Pathways to Equilibria, Pretty Pictures and Diagrams (PPAD)

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Euler graphs

Euler graph = every node has even degree (= number of neighbours)
Euler graphs

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has Eulerian orientation (indegree = outdegree)
Euler graphs ... have tours

Euler graph = every node has even degree (= number of neighbours)

has Eulerian orientation (indegree = outdegree) ... and tour
Euler’s Königsberg bridges problem

The number of odd-degree nodes of a graph is even:
<table>
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![Image of a wire sculpture](image-url)
INTRODUCTION TO

ALGORITHMS

SECOND EDITION

THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN
2-player game: find one Nash equilibrium

$2$-NASH $\in$ PPAD (Polynomial Parity Argument with Direction)

Implicit digraph with indegrees and outdegrees $\leq 1$ is a set of nodes, paths and cycles:

Parity argument: number of sources of paths = number of sinks

Comput. problem: given one source 0, find another source or sink

[Chen/Deng 2006] 2-NASH is PPAD-complete.
Symmetric Nash equilibria of symmetric games

square game matrix \( A = \) payoffs to row player

\[
A = \begin{pmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0
\end{pmatrix}
\]
Symmetric Nash equilibria of symmetric games

**equilibrium**: only optimal strategies are played

\[
A = \begin{bmatrix}
\frac{1}{3} & \frac{2}{3} & 0 \\
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0 \\
\end{bmatrix}
\]
Symmetric Nash equilibria of symmetric games

Plot polytope with strategy weights $z_1, z_2, z_3$

$z \geq 0,$

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$
Symmetric Nash equilibria of symmetric games

with payoffs (scaled to 1) and labels for binding inequalities

\[
\{ z \mid z \geq 0, \ A z \leq 1 \}
\]

\[
A = \begin{pmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0
\end{pmatrix}
\]
Symmetric Nash equilibria of symmetric games

**equilibrium** = completely labeled point

\[
\{ z \mid z \geq 0, \ A z \leq 1 \}
\]

\[
A = \begin{bmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1/6 & 1/3 & 0 \\
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0 \\
\end{bmatrix}
\]
Symmetric Nash equilibria of symmetric games

start path with **artificial equilibrium** \( z=0 \)

\[
\begin{align*}
\{ & z \mid z \geq 0, \ A z \leq 1 \} \\
\begin{pmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{pmatrix} & A = \\
\end{align*}
\]
Symmetric Nash equilibria of symmetric games

start path with artificial equilibrium \( z=0 \), choose e.g.

\[
\begin{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1
2
3
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\end{bmatrix}
\]
Symmetric Nash equilibria of symmetric games

leave facet with label 1, find duplicate label 3

\{ z \mid z \geq 0, \; Az \leq 1 \}

A = \begin{pmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{pmatrix}
Symmetric Nash equilibria of symmetric games

leave facet with old label 3, find duplicate label 2

A =
\[
\begin{bmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0
\end{bmatrix}
\]

{ z \mid z \geq 0, \ Ax \leq 1 \}

missing label ①
Symmetric Nash equilibria of symmetric games

leave facet with old label 2, find duplicate label 3

\[ \{ z \mid z \geq 0, \quad Az \leq 1 \} \]

\[
A = \begin{bmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0
\end{bmatrix}
\]
Symmetric Nash equilibria of symmetric games

leave facet with old label 3, find missing label 1

found label 1

{ z | z \geq 0, A z \leq 1 }

A =
\begin{pmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0
\end{pmatrix}
Symmetric Nash equilibria of symmetric games

equilibria (including artificial equilibrium) = endpoints of paths

\[
\{ \mathbf{z} \mid \mathbf{z} \geq 0, \ A\mathbf{z} \leq 1 \}
\]

\[
A = \begin{bmatrix}
0 & 3 & 0 & 1 \\
2 & 2 & 2 & 1 \\
3 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\]
The castle where each room has at most two doors
The castle where each room has at most two doors
The castle where each room has at most two doors
The castle where each room has at most two doors
Path of “almost completely labeled” edges

two completely labeled vertices
Path of “almost completely labeled” edges

path because at most two neighbours (“doors” in castle)
Path of “almost completely labeled” edges

orientation of edges: 2 on left, 3 on right
Path of “almost completely labeled” edges

opposite orientation (“sign”) of endpoints
Path of “almost completely labeled” edges

equilibrium sign $\ominus$ or $\oplus$ does not depend on path
Path of “almost completely labeled” edges

equilibrium sign $\ominus$ or $\oplus$ does not depend on path
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Path of “almost completely labeled” edges

equilibrium sign $\ominus$ or $\oplus$ does not depend on path
Labeled polytope $P$

Let $a_j \in \mathbb{R}^m$, $\beta_j \in \mathbb{R}$,

$$P = \{ x \in \mathbb{R}^m \mid a_j x \leq \beta_j, \ 1 \leq j \leq n \},$$

let facet $F_j = \{ x \in P \mid a_j x = \beta_j \}$ have label $l(j) \in \{1, \ldots, m\}$.

Assume $P$ is a simple polytope (no $x \in P$ on $> m$ facets)

$\Rightarrow$ each vertex $x$ on $m$ facets $= m$ linearly independent equations.

$x$ completely labeled $\iff \{ l(j) \mid x \in F_j \} = \{1, \ldots, m\}$.
Completely labeled points come in pairs

**Theorem** [Parity Argument]

Let $P$ be a labeled polytope.

Then $P$ has an **even** number of completely labeled vertices.
Completely labeled points come in pairs of opposite sign

**Theorem** [Parity Argument with Direction]

Let $P$ be a labeled polytope.

Then $P$ has an **even** number of completely labeled vertices. Half of these have sign $\ominus$, half have sign $\oplus$. 
Completely labeled points come in pairs of opposite sign

**Theorem** [ Parity Argument with Direction ]

Let $P$ be a labeled polytope.

Then $P$ has an **even** number of completely labeled vertices. Half of these have sign $\ominus$, half have sign $\oplus$.

---

**sign** of completely labeled $x$ is sign of determinant of facet normal vectors: if (e.g.) facet $a_i x = \beta_i$ has label $i = 1, 2, \ldots, m$, then

$$\text{sign}(x) = \text{sign} |a_1 \ a_2 \cdots \ a_m|$$
Lemma

Let $x, y \in \mathbb{R}^m$ be adjacent vertices of a simple polytope $P$. 

Pivoting changes signs
Lemma

Let \( x, y \in \mathbb{R}^m \) be adjacent vertices of a simple polytope \( P \) with facet normals \( c, a_2, \ldots, a_m \) for \( x \) and \( d, a_2, \ldots, a_m \) for \( y \).
Lemma

Let \( x, y \in \mathbb{R}^m \) be adjacent vertices of a simple polytope \( P \) with facet normals \( c, a_2, \ldots, a_m \) for \( x \) and \( d, a_2, \ldots, a_m \) for \( y \).

Then \(|c \ a_2 \cdots a_m|\) and \(|d \ a_2 \cdots a_m|\) have opposite sign.
Pivoting changes signs

Proof:

\[ cx = \beta_0 \]

\[ dy = \beta_1 \]

\[ a_2 x = \beta_2 \]

\[ a_2 y = \beta_2 \]

\[ \vdots \]

\[ \vdots \]

\[ a_m x = \beta_m \]

\[ a_m y = \beta_m \]
Pivoting changes signs

Proof:

\[ cx = \beta_0 \]
\[ dy = \beta_1 \]
\[ a_2 x = \beta_2 \]
\[ a_2 y = \beta_2 \]
\[ \vdots \]
\[ a_m x = \beta_m \]
\[ a_m y = \beta_m \]

Let \((\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)\) with

\[ \gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0 \]
Pivoting changes signs

Proof:

\[ cx = \beta_0 \]
\[ dy = \beta_1 \]
\[ a_2x = \beta_2 \]
\[ a_2y = \beta_2 \]
\[ \vdots \]
\[ a_mx = \beta_m \]
\[ a_my = \beta_m \]

Let \((\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)\) with

\[ \gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0 \]

\[ \Rightarrow \gamma \neq 0, \ \delta \neq 0, \]

\[ (\gamma c + \delta d)x = (\gamma c + \delta d)y \]
Pivoting changes signs

Proof:

\[ cx = \beta_0 \quad cy < \beta_0 \]
\[ dx < \beta_1 \quad dy = \beta_1 \]
\[ a_2 x = \beta_2 \quad a_2 y = \beta_2 \]
\[ \vdots \quad \vdots \]
\[ a_m x = \beta_m \quad a_m y = \beta_m \]

Let \((\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)\) with

\[ \gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0 \]

\[ \Rightarrow \gamma \neq 0, \delta \neq 0, \]

\[ (\gamma c + \delta d)x = (\gamma c + \delta d)y, \quad \gamma(cx - cy) = \delta(dy - dx) \]
Pivoting changes signs

Proof:

\[
\begin{align*}
    cx &= \beta_0 & cy &< \beta_0 \\
    dx &< \beta_1 & dy &= \beta_1 \\
    a_2x &= \beta_2 & a_2y &= \beta_2 \\
    & \vdots & & \vdots \\
    a_mx &= \beta_m & a_my &= \beta_m
\end{align*}
\]

Let \((\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)\) with

\[
\gamma c + \delta d + \alpha_2a_2 + \cdots + \alpha_m a_m = 0
\]

\[\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,\]

\[
(\gamma c + \delta d)x = (\gamma c + \delta d)y, \quad \gamma(cx - cy) = \delta(dy - dx)
\]

\[\Rightarrow \quad \gamma \text{ and } \delta \text{ have same sign}\]
Pivoting changes signs

Proof:

\[ cx = \beta_0 \]
\[ cy < \beta_0 \]
\[ dx < \beta_1 \]
\[ dy = \beta_1 \]
\[ a_2x = \beta_2 \]
\[ a_2y = \beta_2 \]
\[ \vdots \]
\[ a_mx = \beta_m \]
\[ a_my = \beta_m \]

Let \((\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)\) with

\[ \gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0 \]

\[ \Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0, \]

\[ (\gamma c + \delta d)x = (\gamma c + \delta d)y, \quad \gamma(cx - cy) = \delta(dy - dx) \]

\[ \Rightarrow \quad \gamma \text{ and } \delta \text{ have same sign,} \]

\[ |(\gamma c + \delta d) a_2 \cdots a_m| = \gamma |c \ a_2 \cdots a_m| + \delta |d \ a_2 \cdots a_m| = 0 \]
Pivoting changes signs

Proof:

\[
\begin{align*}
\mathbf{c} \mathbf{x} &= \beta_0 & \mathbf{c} \mathbf{y} &< \beta_0 \\
\mathbf{d} \mathbf{x} &< \beta_1 & \mathbf{d} \mathbf{y} &= \beta_1 \\
\mathbf{a}_2 \mathbf{x} &= \beta_2 & \mathbf{a}_2 \mathbf{y} &= \beta_2 \\
\vdots & & \vdots \\
\mathbf{a}_m \mathbf{x} &= \beta_m & \mathbf{a}_m \mathbf{y} &= \beta_m
\end{align*}
\]

Let \((\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)\) with

\[
\gamma \mathbf{c} + \delta \mathbf{d} + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_m \mathbf{a}_m = 0
\]

\[
\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,
\]

\[
(\gamma \mathbf{c} + \delta \mathbf{d}) \mathbf{x} = (\gamma \mathbf{c} + \delta \mathbf{d}) \mathbf{y}, \quad \gamma (\mathbf{c} \mathbf{x} - \mathbf{c} \mathbf{y}) = \delta (\mathbf{d} \mathbf{y} - \mathbf{d} \mathbf{x})
\]

\[
\Rightarrow \quad \gamma \text{ and } \delta \text{ have same sign},
\]

\[
|(\gamma \mathbf{c} + \delta \mathbf{d}) \mathbf{a}_2 \cdots \mathbf{a}_m| = \gamma |\mathbf{c} \mathbf{a}_2 \cdots \mathbf{a}_m| + \delta |\mathbf{d} \mathbf{a}_2 \cdots \mathbf{a}_m| = 0
\]

\[
\Rightarrow \quad |\mathbf{c} \mathbf{a}_2 \cdots \mathbf{a}_m| \text{ and } |\mathbf{d} \mathbf{a}_2 \cdots \mathbf{a}_m| \text{ have opposite sign}, \quad \text{QED.}
\]
General Parity Argument with Direction

Facet normal vectors $a_1 \ a_2 \ a_3 \ c_1 \ c_2 \ c_3$, labels $1 \ 2 \ 3 \ 1 \ 2 \ 3$
General Parity Argument with Direction

Start with $a_1 \ a_2 \ a_3$, sign $\ominus$

\[ a_1 \ a_2 \ a_3 \]

\[ c_1 \ c_2 \ c_3 \]

\[ a_1 \ a_2 \ a_3 \]

\[ a_1 \ a_2 \ a_3 \]
General Parity Argument with Direction

Start with \( a_1 \ a_2 \ a_3 \), sign \( \ominus \), label \( 1 \) missing, \( a_1 \rightarrow c_3 \) gives sign \( \oplus \)

\[
\begin{align*}
\begin{vmatrix}
a_1 & a_2 & a_3 \\
c_3 & a_2 & a_3
\end{vmatrix}
\end{align*}
\]
Switch columns $c_3$ and $a_3$ in determinant: back to sign $\ominus$
General Parity Argument with Direction

next pivot $a_3 \rightarrow c_2$ gives sign $\oplus$
Switch columns $c_2$ and $a_2$ in determinant: back to sign $\ominus$
General Parity Argument with Direction

next pivot $a_2 \rightarrow a_3$ gives sign $\oplus$
General Parity Argument with Direction

Switch columns $a_3$ and $c_3$ in determinant: back to sign $\ominus$
Last pivot $c_3 \rightarrow c_1$ gives sign $\oplus$, opposite to starting sign $\ominus$.
Only need: sign-switching of pivots and column exchanges
Nash equilibria of bimatrix games

Recall: $m \times m$ matrix $C$,

$$P = \{ z \in \mathbb{R}^m \mid -z \leq 0, \quad Cz \leq 1 \}$$

with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$. 

Nash equilibria of bimatrix games

Recall: $m \times m$ matrix $C$,

$$P = \{ z \in \mathbb{R}^m \mid -z \leq 0, \ Cz \leq 1 \}$$

with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$.

Completely labeled $z \neq 0 \iff$

Nash equilibrium $(z, z)$ of game $(C, C^T)$
Nash equilibria of bimatrix games

Recall: $m \times m$ matrix $C$, 

$$P = \{ z \in \mathbb{R}^m \mid -z \leq 0, \ Cz \leq 1 \}$$

with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$.

Completely labeled $z \neq 0 \iff$
Nash equilibrium $(z, z)$ of game $(C, C^\top)$

**Normalize** sign of “artificial equilibrium” 0 to $\ominus$, in general

$$\text{index}(z) = \text{sign}(z) \cdot (-1)^{m+1}$$
Nash equilibria of bimatrix games

Recall: \( m \times m \) matrix \( C \),

\[
P = \{ z \in \mathbb{R}^m \mid -z \leq 0, \quad Cz \leq 1 \}
\]

with \( 2m \) inequalities labeled \( 1, \ldots, m, 1, \ldots, m \).

bimatrix game \( (A, B) \):

\[
C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}, \quad z = (x, y):
\]

Completely labeled \( (x, y) \neq (0, 0) \) \( \iff \)

Nash equilibrium \( (x, y) \) of game \( (A, B) \)
Index of an equilibrium

**Theorem [Shapley 1974]**

A nondegenerate bimatrix game \((A, B)\) has an odd number of equilibria, one more of index \(\oplus\) than of index \(\ominus\).
Theorem [Shapley 1974]

A nondegenerate bimatrix game \((A, B)\) has an odd number of equilibria, one more of index \(\oplus\) than of index \(\ominus\).

\([Proof: Endpoints of pivoting paths have opposite index \(\ominus\) and \(\oplus\).]\)
Index of an equilibrium

**Theorem** [Shapley 1974]

A nondegenerate bimatrix game \((A, B)\) has an odd number of equilibria, one more of index \(\oplus\) than of index \(\ominus\).

\([Proof: Endpoints of pivoting paths have opposite index \(\ominus\) and \(\oplus\).]\)

Equilibria of index \(\oplus\) include every

- pure-strategy equilibrium
- unique equilibrium
- **dynamically stable** equilibrium
Dynamically stable equilibrium: only if $\oplus$
Dynamically stable equilibrium: only if $\oplus$
Dynamically stable equilibrium: only if \(\oplus\)
Dynamically stable equilibrium: only if $\oplus$
Dynamically stable equilibrium: only if $\oplus$
Dynamically stable equilibrium: only if \( ±\)
Literature


Literature – Generalizations


Plan / our results

- Manifolds and oiks [Edmonds]:
Plan / our results

- Manifolds and oiks [Edmonds]: room partitions come in pairs oik!

- We define an orientation for oiks with signs $\pm$

- $2$-oik = Euler graph, room partition = perfect matching
  - Finding a second matching of opposite sign
    - May take exponential time with path-following
    - New polynomial-time algorithm
Oiks and pivoting

**Definition** [Edmonds 2009]

Given: finite set $V$ of nodes. Multiset $\mathcal{R}$ of $d$-element sets of nodes, called **rooms**, is a **$d$-oik** (Euler complex) if every set of $d - 1$ nodes is contained in an **even** number of rooms.
Oiks and pivoting

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Example $d = 2$: Euler graph with nodes in $V$ and edges in $\mathcal{R}$. 
Oiks and pivoting

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Example $d = 2$: Euler graph with nodes in $V$ and edges in $R$.

If every set of $d - 1$ nodes is contained in 0 or 2 rooms then the oik is called a (abstract simplicial pseudo-) manifold.
Oiks and pivoting

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Example $d = 2$: Euler graph with nodes in $V$ and edges in $\mathcal{R}$.

If every set of $d - 1$ nodes is contained in 0 or 2 rooms then the oik is called a (abstract simplicial pseudo-) manifold.

 manifold, $d = 3$  \[ d = 2 \]
Oiks and pivoting

**Definition [Edmonds 2009]**

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Example $d = 2$: Euler graph with nodes in $V$ and edges in $\mathcal{R}$.

If every set of $d - 1$ nodes is contained in 0 or 2 rooms then the oik is called a (abstract simplicial pseudo-) **manifold**.

![Diagram of a 3-oik and a 2-pivoting graph]
Room partitions come in pairs

Given an oik $\mathcal{R}$ with node set $V$, a room partition is a partition of $V$ into rooms.

**Theorem** [Edmonds 2009]

The number of room partitions is even.
Room partition for 3-manifold
Room partition for 3-manifold
$w$-almost room partition
$w$-almost room partition
$w$-almost room partition
$w$-almost room partition
$w$-almost room partition
$w$-almost room partition
$w$-almost room partition
$w$-almost room partition
Found second room partition
[Edmonds/Sanità 2010]: exponentially long path
[Edmonds/Sanità 2010]: exponentially long path
[Edmonds/Sanità 2010]: exponentially long path
[Edmonds/Sanità 2010]: exponentially long path
[Edmonds/Sanità 2010]: exponentially long path
[Edmonds/Sanità 2010]: exponentially long path
6 extra nodes, 12 extra rooms
Path length more than doubles
Path length more than doubles
Forward recursion
Forward recursion
Forward recursion
Forward recursion
Forward recursion
Forward recursion
Forward recursion
Forward recursion
Forward recursion
Forward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Backward recursion
Final steps
Final steps
Final steps
General construction: exponentially long path
Orienting oiks

\[ W = R - \{v\} \text{ for } v \in R \text{ is called a wall of a room } R \]
Orienting oiks

$W = R - \{v\}$ for $v \in R$ is called a wall of a room $R$

A $d$-manifold is orientable if each room has a sign $\oplus$ or $\ominus$ so that any two rooms with a common wall $W$ induce opposite orientation on $W$ ($\iff$ pivoting changes sign).
Orienting oiks

\( W = R - \{ v \} \) for \( v \in R \) is called a wall of a room \( R \)

A \( d \)-manifold is orientable if each room has a sign \( \oplus \) or \( \ominus \) so that any two rooms with a common wall \( W \) induce opposite orientation on \( W \) (\( \Leftrightarrow \) pivoting changes sign).

\[
\begin{array}{c}
1 & 2 & 3 & \oplus & 1 & 4 & 2 & \oplus \\
& & W & & & & \\
1 & 2 & 3 & \ominus & 1 & 2 & 4 & \ominus \\
\end{array}
\]

induces \( 1 \ 2 \ \oplus \), \( 1 \ 2 \ \ominus \) on \( W \)
Orienting oiks

\( W = R - \{ v \} \) for \( v \in R \) is called a wall of a room \( R \).

A \( d \)-manifold is orientable if each room has a sign \( \oplus \) or \( \ominus \) so that any two rooms with a common wall \( W \) induce opposite orientation on \( W \) (\( \iff \) pivoting changes sign).

A \( d \)-oik is orientable if half of the rooms with a common wall \( W \) induce sign \( \oplus \) on \( W \), the other half sign \( \ominus \) on \( W \).
How to orient room partitions?

Example: orientable manifold
How to orient room partitions?

Example: orientable manifold
How to orient room partitions?

Room partition $A, a = \{1, 3, 5\}, \{2, 4, 6\}$
How to orient room partitions?

Room partition $A, a$ : drop node $1$

![Diagram of a labeled polytope with nodes and edges labeled with letters and numbers.](image-url)
How to orient room partitions?

Room partition $A, a$, sign $\oplus$: drop node 1 leads to $c, C$, sign $\ominus$
How to orient room partitions?

Room partition $A, a$, sign $\oplus$: drop node 3
How to orient room partitions?

Room partition $A, a$, sign $\oplus$: drop node 3 leads to $B, b$, sign $\ominus$
How to orient room partitions?

Room partition $c, C$, sign $\ominus$: drop node 5
How to orient room partitions?

Room partition $c, C$, sign $\ominus$: drop node 5
How to orient room partitions?

Room partition $c, C$, sign $\ominus$: drop node 5 leads to $b, B$, sign $\oplus$
How to orient room partitions?

Which sign for \{b, B\}?
How to orient room partitions?

Which sign for \( \{b, B\} \)?  \( \oplus \) for \( b, B \),  \( \ominus \) for \( B, b \)!
How to orient room partitions?

⇒ for odd dimension (here $d = 3$), order of rooms matters: permutations $263\ 154$ (for $b, B$) and $154\ 263$ (for $B, b$) have opposite parity.
Ordered room partitions

**Theorem** [Végh/von Stengel 2012]

Let $\mathcal{R}$ be an oriented $d$-oik with node set $V$. Then the number of ordered room partitions $(R_1, \ldots, R_{|V|/d})$ is even.

Any two ordered room partitions connected by a pivoting path have opposite *sign*, and the respective unordered partitions are distinct.
Ordered room partitions

**Theorem** [Végh/von Stengel 2012]

Let $\mathcal{R}$ be an oriented $d$-oik with node set $V$. Then the number of ordered room partitions $(R_1, \ldots, R_{|V|/d})$ is even.

Any two ordered room partitions connected by a pivoting path have opposite sign, and the respective unordered partitions are distinct.

If $d$ is even, the order of rooms in a room partition is irrelevant.
Ordered room partitions

**Theorem** [Végh/von Stengel 2012]

Let $\mathcal{R}$ be an oriented $d$-oik with node set $V$. Then the number of ordered room partitions $(R_1, \ldots, R_{|V|/d})$ is even.

Any two ordered room partitions connected by a pivoting path have opposite sign, and the respective unordered partitions are distinct.

If $d$ is even, the order of rooms in a room partition is irrelevant.

Proof uses “pivoting systems” with labels $=$ nodes.

Pivoting systems generalize labeled polytopes, *Lemke*’s algorithm, Sperner’s lemma, room partitions in oiks, and more.
Finding a second perfect matching in an Euler graph
Finding a second perfect matching in an Euler graph

1 2 3 4 5 6
Finding a second perfect matching in an Euler graph
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Euler Labeled polytopes Signs Oiks Oriented oiks Matchings
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Euler labeled polytopes
Signs
Oiks
Oriented oiks
Matchings
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Euler Labeled polytopes Signs Oiks Oriented oiks Matchings

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3 4 5 6 4 2
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2 3 5 6 6 4
2 3 4 5 6 4
2 3 4 5 5 6
2 3 3 4 5 6
Finding a second perfect matching in an Euler graph
## Finding a second perfect matching in an Euler graph

### Diagram

![Euler Graph Diagram](image)

### Table

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Finding a second perfect matching in an Euler graph

Graph:

- Nodes: 1, 2, 3, 4, 5, 6
- Edges: 1-2, 2-3, 3-4, 4-5, 5-1, 1-6, 6-4, 4-3, 3-5, 5-6, 6-1
- Directed edges:
  - 1-2, 2-3, 3-4, 4-5, 5-1
  - 1-6, 6-4, 4-3, 3-5, 5-6, 6-1

Matching:

- Set of edges that form a perfect matching:
  - 1-2, 2-3, 3-4, 4-5, 5-1

- Another possible perfect matching:
  - 1-6, 6-4, 4-3, 3-5, 5-6, 6-1

- Eulerian graph:
  - All nodes have even degree
  - Every edge is used exactly once in a cycle

- Oriented oiks:
  - Directed edges

- Signs:
  - Positive (green): 1-2, 2-3, 3-4, 4-5, 5-1
  - Negative (red): 1-6, 6-4, 4-3, 3-5, 5-6, 6-1
Finding a second perfect matching in an Euler graph

\[ \begin{align*}
&+ \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
&- \quad 2 \quad 1 \quad 3 \quad 4 \quad 5 \quad 6
\end{align*} \]
Finding a second perfect matching in an Euler graph
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A computational problem

**Input:** Graph \((V, \mathcal{R})\) with Eulerian orientation and perfect matching of sign \(\oplus\).

**Output:** A perfect matching with sign \(\ominus\).
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— but may take **exponential** time in general [Morris 1994]
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— but may take **exponential** time in general [Morris 1994]

**Note:** A second matching can be found in polynomial time [Edmonds 1965], but not with sign \(\ominus\).

Related difficult problem: Pfaffian orientations of graphs.
Given an oriented graph and a perfect matching $M$, a sign-switching cycle is a cycle $C$ with every other edge in $M$ and an even number of forward-pointing edges.

$\Rightarrow M \triangle C$ is a matching of opposite sign to $M$. 
Finding a SSC in near-linear time

Two **reductions** which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges
Finding a SSC in near-linear time

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Two reductions which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges

2. **delete** directed cycle of unmatched edges

![Diagram showing the process of contracting and deleting nodes and edges to find a SSC](image-url)
Finding a SSC in near-linear time

Two **reductions** which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges

2. **delete** directed cycle of **unmatched** edges
Finding a SSC in near-linear time

Two *reductions* which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges

2. **delete** directed cycle of *unmatched* edges
Finding a SSC in near-linear time

Two reductions which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges

2. **delete** directed cycle of unmatched edges

until trivial SSC found
Finding a SSC in near-linear time

Two reductions which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges

![Diagram showing node contraction]

2. **delete** directed cycle of unmatched edges

until trivial SSC found

![Diagram showing cycle deletion]
Finding a SSC in near-linear time

Two reductions which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges

2. **delete** directed cycle of unmatched edges

until trivial SSC found, re-insert contracted edge pairs
Finding a SSC in near-linear time

Two reductions which preserve Euler and matching property:

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Finding a SSC in near-linear time

Two **reductions** which preserve Euler and matching property:

1. **contract** node of indegree = outdegree = 1 with its two edges

![Diagram of node contract](image)

2. **delete** directed cycle of unmatched edges

![Diagram of node delete](image)

until trivial SSC found, re-insert contracted edge pairs, switch.
Summary of results

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- Orienting oiks and (ordered) room partitions.

[Morris 1994], [Casetti/Merschen/von Stengel 2010].
Summary of results

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- If pivoting is sign-switching (orientability) \[\Rightarrow\] endpoints of paths have opposite signs \(\oplus\), \(\ominus\)
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- Opposite-signed matching in Euler graph found in linear time.
- Exponentially long paths for matchings in Euler graph emulate exponentially long Lemke–Howson paths in games [Morris 1994], [Casetti/Merschen/von Stengel 2010].