Pathways to Equilibria, Pretty Pictures and Diagrams (PPAD)

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joint work with:
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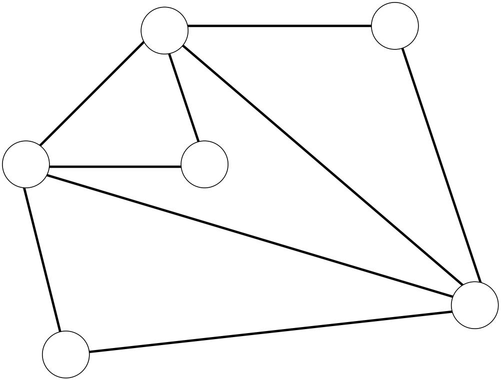
Euler graphs

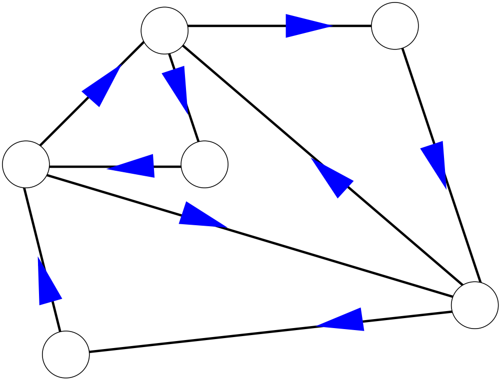
Euler graph = every node has even degree (= number of neighbours)



Euler graphs

Euler graph = every node has even degree (= number of neighbours)

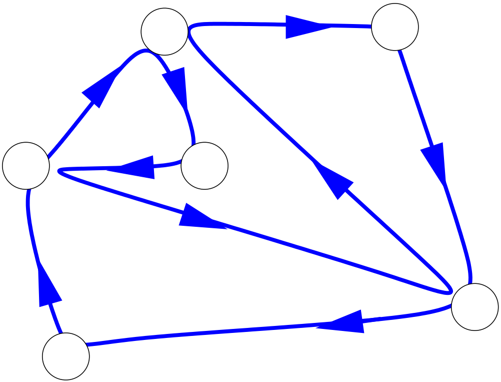
has Eulerian orientation (indegree = outdegree)



Euler graphs ... have tours

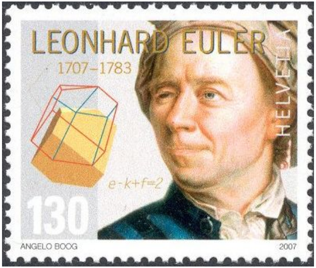
Euler graph = every node has even degree (= number of neighbours)

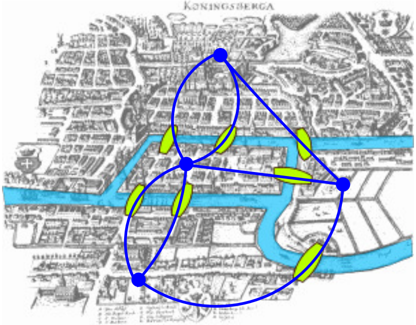
has Eulerian orientation (indegree = outdegree) ... and tour

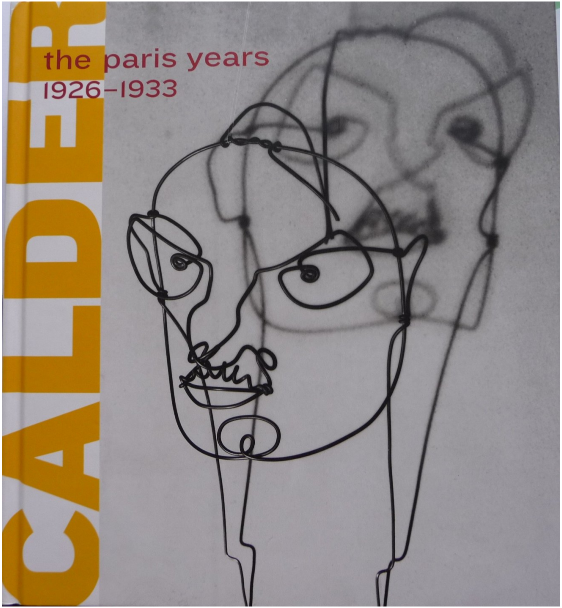


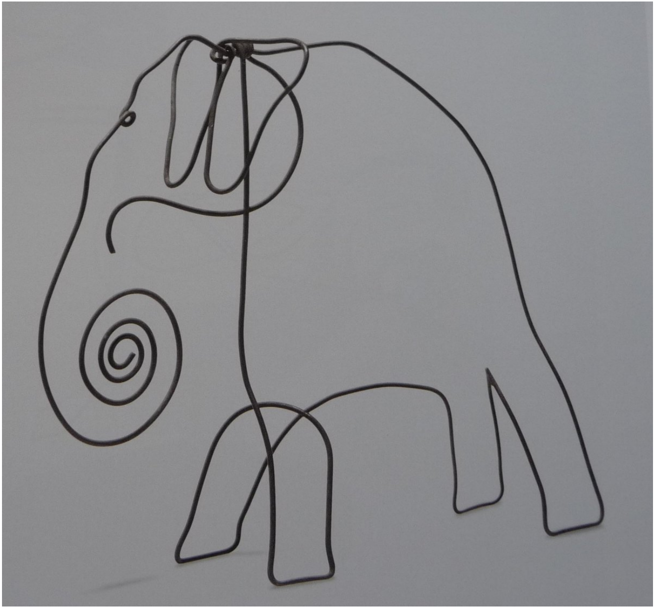
Euler's Königsberg bridges problem

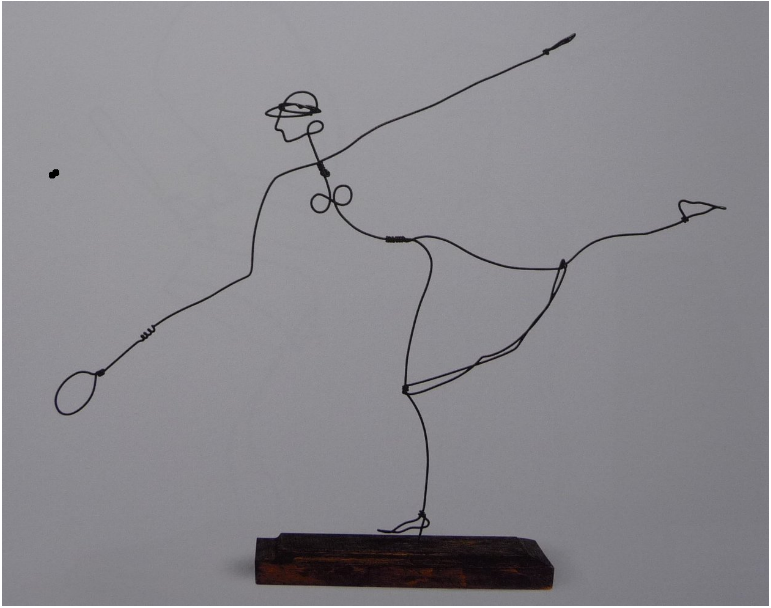
The number of odd-degree nodes of a graph is even:

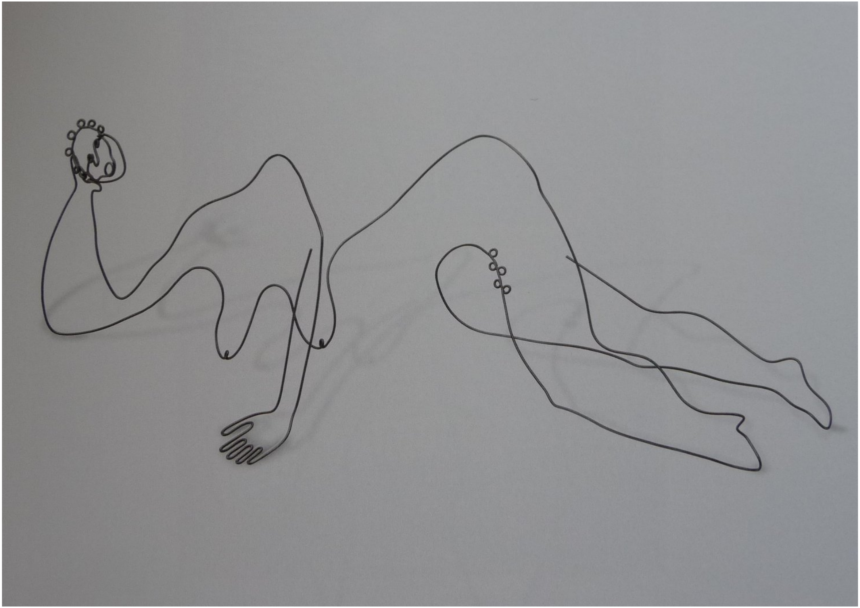


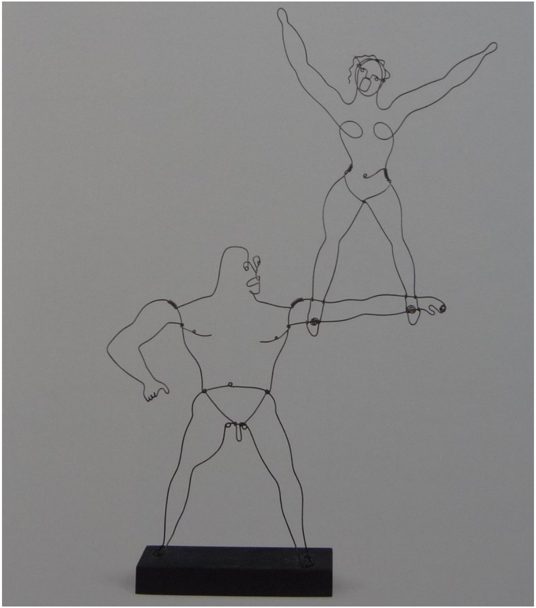


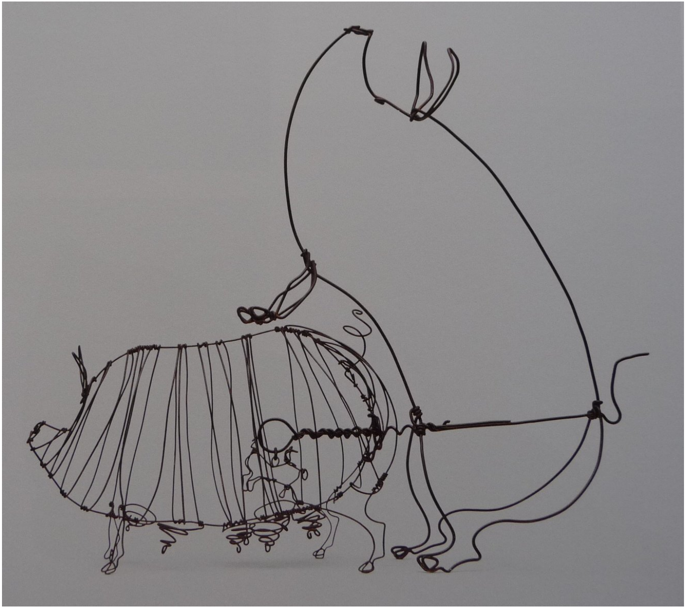

CALDER
the paris years
1926–1933

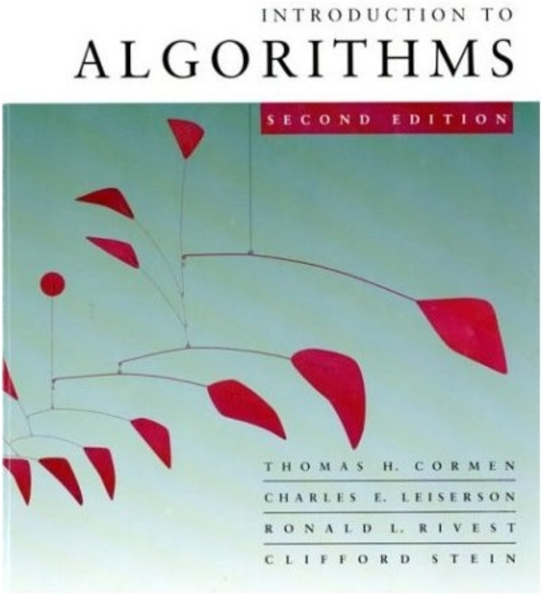











INTRODUCTION TO
ALGORITHMS
SECOND EDITION
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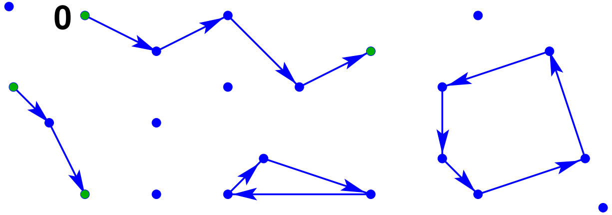
2-player game: find one Nash equilibrium

2-NASH $\in$ PPAD (Polynomial Parity Argument with Direction)

Implicit digraph with indegrees and outdegrees $\leq$ **1** is a set of [nodes], paths and cycles:



Parity argument: number of **sources** of paths = number of **sinks**

Comput. problem: given one source **0**, find another source or sink

[Chen/Deng 2006] 2-NASH is PPAD-complete.

Symmetric Nash equilibria of symmetric games

square game matrix A = payoffs to row player

$$A = \boxed{\begin{matrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{matrix}}$$

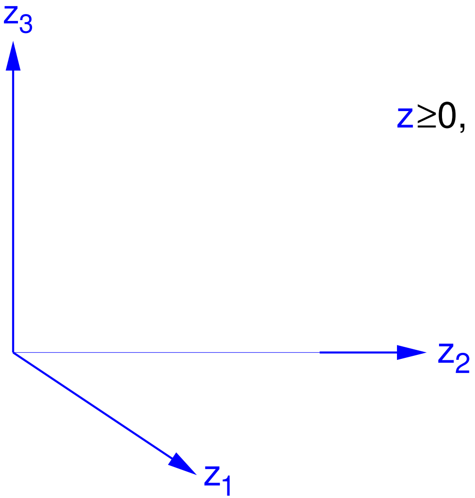
Symmetric Nash equilibria of symmetric games

equilibrium: only optimal strategies are played

$$A = \begin{array}{ccc|c} 1/3 & 2/3 & \mathbf{0} & \\ \hline 0 & 3 & 0 & \mathbf{2} \\ 2 & 2 & 2 & \mathbf{2} \\ 3 & 0 & 0 & 1 \\ \hline \end{array}$$

Symmetric Nash equilibria of symmetric games

plot polytope with strategy weights z_1, z_2, z_3

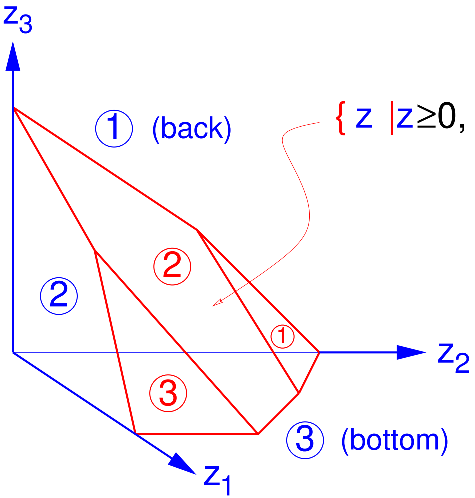


$z \geq 0$,

$$A = \begin{array}{c} \\ \end{array}\begin{array}{ccc} z_1 & z_2 & z_3 \\ \hline 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{array}$$

Symmetric Nash equilibria of symmetric games

with payoffs (scaled to 1) and **labels** for binding inequalities

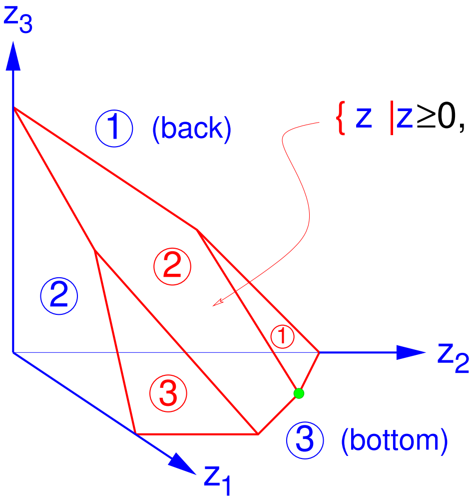


$\{ z \mid z \geq 0, Az \leq 1 \}$

$$A = \begin{array}{ccc} z_1 & z_2 & z_3 \\ \hline 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{array}$$

Symmetric Nash equilibria of symmetric games

equilibrium = **completely labeled** point

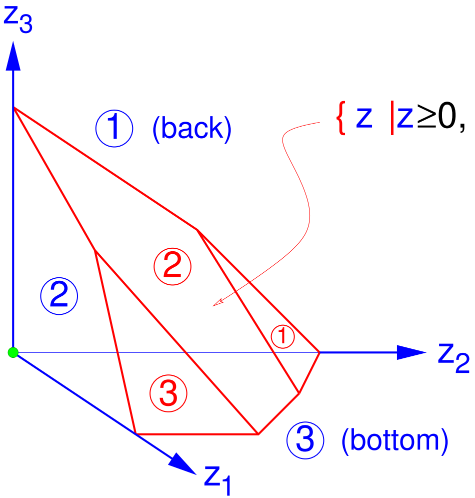


$\{ z \mid z \geq 0,\ Az \leq 1 \}$

$$\begin{array}{cc} & \begin{array}{ccc} 1/6 & 1/3 & \mathbf{0} \end{array} & \\ A = & \boxed{\begin{array}{ccc} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{array}} & \begin{array}{c} \mathbf{1} \\ \mathbf{1} \\ 1/2 \end{array} \end{array}$$

Symmetric Nash equilibria of symmetric games

start path with **artificial equilibrium** $z=\mathbf{0}$

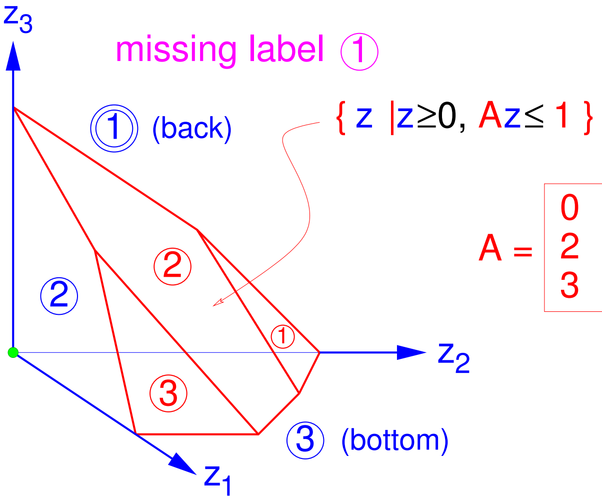


$\{ z \mid z \geq 0,\ Az \leq \mathbf{1} \}$

$$A = \boxed{\begin{matrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{matrix}}$$

Symmetric Nash equilibria of symmetric games

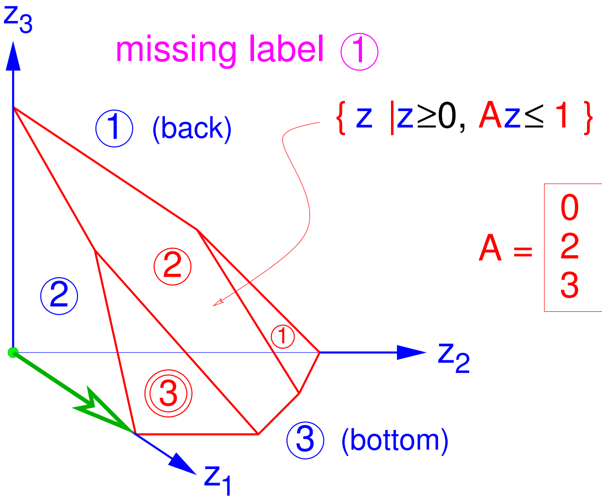
start path with **artificial equilibrium** $z = \mathbf{0}$, choose e.g.



$$A = \boxed{\begin{matrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{matrix}}$$

Symmetric Nash equilibria of symmetric games

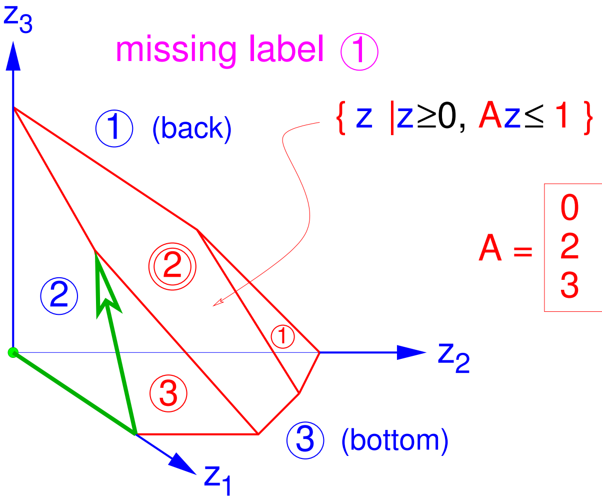
leave facet with label **1**, find duplicate label **3**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

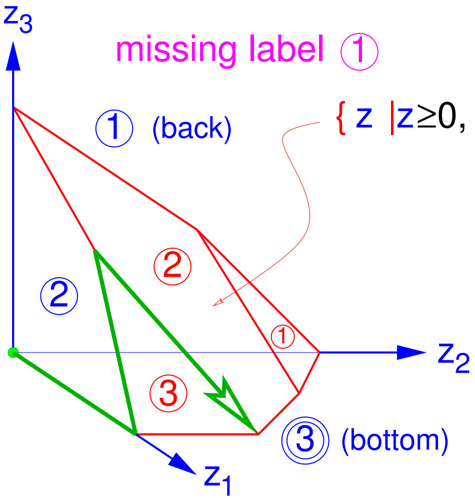
leave facet with old label **3**, find duplicate label **2**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

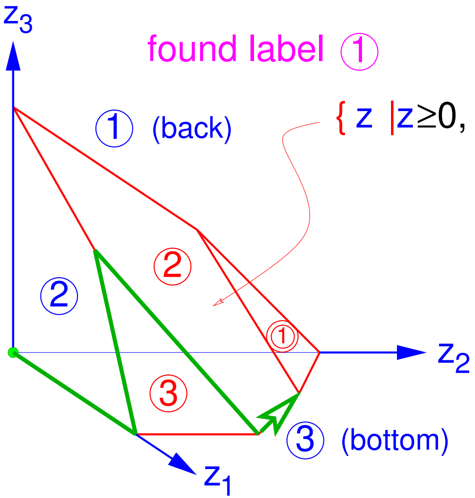
leave facet with old label **2**, find duplicate label **3**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

leave facet with old label **3**, find missing label **1**

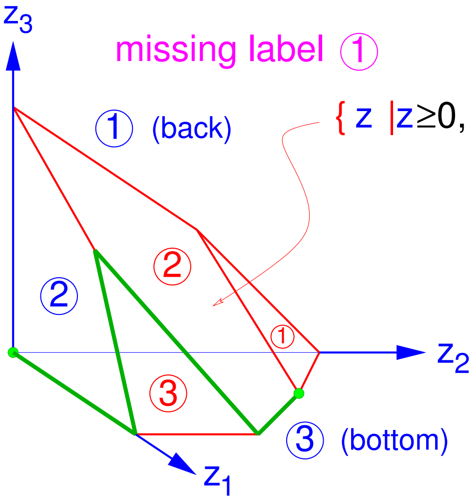


$\{ z \mid z \geq 0,\ Az \leq 1 \}$

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

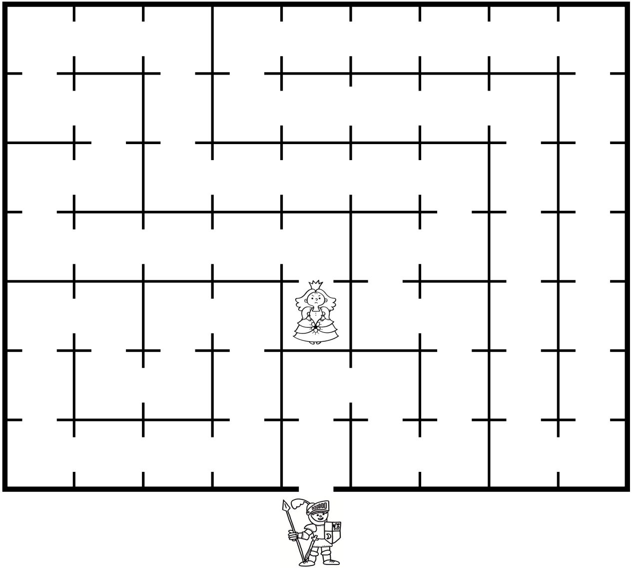
equilibria (including artificial equilibrium) = endpoints of paths



$\{\, z \mid z \geq 0,\ Az \leq \mathbf{1} \,\}$

$$A = \begin{array}{c} \begin{array}{ccc} 1/6 & 1/3 & \mathbf{0} \end{array} \\ \boxed{\begin{array}{ccc} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{array}} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{1} \\ 1/2 \end{array}$$

The castle where each room has at most two doors



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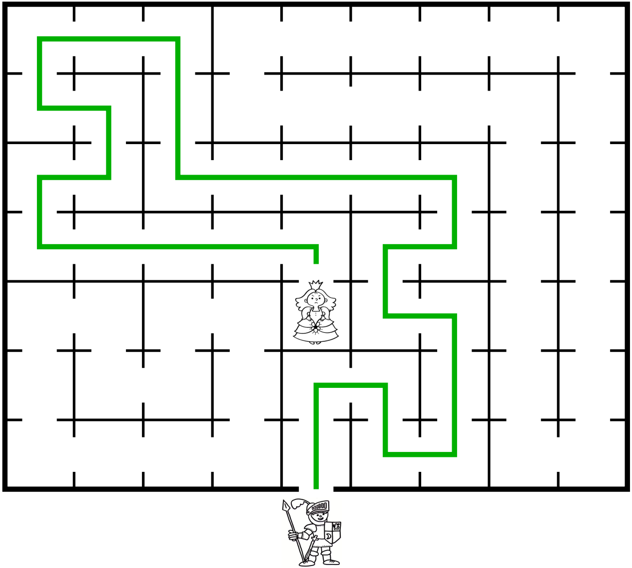

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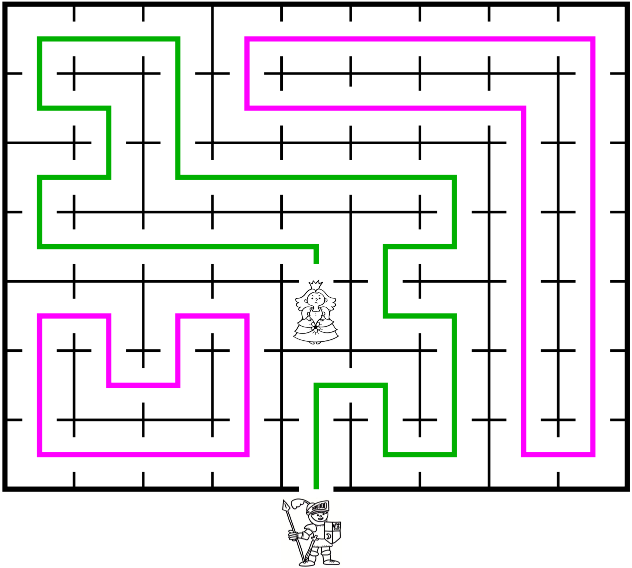
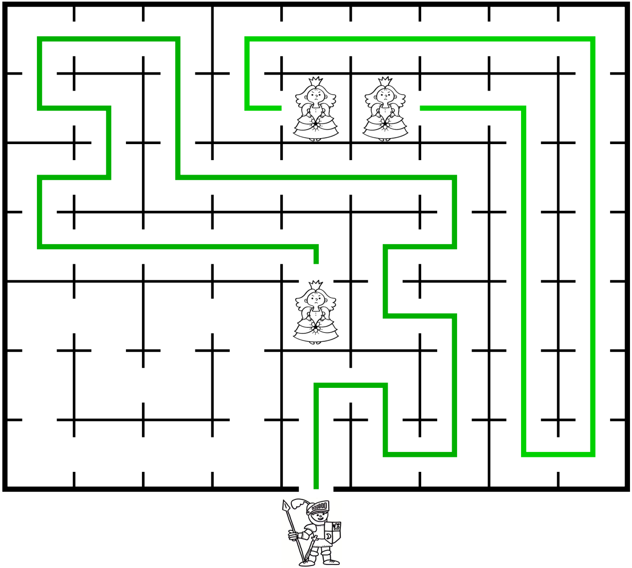
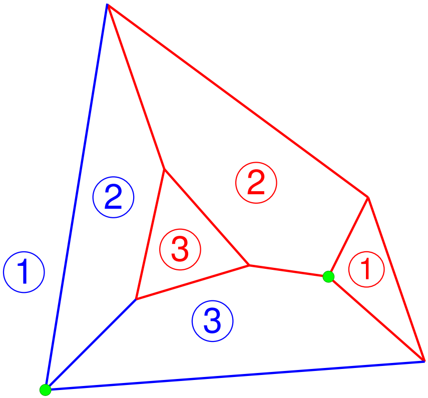


The castle where each room has at most two doors



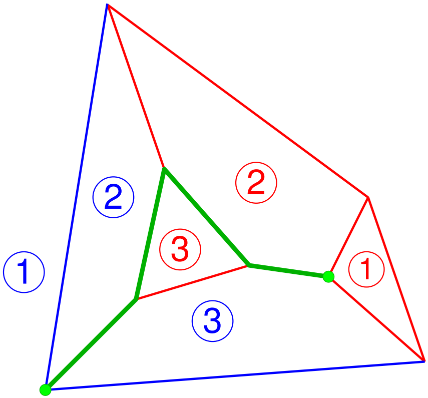
Path of “almost completely labeled” edges

two completely labeled vertices



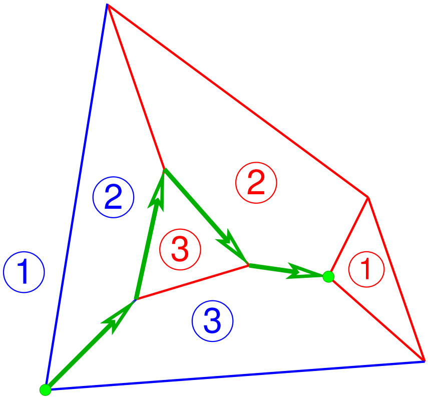
Path of “almost completely labeled” edges

path because at most two neighbours (“doors” in castle)



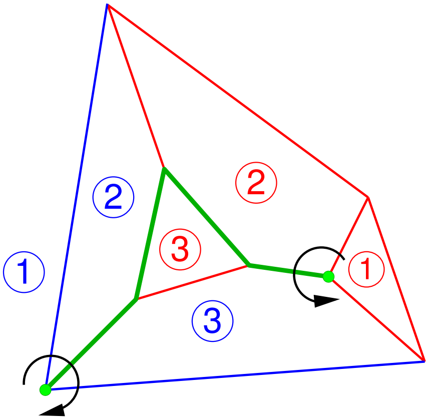
Path of “almost completely labeled” edges

orientation of edges: **2** on left, **3** on right



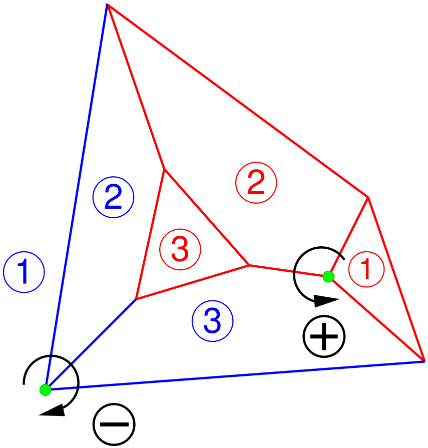
Path of “almost completely labeled” edges

opposite orientation (“sign”) of endpoints



Path of “almost completely labeled” edges

equilibrium **sign** ⊖ or ⊕ does not depend on path



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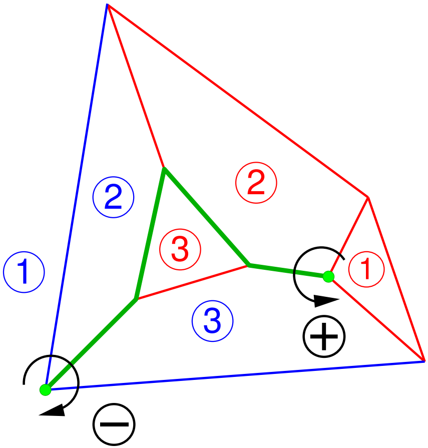

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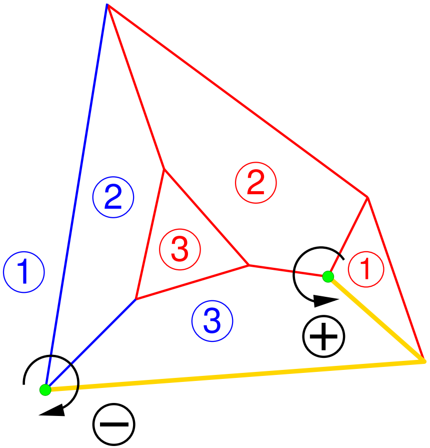

Path of "almost completely labeled" edges

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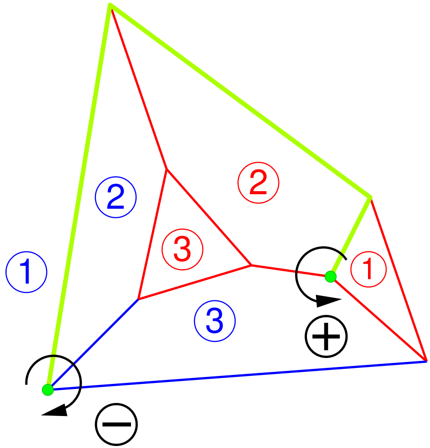
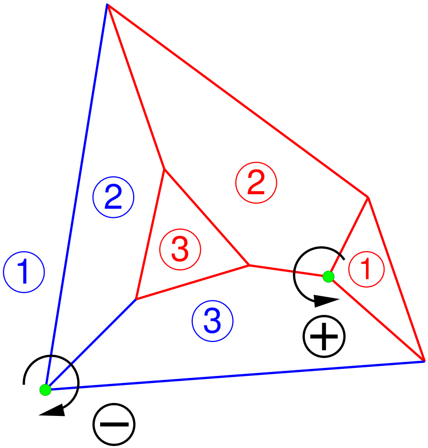


Path of "almost completely labeled" edges

equilibrium **sign** ⊖ or ⊕ does not depend on path



Labeled polytope P

Let $a_j \in \mathbb{R}^m$, $\beta_j \in \mathbb{R}$,

$$P = \{x \in \mathbb{R}^m \mid a_j x \le \beta_j, \ \ 1 \le j \le n\},$$

let **facet** $F_j = \{ x \in P \mid a_j x = \beta_j \}$ have

label $l(j) \in \{1, \ldots, m\}$.

Assume P is a **simple** polytope (no $x \in P$ on $> m$ facets)

$\Rightarrow$ each vertex x on m facets = m linearly independent equations.

x **completely labeled** $\Leftrightarrow \{l(j) \mid x \in F_j\} = \{1, \ldots, m\}$.

Completely labeled points come in pairs

Theorem [Parity Argument]

Let ***P*** be a labeled polytope.

Then ***P*** has an **even** number of completely labeled vertices.

Completely labeled points come in pairs of opposite sign

Theorem [Parity Argument with Direction]

Let $\boldsymbol{P}$ be a labeled polytope.

Then $\boldsymbol{P}$ has an **even** number of completely labeled vertices. Half of these have **sign** $\ominus$, half have sign $\oplus$.

Completely labeled points come in pairs of opposite sign

Theorem [Parity Argument with Direction]

Let $\boldsymbol{P}$ be a labeled polytope.

Then $\boldsymbol{P}$ has an **even** number of completely labeled vertices.
Half of these have **sign** $\ominus$, half have sign $\oplus$.

---

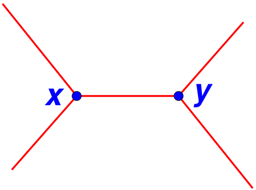
sign of completely labeled $\boldsymbol{x}$ is sign of determinant of facet normal vectors: if (e.g.) facet $\boldsymbol{a_i x} = \beta_i$ has label $\boldsymbol{i = 1, 2, \ldots, m}$, then

$$\boldsymbol{sign(x) = sign\ |a_1\ a_2 \cdots a_m|}$$

Pivoting changes signs

Lemma

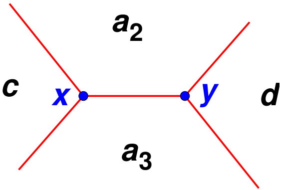
Let $x, y \in \mathbb{R}^m$ be adjacent vertices of a simple polytope P



Pivoting changes signs

Lemma

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}$ be adjacent vertices of a simple polytope $\boldsymbol{P}$ with facet normals $\boldsymbol{c}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{x}$ and $\boldsymbol{d}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{y}$.



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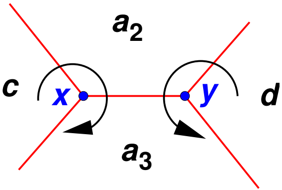
Then $|\boldsymbol{c}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_m|$ and $|\boldsymbol{d}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_m|$ have opposite sign.



Pivoting changes signs

Proof :

$$\begin{array}{ll} c\mathbf{x} = \beta_0 & \\ & d\mathbf{y} = \beta_1 \\ a_2\mathbf{x} = \beta_2 & a_2\mathbf{y} = \beta_2 \\ \vdots & \vdots \\ a_m\mathbf{x} = \beta_m & a_m\mathbf{y} = \beta_m \end{array}$$

Pivoting changes signs

Proof :

$$
\begin{array}{rcl}
c\mathbf{x} = \beta_0 & & \\
 & & d\mathbf{y} = \beta_1 \\
a_2\mathbf{x} = \beta_2 & & a_2\mathbf{y} = \beta_2 \\
\vdots & & \vdots \\
a_m\mathbf{x} = \beta_m & & a_m\mathbf{y} = \beta_m
\end{array}
$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

Pivoting changes signs

Proof :

$$
\begin{array}{rcl}
\mathbf{c}\mathbf{x} = \beta_0 & & \\
& & \mathbf{d}\mathbf{y} = \beta_1 \\
\mathbf{a}_2\mathbf{x} = \beta_2 & & \mathbf{a}_2\mathbf{y} = \beta_2 \\
\vdots & & \vdots \\
\mathbf{a}_m\mathbf{x} = \beta_m & & \mathbf{a}_m\mathbf{y} = \beta_m
\end{array}
$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma \mathbf{c} + \delta \mathbf{d} + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_m \mathbf{a}_m = \mathbf{0}$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma \mathbf{c} + \delta \mathbf{d})\mathbf{x} = (\gamma \mathbf{c} + \delta \mathbf{d})\mathbf{y}$$

Pivoting changes signs

Proof :

$$\begin{array}{cc} cx = \beta_0 & \boxed{cy < \beta_0} \\ \boxed{dx < \beta_1} & dy = \beta_1 \\ a_2x = \beta_2 & a_2y = \beta_2 \\ \vdots & \vdots \\ a_mx = \beta_m & a_my = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d)x = (\gamma c + \delta d)y, \qquad \gamma(cx - cy) = \delta(dy - dx)$$

Pivoting changes signs

Proof :

$$
\begin{array}{cc}
\boldsymbol{c}\boldsymbol{x} = \beta_0 & \boxed{\boldsymbol{c}\boldsymbol{y} < \beta_0} \\
\boxed{\boldsymbol{d}\boldsymbol{x} < \beta_1} & \boldsymbol{d}\boldsymbol{y} = \beta_1 \\
\boldsymbol{a}_2\boldsymbol{x} = \beta_2 & \boldsymbol{a}_2\boldsymbol{y} = \beta_2 \\
\vdots & \vdots \\
\boldsymbol{a}_m\boldsymbol{x} = \beta_m & \boldsymbol{a}_m\boldsymbol{y} = \beta_m
\end{array}
$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma \boldsymbol{c} + \delta \boldsymbol{d} + \alpha_2 \boldsymbol{a}_2 + \cdots + \alpha_m \boldsymbol{a}_m = \boldsymbol{0}$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma \boldsymbol{c} + \delta \boldsymbol{d})\boldsymbol{x} = (\gamma \boldsymbol{c} + \delta \boldsymbol{d})\boldsymbol{y}, \qquad \gamma(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{c}\boldsymbol{y}) = \delta(\boldsymbol{d}\boldsymbol{y} - \boldsymbol{d}\boldsymbol{x})$$

$\Rightarrow \quad \gamma$ and δ have same sign

Pivoting changes signs

Proof :

$$\begin{array}{cc} c\mathbf{x} = \beta_0 & \boxed{c\mathbf{y} < \beta_0} \\ \boxed{d\mathbf{x} < \beta_1} & d\mathbf{y} = \beta_1 \\ a_2\mathbf{x} = \beta_2 & a_2\mathbf{y} = \beta_2 \\ \vdots & \vdots \\ a_m\mathbf{x} = \beta_m & a_m\mathbf{y} = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d)\mathbf{x} = (\gamma c + \delta d)\mathbf{y}, \qquad \gamma(c\mathbf{x} - c\mathbf{y}) = \delta(d\mathbf{y} - d\mathbf{x})$$

$\Rightarrow \quad \gamma$ and δ have same sign,

$$|(\gamma c + \delta d)\ a_2 \cdots a_m| = \gamma\ |c\ a_2 \cdots a_m| + \delta\ |d\ a_2 \cdots a_m| = 0$$

Pivoting changes signs

Proof :

$$
\begin{array}{cc}
cx = \beta_0 & \boxed{cy < \beta_0} \\
\boxed{dx < \beta_1} & dy = \beta_1 \\
a_2 x = \beta_2 & a_2 y = \beta_2 \\
\vdots & \vdots \\
a_m x = \beta_m & a_m y = \beta_m
\end{array}
$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d)x = (\gamma c + \delta d)y, \qquad \gamma(cx - cy) = \delta(dy - dx)$$

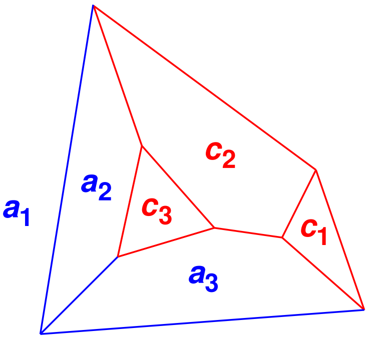
$\Rightarrow \quad \gamma$ and δ have same sign,

$$|(\gamma c + \delta d)\ a_2 \cdots a_m| = \gamma\ |c\ a_2 \cdots a_m| + \delta\ |d\ a_2 \cdots a_m| = 0$$

$\Rightarrow \quad |c\ a_2 \cdots a_m|$ and $|d\ a_2 \cdots a_m|$ have opposite sign, QED.

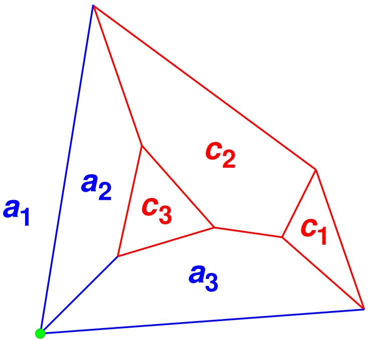
General Parity Argument with Direction

Facet normal vectors $\boldsymbol{a}_1$ $\boldsymbol{a}_2$ $\boldsymbol{a}_3$ $\boldsymbol{c}_1$ $\boldsymbol{c}_2$ $\boldsymbol{c}_3$, labels **1 2 3 1 2 3**



General Parity Argument with Direction

Start with a_1 a_2 a_3, sign ⊖

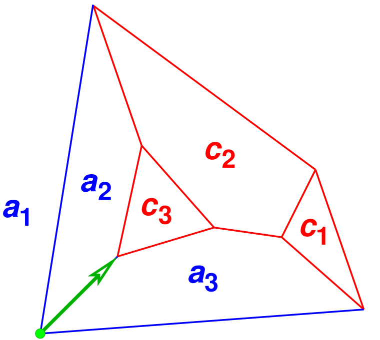


⊖

$|\boxed{a_1}\ a_2\ a_3|$

General Parity Argument with Direction

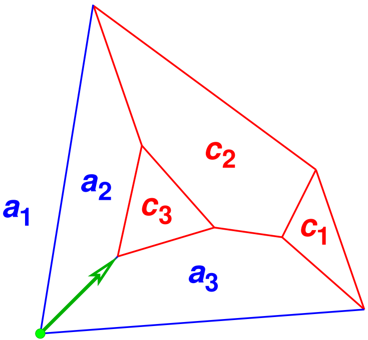
Start with $a_1\ a_2\ a_3$, sign $\ominus$, label **1** missing, $a_1 \to c_3$ gives sign $\oplus$



$$\overset{\ominus}{|\boxed{a_1}\ a_2\ a_3|} \longrightarrow \overset{\oplus}{|\boxed{c_3}\ a_2\ a_3|}$$

General Parity Argument with Direction

Switch columns c_3 and a_3 in determinant: back to sign $\ominus$



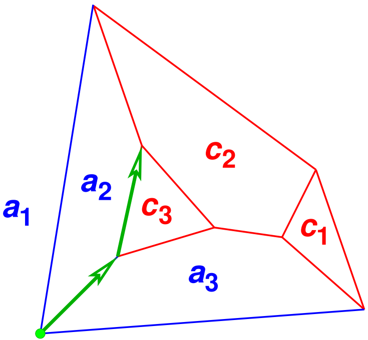
$$\ominus \qquad\qquad \oplus$$

$$\left|\boxed{a_1}\ a_2\ a_3\right| \longrightarrow \left|\boxed{c_3}\ a_2\ a_3\right|$$

$$\left|\boxed{a_3}\ a_2\ c_3\right| \longleftrightarrow \left|\boxed{c_3}\ a_2\ a_3\right|$$

General Parity Argument with Direction

next pivot $a_3 \rightarrow c_2$ gives sign $\oplus$



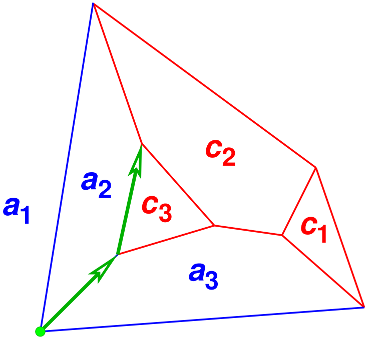
$\ominus$ $\oplus$

$$|\boxed{a_1}\ a_2\ a_3| \longrightarrow |\boxed{c_3}\ a_2\ a_3|$$

$$|\boxed{a_3}\ a_2\ c_3| \longrightarrow |\boxed{c_2}\ a_2\ c_3|$$

General Parity Argument with Direction

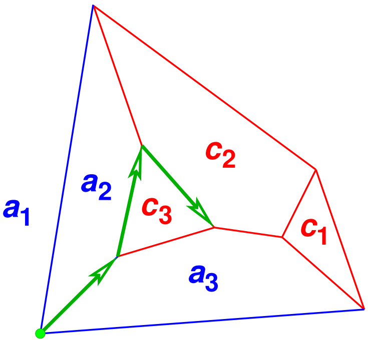
Switch columns c_2 and a_2 in determinant: back to sign ⊖



⊖		⊕
$\left\lvert \boxed{a_1}\ a_2\ a_3 \right\rvert$	$\longrightarrow$	$\left\lvert \boxed{c_3}\ a_2\ a_3 \right\rvert$
$\left\lvert \boxed{a_3}\ a_2\ c_3 \right\rvert$	$\longrightarrow$	$\left\lvert \boxed{c_2}\ a_2\ c_3 \right\rvert$
$\left\lvert \boxed{a_2}\ c_2\ c_3 \right\rvert$		

General Parity Argument with Direction

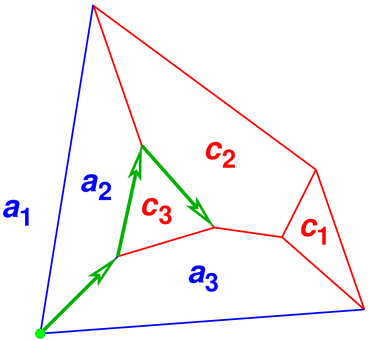
next pivot $a_2 \to a_3$ gives sign $\oplus$



$$\begin{array}{ccc} \ominus & & \oplus \\ |\boxed{a_1}\ a_2\ a_3| & \longrightarrow & |\boxed{c_3}\ a_2\ a_3| \\ |\boxed{a_3}\ a_2\ c_3| & \longrightarrow & |\boxed{c_2}\ a_2\ c_3| \\ |\boxed{a_2}\ c_2\ c_3| & \longrightarrow & |\boxed{a_3}\ c_2\ c_3| \end{array}$$

General Parity Argument with Direction

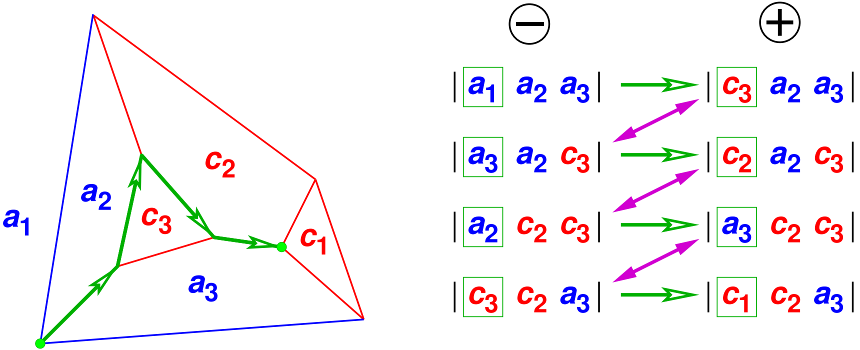
Switch columns a_3 and c_3 in determinant: back to sign $\ominus$



$$
\begin{array}{ccc}
\ominus & & \oplus \\
|\boxed{a_1}\ a_2\ a_3| & \longrightarrow & |\boxed{c_3}\ a_2\ a_3| \\
|\boxed{a_3}\ a_2\ c_3| & \longrightarrow & |\boxed{c_2}\ a_2\ c_3| \\
|\boxed{a_2}\ c_2\ c_3| & \longrightarrow & |\boxed{a_3}\ c_2\ c_3| \\
|\boxed{c_3}\ c_2\ a_3| & &
\end{array}
$$

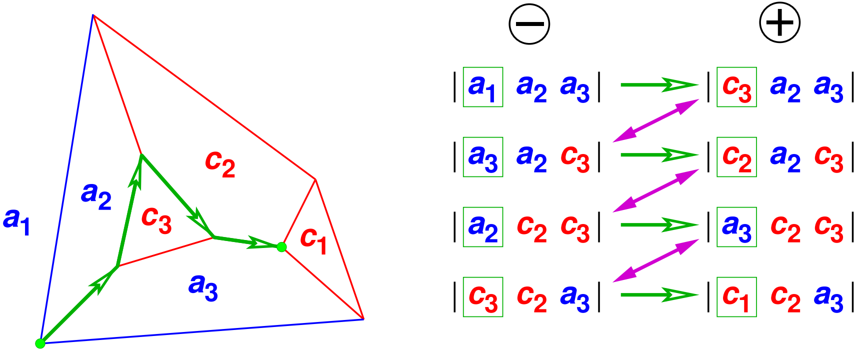
General Parity Argument with Direction

Last pivot $c_3 \to c_1$ gives sign $\oplus$, opposite to starting sign $\ominus$.



General Parity Argument with Direction

Only need: sign-switching of **pivots** and **column exchanges**



Nash equilibria of bimatrix games

Recall: $\boldsymbol{m} \times \boldsymbol{m}$ matrix $\boldsymbol{C}$,

$$\boldsymbol{P} = \{\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{m}} \mid -\boldsymbol{z} \le \mathbf{0}, \quad \boldsymbol{C}\boldsymbol{z} \le \mathbf{1}\ \}$$

with $\mathbf{2}\boldsymbol{m}$ inequalities labeled $\mathbf{1}, \ldots, \boldsymbol{m}$, $\mathbf{1}, \ldots, \boldsymbol{m}$.

Nash equilibria of bimatrix games

Recall: $m \times m$ matrix C,

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with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$.

Completely labeled $z \neq 0 \quad \Leftrightarrow$

Nash equilibrium (z, z) of game $(C, C^\top)$

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Nash equilibrium (z, z) of game $(C, C^\top)$

Normalize sign of "artificial equilibrium" $\mathbf{0}$ to $\ominus$, in general

$$index(z) = sign(z) \cdot (-1)^{m+1}$$

Nash equilibria of bimatrix games

Recall: $m \times m$ matrix C,

$$P = \{ z \in \mathbb{R}^m \mid -z \le 0, \quad Cz \le 1 \}$$

with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$.

bimatrix game (A, B):

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}, \quad z = (x, y) :$$

Completely labeled $(x, y) \neq (0, 0) \quad \Leftrightarrow$

Nash equilibrium (x, y) of game (A, B)

Index of an equilibrium

Theorem [Shapley 1974]

A nondegenerate bimatrix game (A, B) has an odd number of equilibria, one more of index $\oplus$ than of index $\ominus$.

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[*Proof:* Endpoints of pivoting paths have opposite index $\ominus$ and $\oplus$.]

Index of an equilibrium

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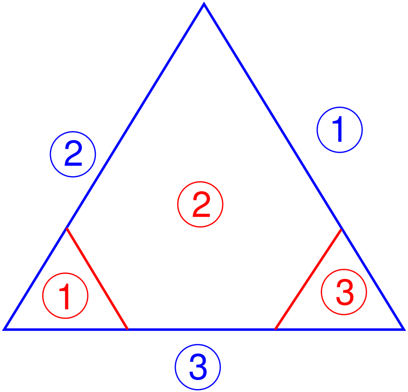
[*Proof:* Endpoints of pivoting paths have opposite index $\ominus$ and $\oplus$.]

Equilibria of index $\oplus$ include every

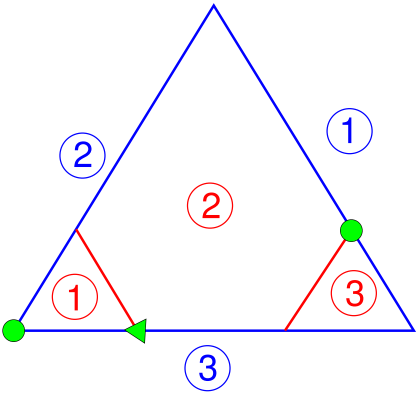
- pure-strategy equilibrium
- unique equilibrium
- **dynamically stable** equilibrium

Dynamically stable equilibrium: only if ⊕



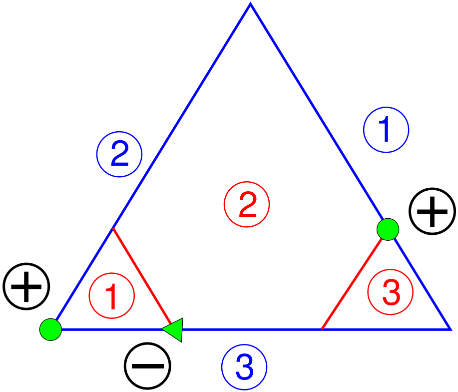
$$\begin{array}{c|ccc|}
 & 1 & 2 & 3 \\
\hline
1 & \mathbf{3} & \mathbf{0} & \mathbf{0} \\
2 & \mathbf{2} & \mathbf{2} & \mathbf{2} \\
3 & \mathbf{0} & \mathbf{3} & \mathbf{0} \\
\hline
\end{array}$$

Dynamically stable equilibrium: only if $\oplus$



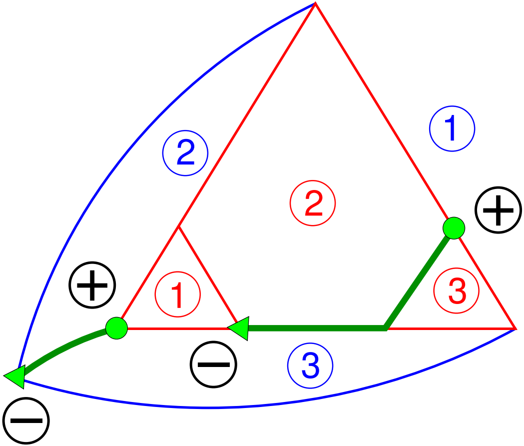
	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Dynamically stable equilibrium: only if $\oplus$



	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Dynamically stable equilibrium: only if ⊕



	1	2	3
1	**3**	**0**	**0**
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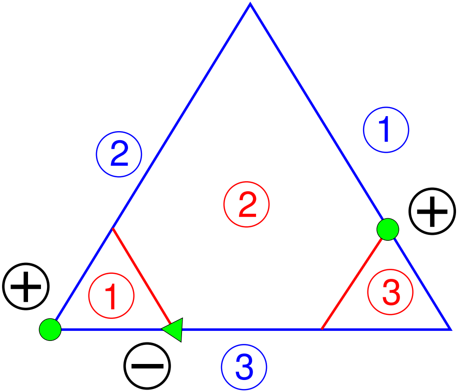

$$
\begin{array}{c|ccc|}
 & 1 & 2 & 3 \\
\hline
1 & \mathbf{3} & \mathbf{0} & \mathbf{0} \\
2 & \mathbf{2} & \mathbf{2} & \mathbf{2} \\
3 & \mathbf{0} & \mathbf{3} & \mathbf{0} \\
\hline
\end{array}
$$

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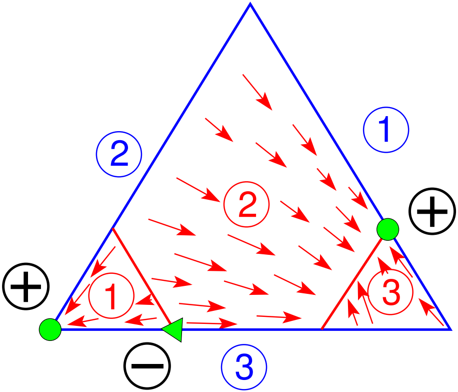

	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Literature

[Lemke/Howson 1964] Equilibrium points of bimatrix games. *J. SIAM* **12**, 413–423: Path-following to find Nash equilibrium.

[Shapley 1974] A note on the Lemke–Howson algorithm. *Math. Prog. Study* **1**, 175–189: Endpoints of paths have opposite index.

Literature – Generalizations

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[Lemke/Grotzinger 1976] On generalizing Shapley's index theory to labelled pseudomanifolds. *Math. Progr.* **10**, 245–262.

[Todd 1976] Orientation in complementary pivot algorithms. *Math. Oper. Res.* **1**, 54–66: Abstract pivoting with “duoids”.

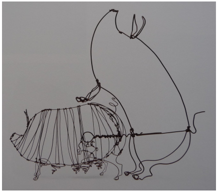
[Edmonds 2009] Euler complexes. “Oiks”, room partitions.

[Edmonds/Gaubert/Gurvich 2010] Sperner oiks. *Elec. Notes Discr. Math.* **36**, 1273–1280: Replacing labels by Sperner oik rooms.

Plan / our results

- Manifolds and **oiks** [Edmonds] :

Plan / our results



- Manifolds and **oiks** [Edmonds] :
  room partitions come in pairs ***oik*!**

- we define an **orientation** for oiks with signs ⊕ ⊖

- **2**-oik = Euler graph, room partition = perfect matching

  finding a second matching of opposite sign

  — may take exponential time with path-following

  — new polynomial-time algorithm

Oiks and pivoting

Definition [Edmonds 2009]

Given: finite set V of nodes. Multiset $\mathcal{R}$ of d-element sets of nodes, called **rooms**, is a ***d*-oik** (Euler complex) if every set of $d-1$ nodes is contained in an **even** number of rooms.

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Example $d = 2$: Euler graph with nodes in V and edges in $\mathcal{R}$.

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If every set of $d-1$ nodes is contained in **0** or **2** rooms then the oik is called a (abstract simplicial pseudo-) **manifold.**

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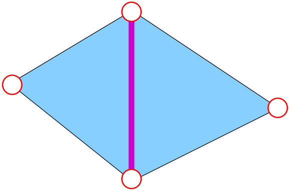
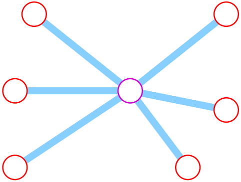
manifold, $d = 3$



$d = 2$



Oiks and pivoting

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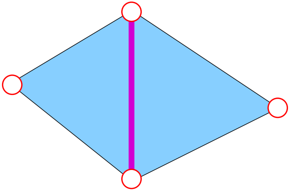
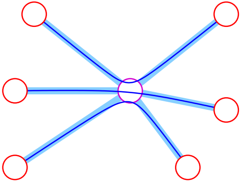
manifold, $d = 3$ **pivoting** $d = 2$





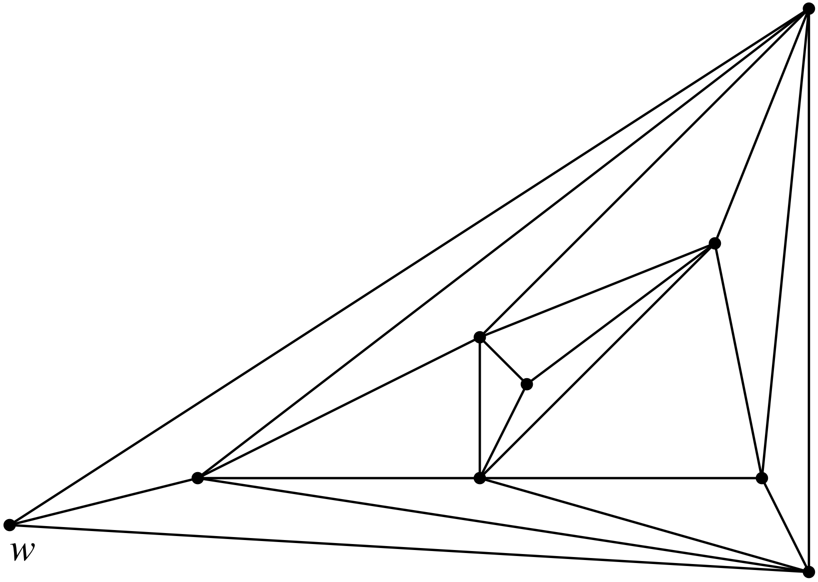
Room partitions come in pairs

Given an oik $\mathcal{R}$ with node set V,
a **room partition** is a partition of V into rooms.

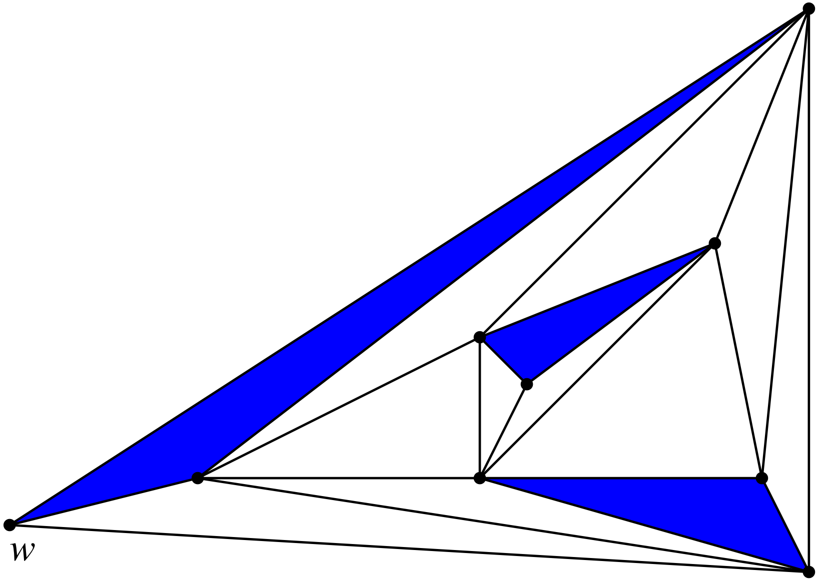
Theorem [Edmonds 2009]

The number of room partitions is **even.**

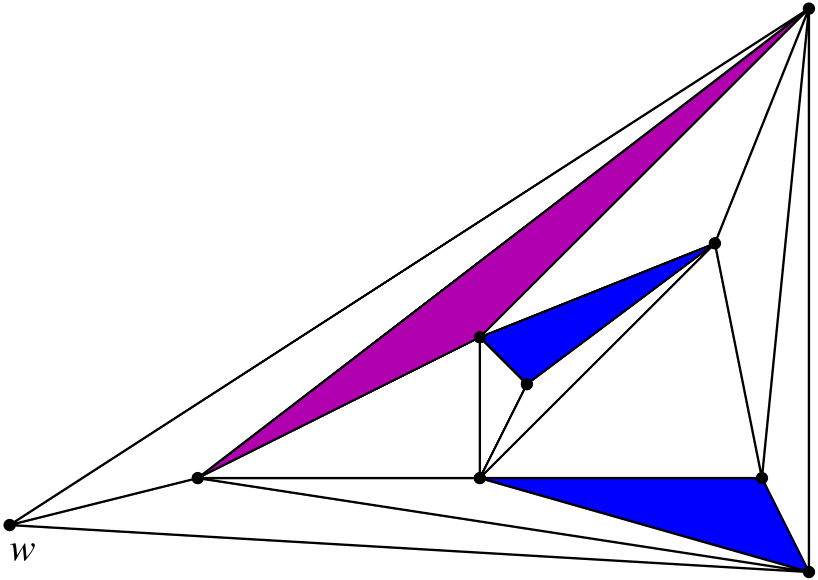
Room partition for **3**-manifold



Room partition for 3-manifold



w-almost room partition



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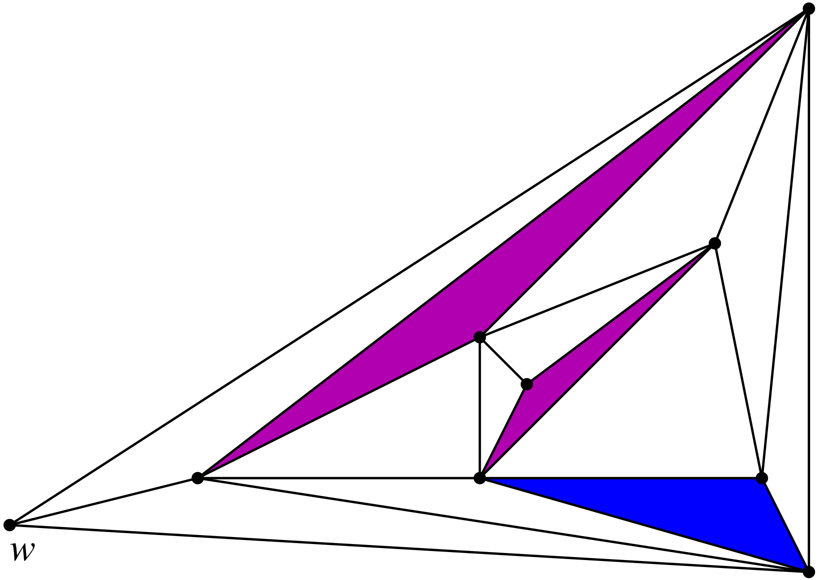
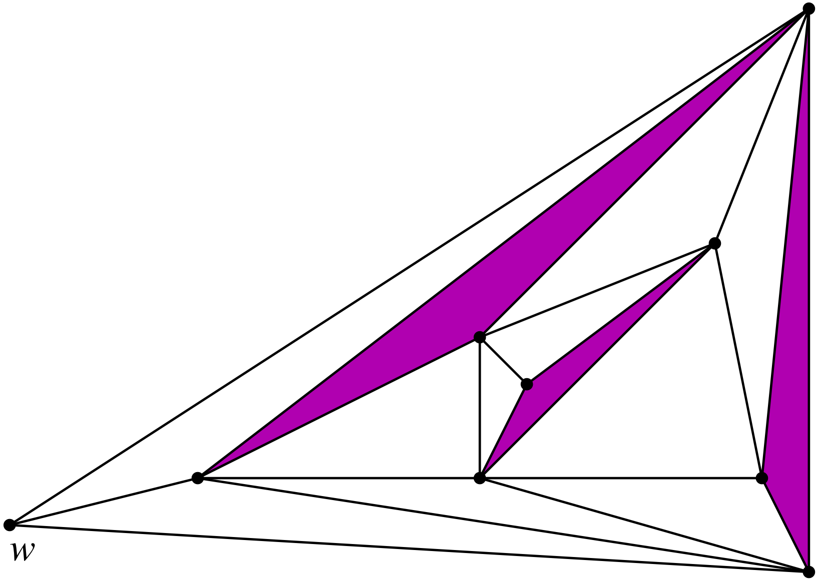
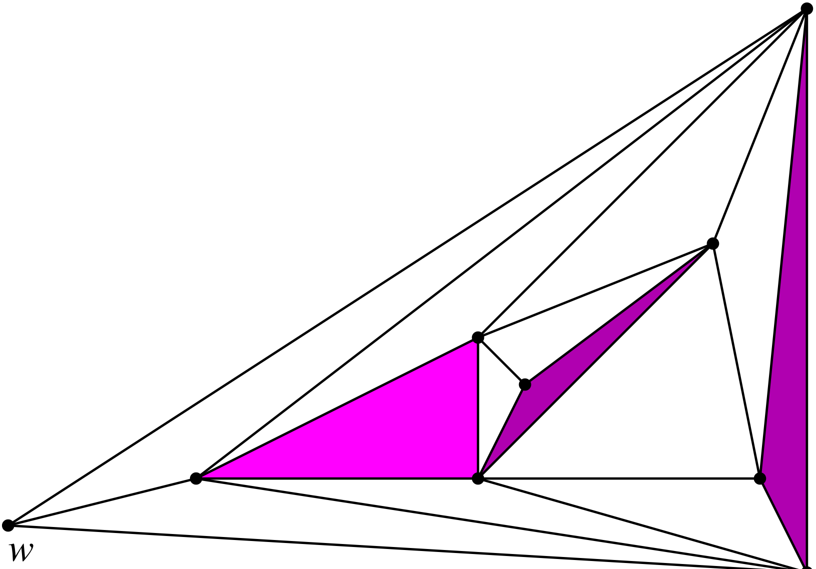


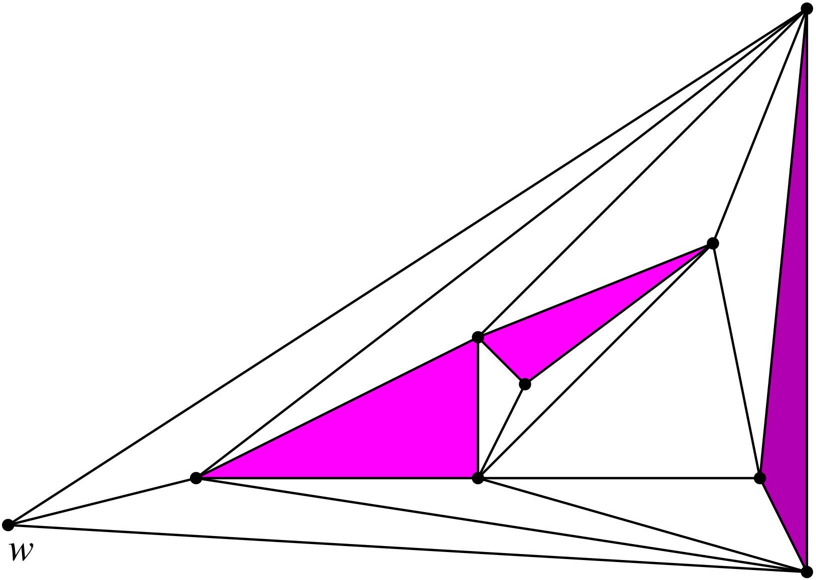
w-almost room partition



w-almost room partition



w-almost room partition



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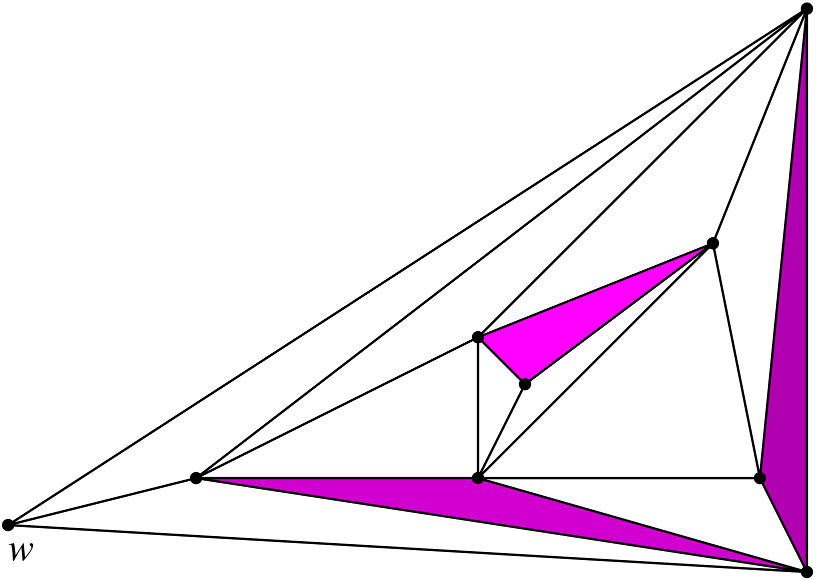
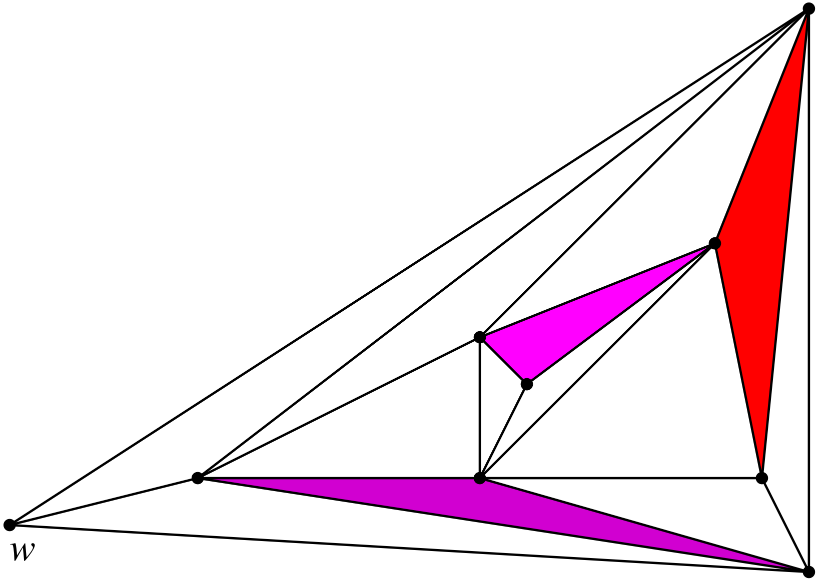
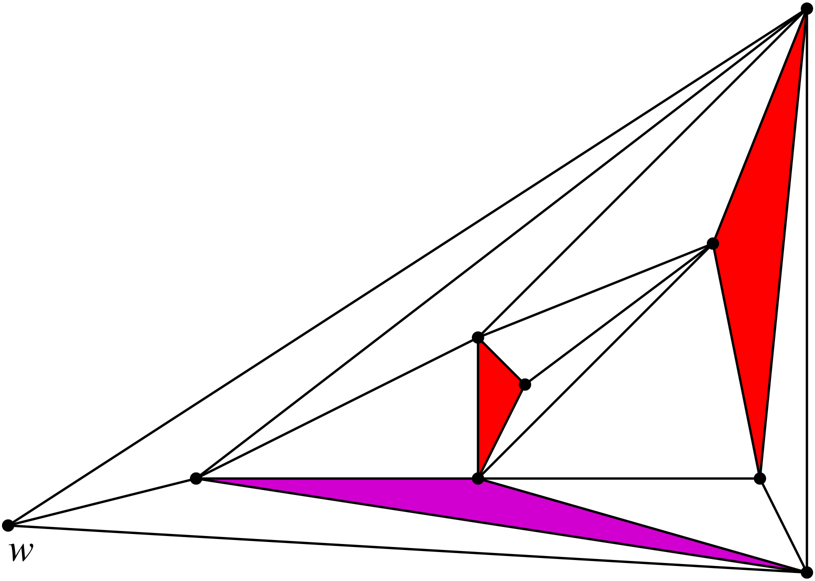


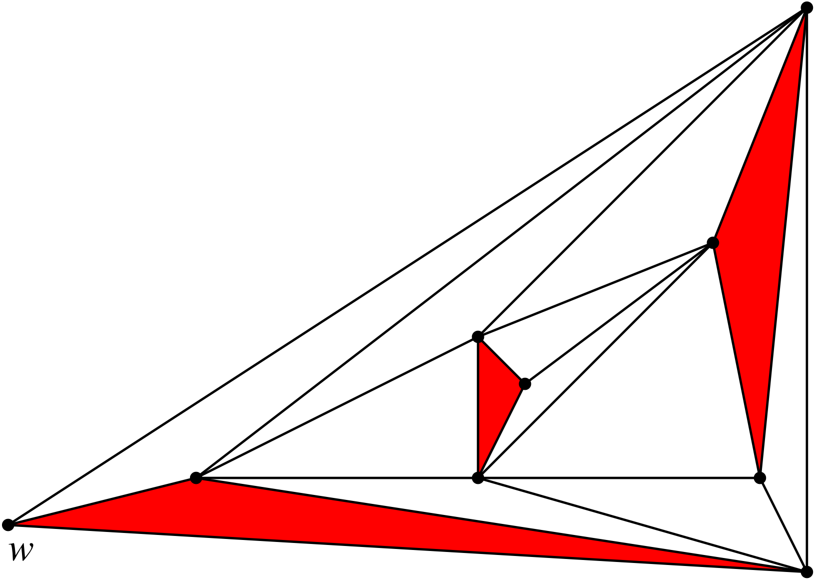
w-almost room partition



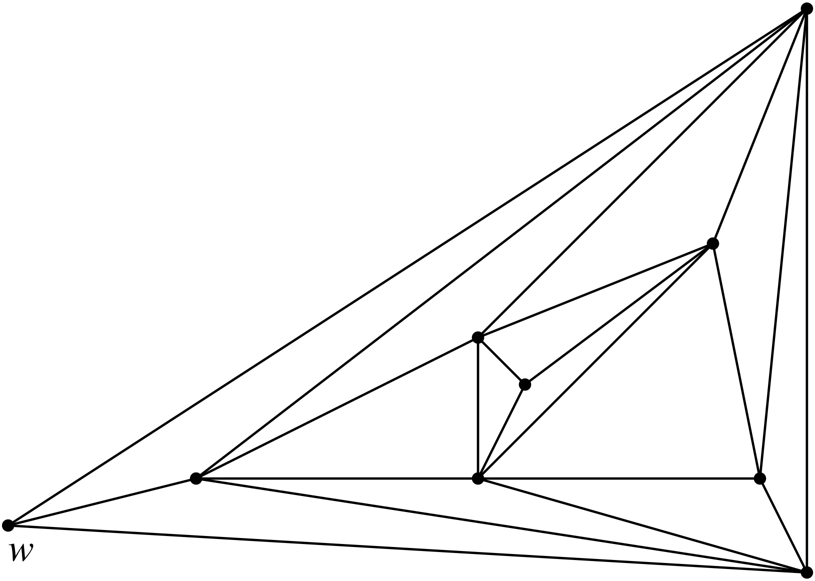
w-almost room partition



Found second room partition



[Edmonds/Sanità 2010]: exponentially long path



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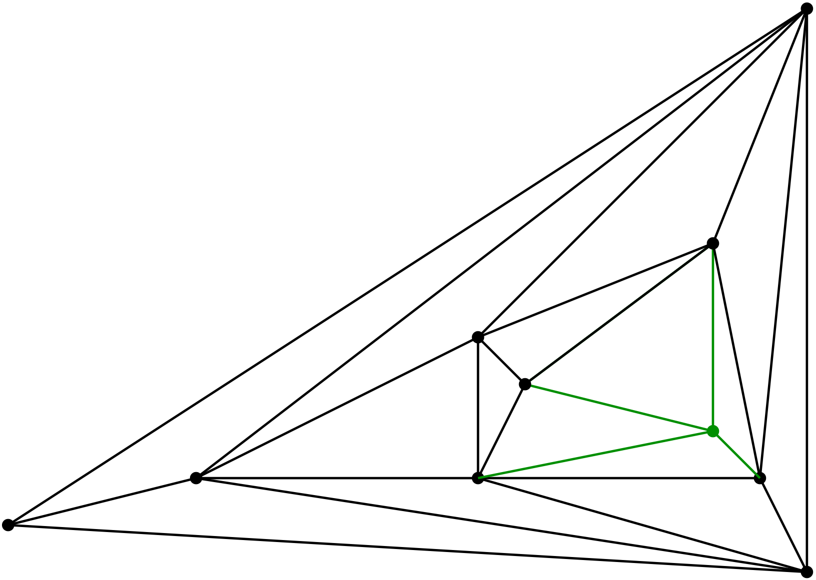

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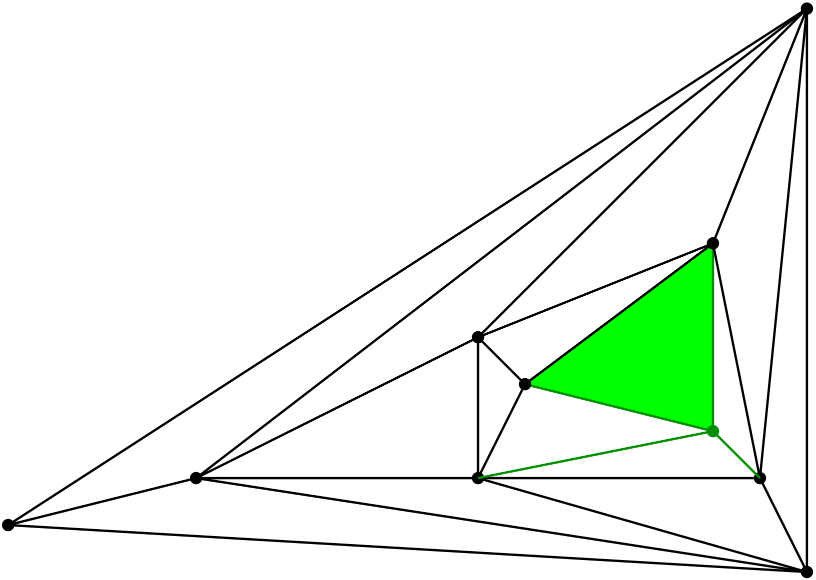

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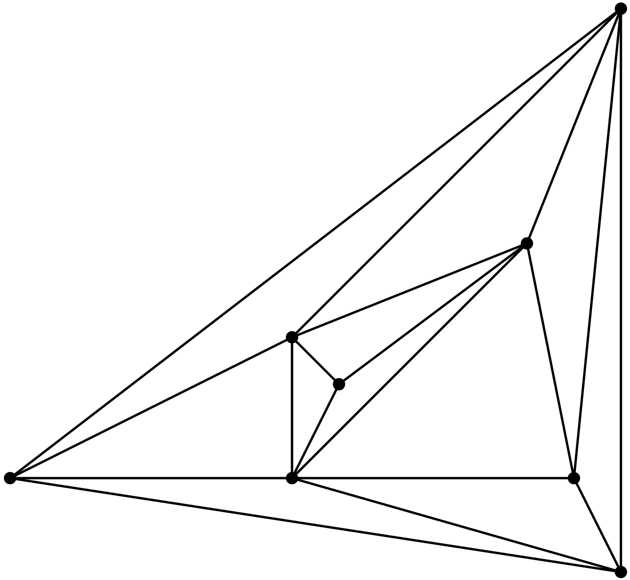

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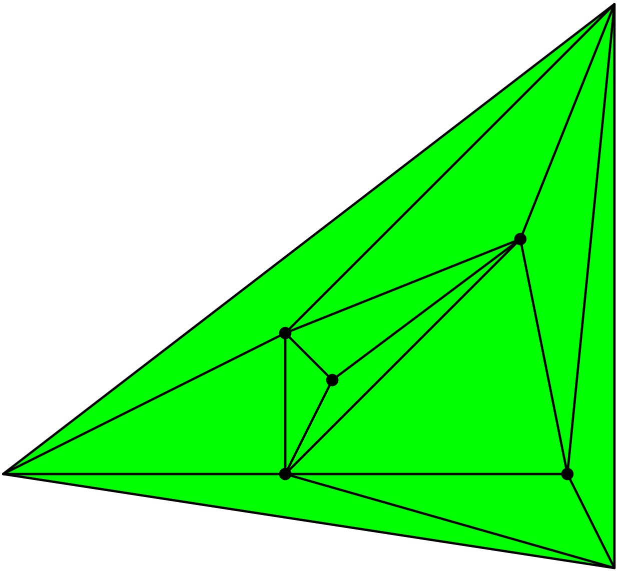

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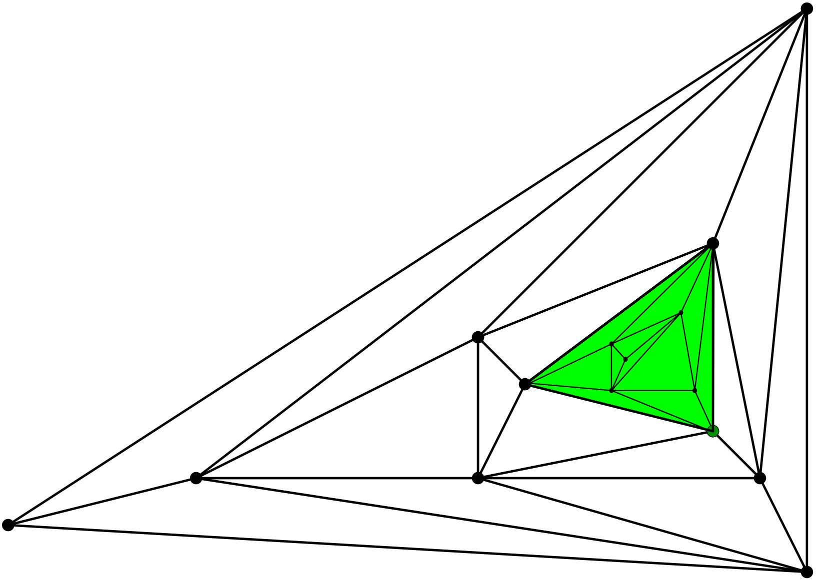
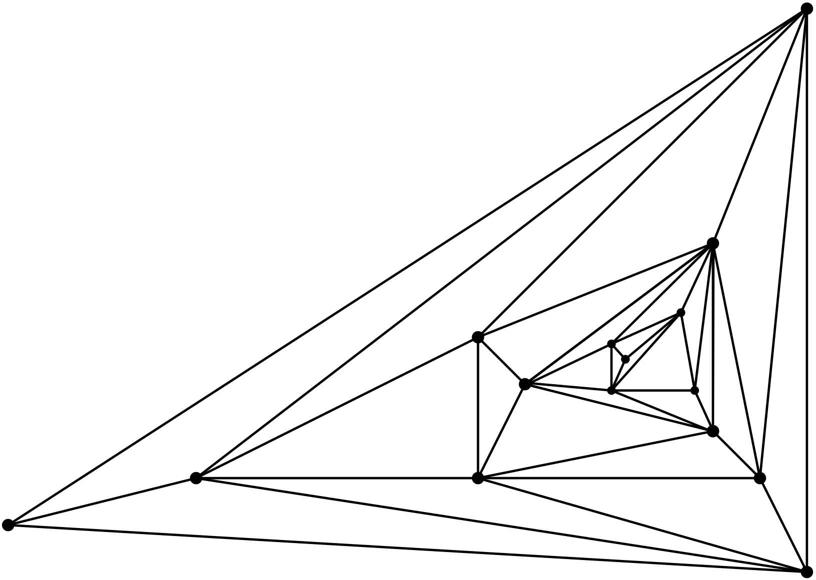
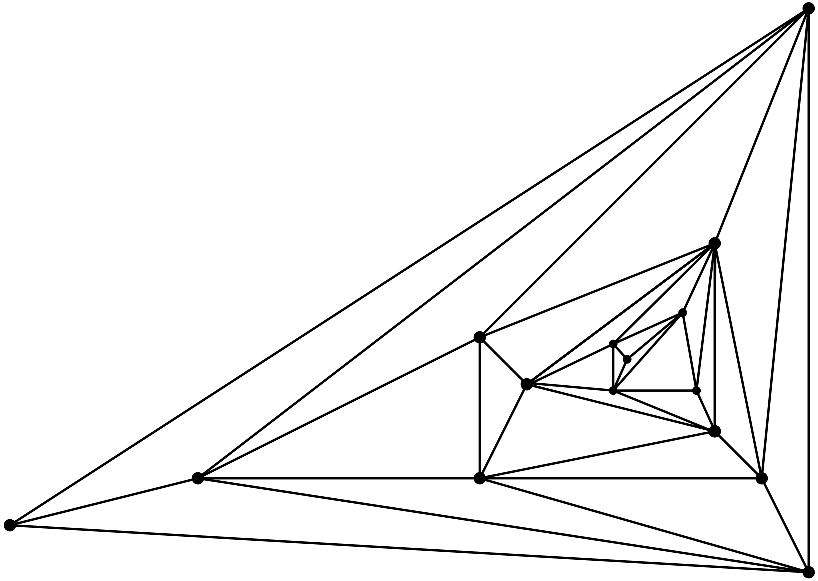


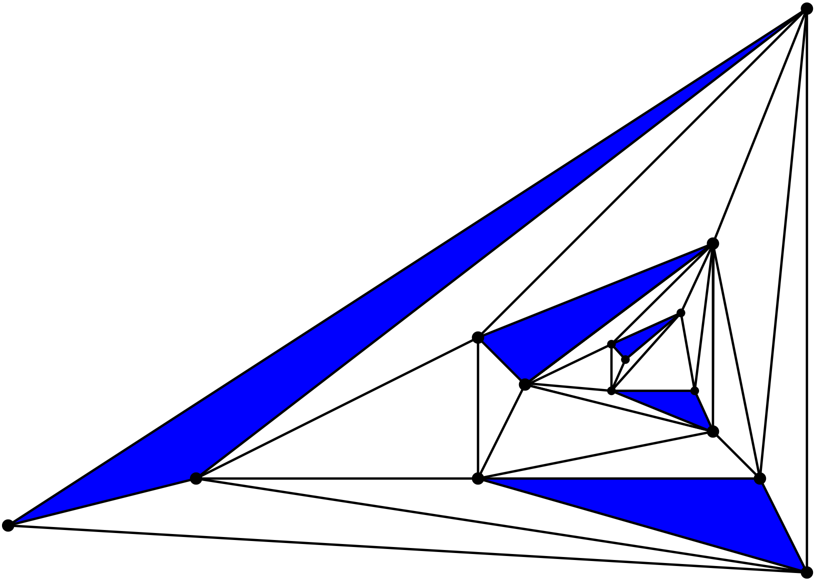
6 extra nodes, **12** extra rooms



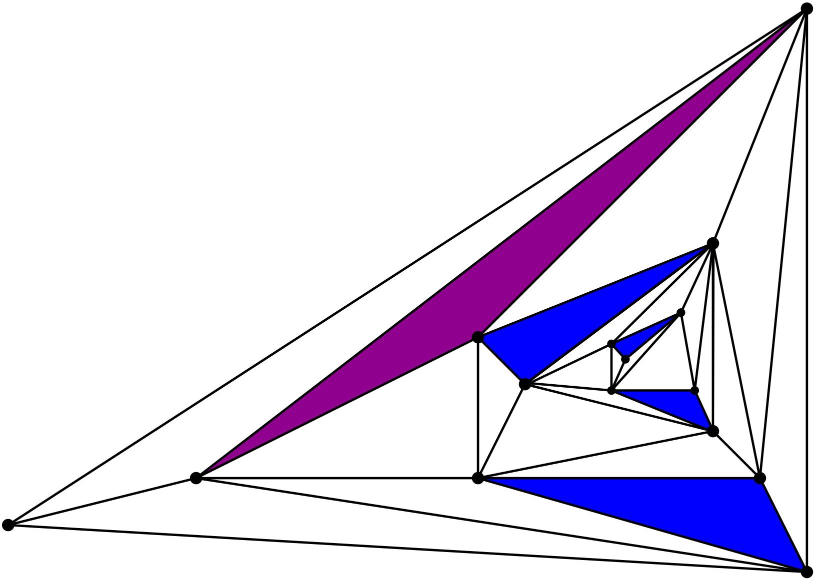
Path length more than doubles



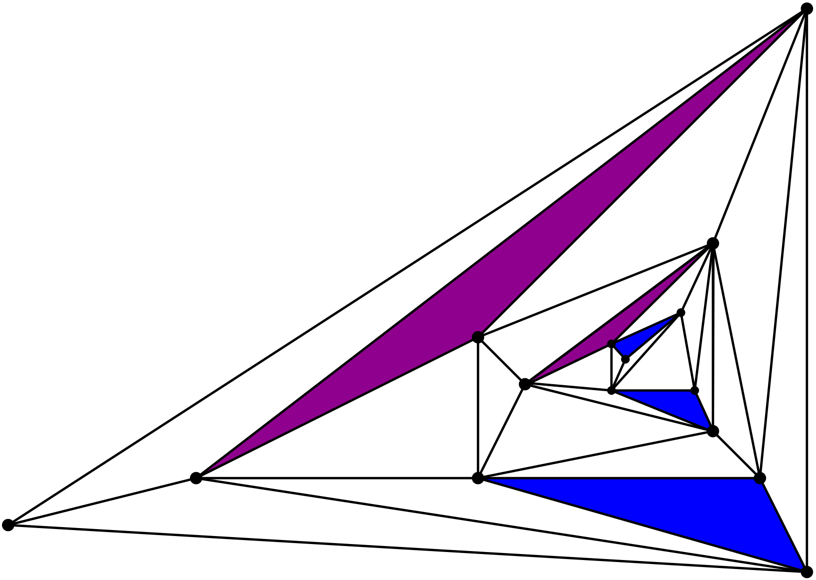
Path length more than doubles



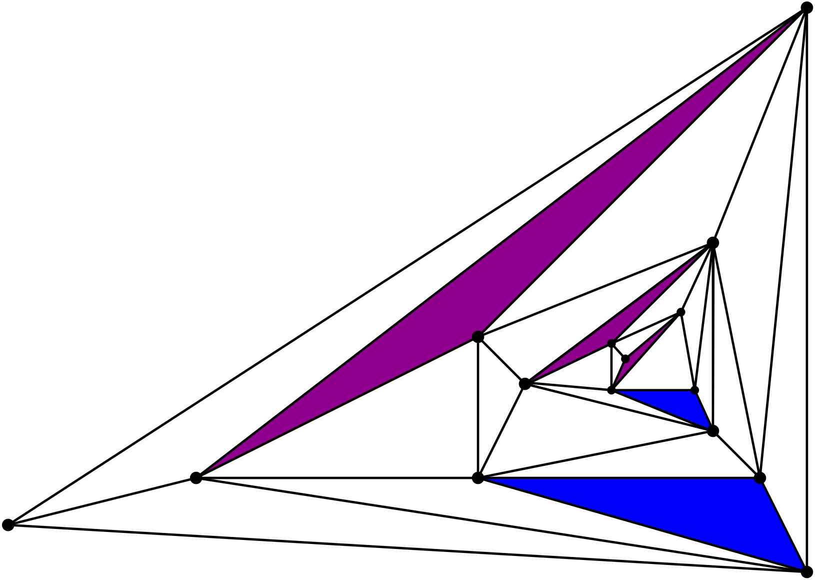
Forward recursion



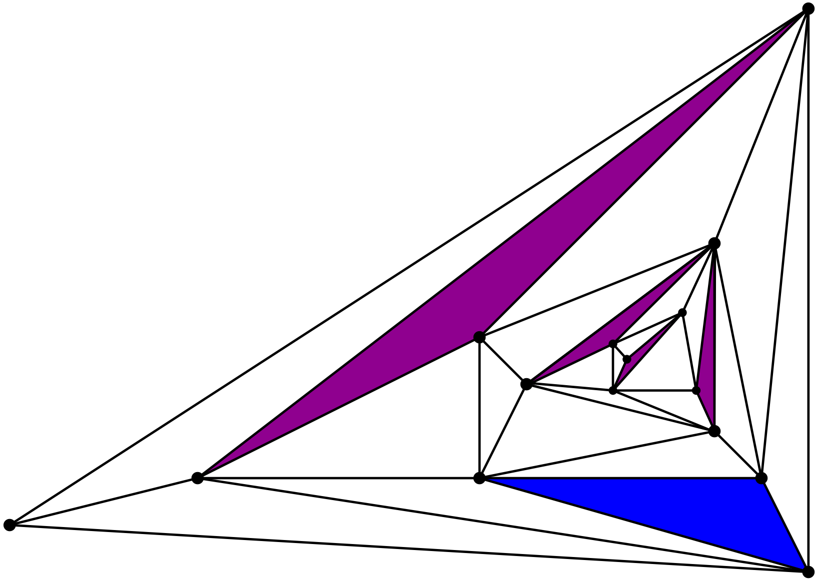
Forward recursion



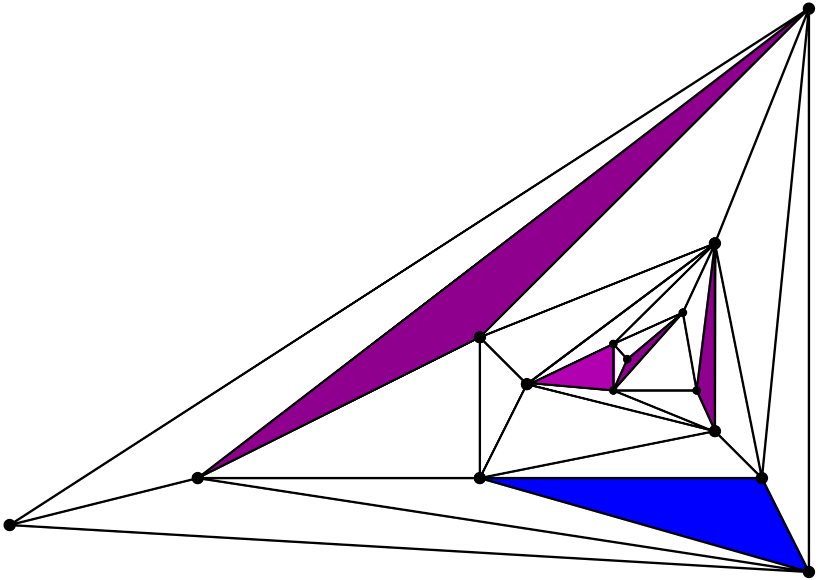
Forward recursion



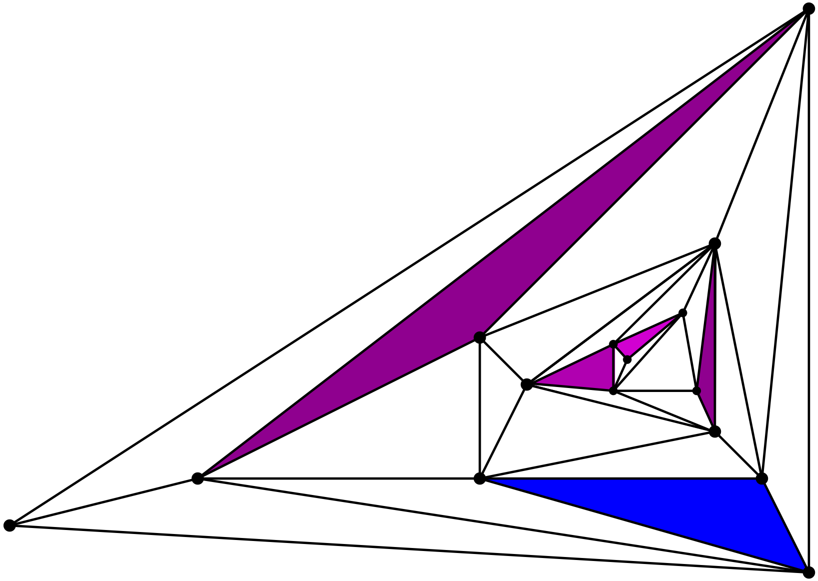
Forward recursion



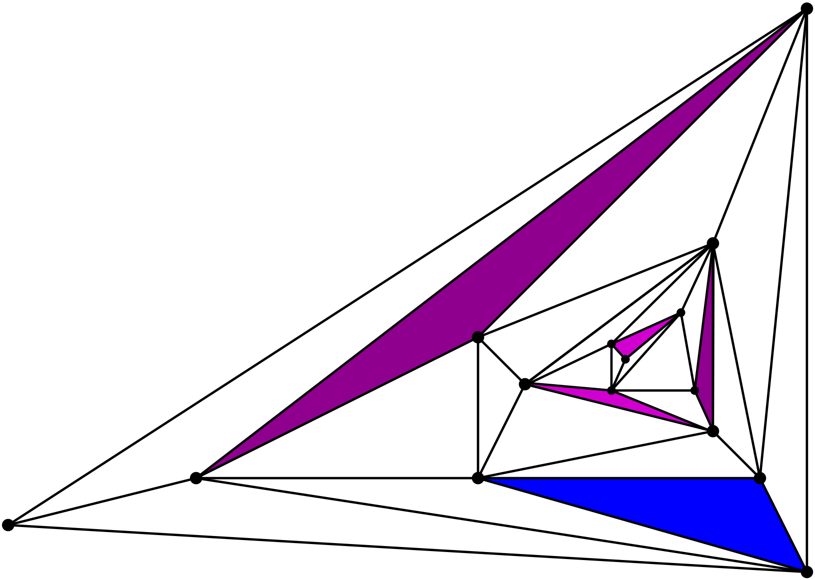
Forward recursion



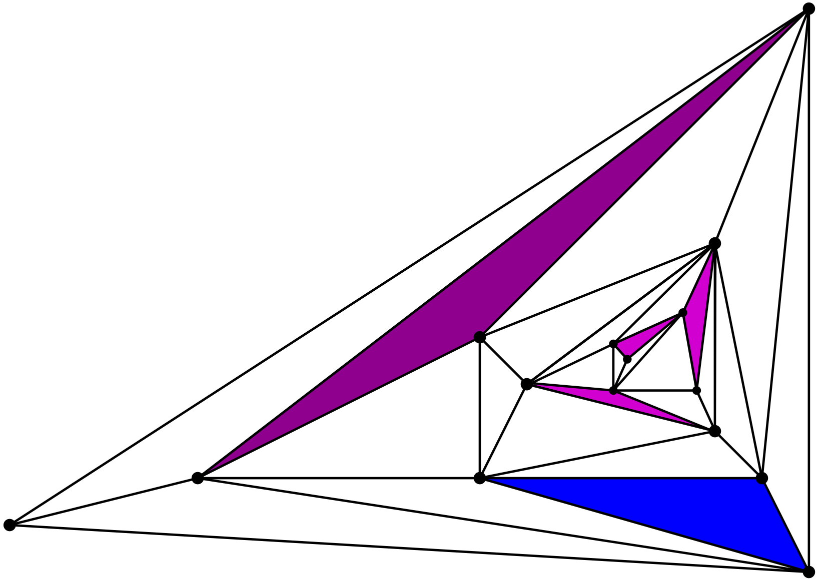
Forward recursion



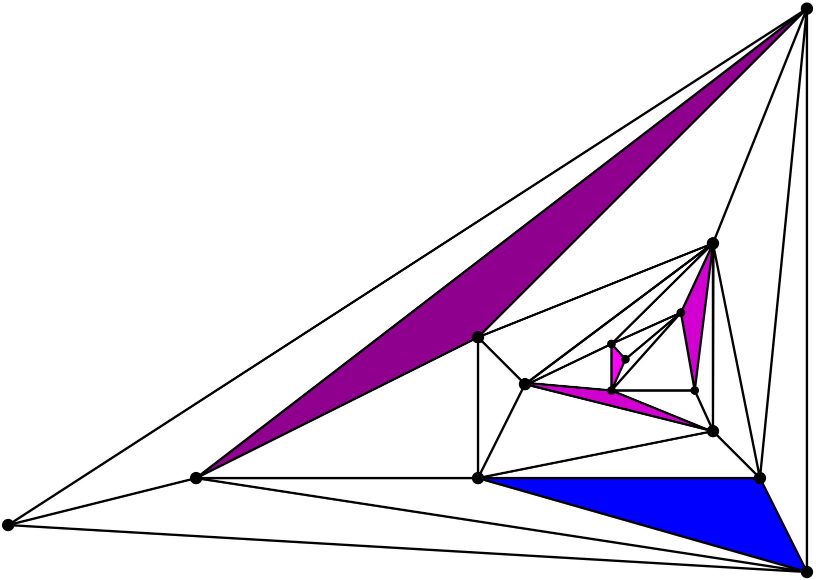
Forward recursion



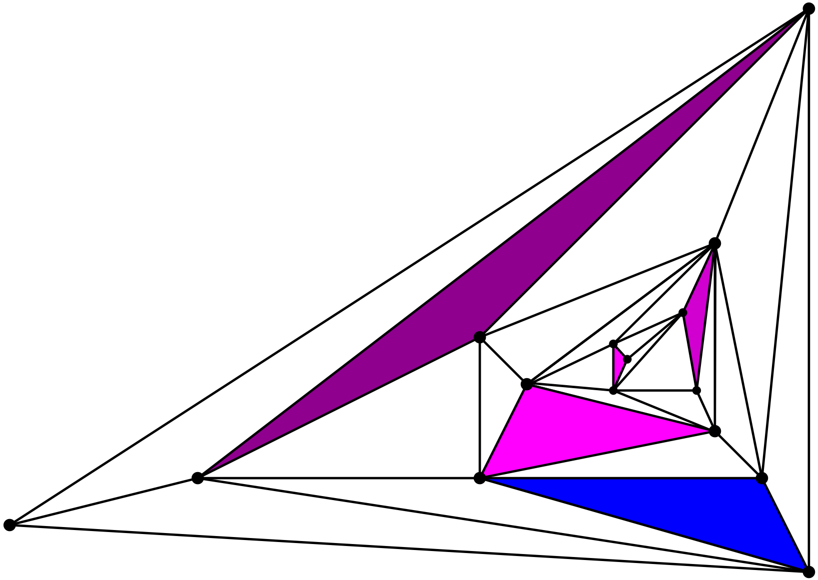
Forward recursion



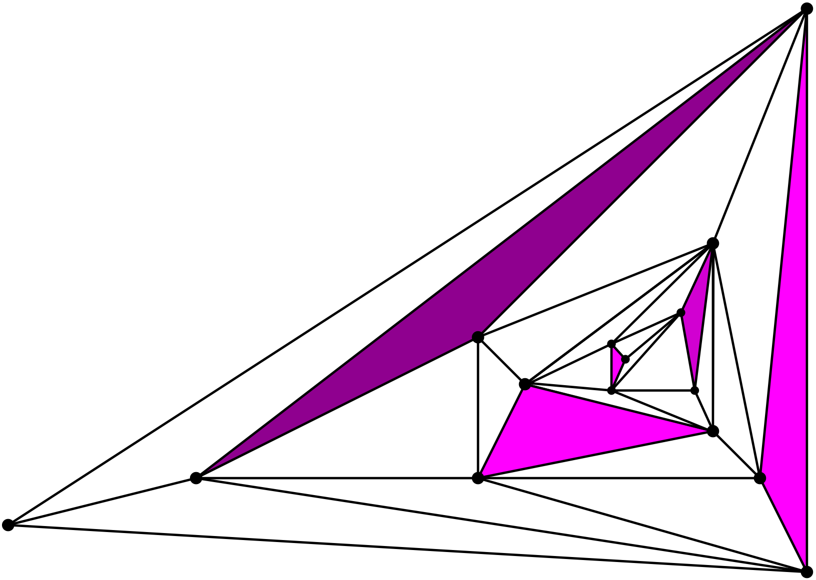
Forward recursion



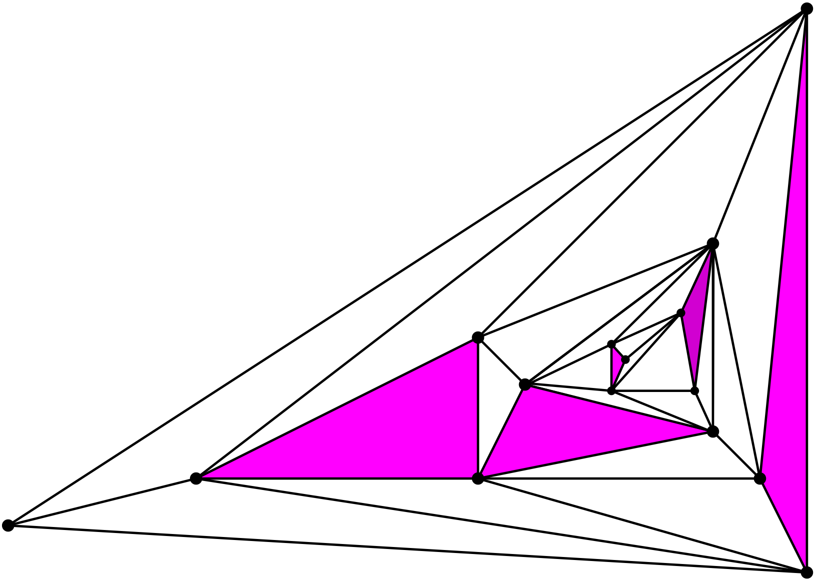
Forward recursion



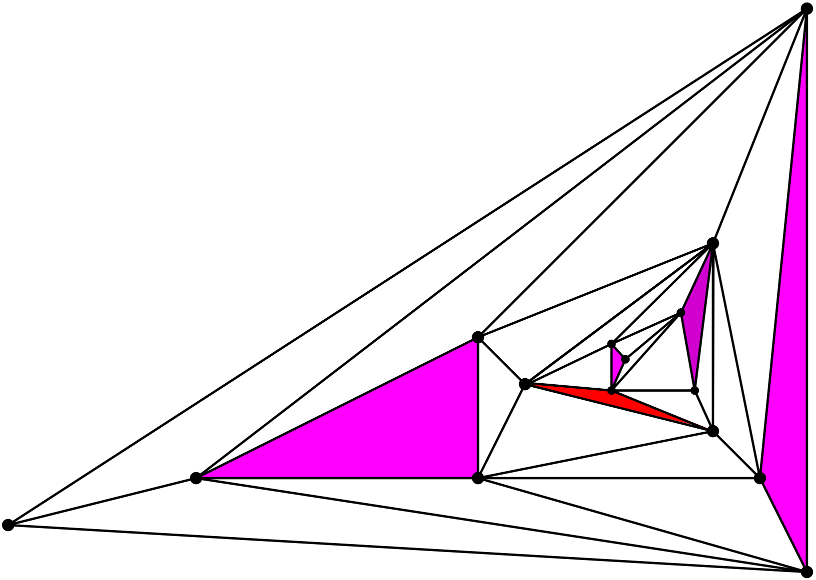
Backward recursion



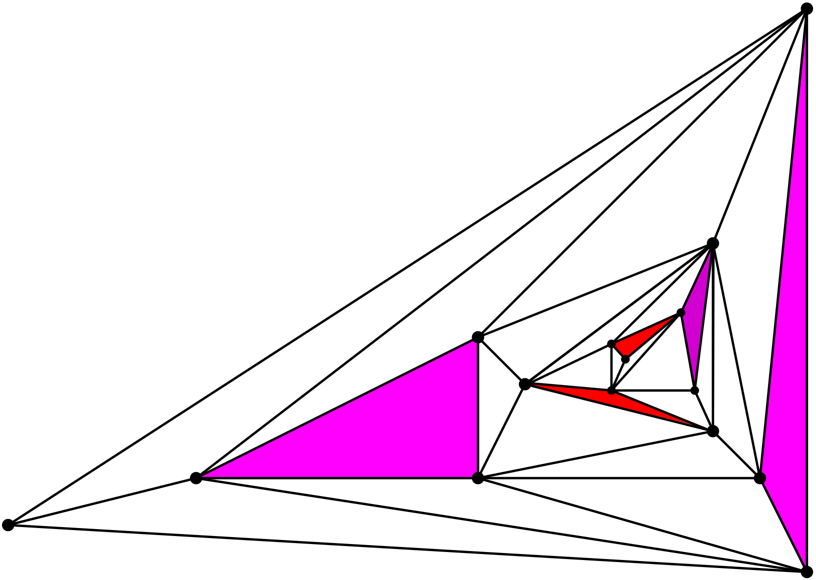
Backward recursion



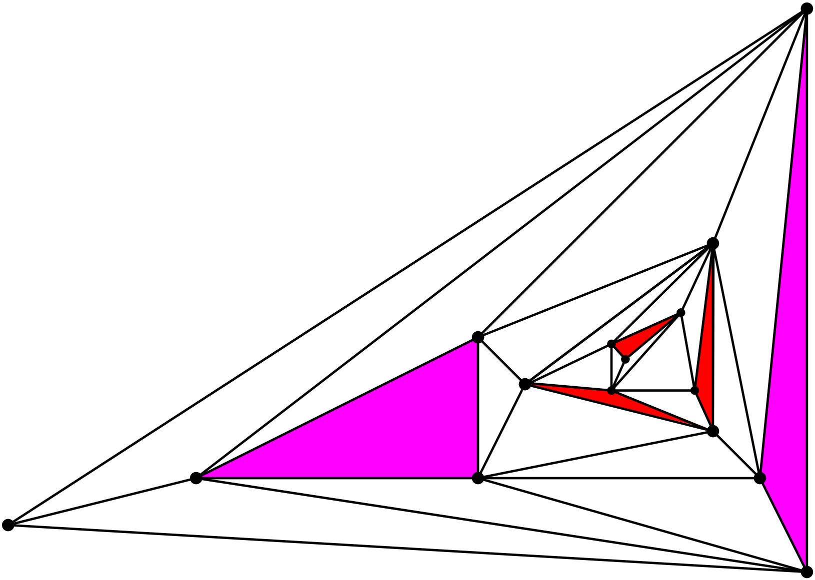
Backward recursion



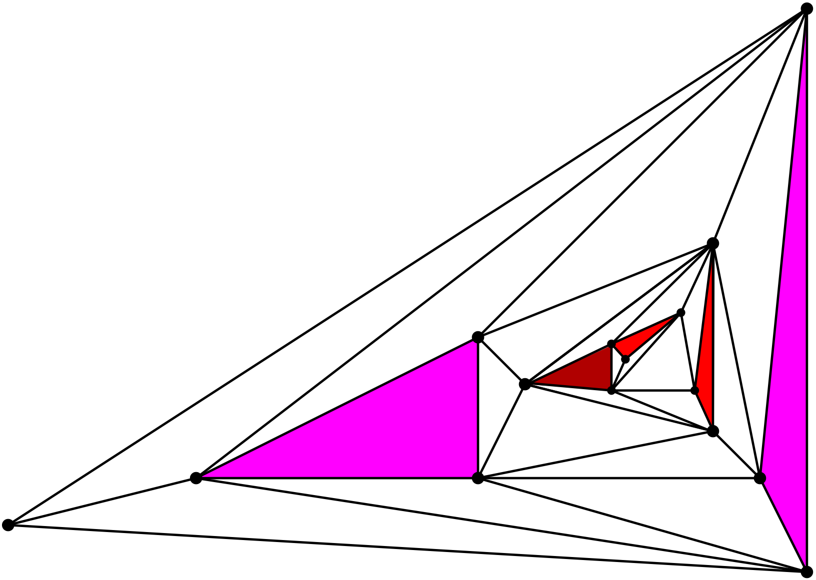
Backward recursion



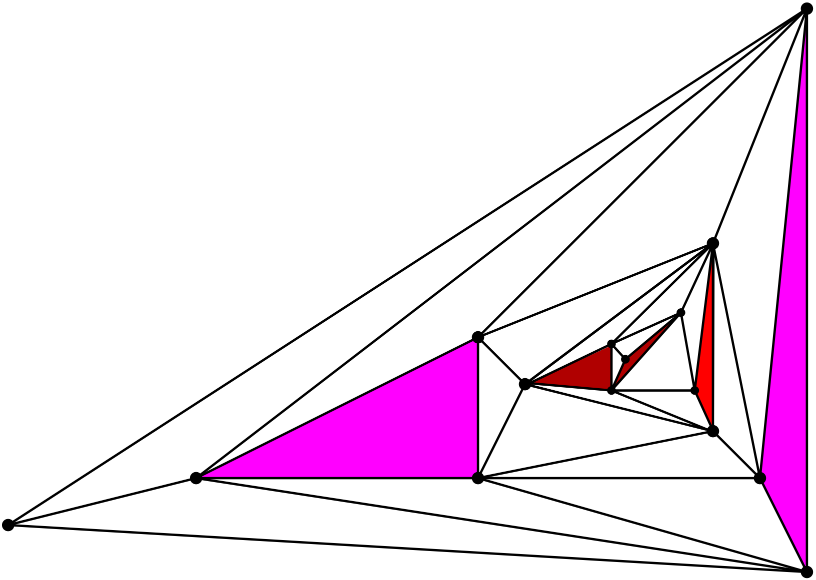
Backward recursion



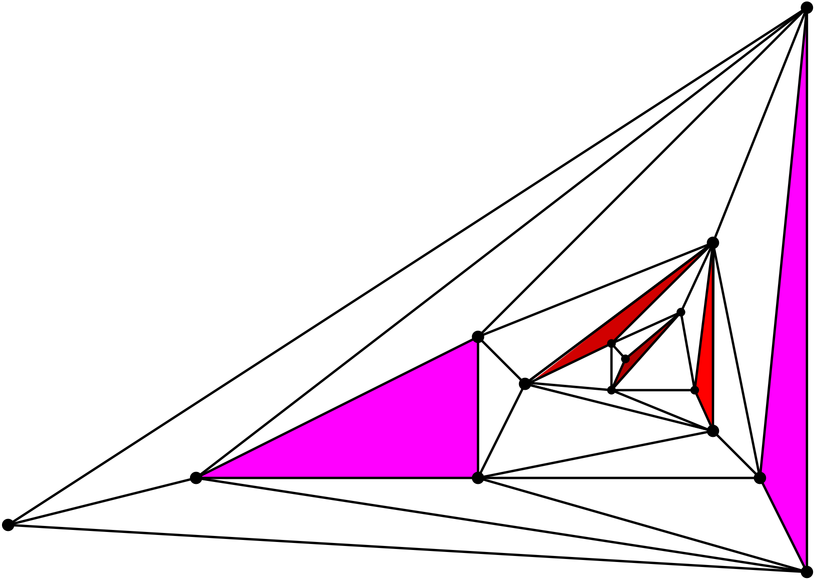
Backward recursion



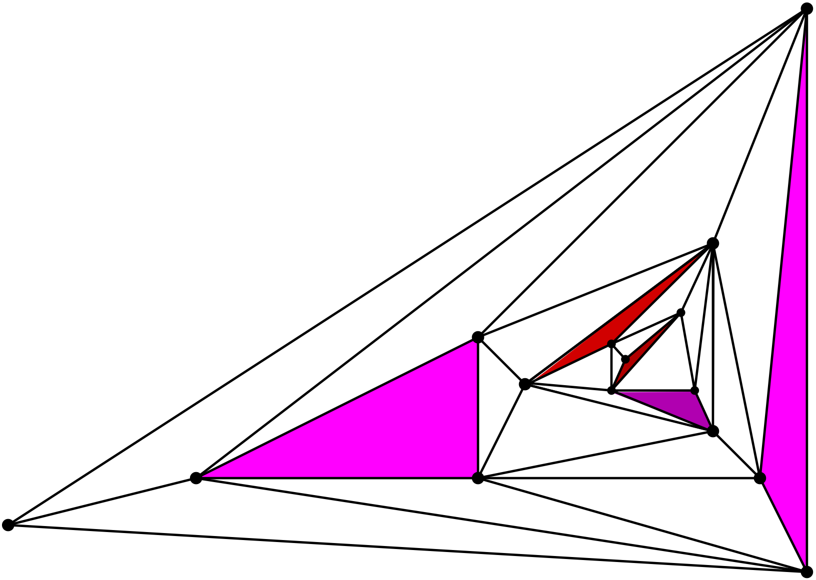
Backward recursion



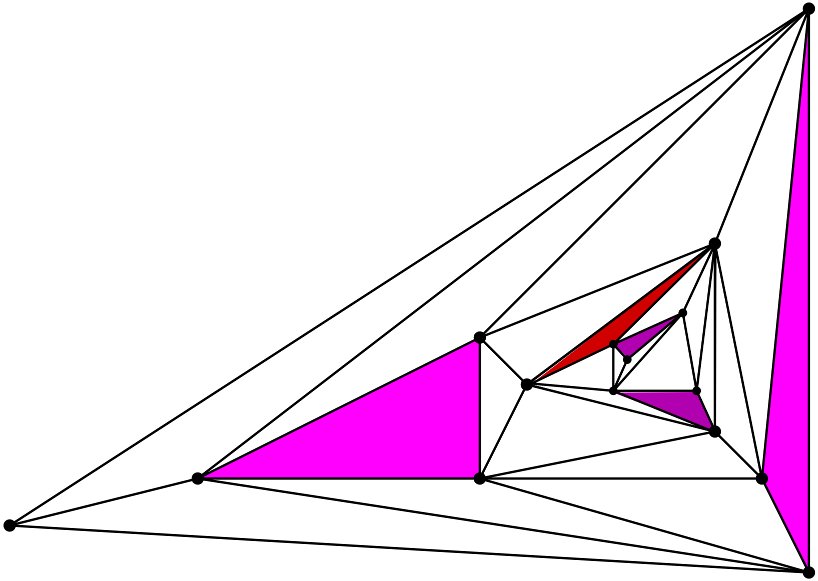
Backward recursion



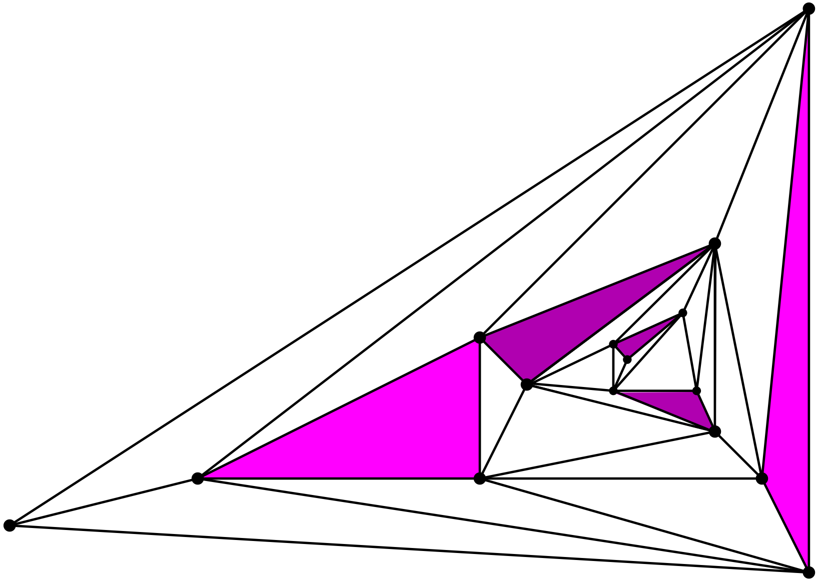
Backward recursion



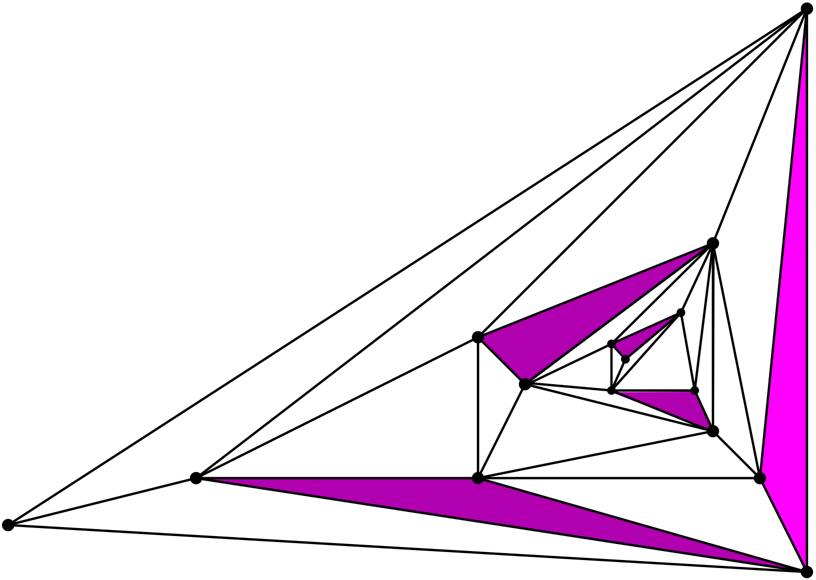
Backward recursion



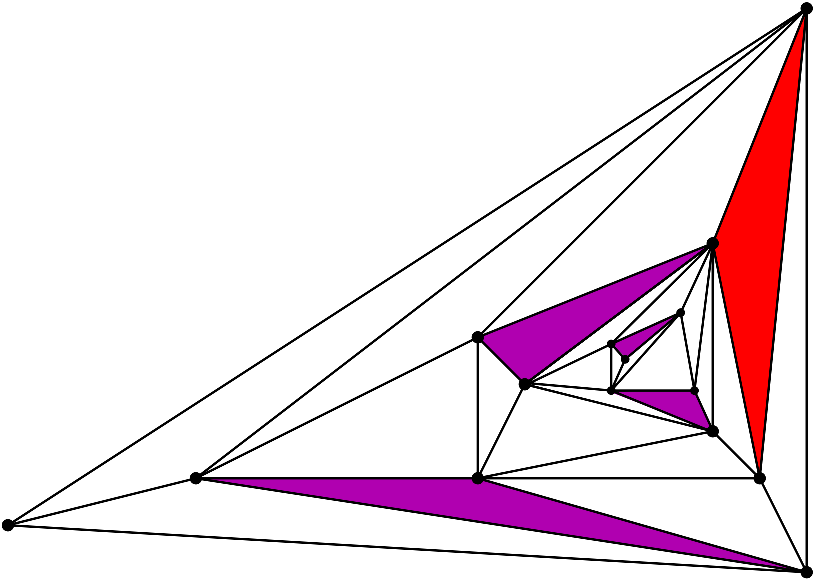
Backward recursion



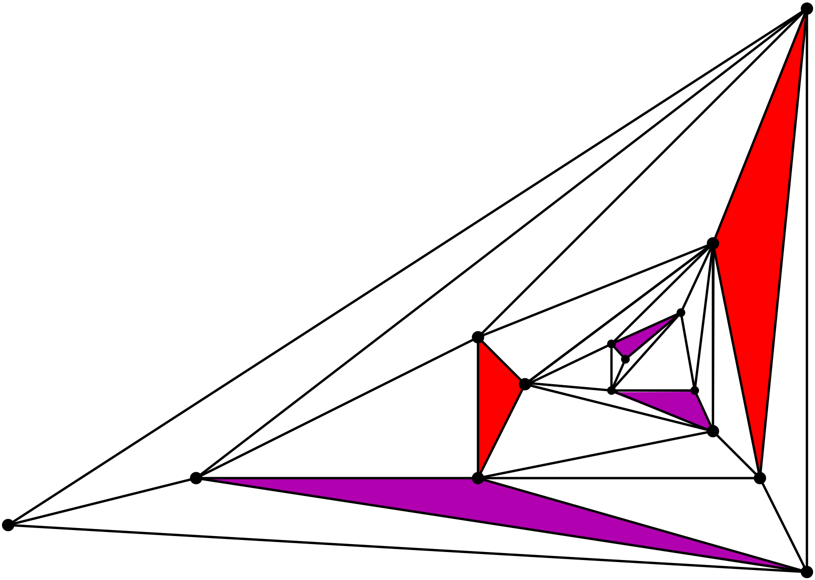
Final steps



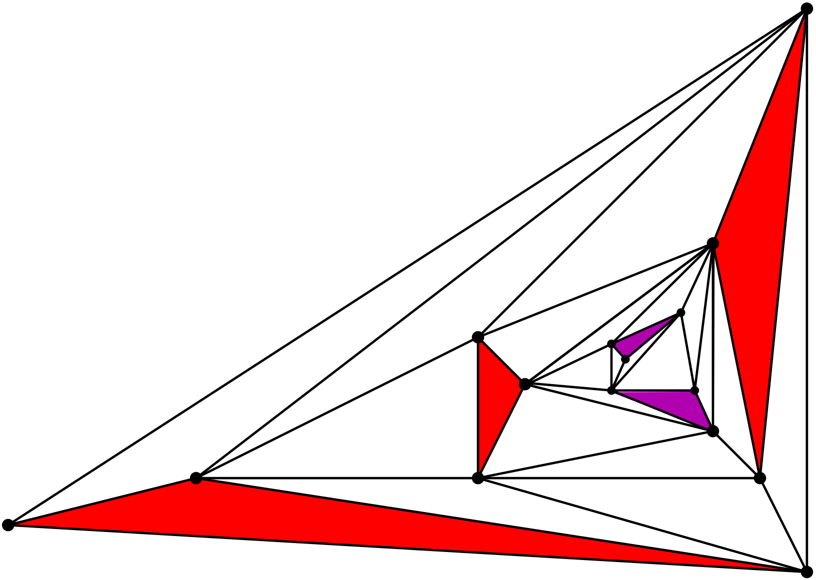
Final steps



Final steps



General construction: exponentially long path



Orienting oiks

$W = R - \{v\}$ for $v \in R$ is called a **wall** of a room R

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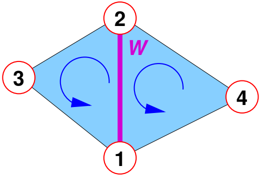
A d-manifold is **orientable** if each room has a sign $\oplus$ or $\ominus$ so that any two rooms with a common wall W induce **opposite** orientation on W ($\Leftrightarrow$ pivoting changes sign).



1 2 3 $\oplus$ 1 4 2 $\oplus$

= 1 2 4 $\ominus$

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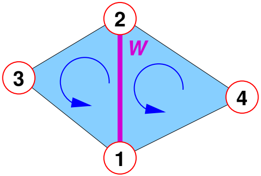


1 2 3 $\oplus$ **1 4 2** $\oplus$

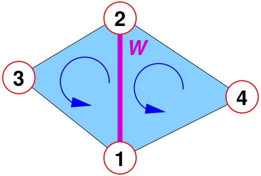
= 1 2 4 $\ominus$

induces 1 2 $\oplus$ **, 1 2** $\ominus$ **on** W

Orienting oiks

$W = R - \{v\}$ for $v \in R$ is called a **wall** of a room R

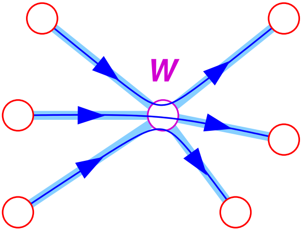
A d-manifold is **orientable** if each room has a sign $\oplus$ or $\ominus$ so that any two rooms with a common wall W induce **opposite** orientation on W ($\Leftrightarrow$ pivoting changes sign).



1 2 3 $\oplus$ 1 4 2 $\oplus$

= 1 2 4 $\ominus$

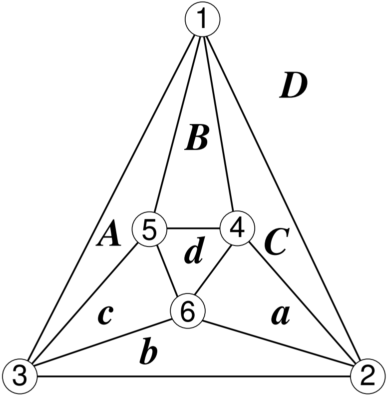
induces 1 2 $\oplus$, 1 2 $\ominus$ on W



A d-**oik** is **orientable** if half of the rooms with a common wall W induce sign $\oplus$ on W, the other half sign $\ominus$ on W.

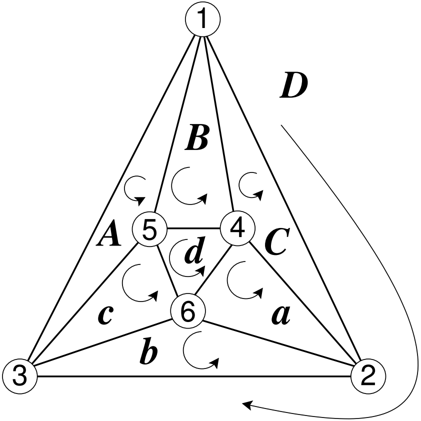
How to orient room partitions?

Example: orientable manifold



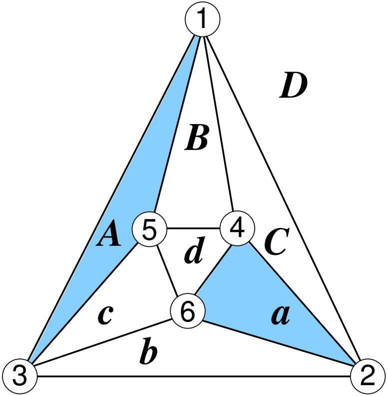
How to orient room partitions?

Example: orientable manifold



How to orient room partitions?

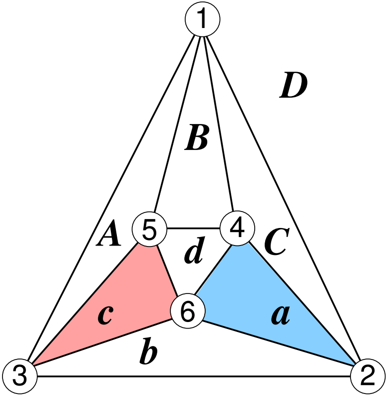
Room partition $\boldsymbol{A}, \boldsymbol{a} \;=\; \{1,3,5\},\; \{2,4,6\}$

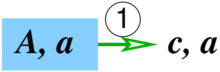


$\boldsymbol{A}, \boldsymbol{a}$

How to orient room partitions?

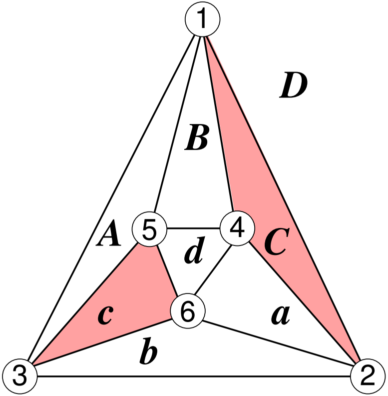
Room partition ***A*, *a*** : drop node **1**

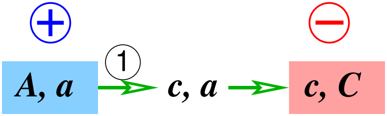




How to orient room partitions?

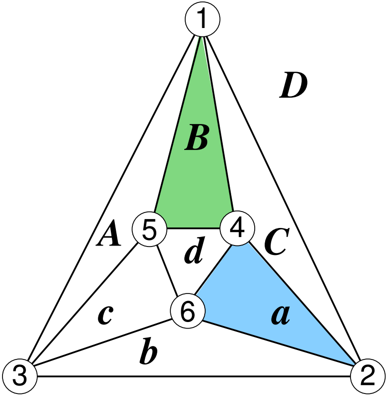
Room partition $\boldsymbol{A}$, $\boldsymbol{a}$, sign ⊕: drop node **1** leads to $\boldsymbol{c}$, $\boldsymbol{C}$, sign ⊖

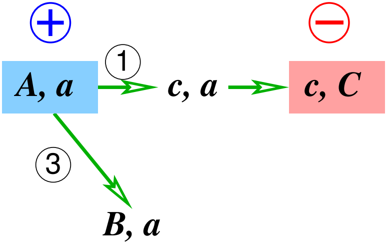




How to orient room partitions?

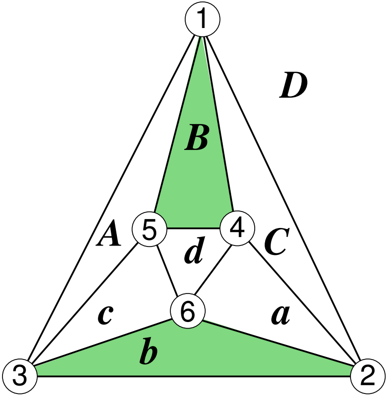
Room partition ***A***, ***a***, sign ⊕: drop node **3**

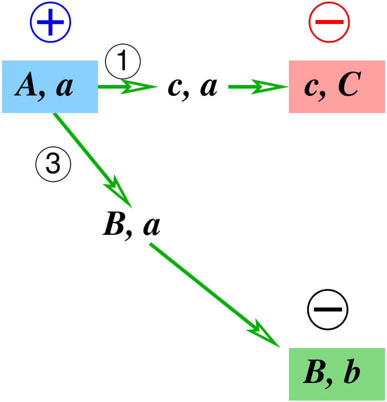




How to orient room partitions?

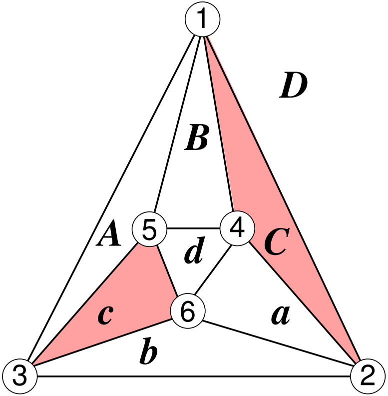
Room partition ***A***, ***a***, sign ⊕: drop node **3** leads to ***B***, ***b***, sign ⊖

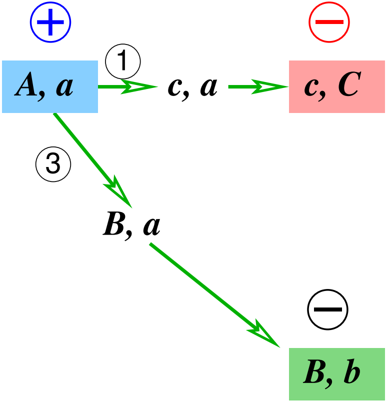




How to orient room partitions?

Room partition ***c***, ***C***, sign ⊖: drop node **5**





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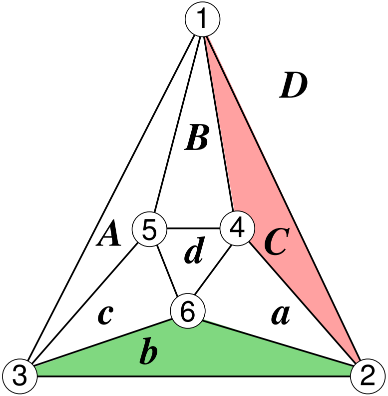
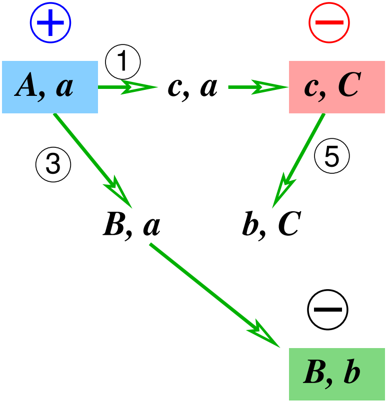
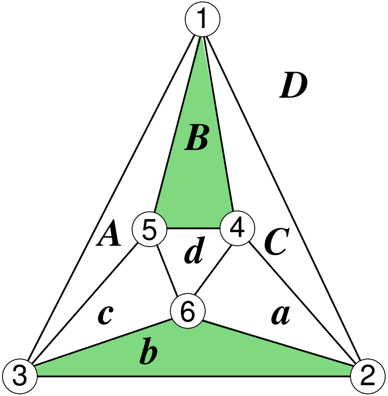
Room partition ***c***, ***C***, sign ⊖: drop node **5**

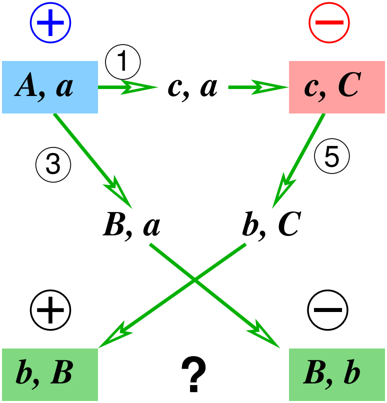




How to orient room partitions?

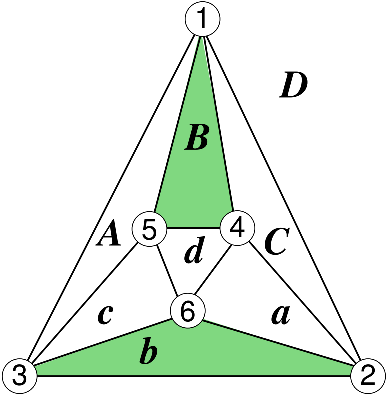
Room partition ***c*, *C***, sign ⊖: drop node **5** leads to ***b*, *B***, sign ⊕

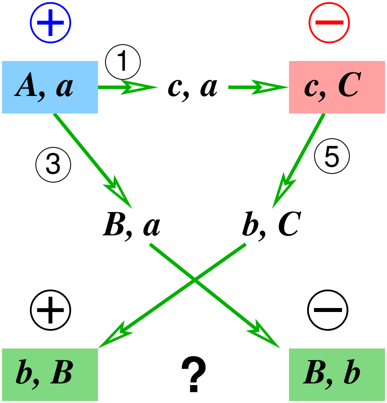




How to orient room partitions?

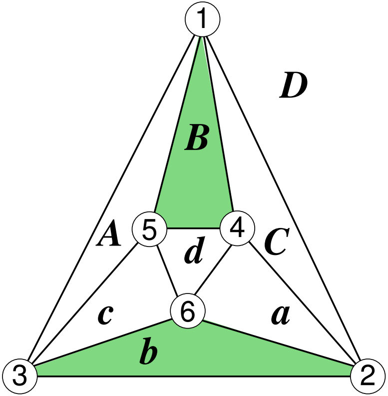
Which sign for $\{b, B\}$?

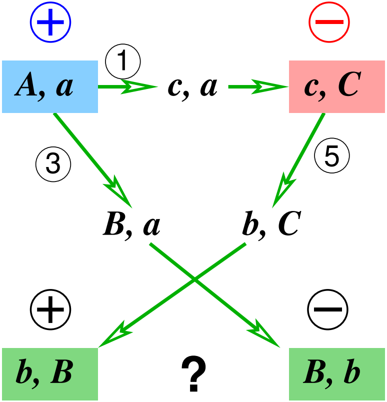




How to orient room partitions?

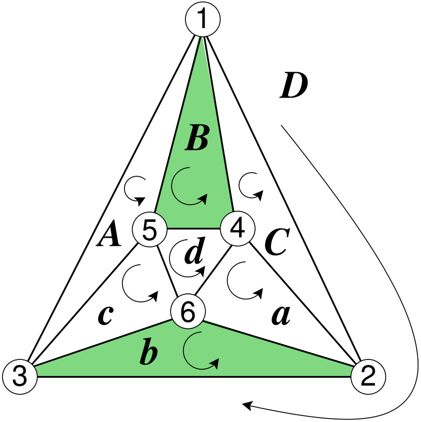
Which sign for $\{b, B\}$? ⊕ for b, B, ⊖ for B, b !





How to orient room partitions?

$\Rightarrow$ for odd dimension (here $d = 3$), order of rooms matters: permutations **263 154** (for ***b***, ***B***) and **154 263** (for ***B***, ***b***) have opposite parity.



$\ominus$

b*, *B

$\oplus$

B*, *b

Ordered room partitions

Theorem [Végh/von Stengel 2012]

Let $\mathcal{R}$ be an oriented $\boldsymbol{d}$-oik with node set $\boldsymbol{V}$. Then the number of **ordered room partitions** $(R_1, \ldots, R_{|V|/d})$ is **even.**

Any two ordered room partitions connected by a pivoting path have opposite **sign**, and the respective unordered partitions are distinct.

Ordered room partitions

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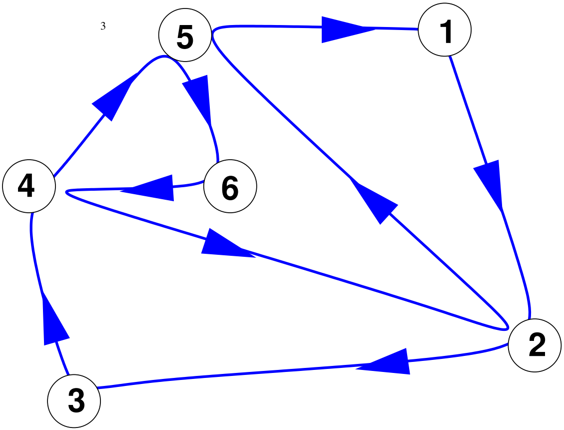
Proof uses "pivoting systems" with **labels** = nodes.

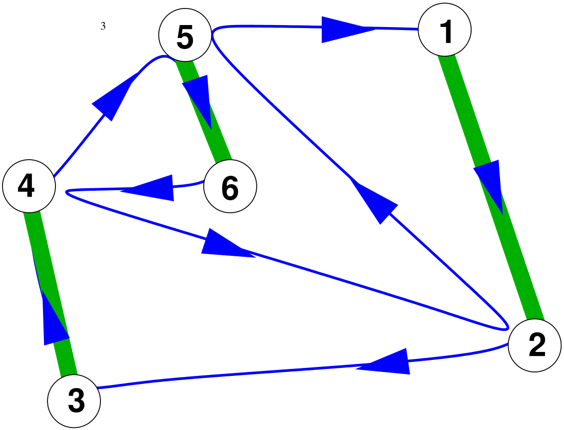
Pivoting systems generalize labeled polytopes, Lemke's algorithm, Sperner's lemma, room partitions in oiks, and more.

Finding a second perfect matching in an Euler graph



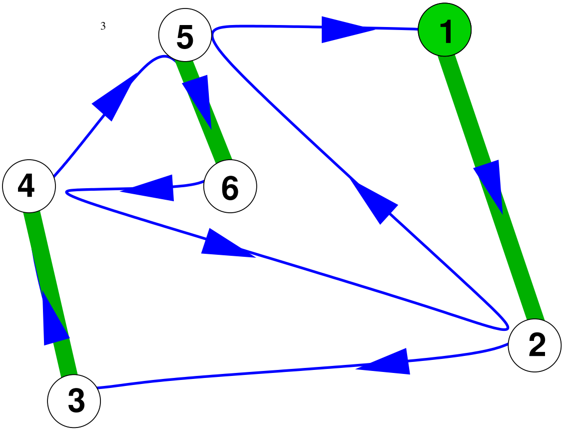
Finding a second perfect matching in an Euler graph

1 2   3 4   5 6



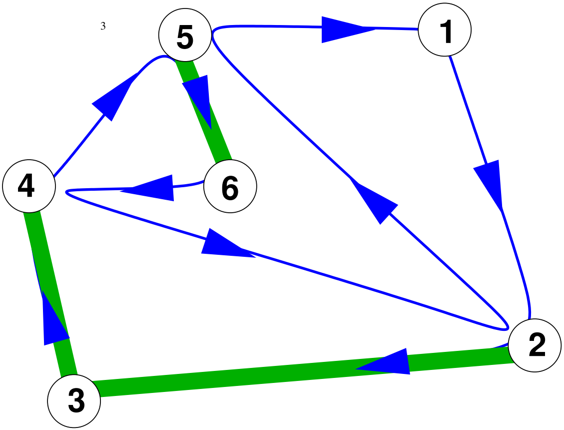
Finding a second perfect matching in an Euler graph

1 2 3 4 5 6



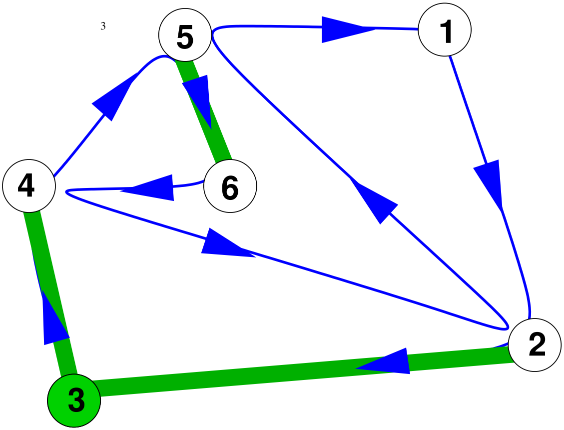
Finding a second perfect matching in an Euler graph

$\boxed{1}$ 2	3 4	5 6
2 3	3 4	5 6

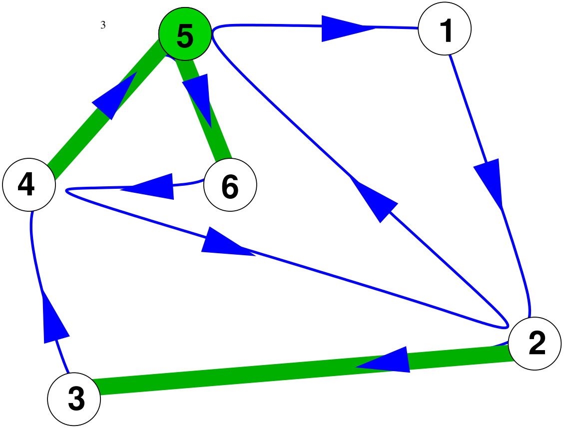


Finding a second perfect matching in an Euler graph

$\boxed{1}$ 2	3 4	5 6
2 3	$\boxed{3}$ 4	5 6

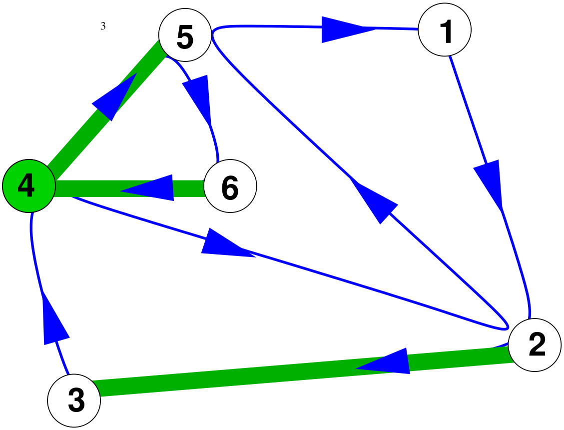


Finding a second perfect matching in an Euler graph



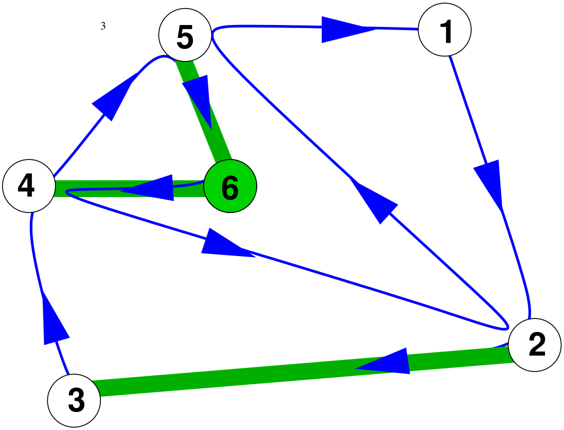
$\boxed{1}$	2	3	4	5	6
2	3	$\boxed{3}$	4	5	6
2	3	4	5	$\boxed{5}$	6

Finding a second perfect matching in an Euler graph



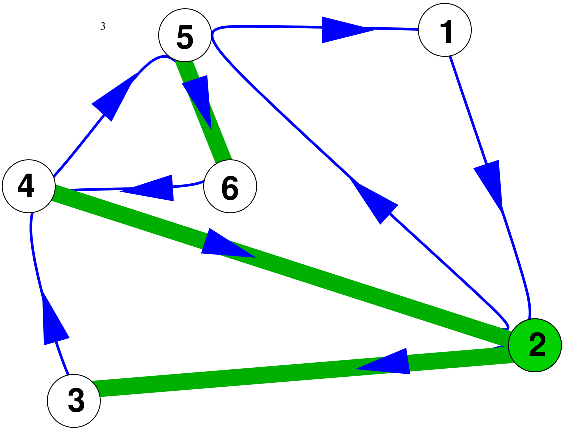
$\boxed{1}$ 2	3 4	5 6
2 $\underline{3}$	$\boxed{3}$ 4	5 6
2 3	4 $\underline{5}$	$\boxed{5}$ 6
2 3	$\boxed{4}$ 5	6 $\underline{4}$

Finding a second perfect matching in an Euler graph



1 2	3 4	5 6
2 3	3 4	5 6
2 3	4 5	5 6
2 3	4 5	6 4
2 3	5 6	6 4

Finding a second perfect matching in an Euler graph



$\boxed{1}$ 2	3 4	5 6
2 $\underline{3}$	$\boxed{3}$ 4	5 6
2 3	4 $\underline{5}$	$\boxed{5}$ 6
2 3	$\boxed{4}$ 5	6 $\underline{4}$
2 3	5 $\underline{6}$	$\boxed{6}$ 4
$\boxed{2}$ 3	5 6	4 $\underline{2}$

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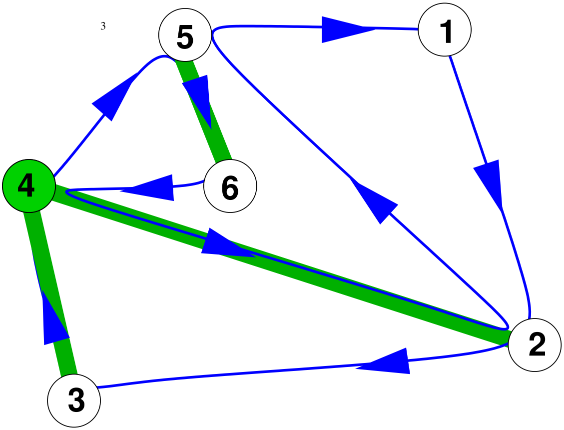

$\boxed{1}\ 2$	$3\ 4$	$5\ 6$
$2\ \underline{3}$	$\boxed{3}\ 4$	$5\ 6$
$2\ 3$	$4\ \underline{5}$	$\boxed{5}\ 6$
$2\ 3$	$\boxed{4}\ 5$	$6\ \underline{4}$
$2\ 3$	$5\ \underline{6}$	$\boxed{6}\ 4$
$\boxed{2}\ 3$	$5\ 6$	$4\ \underline{2}$
$\underline{3}\ 4$	$5\ 6$	$\boxed{4}\ 2$

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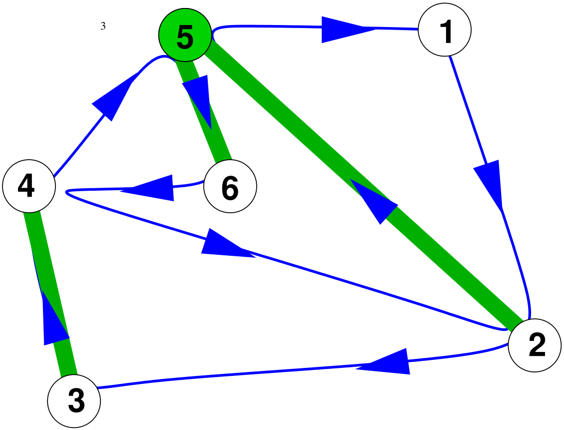

$\boxed{1}$ 2	3 4	5 6
2 $\underline{3}$	$\boxed{3}$ 4	5 6
2 3	4 $\underline{5}$	$\boxed{5}$ 6
2 3	$\boxed{4}$ 5	6 $\underline{4}$
2 3	5 $\underline{6}$	$\boxed{6}$ 4
$\boxed{2}$ 3	5 6	4 $\underline{2}$
$\underline{3}$ 4	5 6	$\boxed{4}$ 2
3 4	$\boxed{5}$ 6	2 $\underline{5}$

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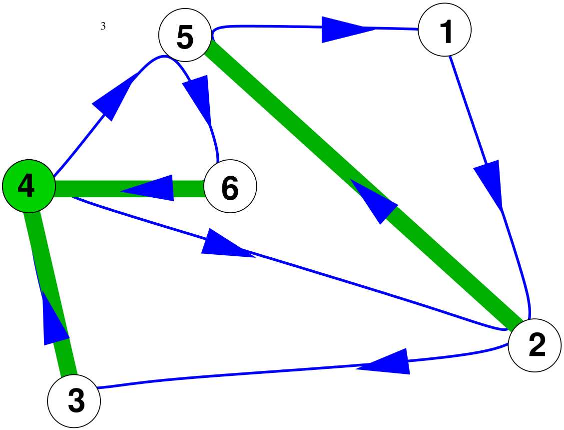

$\boxed{1}$ 2	3 4	5 6
2 3	$\boxed{3}$ 4	5 6
2 3	4 5	$\boxed{5}$ 6
2 3	$\boxed{4}$ 5	6 4
2 3	5 6	$\boxed{6}$ 4
$\boxed{2}$ 3	5 6	4 2
3 4	5 6	$\boxed{4}$ 2
3 4	$\boxed{5}$ 6	2 5
3 $\boxed{4}$	6 4	2 5

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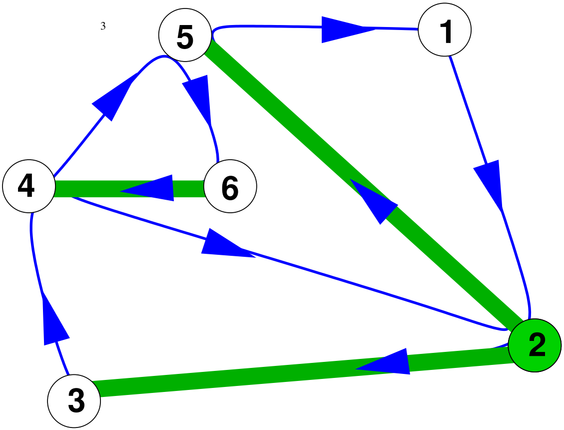

$\boxed{1}$ 2	3 4	5 6
2 3	$\boxed{3}$ 4	5 6
2 3	4 5	$\boxed{5}$ 6
2 3	$\boxed{4}$ 5	6 4
2 3	5 6	$\boxed{6}$ 4
$\boxed{2}$ 3	5 6	4 2
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2 3	6 4	$\boxed{2}$ 5

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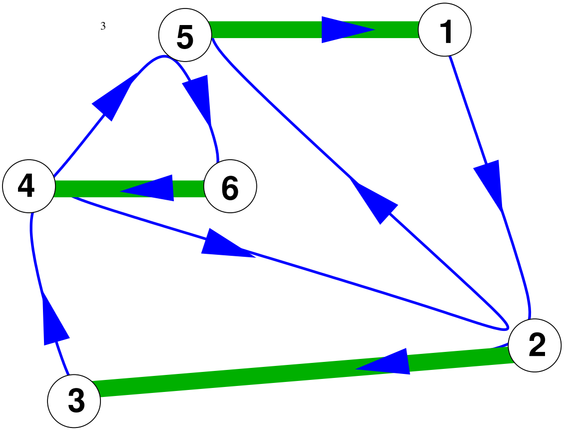

$\boxed{1}$ 2	3 4	5 6
2 3	$\boxed{3}$ 4	5 6
2 3	4 5	$\boxed{5}$ 6
2 3	$\boxed{4}$ 5	6 4
2 3	5 6	$\boxed{6}$ 4
$\boxed{2}$ 3	5 6	4 2
3 4	5 6	$\boxed{4}$ 2
3 4	$\boxed{5}$ 6	2 5
3 $\boxed{4}$	6 4	2 5
2 3	6 4	$\boxed{2}$ 5
2 3	6 4	5 1

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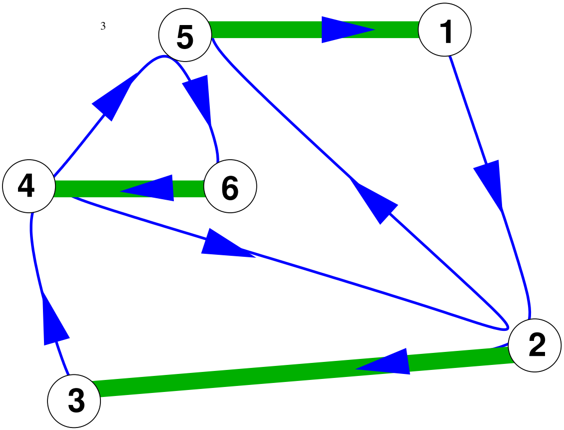
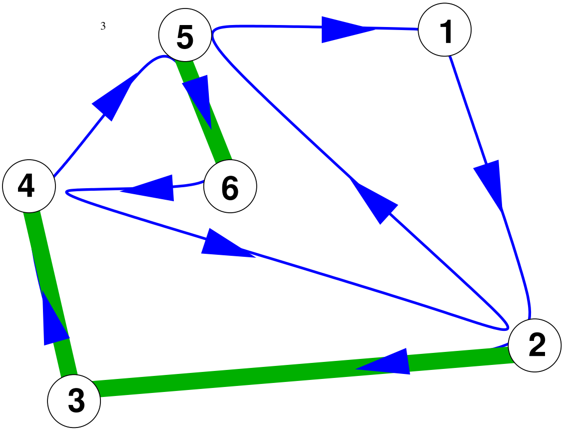


⊕	$\boxed{1}\,2$	$3\,4$	$5\,6$
	$2\,\underline{3}$	$\boxed{3}\,4$	$5\,6$
	$2\,3$	$4\,\underline{5}$	$\boxed{5}\,6$
	$2\,3$	$\boxed{4}\,5$	$6\,\underline{4}$
	$2\,3$	$5\,\underline{6}$	$\boxed{6}\,4$
	$\boxed{2}\,3$	$5\,6$	$4\,\underline{2}$
	$\underline{3}\,4$	$5\,6$	$\boxed{4}\,2$
	$3\,4$	$\boxed{5}\,6$	$2\,\underline{5}$
	$3\,\boxed{4}$	$6\,\underline{4}$	$2\,5$
	$\underline{2}\,3$	$6\,4$	$\boxed{2}\,5$
⊖	$2\,3$	$6\,4$	$5\,\underline{1}$

Finding a second perfect matching in an Euler graph

$\oplus$ $\boxed{1}\,2$ $3\,4$ $5\,6$

$\ominus$ $2\,\underline{1}$ $3\,4$ $5\,6$



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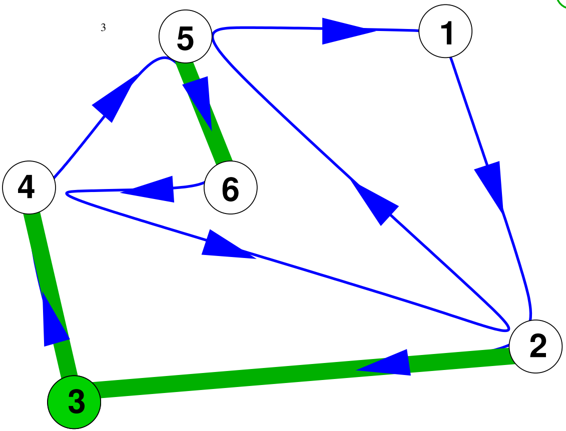
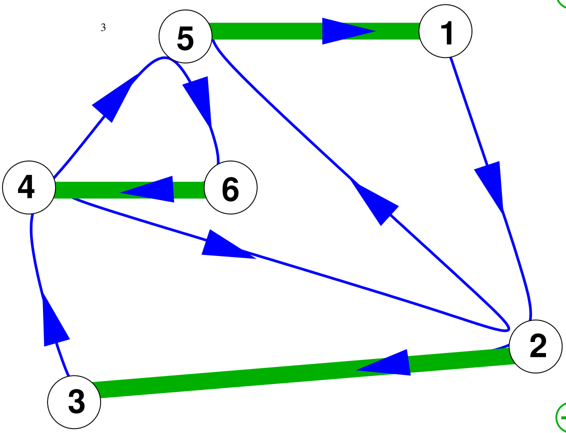
⊕ ⊖	2 3	$\boxed{1}$ 4	5 6



Finding a second perfect matching in an Euler graph

$\oplus$ $\boxed{1}\,2\quad 3\,4\quad 5\,6$

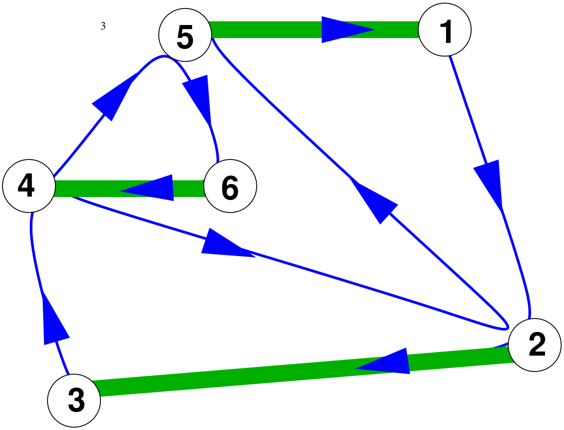
$\oplus\ \ominus$ $2\,\underline{3}\quad \boxed{1}\,4\quad 5\,6$



$\oplus$ $\underline{2}\,3\quad 6\,4\quad \boxed{2}\,5$

$\ominus$ $2\,3\quad 6\,4\quad 5\,\underline{1}$

Finding a second perfect matching in an Euler graph



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	3 4	$\boxed{5}$ 6	2 $\underline{5}$
	3 $\boxed{4}$	6 $\underline{4}$	2 5
	$\underline{2}$ 3	6 4	$\boxed{2}$ 5
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A computational problem

Input: Graph $(V, \mathcal{R})$ with Eulerian orientation and perfect matching of sign $\oplus$.

Output: A perfect matching with sign $\ominus$.

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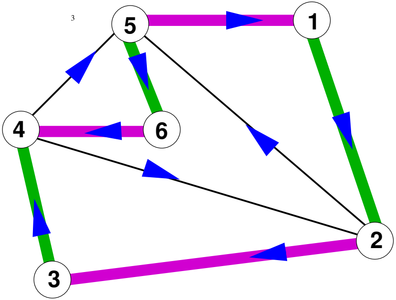
Note: A second matching can be found in polynomial time [Edmonds 1965], but not with sign $\ominus$.

Related difficult problem: Pfaffian orientations of graphs.

Sign-switching cycle (SSC)

Given an oriented graph and a perfect matching $\boldsymbol{M}$, a **sign-switching cycle** is a cycle $\boldsymbol{C}$ with every other edge in $\boldsymbol{M}$ and an **even** number of forward-pointing edges.

$\Rightarrow$ $\boldsymbol{M} \triangle \boldsymbol{C}$ is a matching of opposite sign to $\boldsymbol{M}$.



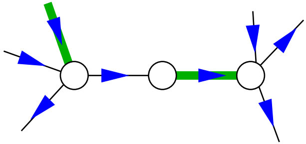
⊕ **$\boxed{1}$ 2 3 4 5 6**

⊖ **2 3 6 4 5 1**

Finding a SSC in near-linear time

Two **reductions** which preserve Euler and matching property:

1. contract node of indegree = outdegree = **1** with its two edges



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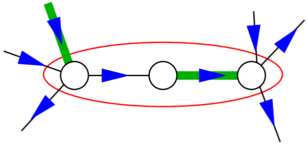

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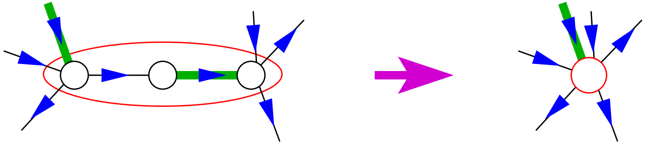

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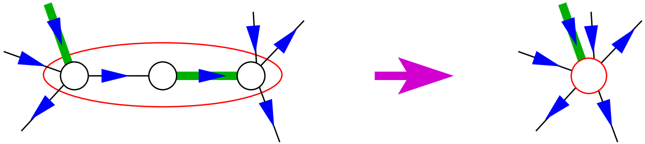
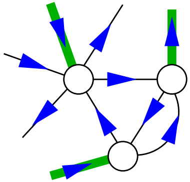


2. delete directed cycle of **unmatched** edges



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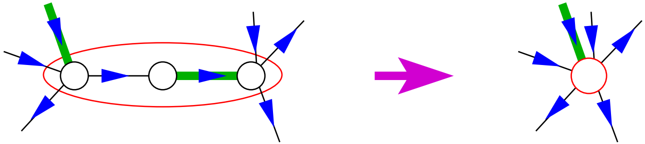

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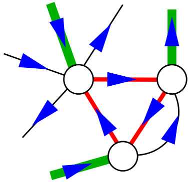

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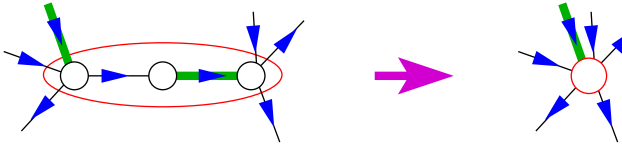

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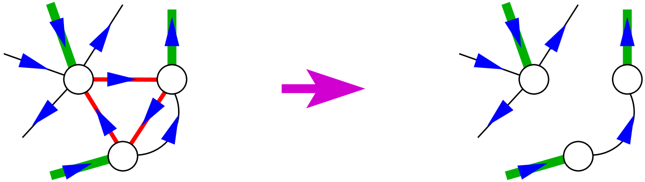

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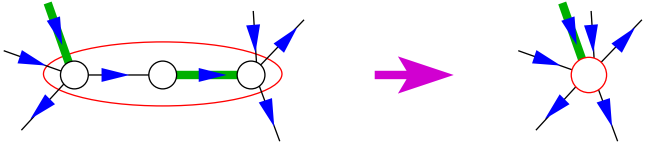

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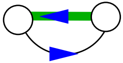

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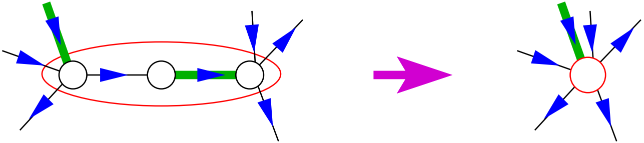

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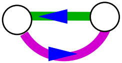

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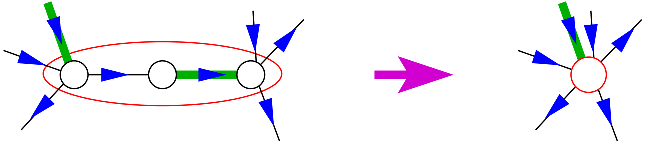
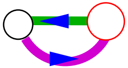


2. delete directed cycle of **unmatched** edges

until trivial SSC found, re-insert contracted edge pairs



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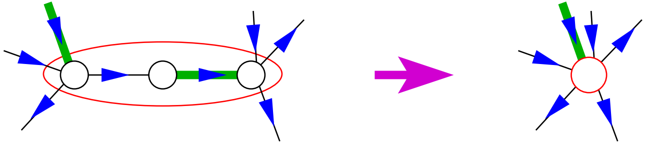

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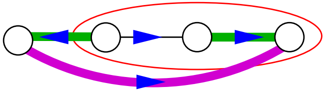

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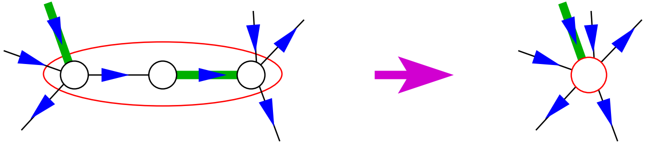
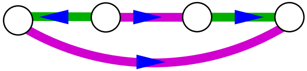


2. delete directed cycle of **unmatched** edges

until trivial SSC found, re-insert contracted edge pairs, switch.



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- exponentially long paths for matchings in Euler graph emulate exponentially long Lemke–Howson paths in **games** [Morris 1994], [Casetti/Merschen/von Stengel 2010].