \varepsilon_0 + \varepsilon_0$$

Term by term? $\quad \nu_1 = 1 - \varepsilon_1 > \varepsilon_0 \quad$ yes

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e.g. $\varepsilon_1 = 1/8$ $\varepsilon_0 = \nu_0 = 1/2$

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e.g. $\varepsilon_1 = 1/8 \quad \varepsilon_0 = \nu_0 = 1/2$

But bit by bit:

$$\begin{aligned}
\varepsilon_1\nu_1\nu_1 + \nu_1 &= (1-\nu_1)\nu_1\nu_1 + \nu_1 \\
&= \nu_1\nu_1 + \nu_1(-\nu_1\nu_1 + 1) \\
&> \nu_1\nu_1 + \varepsilon_0(-\nu_1\nu_1 + 1) \\
&= (1-\varepsilon_0)\nu_1\nu_1 + \varepsilon_0 \\
&> (1-\varepsilon_0)\varepsilon_0\nu_1 + \varepsilon_0 \\
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&= \nu_0\varepsilon_0\varepsilon_0 + \varepsilon_0
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... can be done generally

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&= (1 - \varepsilon_0)\nu_1 \nu_1 + \varepsilon_0 \\
&> (1 - \varepsilon_0)\varepsilon_0 \nu_1 + \varepsilon_0 \\
&> (1 - \varepsilon_0)\varepsilon_0 \varepsilon_0 + \varepsilon_0 \\
&= \nu_0 \varepsilon_0 \varepsilon_0 + \varepsilon_0
\end{aligned}
$$

... can be done generally, even with different errors per bit.

Consistent decoding too strong?

Consistent decoding

$$i \in \{k \in \Omega \mid y' \in Y_k\} \subseteq \{k \in \Omega \mid y \in Y_k\}$$

$$\Rightarrow \quad d(y', i) \geq d(y, i)$$

Consistent decoding too strong?

Consistent decoding implies

$$i \in \{k \in \Omega \mid y' \in Y_k\} = \{k \in \Omega \mid y \in Y_k\}$$

$$\Rightarrow \quad d(y', i) = d(y, i)$$

same sets of tied states $\Rightarrow$ same decoding probabilities

An ambiguous code

Same codeword for both states

$\Rightarrow$ received $\boldsymbol{y}$ is completely uninformative

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[*sarc**h**asm = the gap between the sender making a sarcastic remark and the receiver who does not get it*]

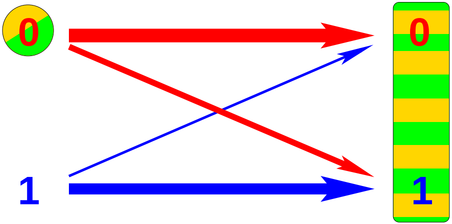
An ambiguous code

Same codeword for both states

⇒ received ***y*** is completely uninformative

⇒ consistent decoding must not distinguish received signals

$\boldsymbol{x}^0 = \boldsymbol{x}^1 =$



$= \boldsymbol{Y}_0 = \boldsymbol{Y}_1$

An ambiguous code

Same codeword for both states

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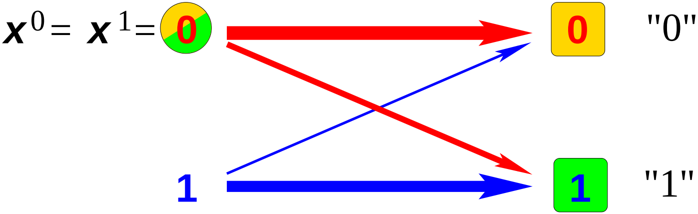


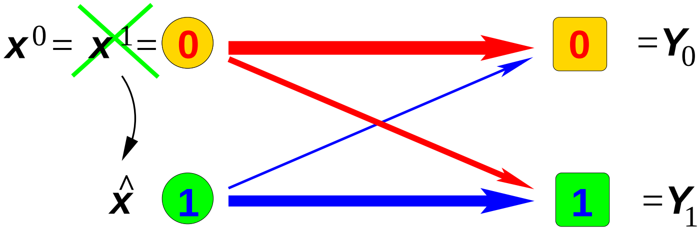
Inconsistent tie breaking

An ambiguous code

Same codeword for both states

⇒ received $\boldsymbol{y}$ is completely uninformative

⇒ consistent decoding must not distinguish received signals



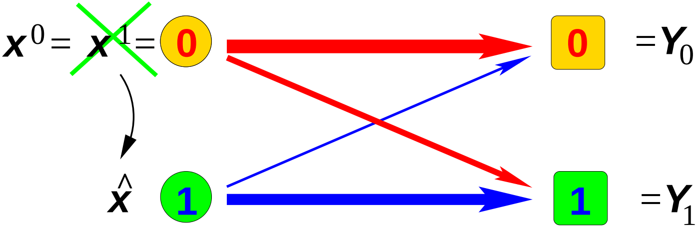
Inconsistent tie breaking ⇒ sender deviates

An ambiguous code ... may evolve?

Same codeword for both states

$\Rightarrow$ received ***y*** is completely uninformative

$\Rightarrow$ consistent decoding must not distinguish received signals



Inconsistent tie breaking $\Rightarrow$ sender deviates ... to better code.

Over-interpreting **ambiguous** signals allows codes to **evolve**?

Questions – answers

1. Is every code a Nash code? – no
2. Is some code a Nash code? – yes, receiver-optimal code is Nash
3. Do small alphabets allow for more Nash codes? – yes, every consistently decoded binary code is Nash

Questions – answers – more questions

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4. Future topic:

   noisy channel as model of **ambiguity**

   $\Rightarrow$ sender may **deviate**

   $\Rightarrow$ let code **evolve**

Nash-stable channels

Definition : A channel (defined by its transition probabilities) is **Nash-stable** if, for **any** generic prior, **any** assignment of states to channel inputs defines a Nash code.

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Example : binary channel.

Theorem : The product of Nash-stable channels (with independent errors) is Nash-stable.

Question : Computational complexity of recognizing that a channel is Nash-stable.

A channel that is not Nash-stable

$q_i V_i$	$p(y\|x)$		y = 0	y = 1	y = 2
3.0	x	0	0.7	0.15	0.15
2.0		1	0.25	0.5	0.25
		2	0.2	0.2	0.6

A new Nash-stable channel

$q_i V_i$

$p(y\|x)$		y = 0	y = 1	y = 2
x	0	**4/7**	**1/7**	**2/7**
	1	**2/7**	**4/7**	**1/7**
	2	**1/7**	**2/7**	**4/7**

A new Nash-stable channel

$q_i V_i$	$p(y\|x)$	y = 0	1	2
1.0	0	**4/7**	**1/7**	**2/7**
1.9	x 1	**2/7**	**4/7**	**1/7**
	2	**1/7**	**2/7**	**4/7**

A new Nash-stable channel

$q_i V_i$	$p(y\|x)$		y	
		0	1	2
1.0	0	**4/7**	**1/7**	**2/7**
2.0	x 1	**2/7**	**4/7**	**1/7**
	2	**1/7**	**2/7**	**4/7**

A new Nash-stable channel

forbidden : non-generic prior with non-monotonic decoding

$q_i V_i$	$p(y\|x)$		y = 0	y = 1	y = 2
1.0		0	**4/7**	**1/7**	**2/7**
2.0	x	1	**2/7**	**4/7**	**1/7**
		2	**1/7**	**2/7**	**4/7**

A new Nash-stable channel

$q_i V_i$	$p(y\|x)$		y	
		0	1	2
1.0	0	**4/7**	**1/7**	**2/7**
2.1	x 1	**2/7**	**4/7**	**1/7**
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1.0		0	**4/7**	**1/7**	**2/7**
2.1	x	1	**2/7**	**4/7**	**1/7**
		2	**1/7**	**2/7**	**4/7**

Does it suffice to test **two** states and their possible priors as channel inputs?

This can be done in polynomial time.

Or is recognizing Nash-stability co-NP-complete?

Thank you!