Pathways to Equilibria, Pretty Pictures and Diagrams (PPAD)

Bernhard von Stengel

Department of Mathematics
London School of Economics

Jack Edmonds

Équipe Combinatiore & Optimisation
University of Paris 6

Overview

- Equilibria and fixed points exist via combinatorial, path-following "parity argument" with direction
- Computational complexity class PPAD
- Exposition of these concepts

Goal:

"canonical" unifying view via
oriented combinatorial manifolds
(work in progress)

Sperner's Lemma

- simplex S with r vertices, triangulation T
- each vertex of T has color in {1, ..., r}
- color of a vertex of S **not** found on opposite facet (Sperner condition).

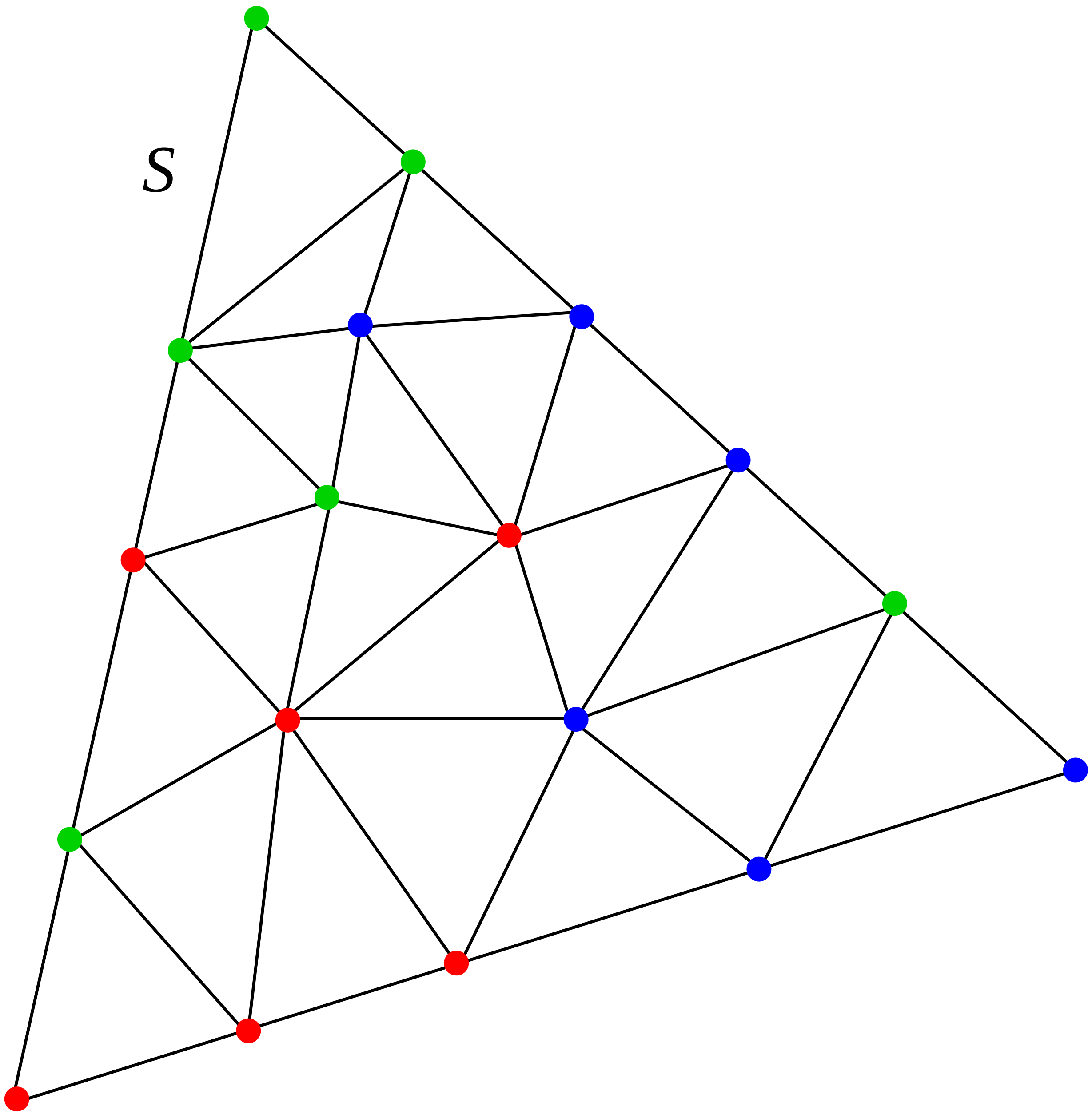
Theorem:

T has an **odd** number of panchromatic simplices.

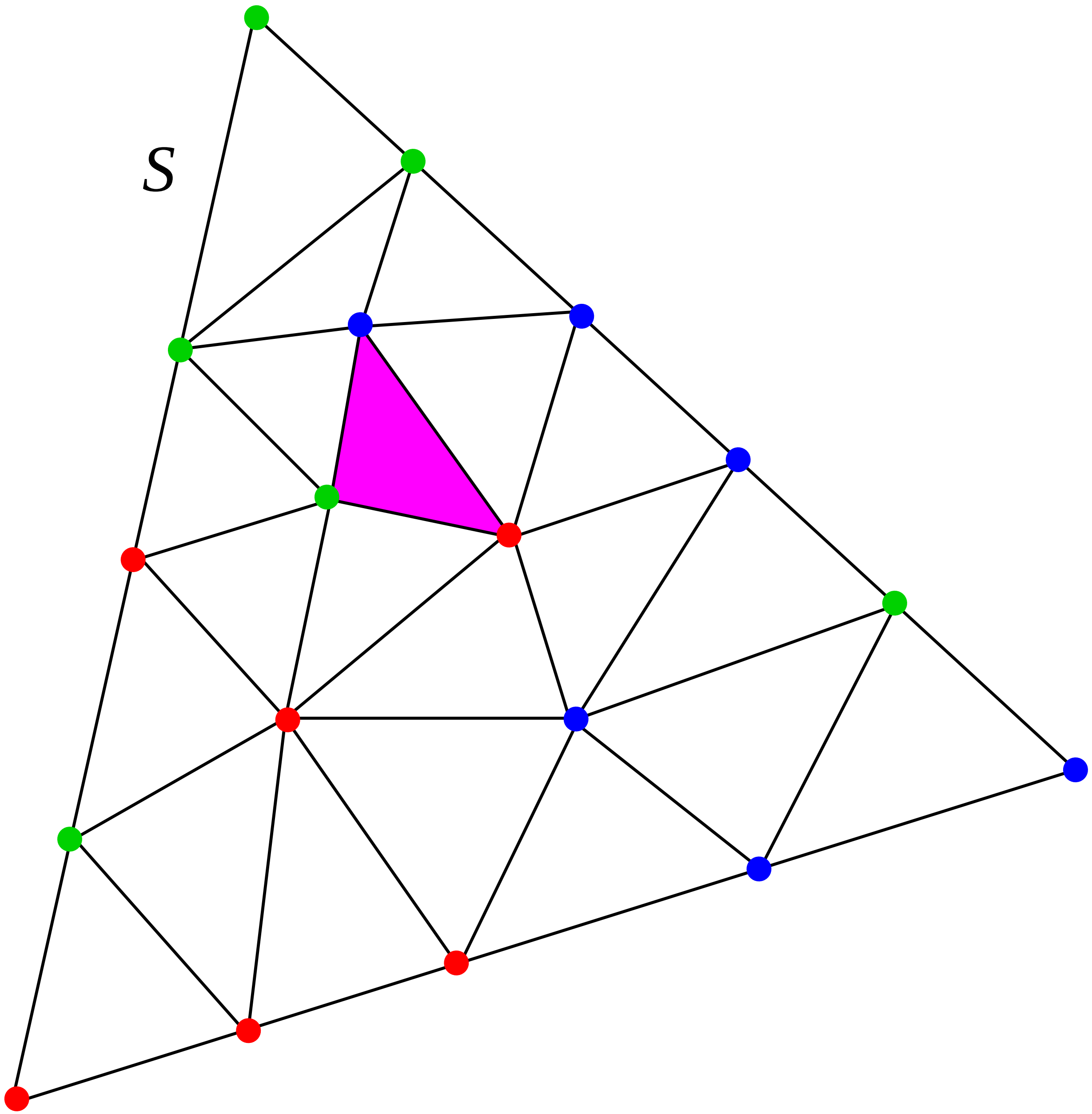
Proof:

Inductive hypothesis (lower dimension): each facet has an odd number of panchromatic simplices on that facet.

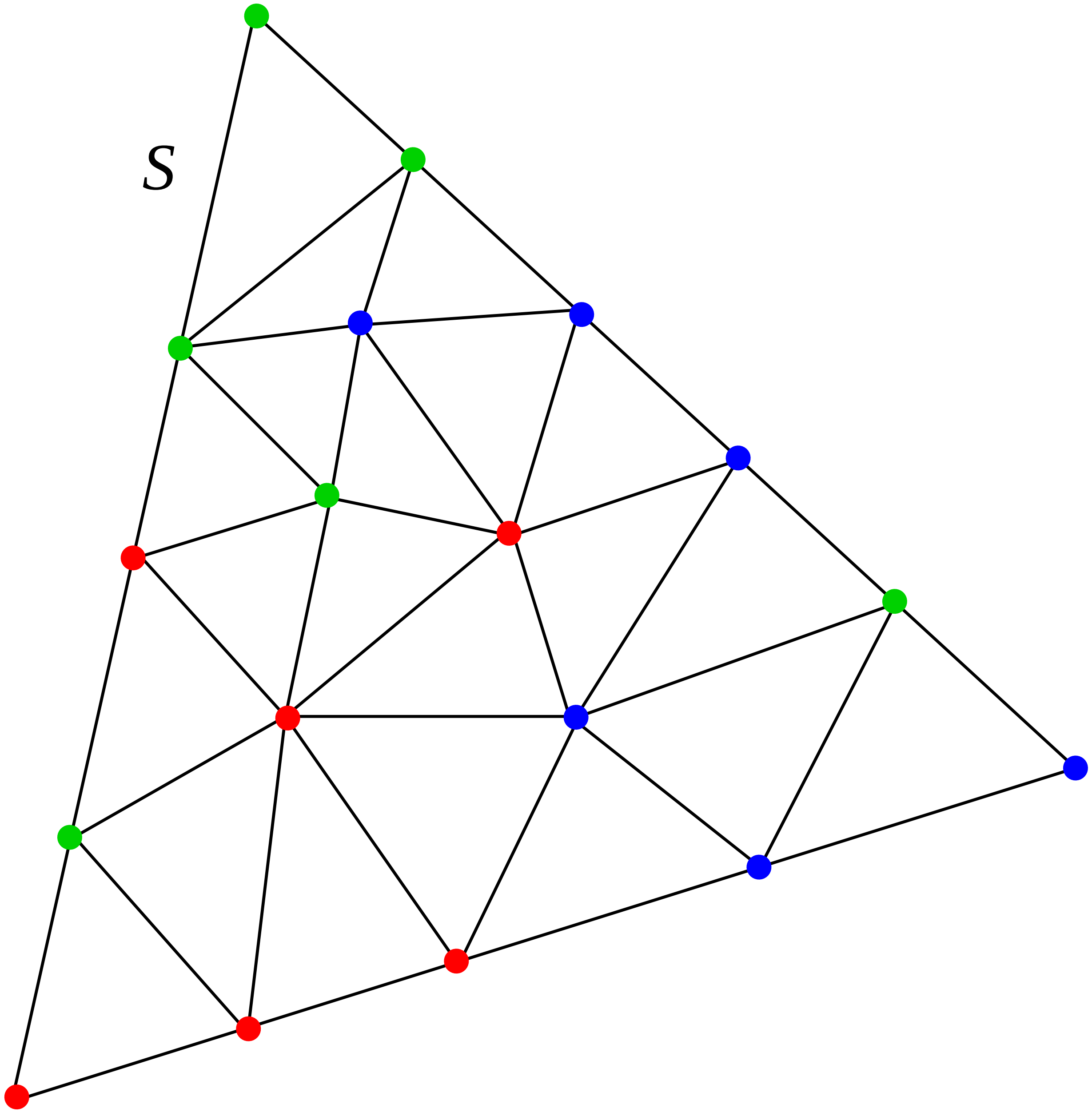
Simplex S with triangulation T and colors



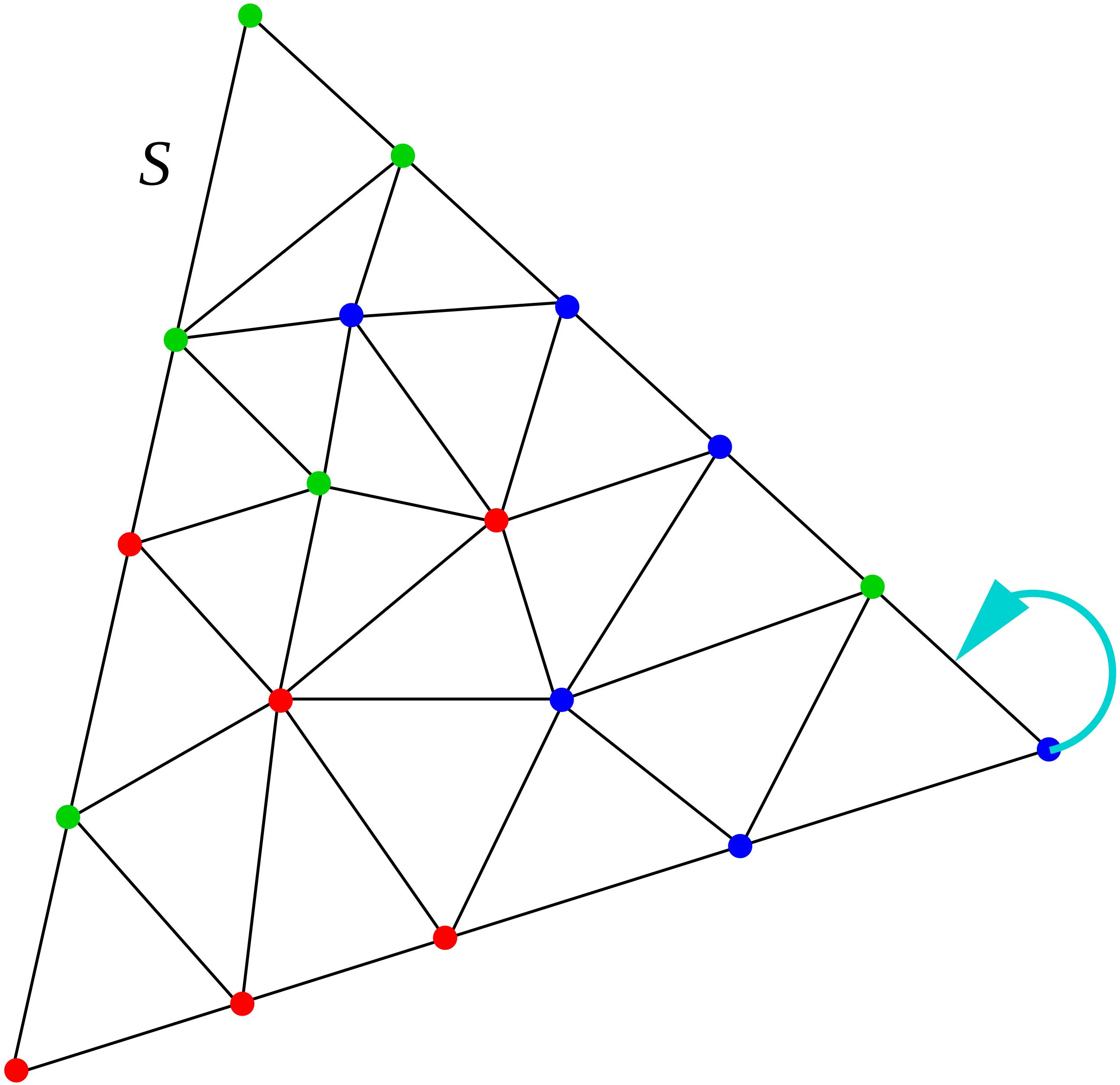
1 panchromatic simplex



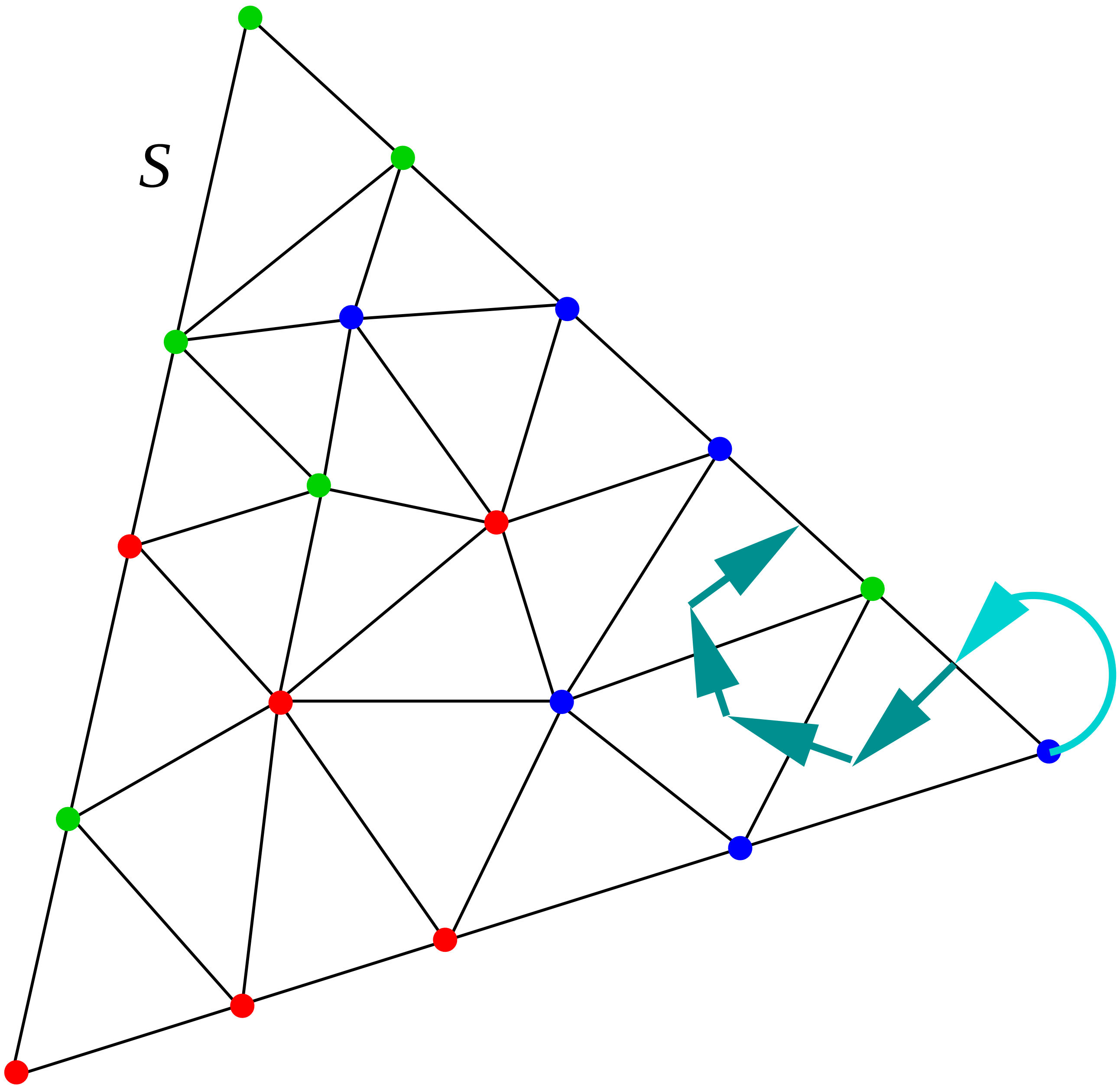
start at blue vertex, look for green



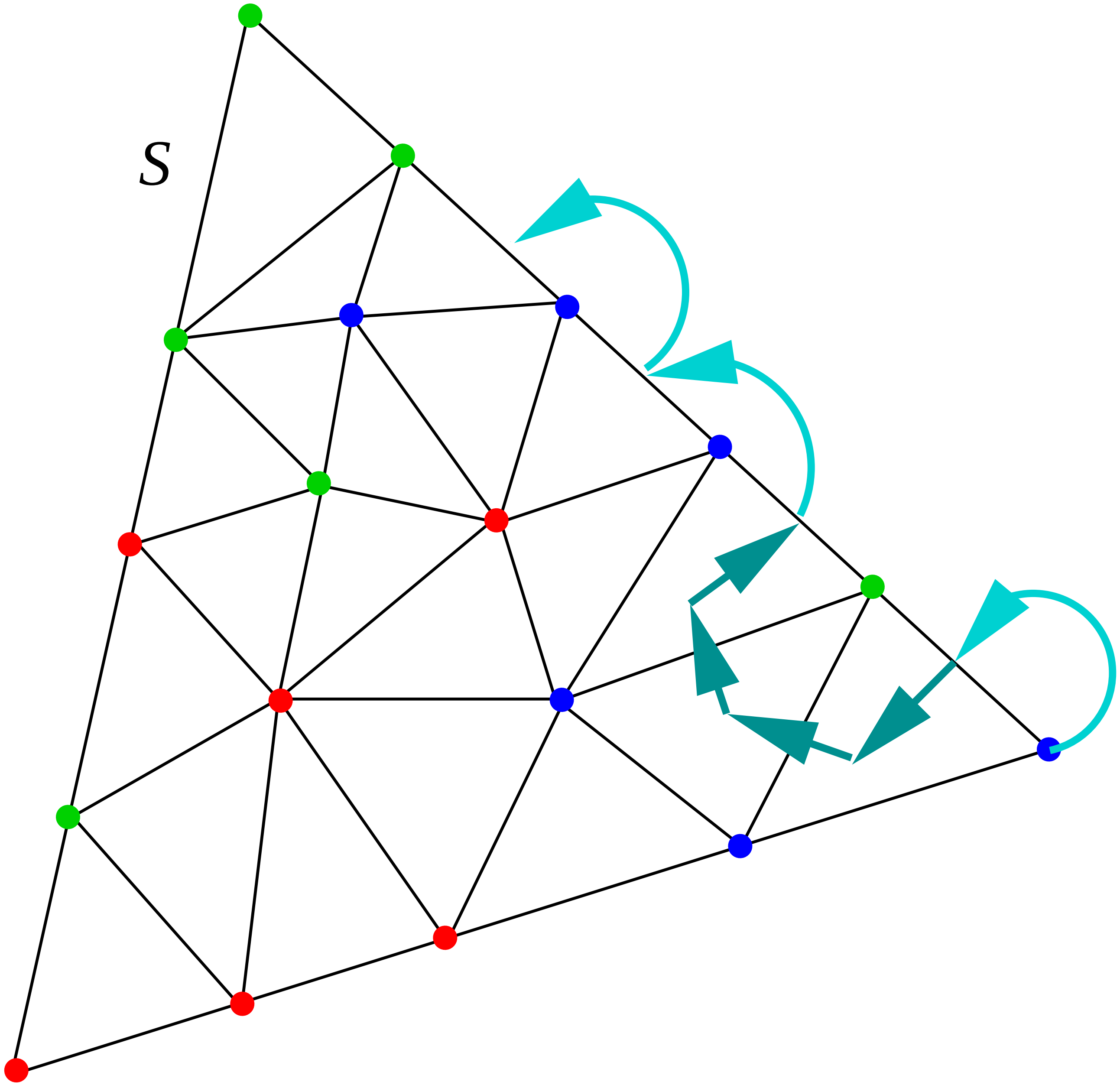
found green, look for red



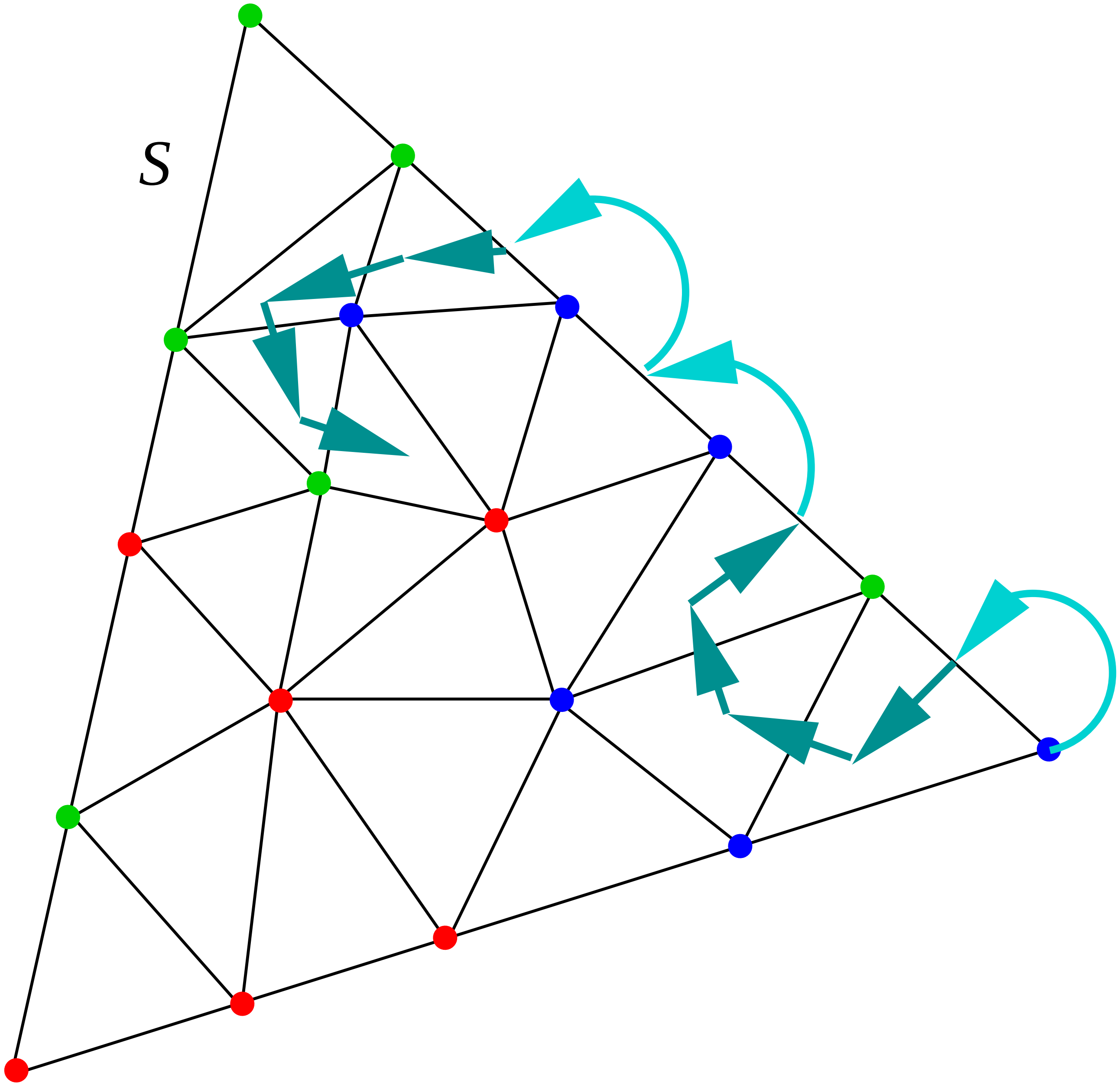
got back to lower dimension



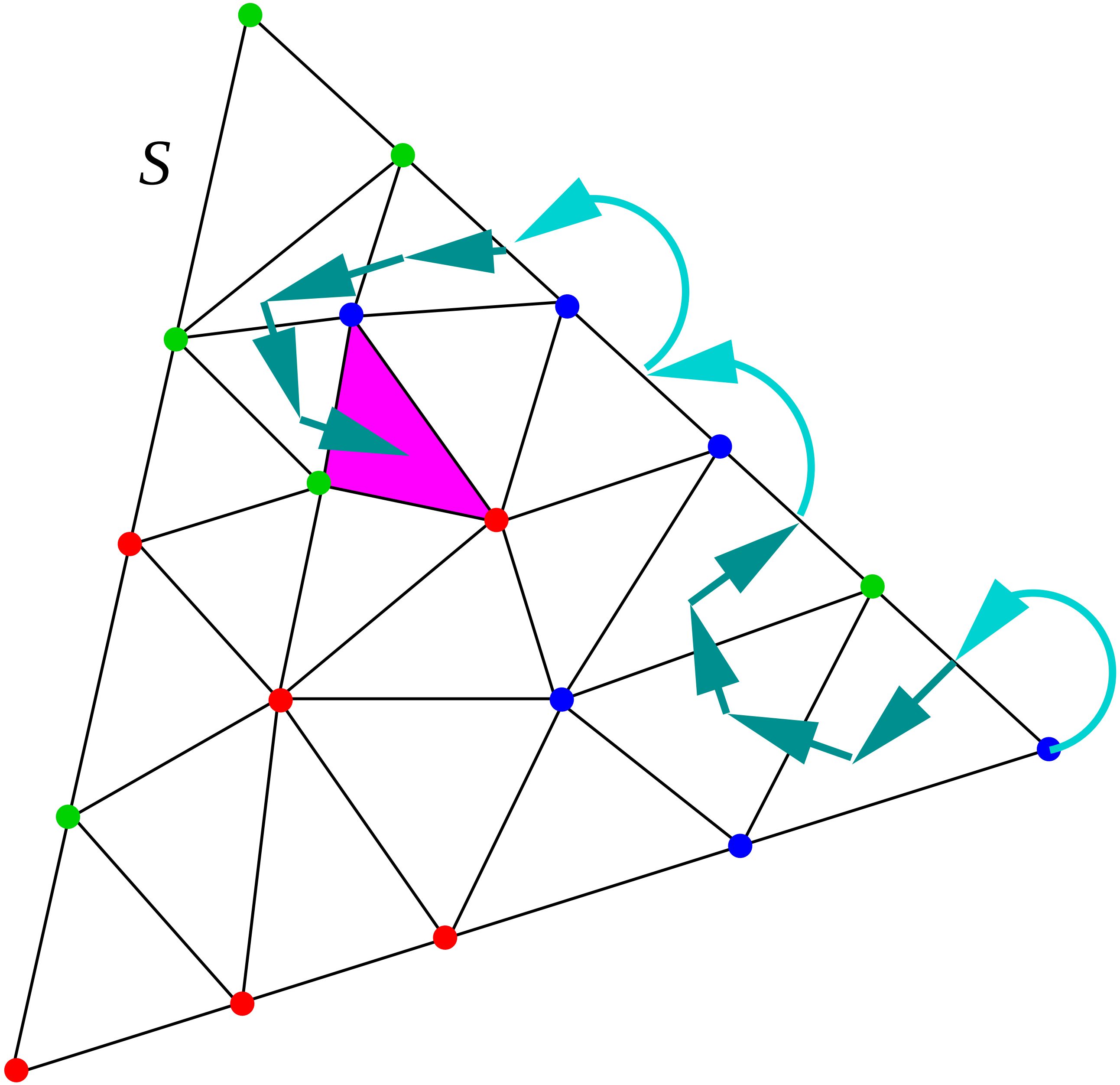
look for green



found green, look for red



found red, panchromatic simplex found



Brouwer Fixed Points via Sperner

Brouwer's FPT: Unit simplex S, continuous $f: S \to S$.
Then f has a **fixed point**: $f(\mathbf{s}) = \mathbf{s}$ for some $\mathbf{s} \in S$.

Proof:
$x = (x_1, \ldots, x_r)$, $f(x) = (f_1(x), \ldots, f_r(x))$.

Triangulate S, each vertex x gets some color
$i \in \{1, \ldots, r\}$ if $x_i > 0$ and $x_i \geq f_i(x)$.

Panchromatic simplex via Sperner:
vertices are "approximately fixed" points.
Take finer and finer triangulations and convergent subsequence of (e.g., blue) vertices of panchromatic simplices: Limit is fixed point.

Combinatorial manifolds

Given: **rank** r

collection M of r-element sets called **rooms**
(also: abstract simplicial complex M, rooms = simplices)

set of **vertices** $V = \bigcup M$

wall = room without a vertex v (wall "opposite" v)

any wall belongs to exactly 2 rooms
(i.e. any (r–1)-set of vertices belongs to 0 or 2 rooms)

Call M a **manifold**.

Abstract Sperner

- Manifold M of rank r
- each vertex v has label (color) in {1, . . ., r}
- call a set of vertices **panchromatic** if no two of its vertices have the same color.

Theorem:

M has an **even** number of panchromatic rooms.

Abstract Sperner

- Manifold M of rank r
- each vertex v has label (color) in {1, . . ., r}
- call a set of vertices **panchromatic** if no two of its vertices have the same color.

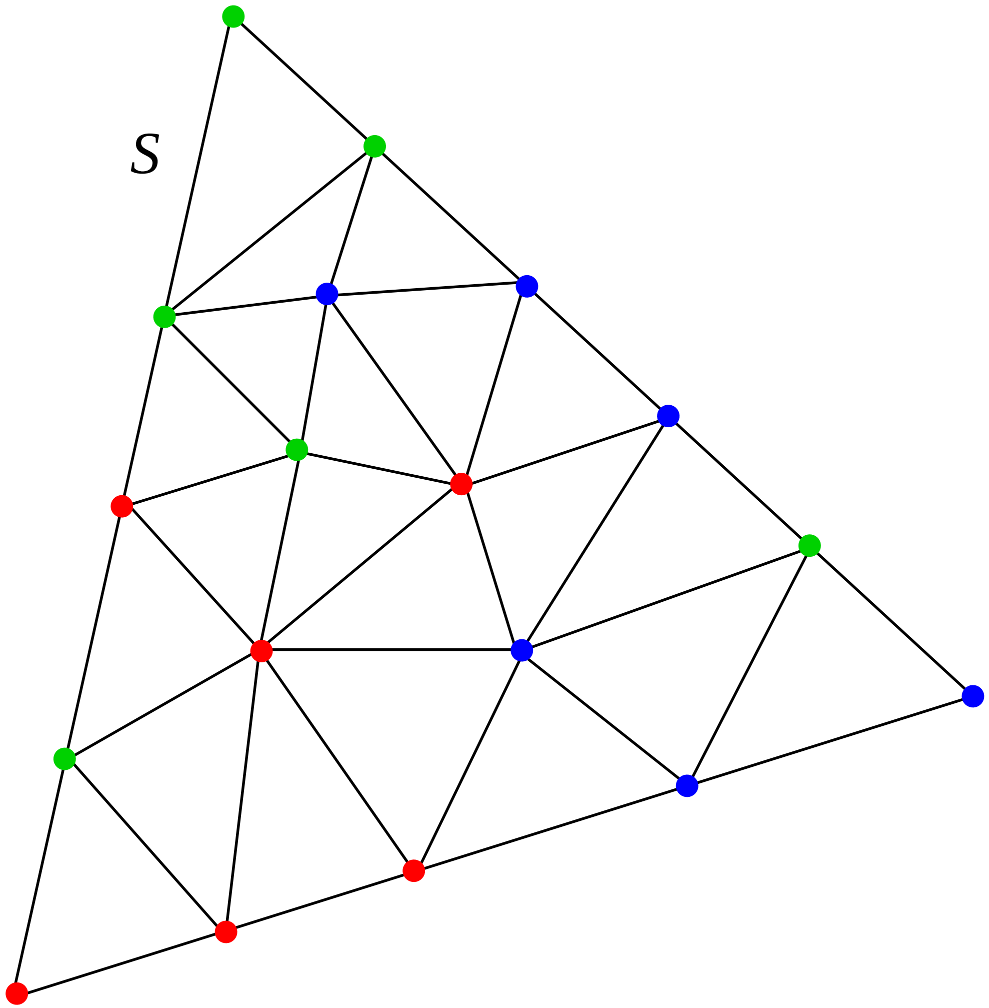
Theorem:

M has an **even** number of panchromatic rooms.

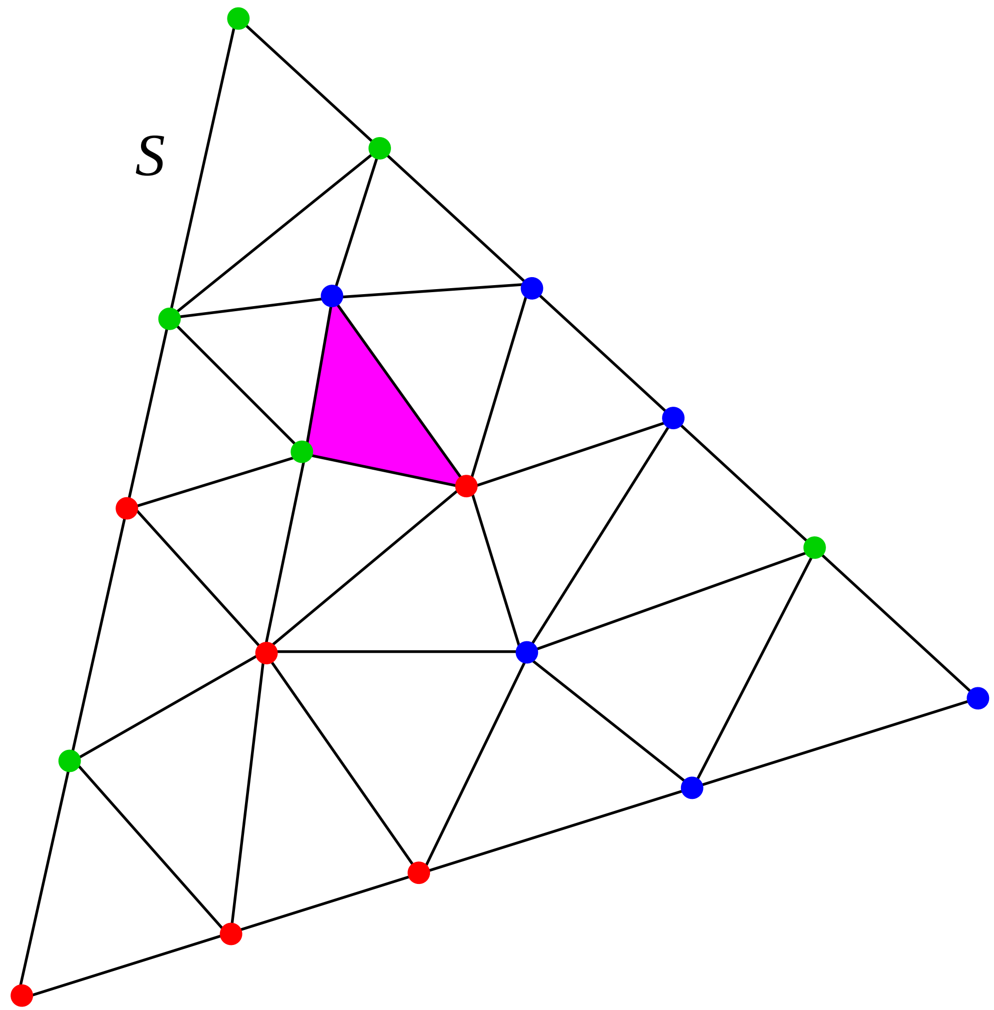
Corollary:

Sperner's Lemma.

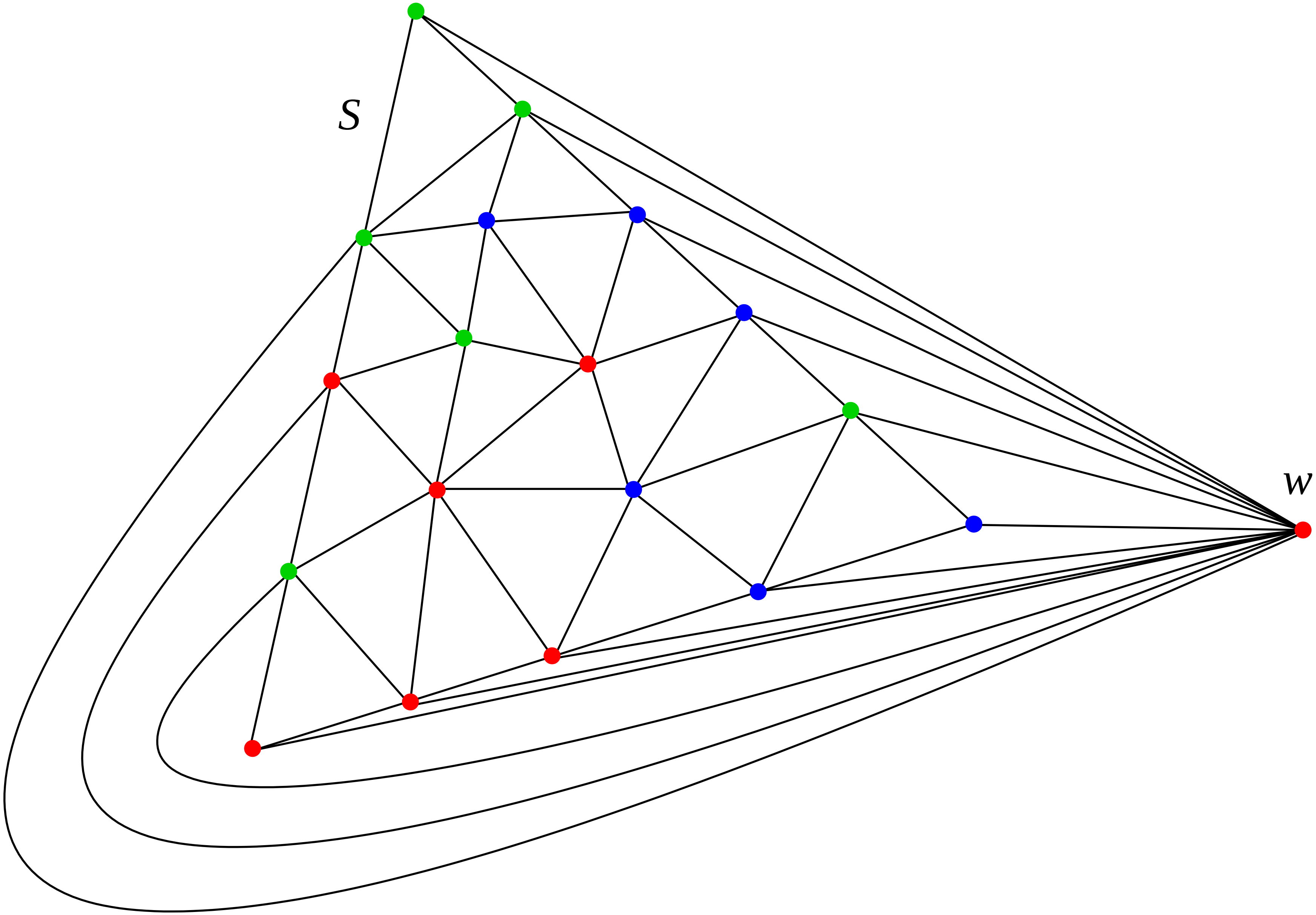
Sperner's lemma via Abstract Sperner



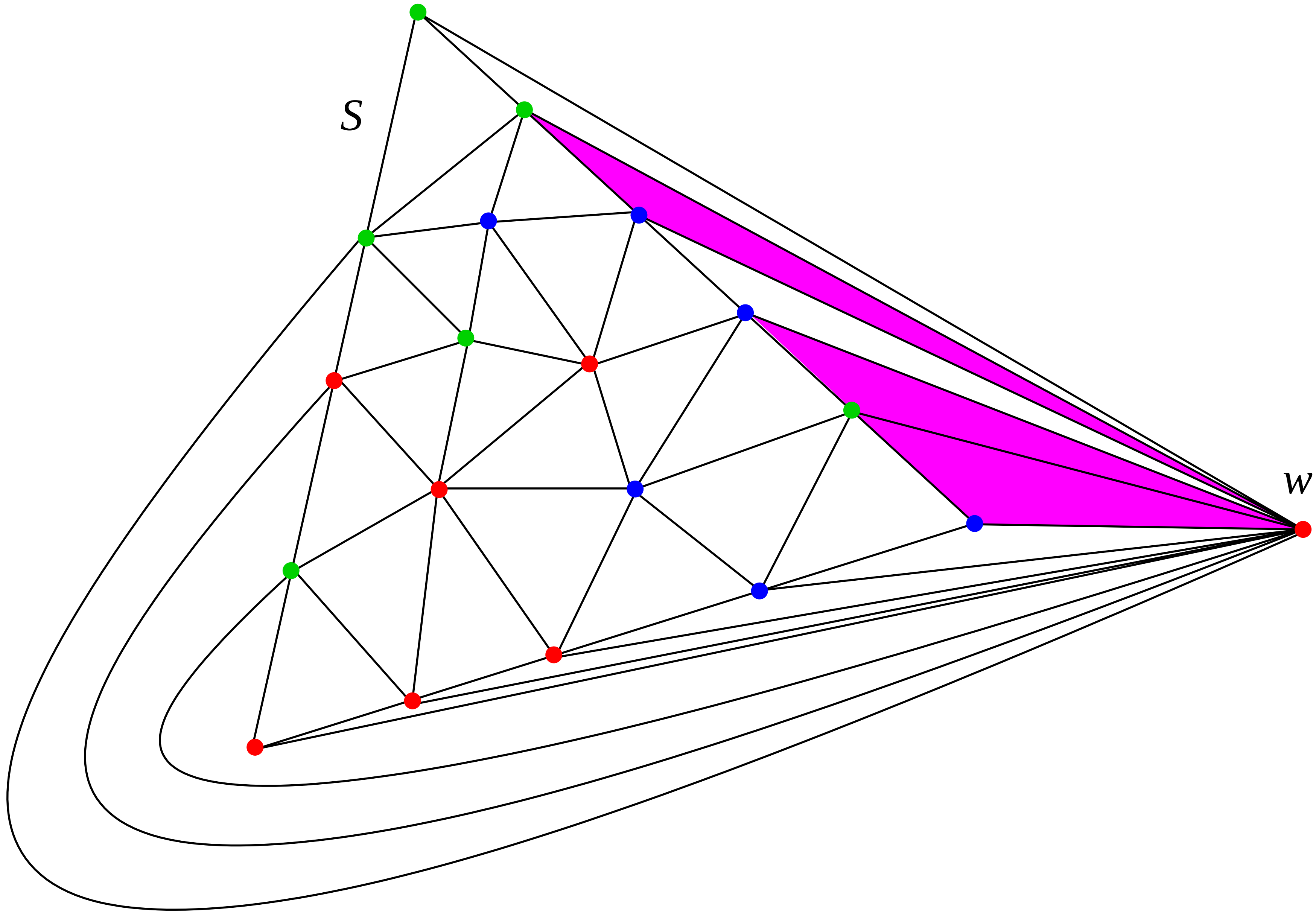
Want to show:
Odd number of panchromatic simplices



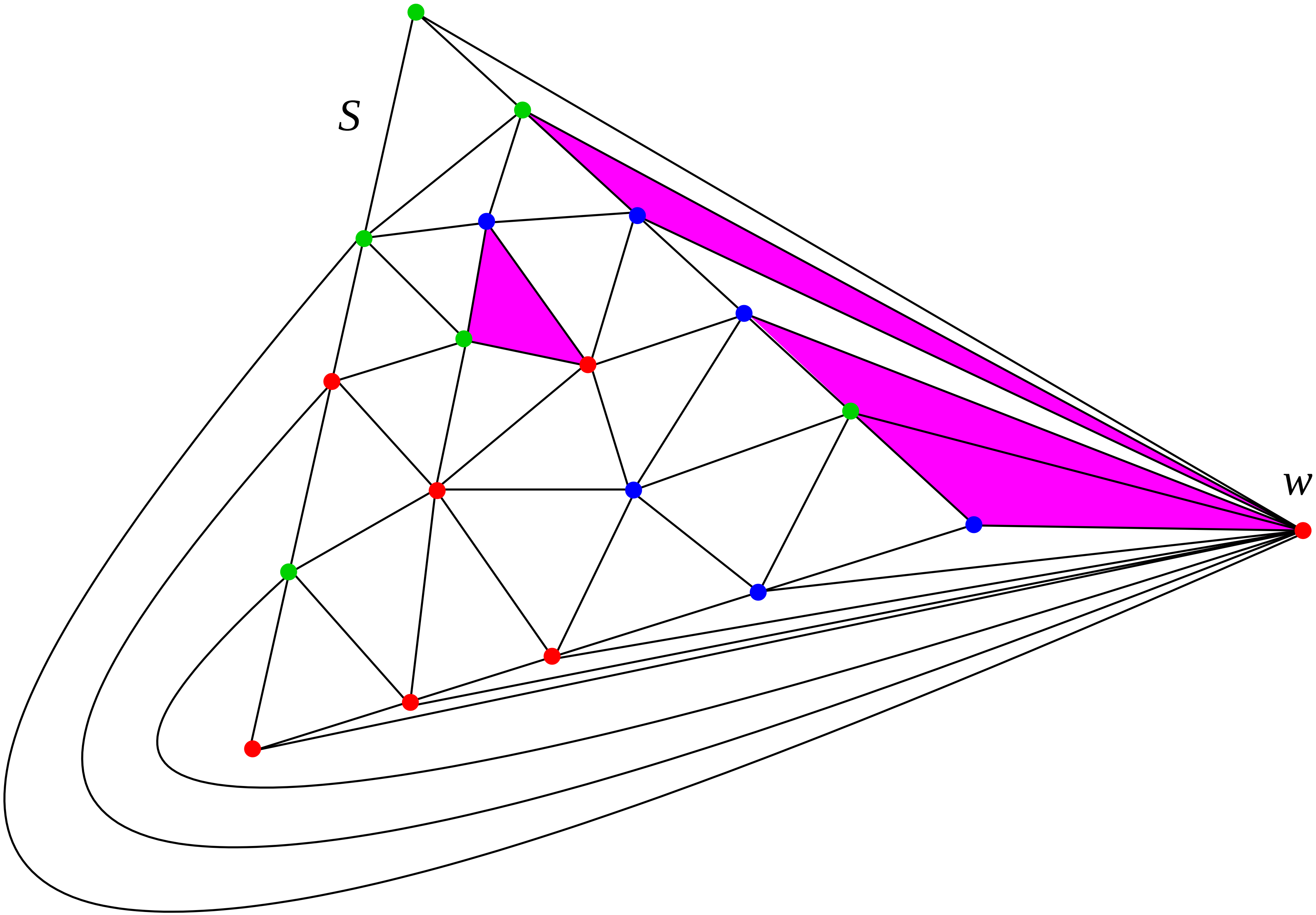
Add new vertex w, any color, connect to outside vertices to get manifold



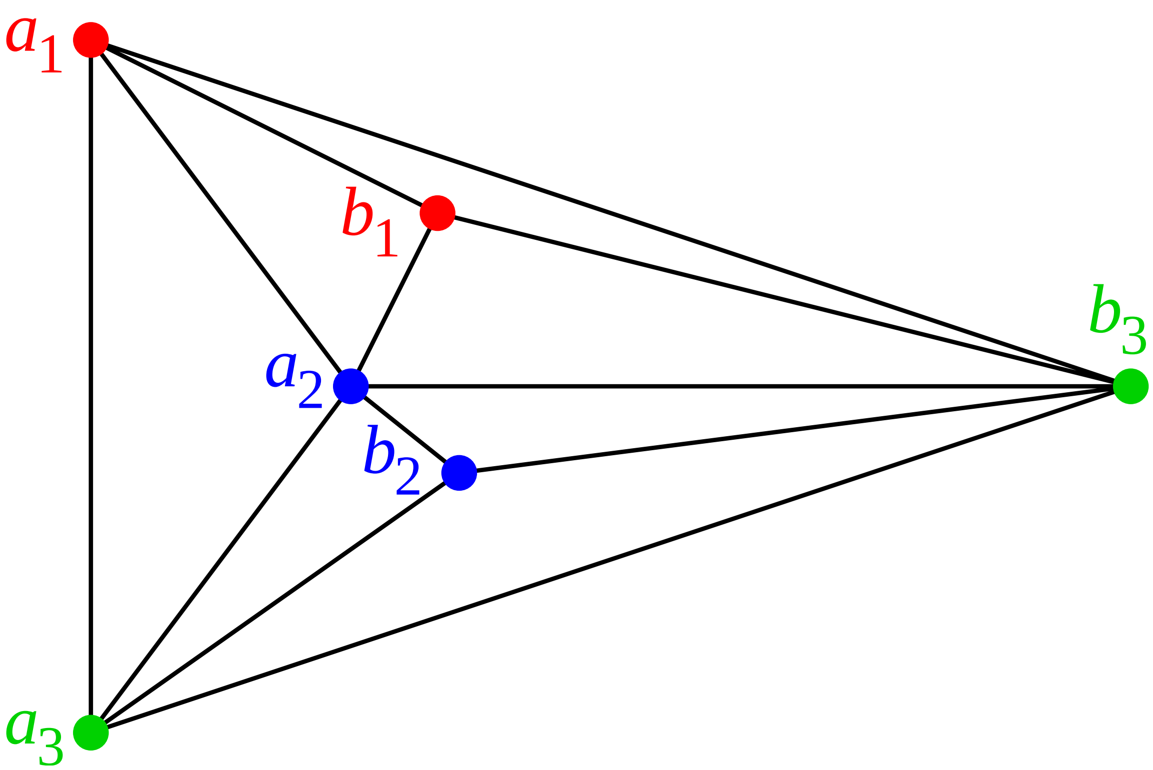
By induction: odd # of panchromatic rooms that contain w (outside rooms)



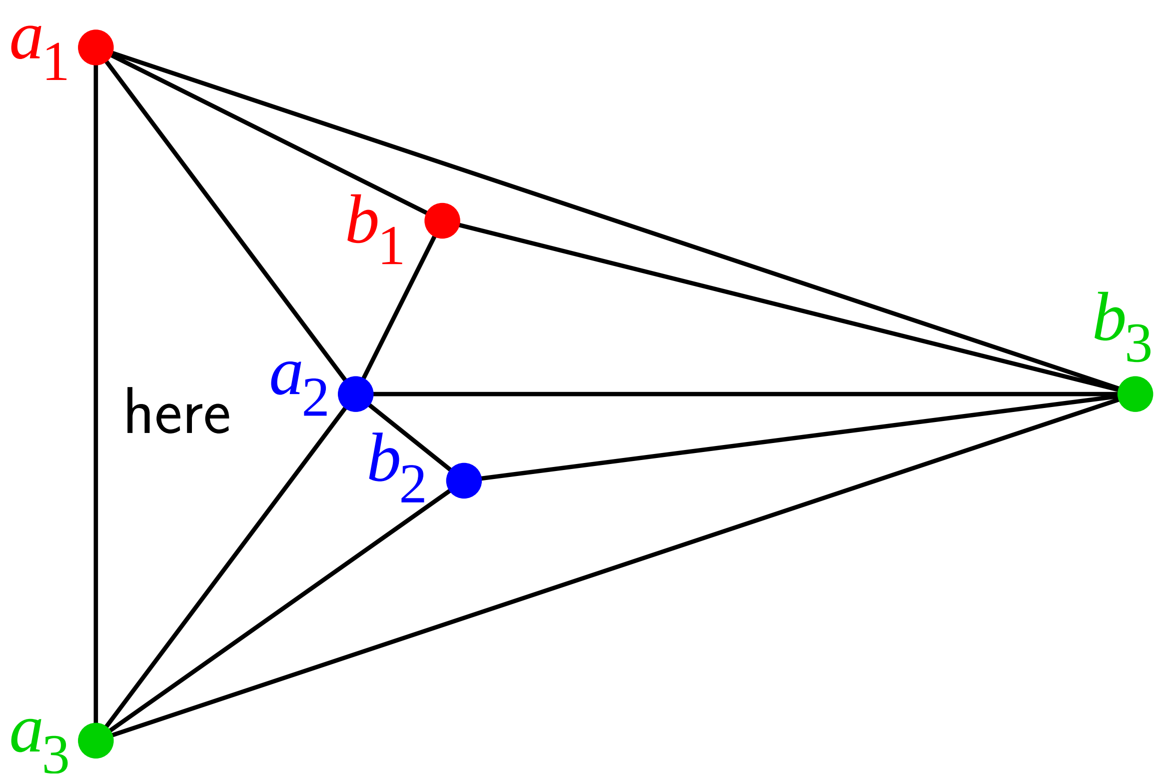
By Abstract Sperner (even total #): odd # of inside panchromatic rooms



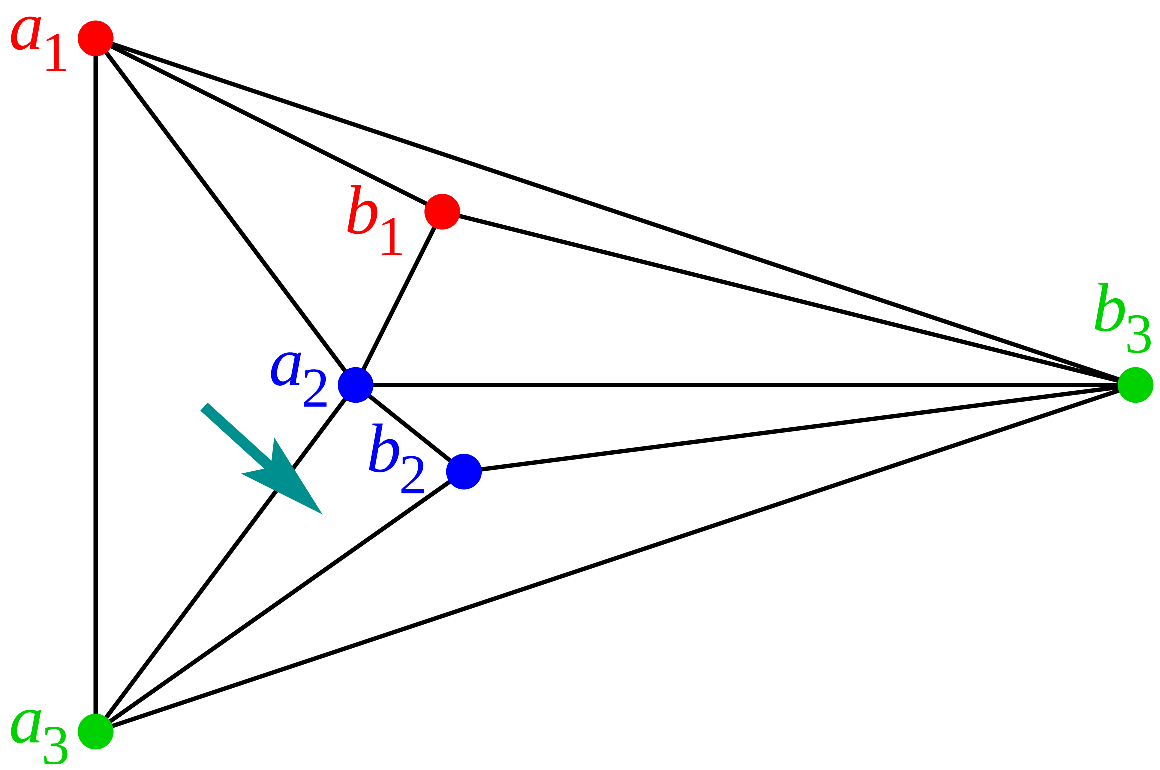
Abstract Sperner: Proof



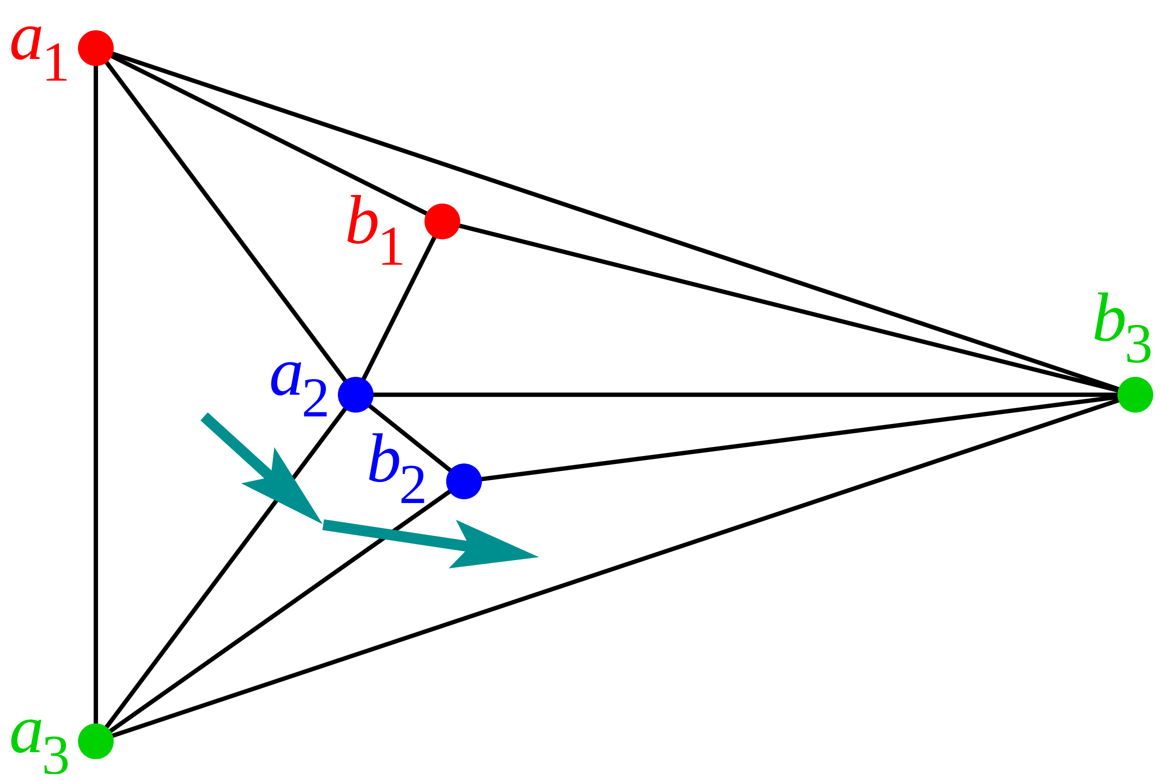
start at panchromatic room



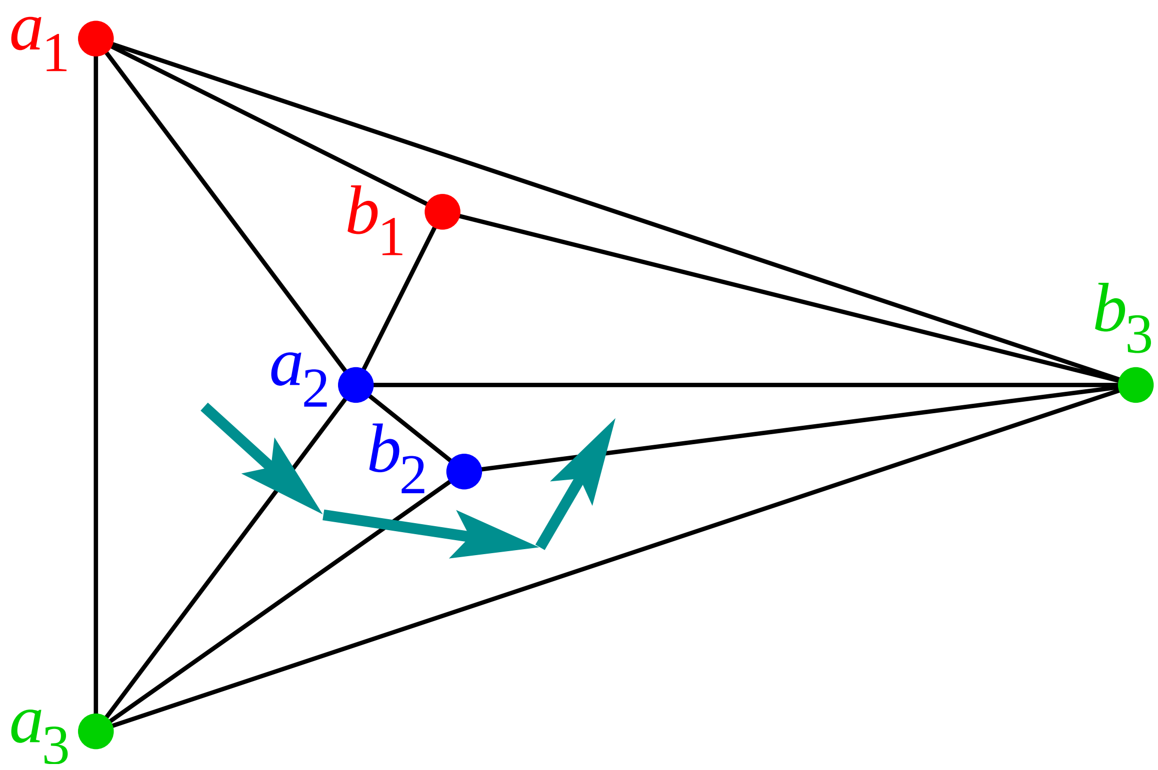
allow missing color 1



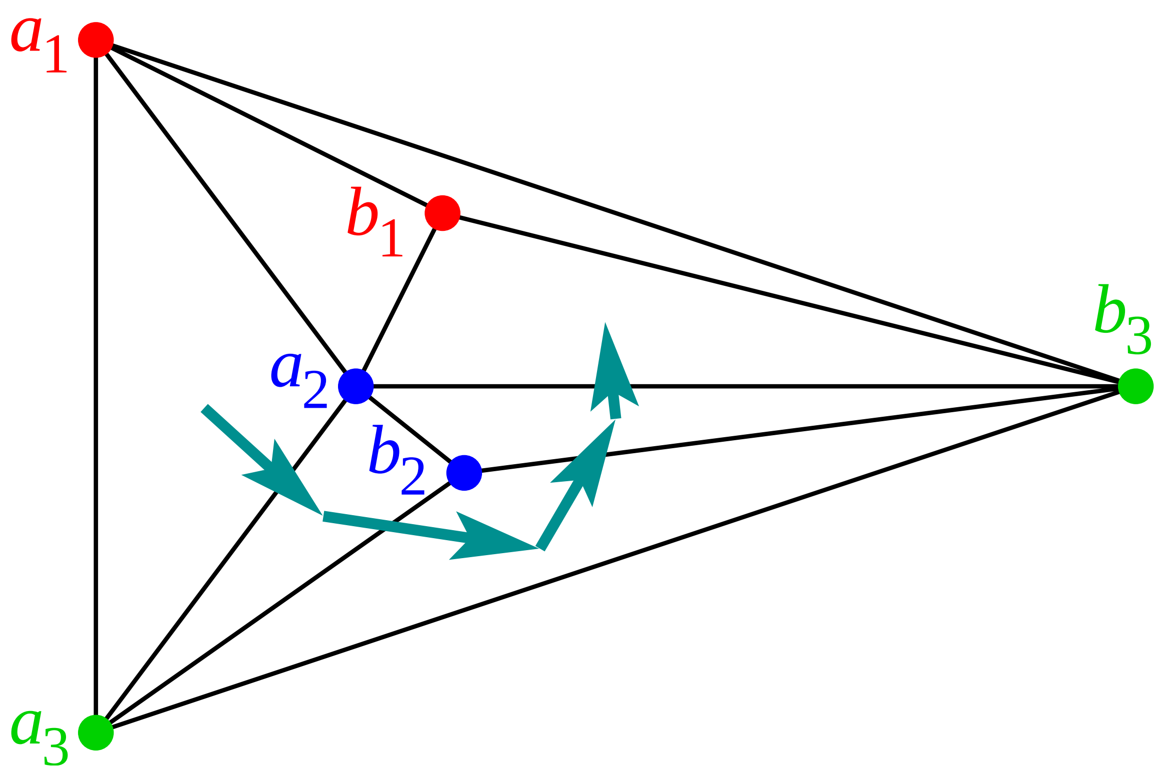
allow missing color 1



allow missing color 1



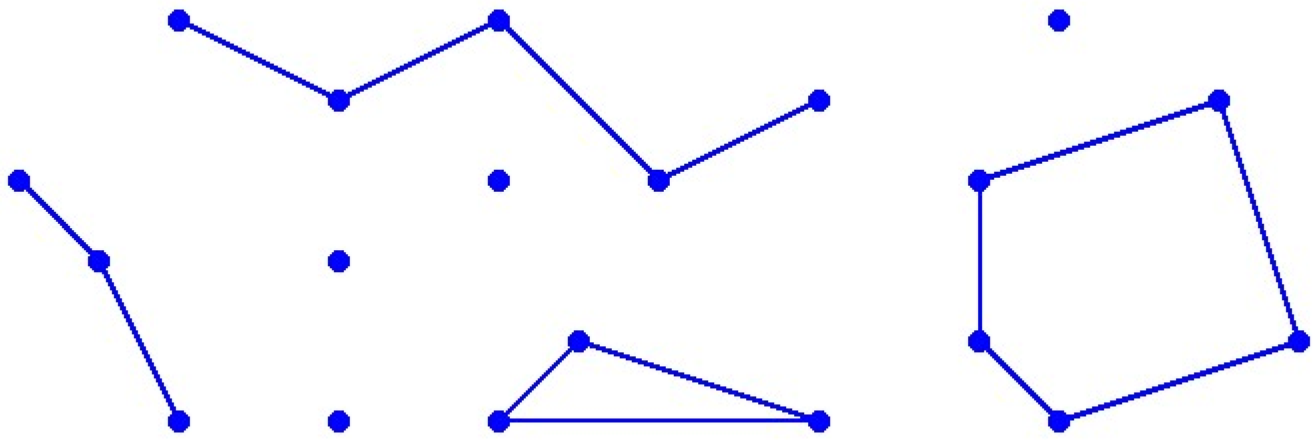
find color 1, done!



The Parity Argument (PA)

Given: Implicit graph G of degree at most 2 (every node has at most 2 neighbors).

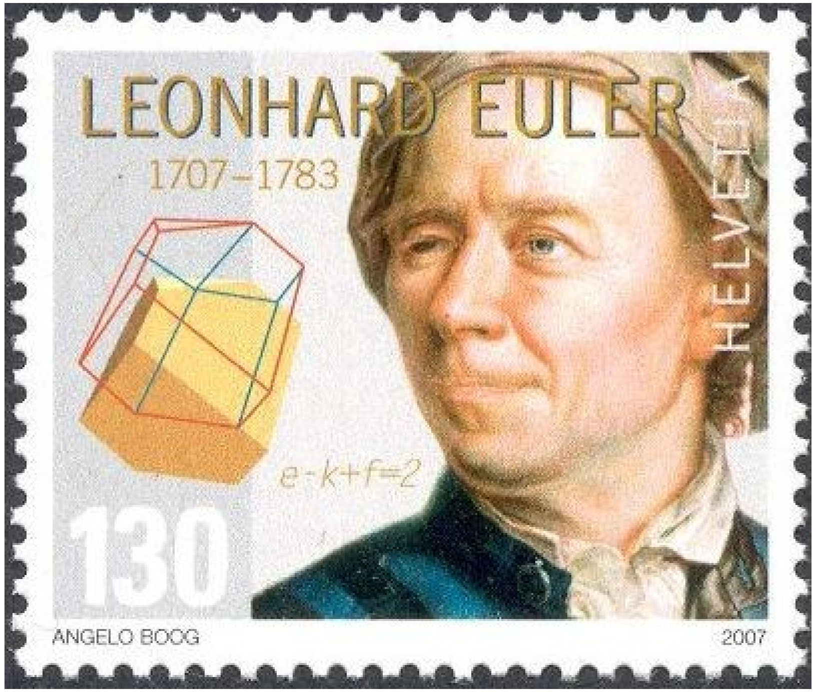
Then G is a collection of paths and cycles:

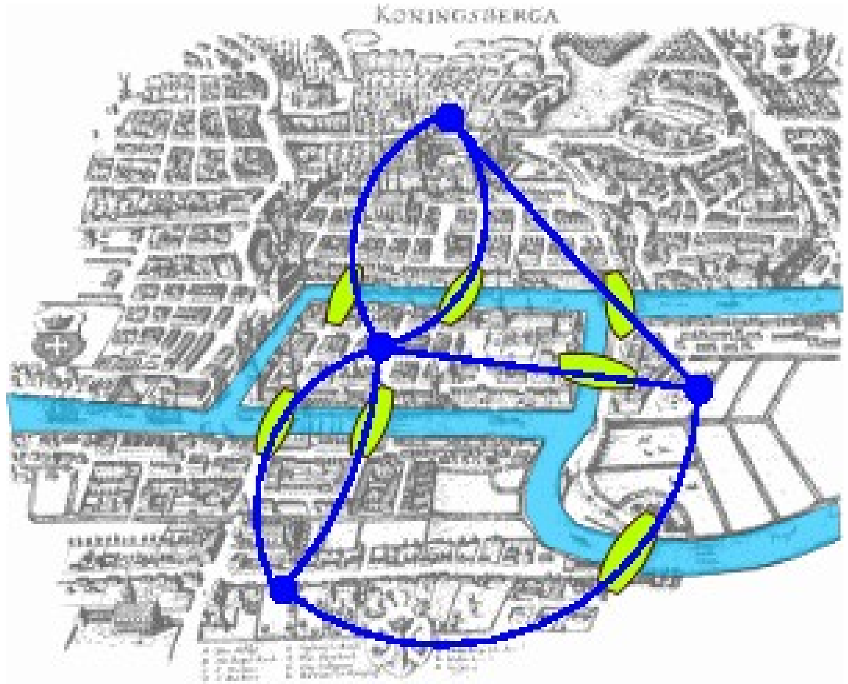


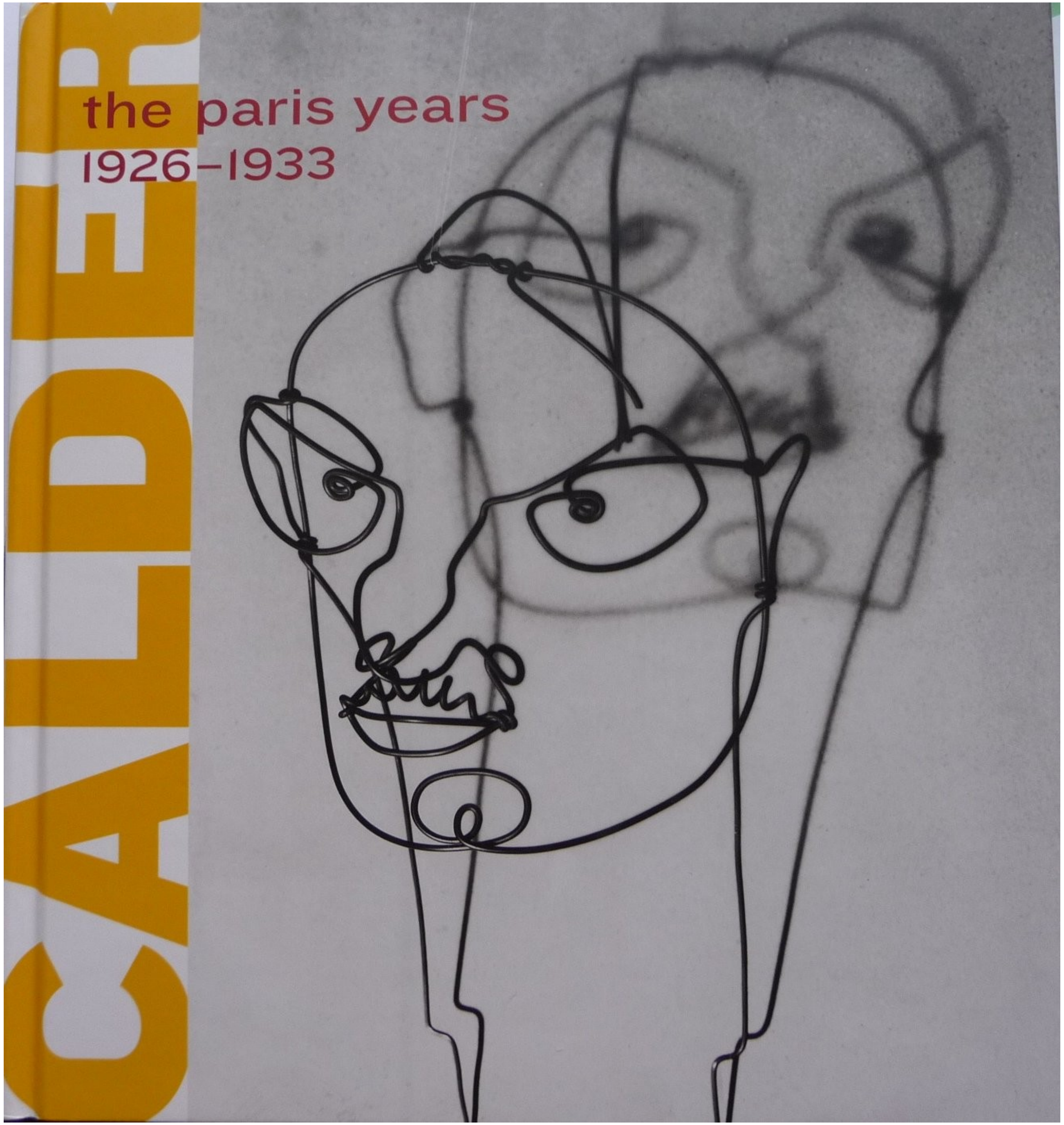
The number of degree-1 nodes (endpoints of paths) is **even**.

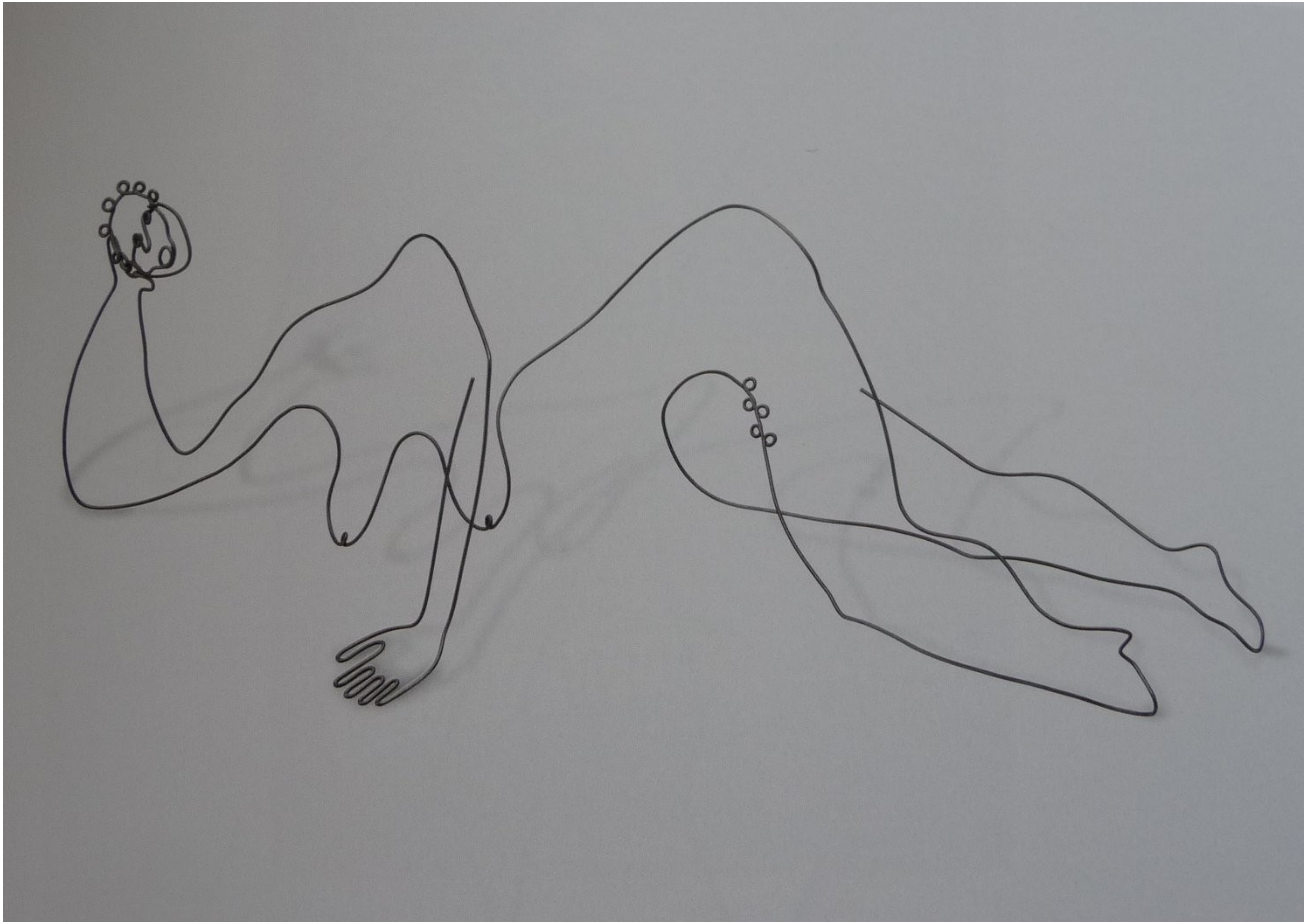
More generally (Euler)

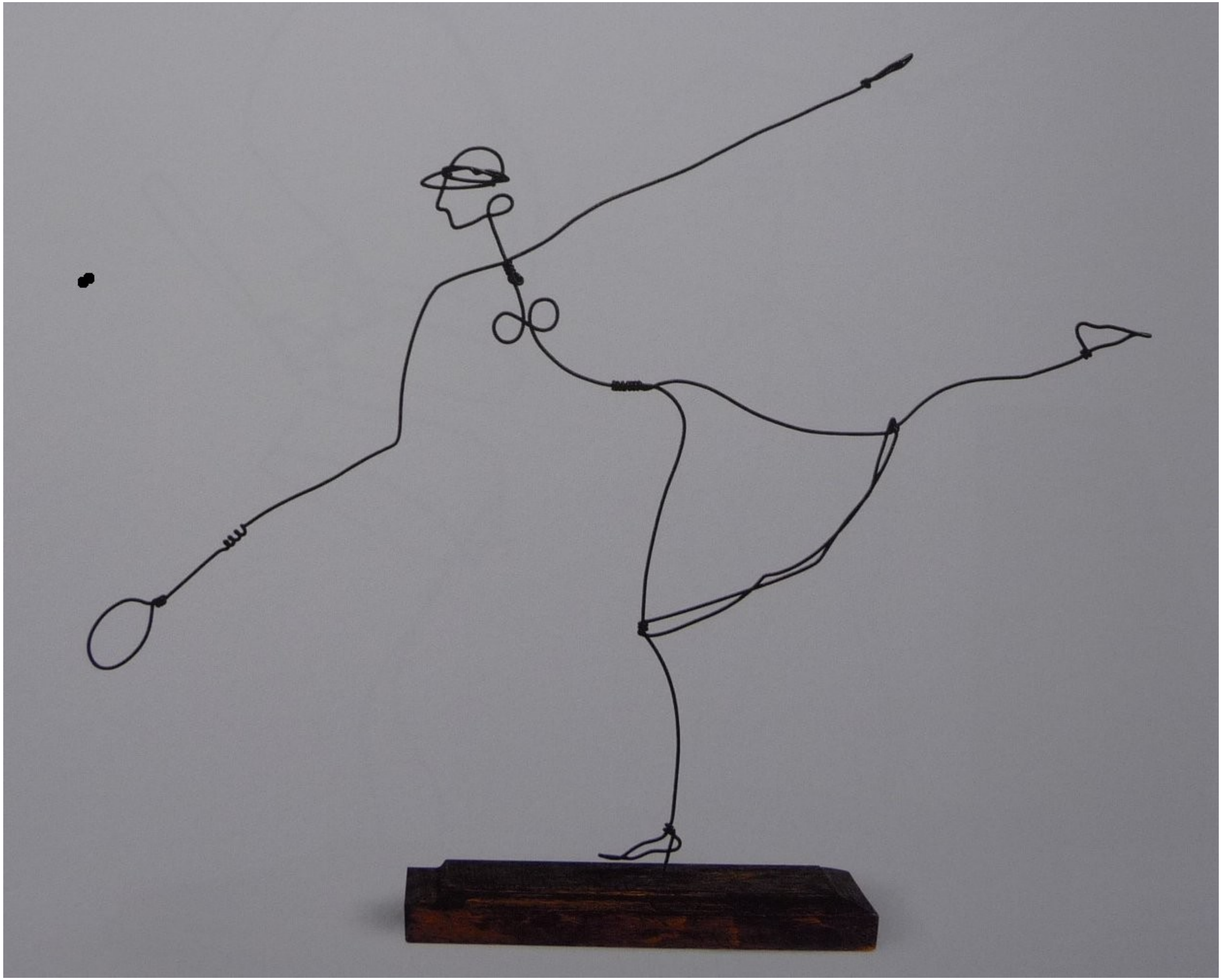
The number of odd-degree nodes of a graph is even:

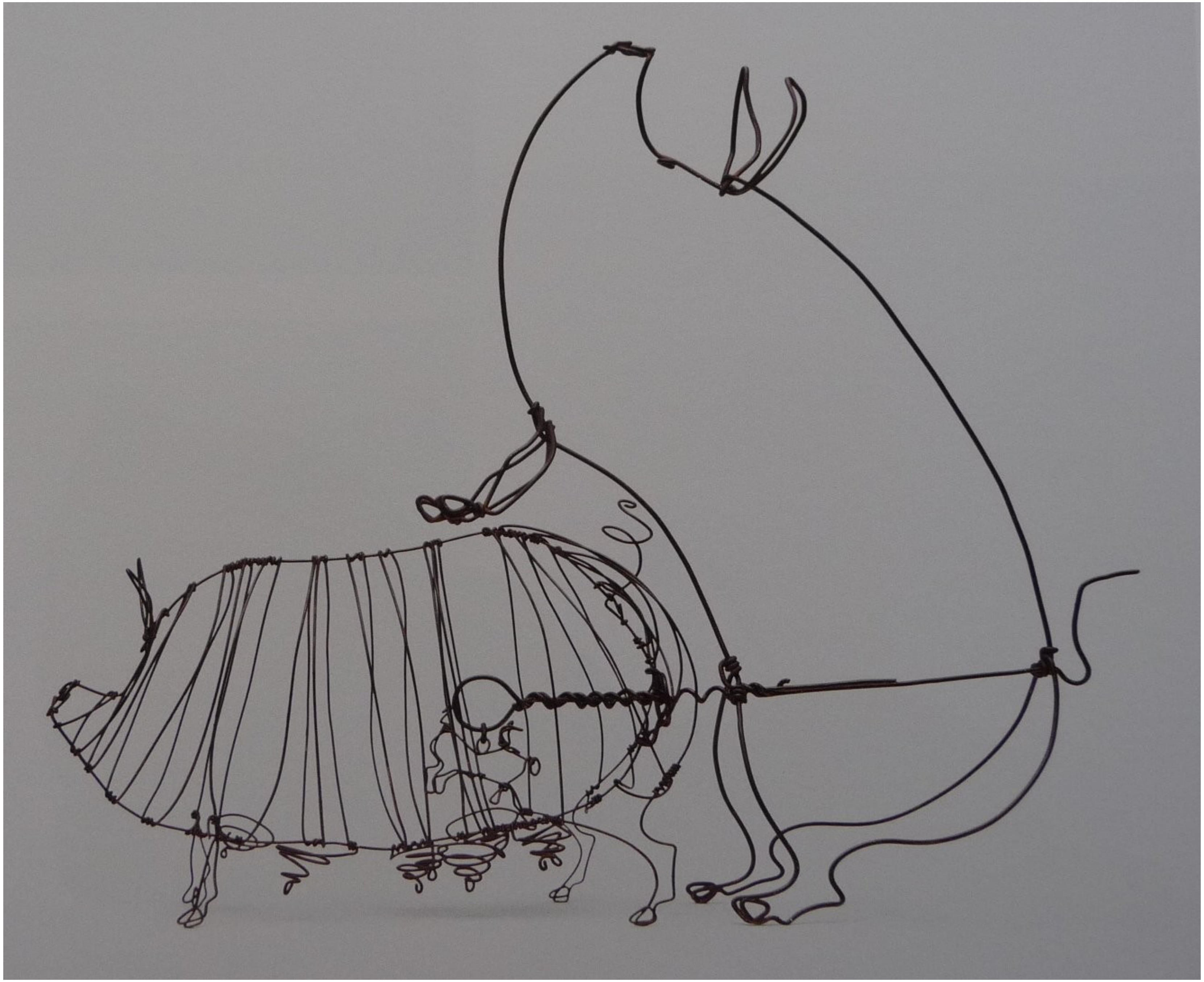





CALDER
the paris years
1926–1933

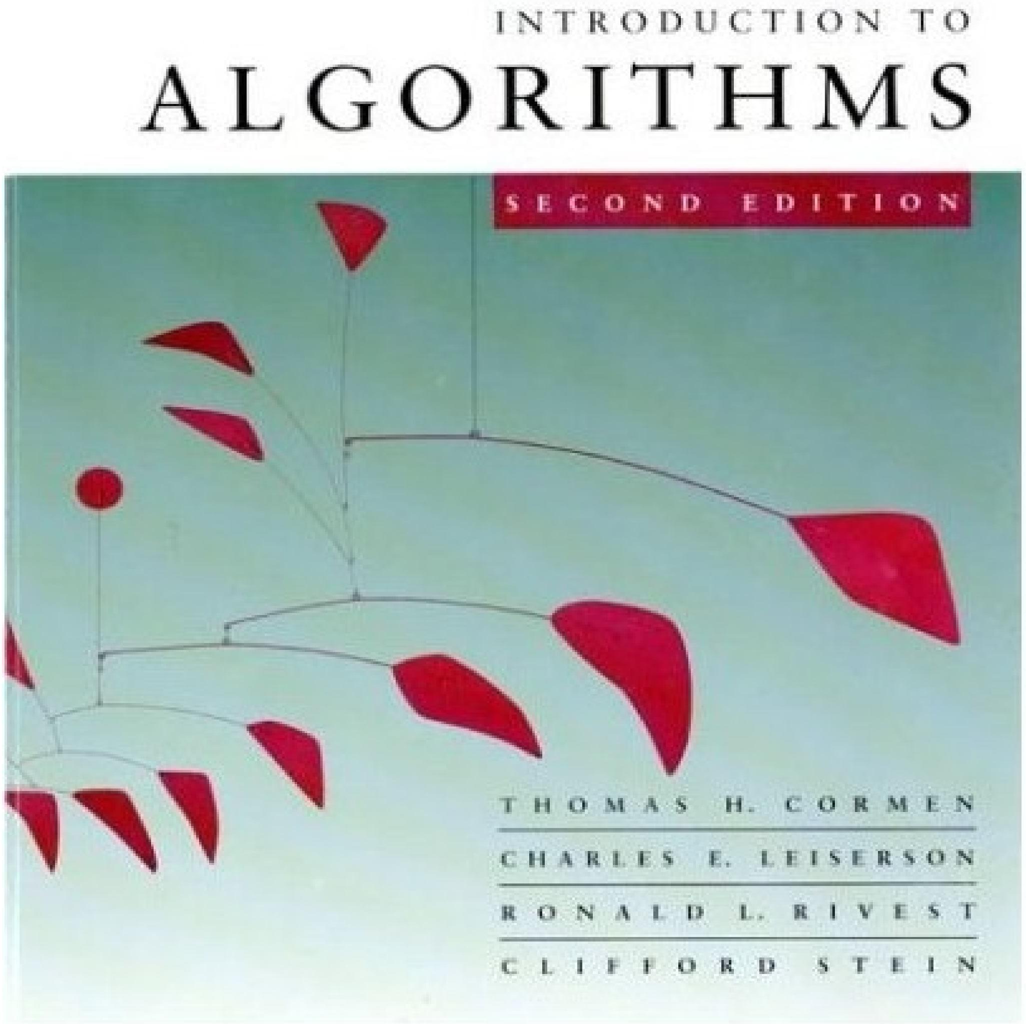






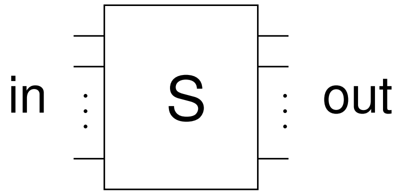
Schweinkram (filth)!

The computational complexity view



Implicit graph via Boolean circuits

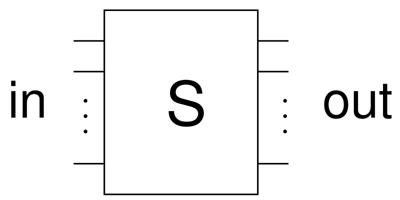
Successor circuit



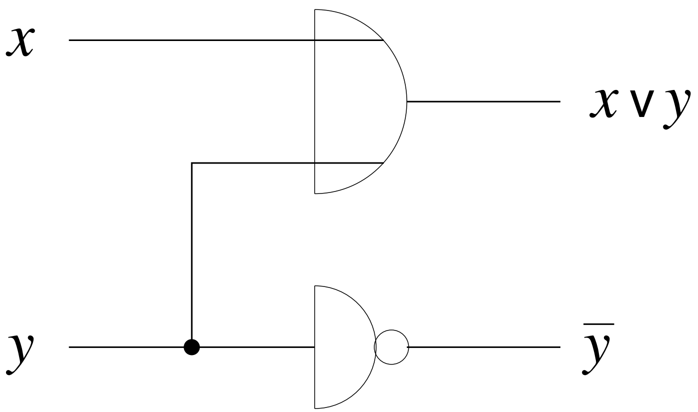
x,y	S(x,y)
00	01
01	10
10	11
11	10

Implicit graph via Boolean circuits

Successor
circuit

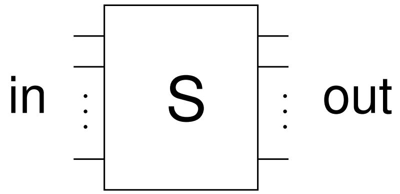


x,y	S(x,y)
00	01
01	10
10	11
11	10

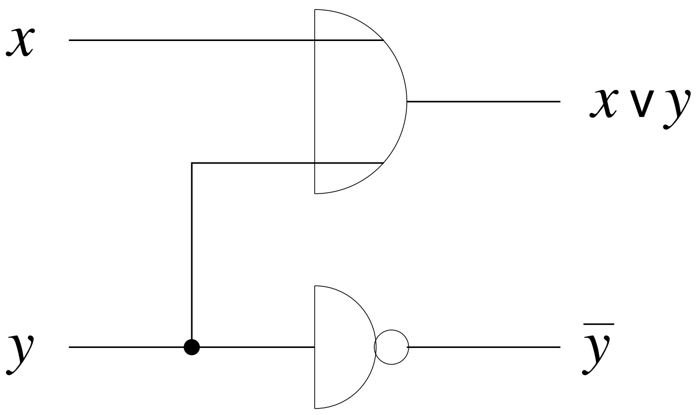


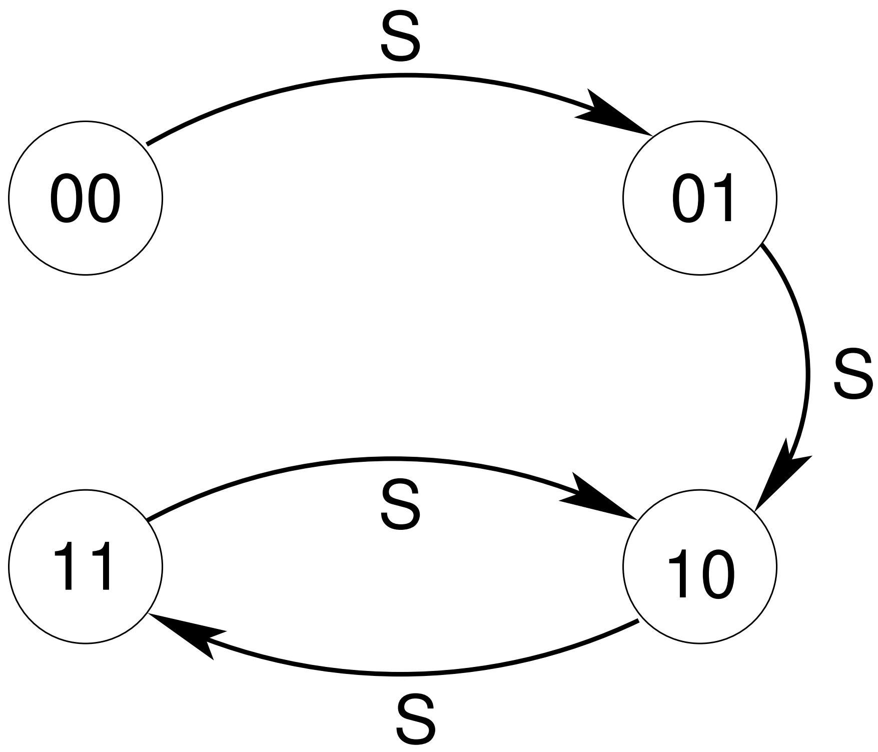
Implicit graph via Boolean circuits

Successor circuit



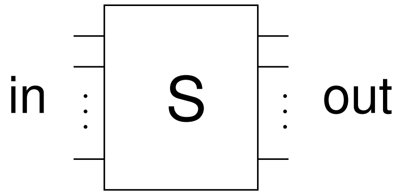
x,y	S(x,y)
00	01
01	10
10	11
11	10





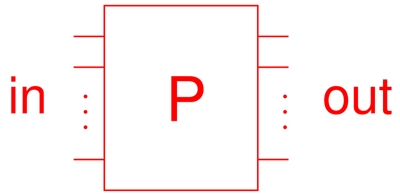
Implicit graph via Boolean circuits

Successor circuit



x,y	S(x,y)
00	01
01	10
10	11
11	10

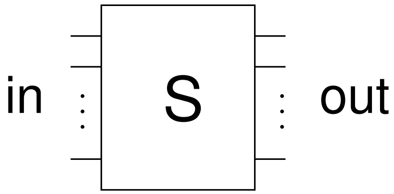
Predecessor circuit



x,y	P(x,y)
00	11
01	00
10	01
11	00

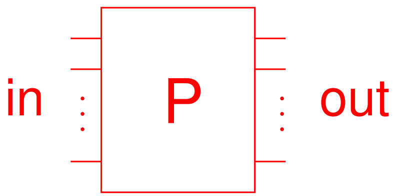
Implicit graph via Boolean circuits

Successor circuit

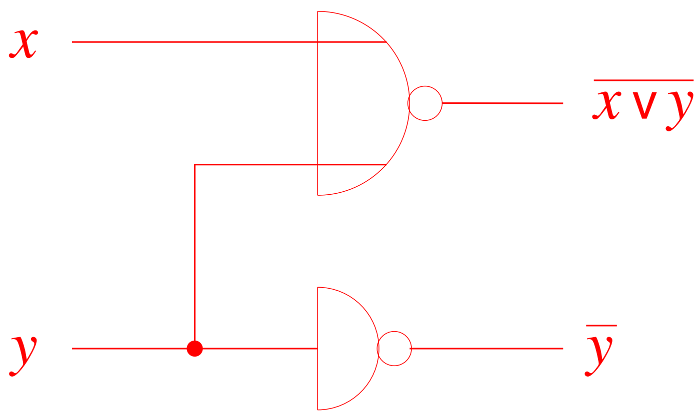


x,y	S(x,y)
00	01
01	10
10	11
11	10

Predecessor circuit

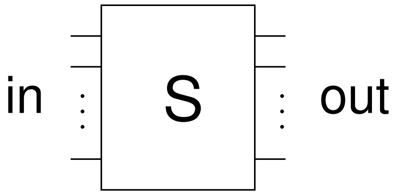


x,y	P(x,y)
00	11
01	00
10	01
11	00



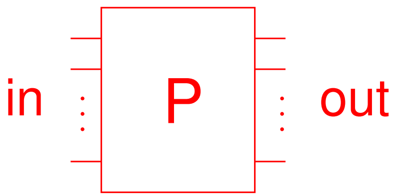
Implicit graph via Boolean circuits

Successor circuit

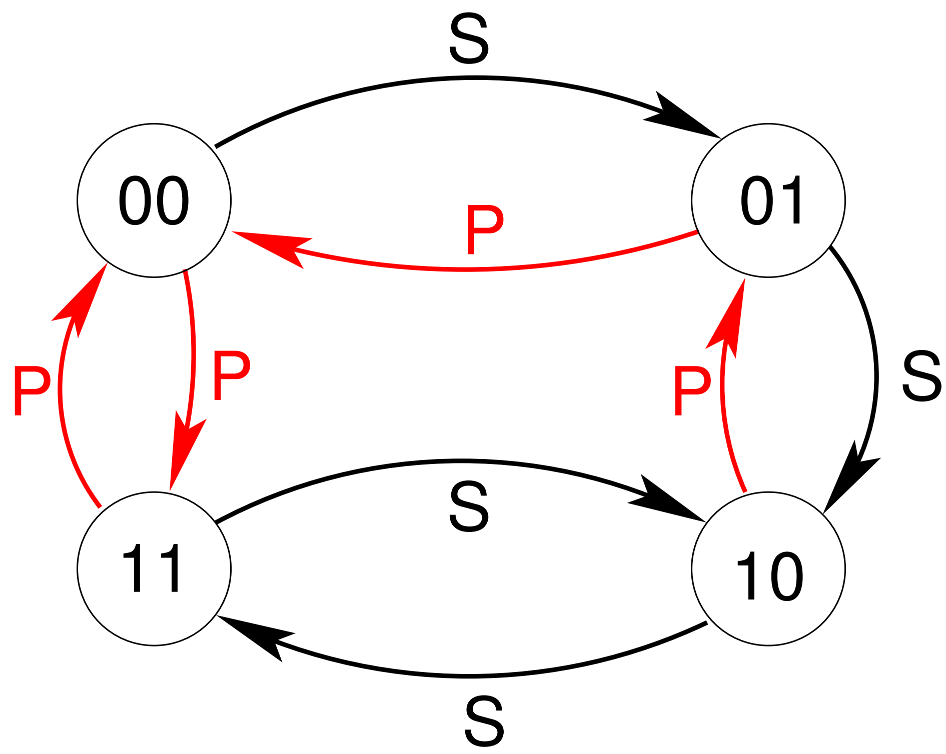


x,y	S(x,y)
00	01
01	10
10	11
11	10

Predecessor circuit



x,y	P(x,y)
00	11
01	00
10	01
11	00

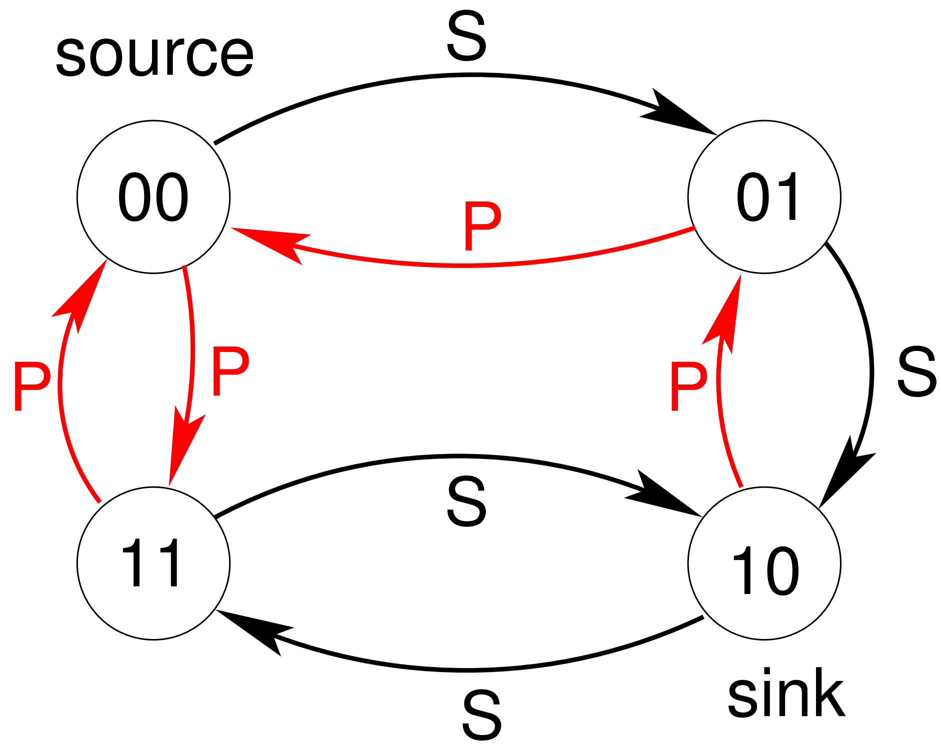


Sources and Sinks

u predecessor of v

$\Leftrightarrow$ v successor of u

$\Leftrightarrow$ v = S(u), u = P(v)



u source $\Leftrightarrow$ $S(P(u)) \neq u$

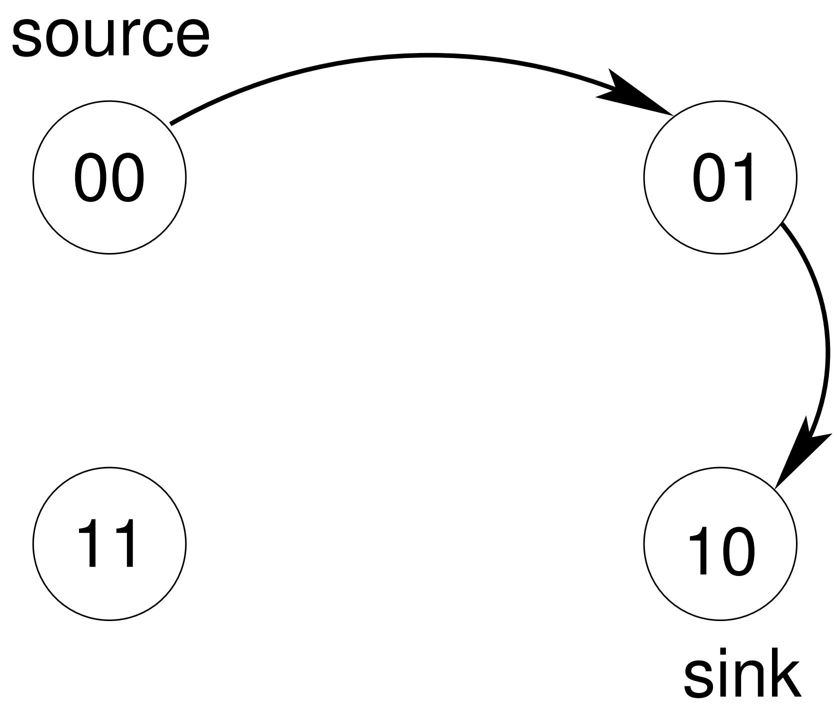
v sink $\Leftrightarrow$ $P(S(v)) \neq v$

Sources and Sinks

u predecessor of v

$\Longleftrightarrow$ v successor of u

$\Longleftrightarrow$ $v = S(u), \ u = P(v)$



u source $\Longleftrightarrow$ $S(P(u)) \neq u$

v sink $\Longleftrightarrow$ $P(S(v)) \neq v$

The problem End–Of–the–Line (EOL)

Input:

circuits S, P: $2^n \rightarrow 2^n$

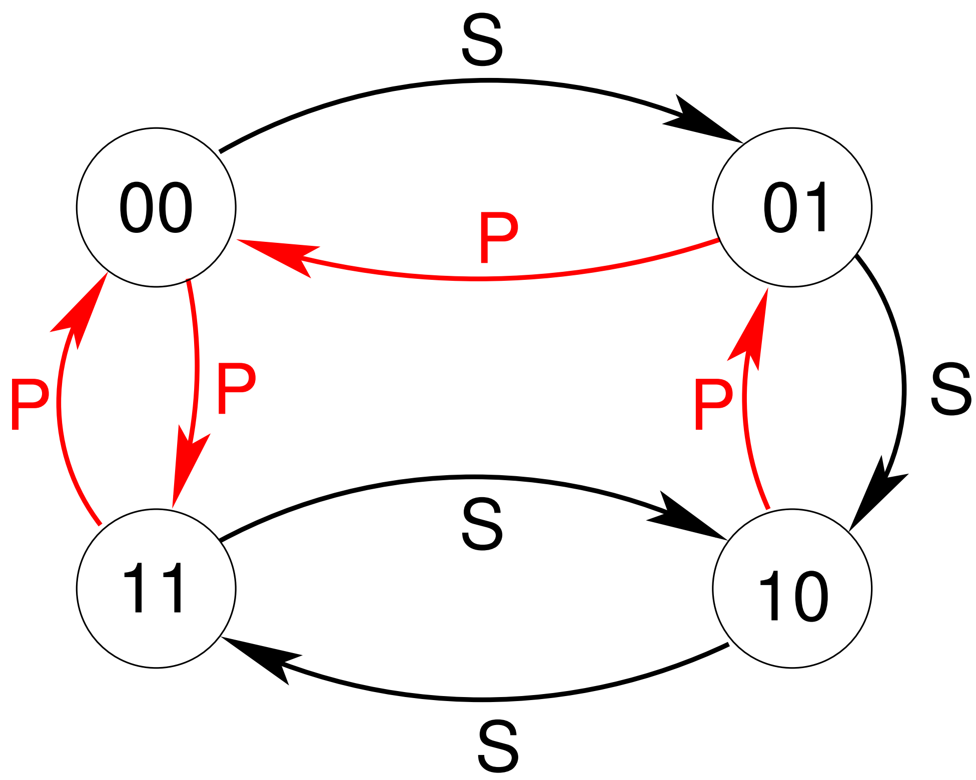
polynomial size in n

source 0^n

Output:

Any sink, or

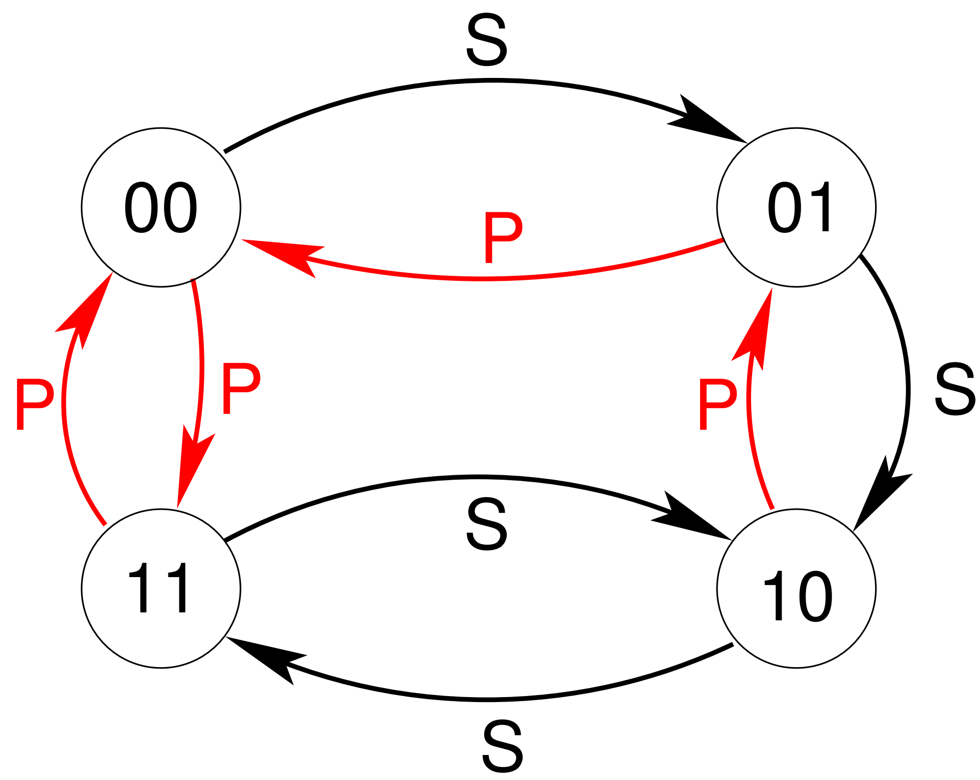
source other than 0^n



The problem End–Of–the–Line (EOL)

Input:
circuits S, P: $2^n \to 2^n$
polynomial size in n
source 0^n

Output:
Any sink, or
source other than 0^n



PPAD = any instances of EOL

"polynomial parity argument with direction"

[PaPADimitriou 1994]

PPAD-completeness

A computational problem is **PPAD-complete** if EOL can be reduced to it.

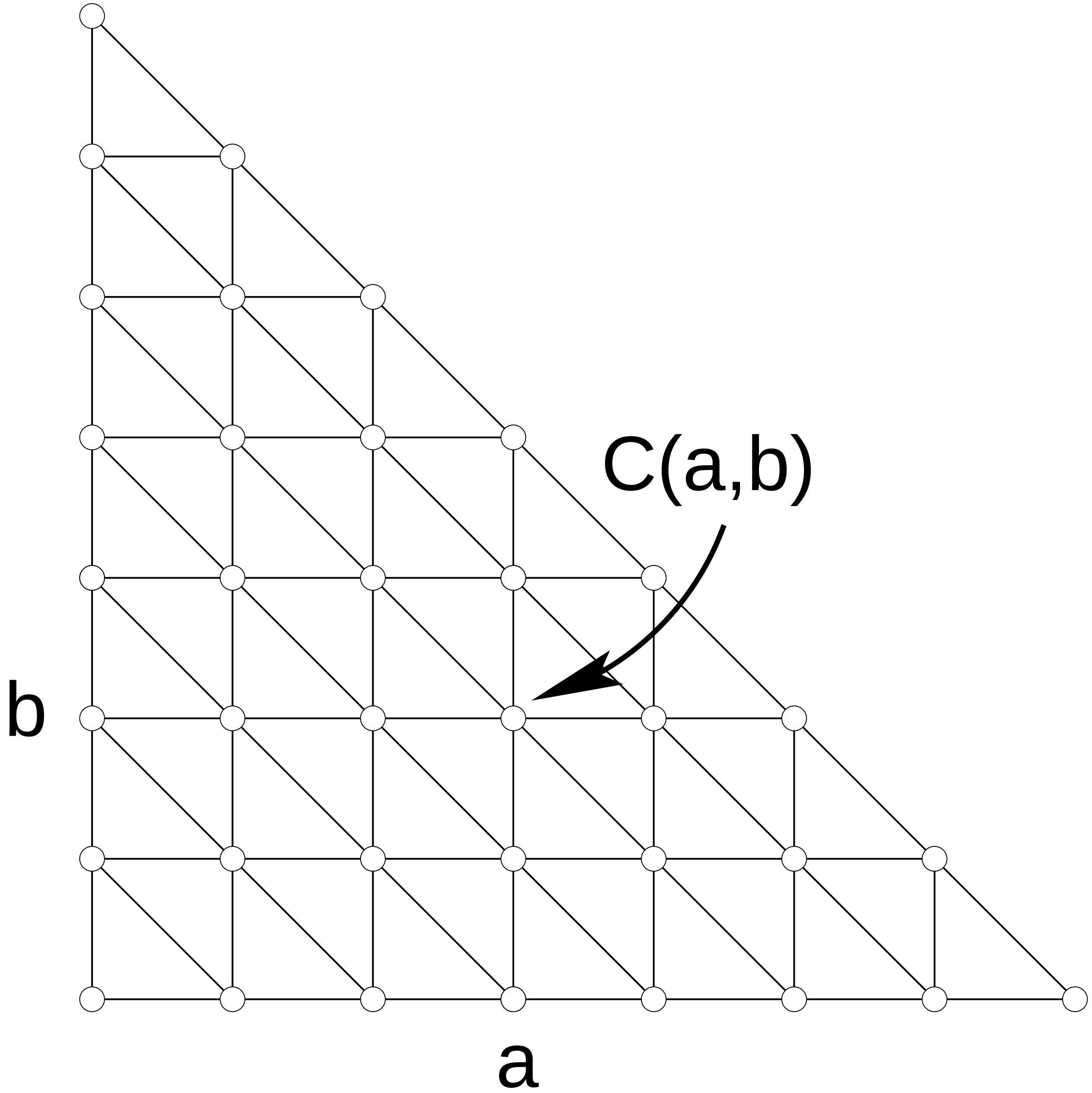
[Chen & Deng 2006]:
2D-SPERNER is PPAD-complete.

Problem 2D-SPERNER:

Input: circuit C: $\mathbf{2}^n \times \mathbf{2}^n \rightarrow$ {00, 01, 10}
C(a,b) = color of grid point (a,b)
of triangulation.

Output: panchromatic triangle or violation of Sperner condition.

Circuit C = input to 2D-SPERNER



Nash equilibria of bimatrix games

[Chen & Deng 2005]:
2-NASH is PPAD-complete.

Problem 2-NASH:

Input: 2-player game (A,B) in strategic form with integer payoffs.

Output: One Nash equilibrium of (A,B).

Nash equilibria of bimatrix games

Why is 2-NASH in PPAD?
Lemke-Howson algorithm!

We will consider here:

- unit-vector games (U,B)
- are as general as any bimatrix games
- define labeled (= vertex-colored) **polytope**
- defines **manifold** with:

rooms	=	polytope facets,
panchromatic rooms	=	equilibria of (U,B),
Abstract Sperner	=	Lemke-Howson

- in PPA**D**: [Shapley 1974], oriented manifold

Simplicial polytopes P and games

Given: **P** = conv $\{-e_1, \ldots, -e_r, b_1, \ldots, b_n\}$ with **colors**

- **i** for negative unit vector $-e_{\mathbf{i}}$, **i** = 1,...,r
- **c**(j) $\in \{1, \ldots, r\}$ for each r-vector $b_j > 0$, j = 1,...,n.

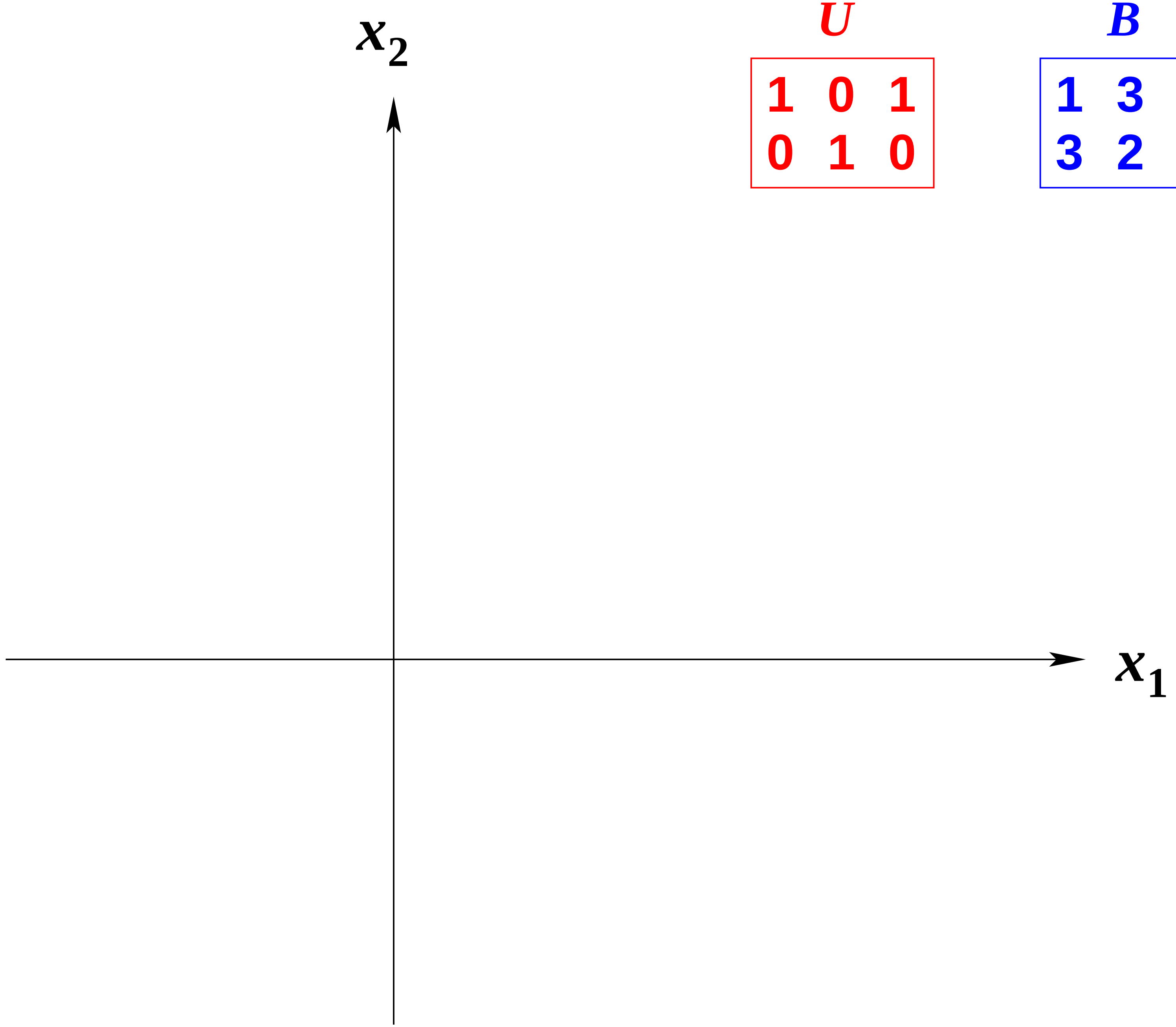
Then: panchromatic **facets** of **P**

⇔ Nash equilibria of the $r \times n$ "unit vector" game

$([\, e_{\mathbf{c}(1)} \ldots e_{\mathbf{c}(n)} \,], [\, B_1 \ldots B_n \,])$

where $B_j = b_j \,/\, (1 + \|\, b_j \,\|_1)$, j = 1,...,n.

Polytope P from unit-vector game



U

1	0	1
0	1	0

B

1	3	4
3	2	1

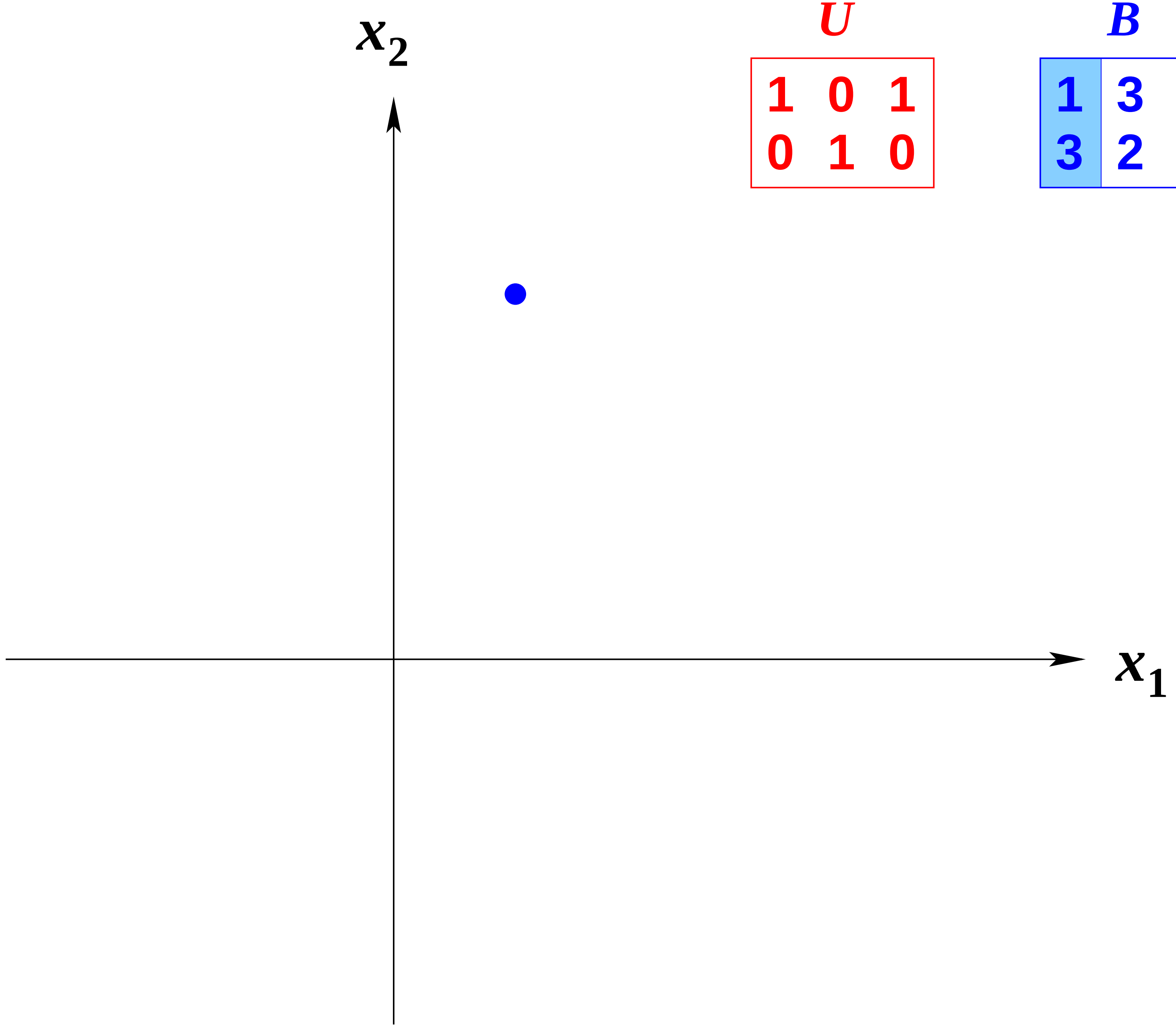
first column of B

U

1	0	1
0	1	0

B

1	3	4
3	2	1



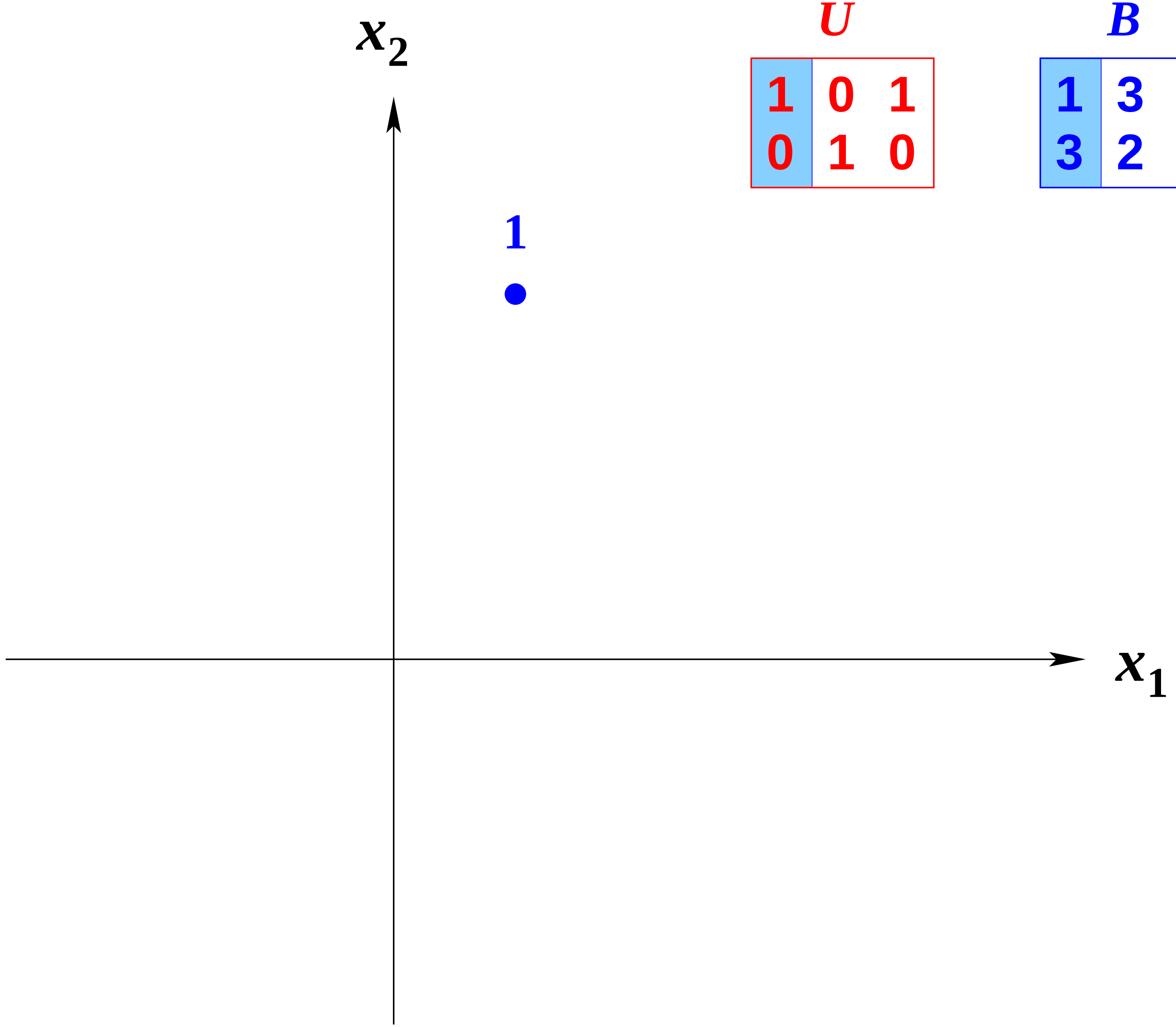
first column of B with color (label) 1

U

1	0	1
0	1	0

B

1	3	4
3	2	1



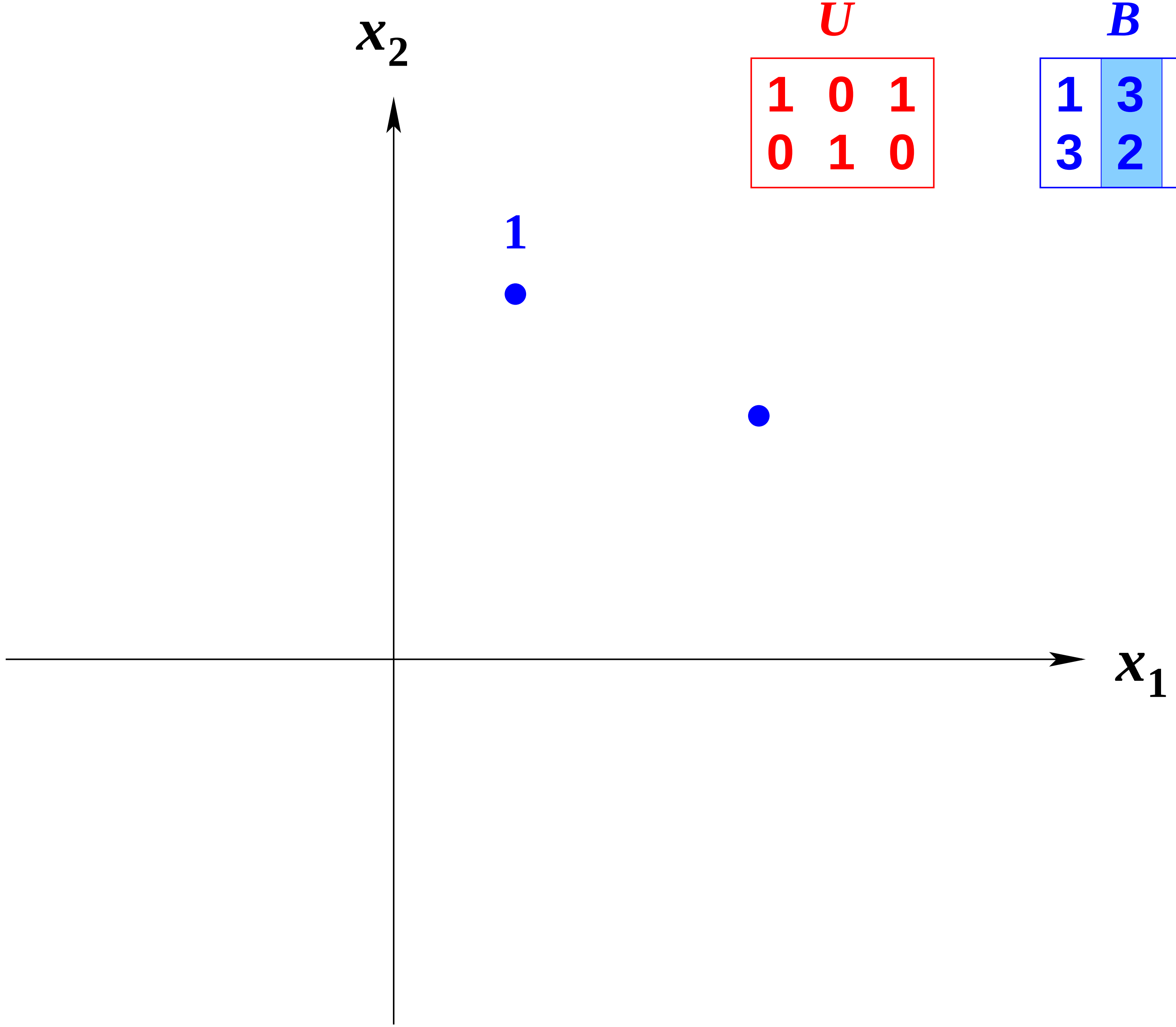
second column of B

U

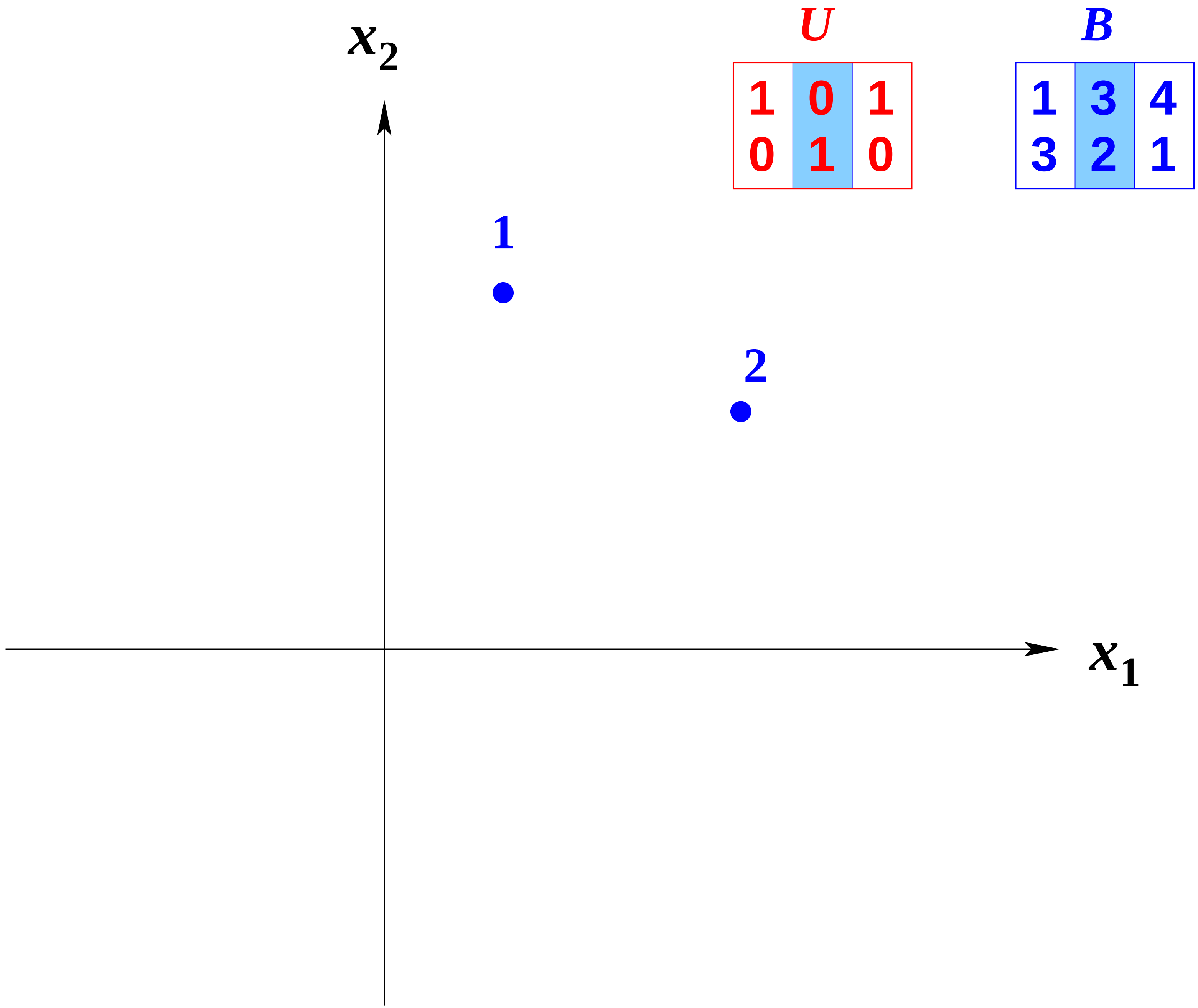
1	0	1
0	1	0

B

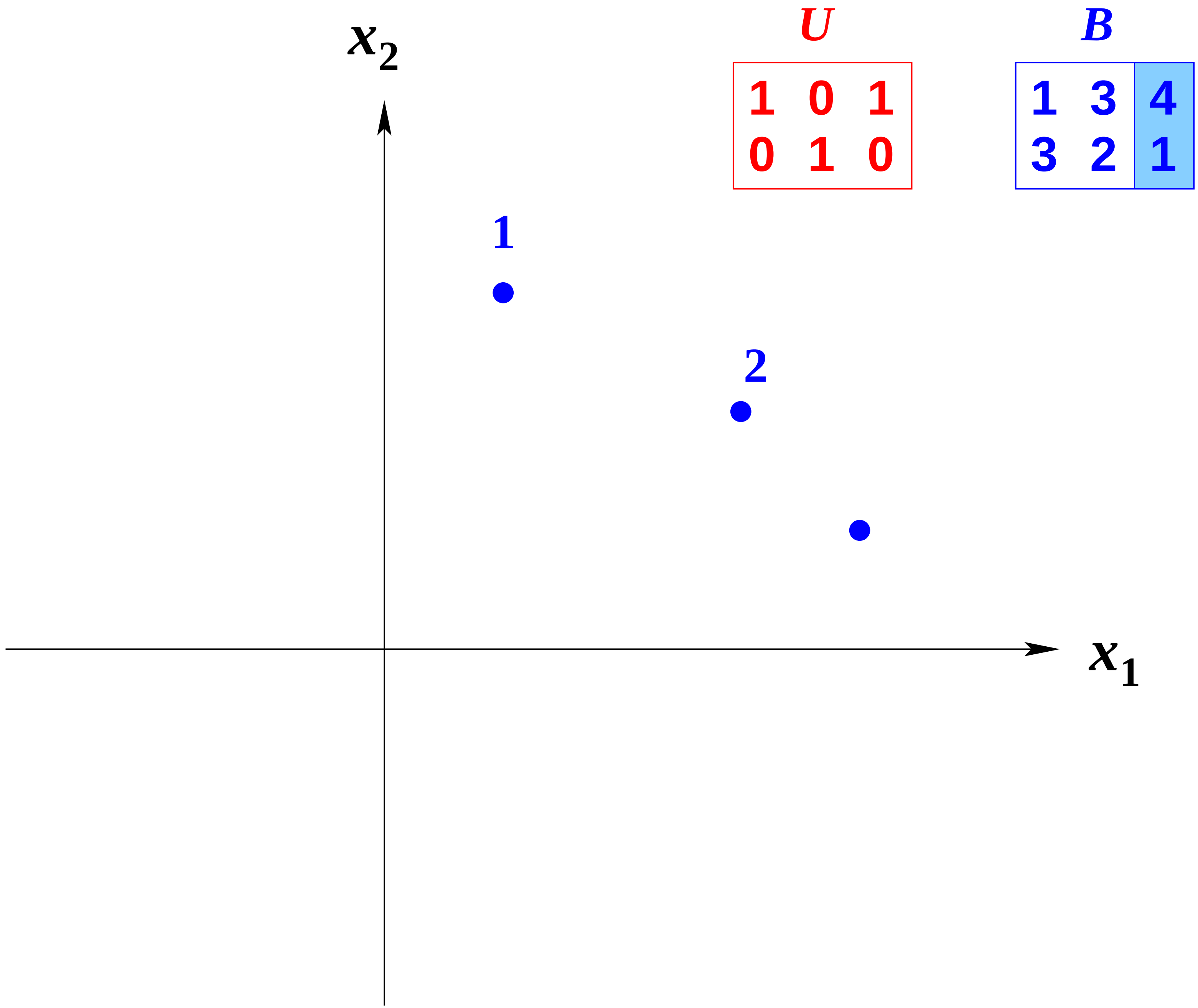
1	3	4
3	2	1



second column of B with color 2



third column of B



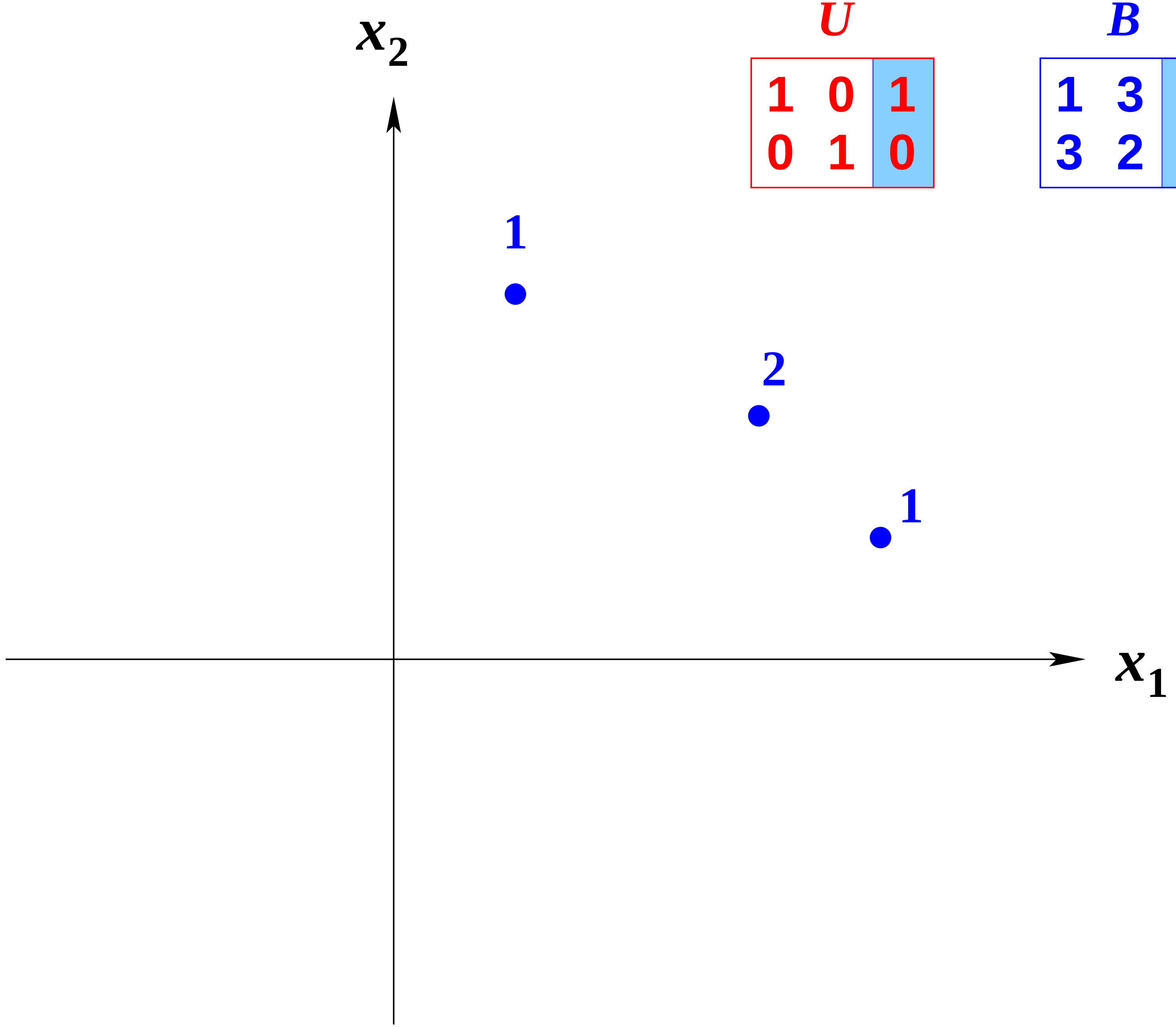
third second column of B with color 1

U

1	0	1
0	1	0

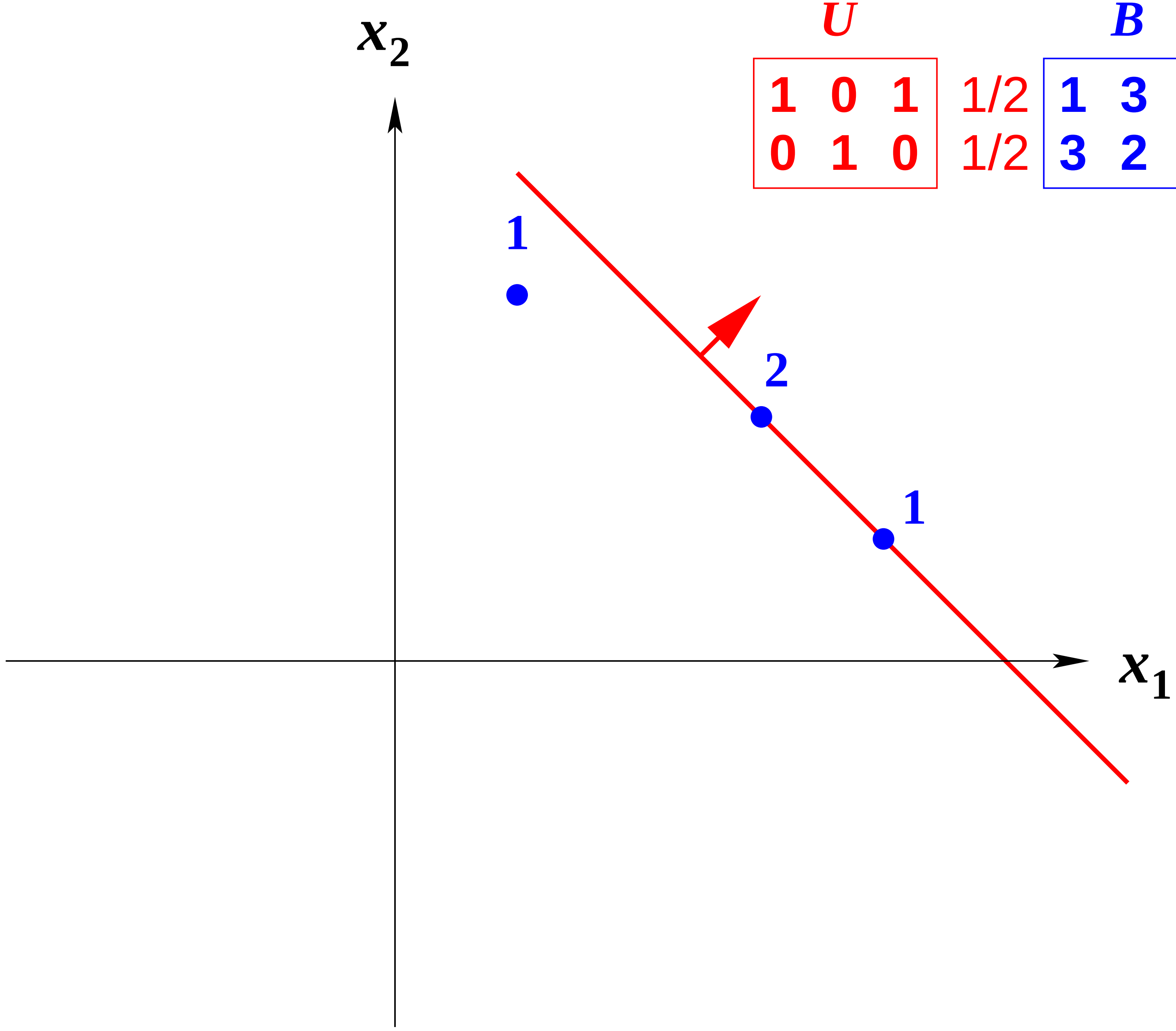
B

1	3	4
3	2	1

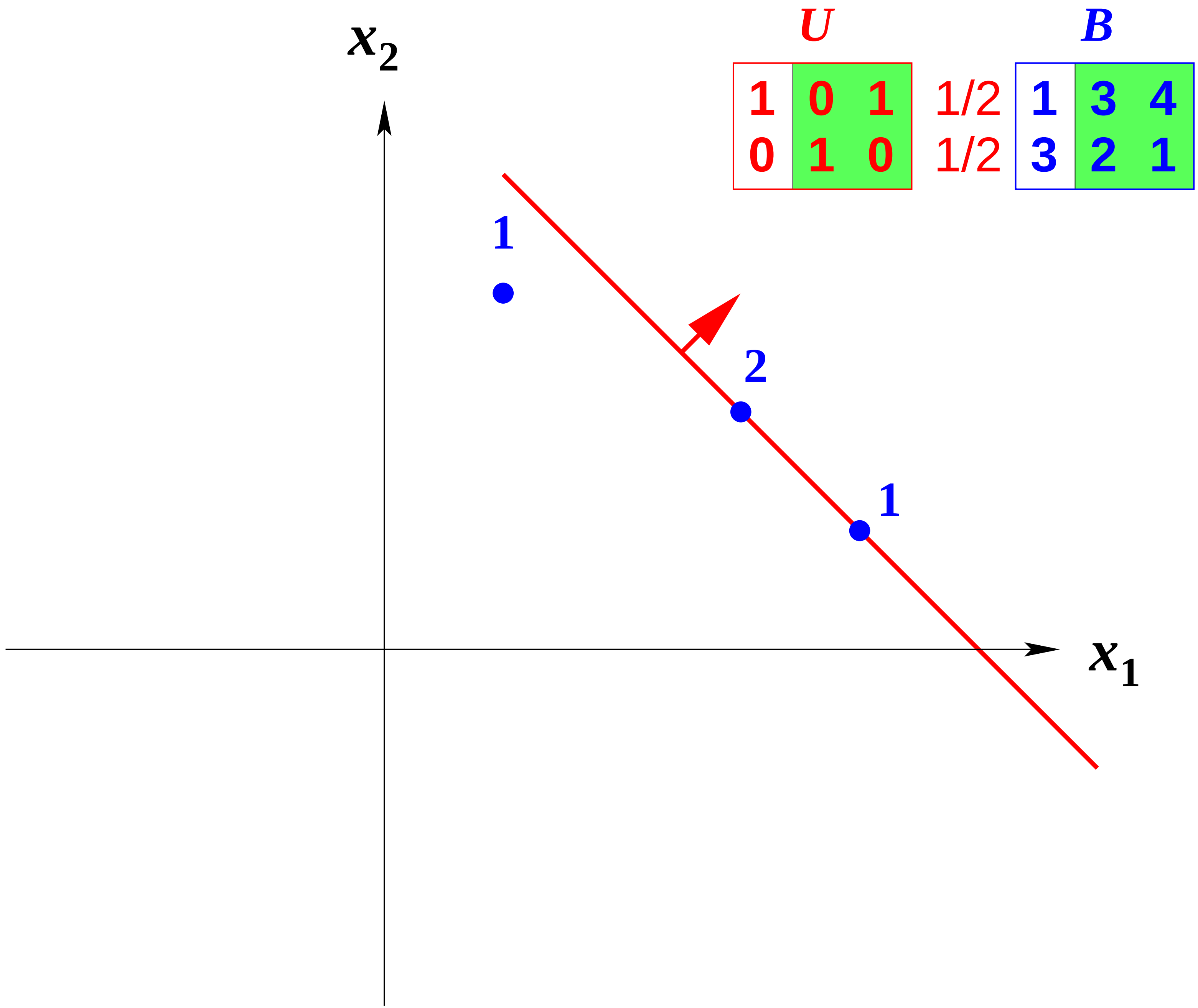


mixed row strategy = normal vector

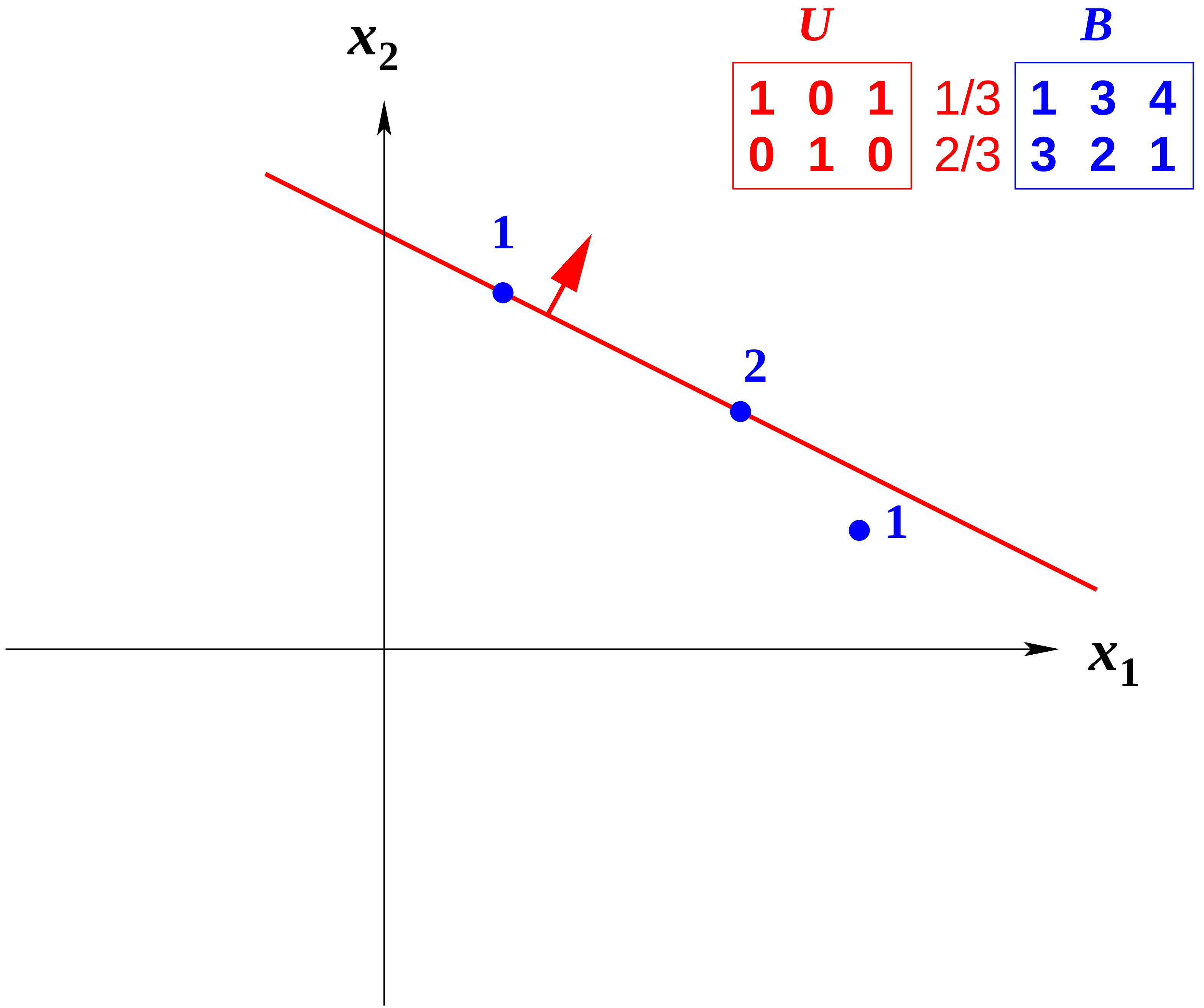
U				*B*		
1	0	1	1/2	1	3	4
0	1	0	1/2	3	2	1



(1/2, 1/2) is equilibrium strategy

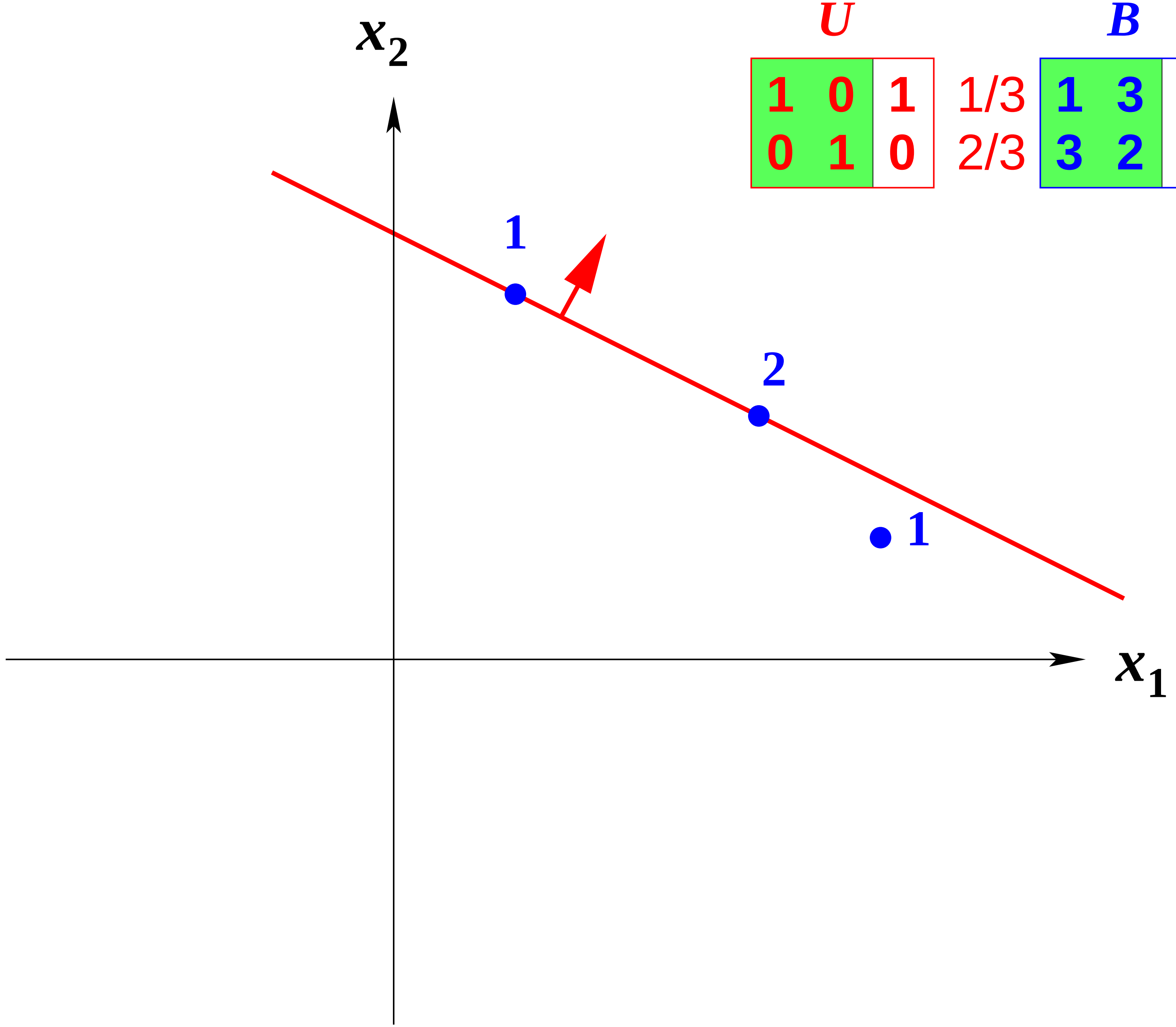


(1/3, 2/3) as normal vector

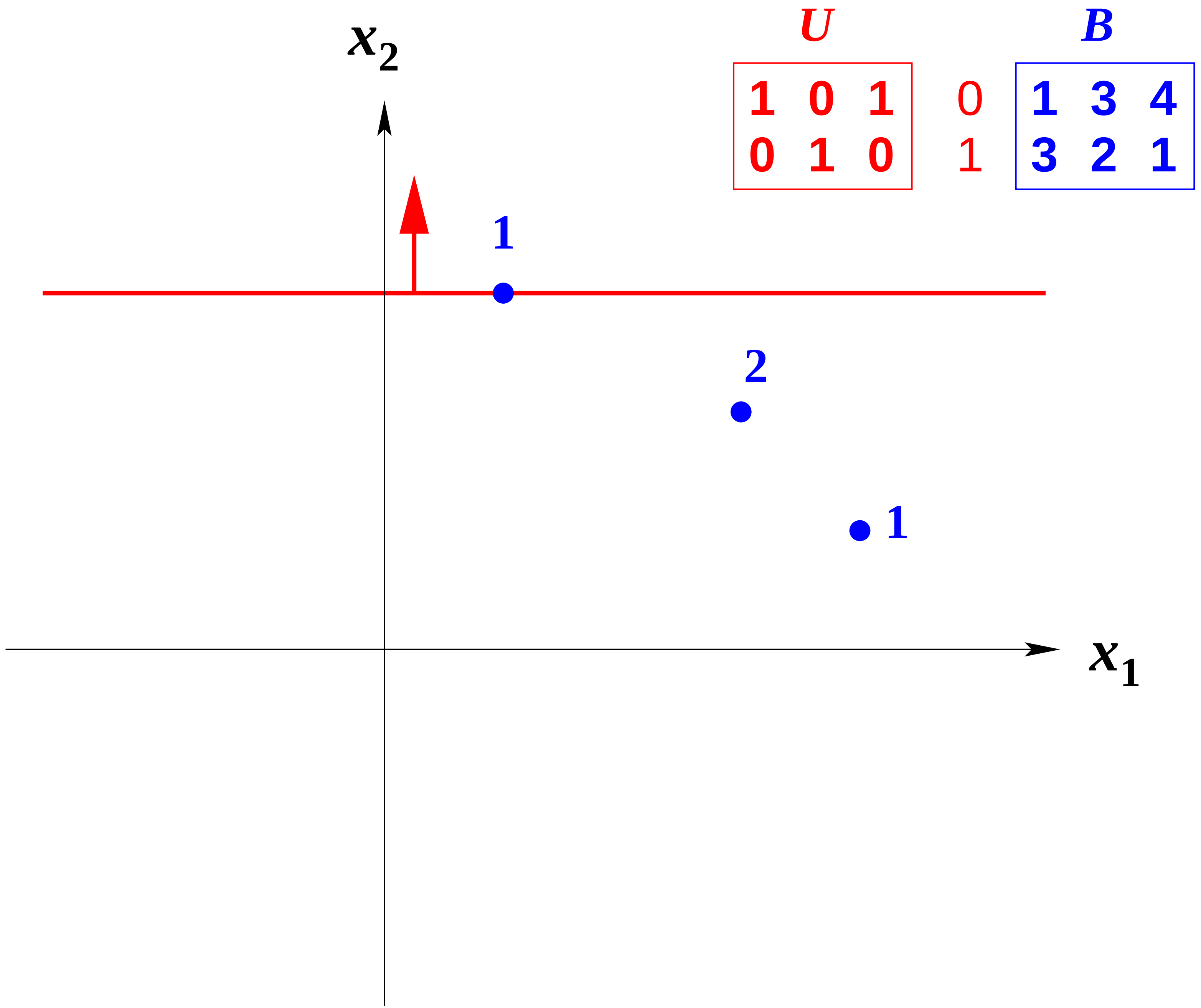


(1/3, 2/3) is equilibrium strategy

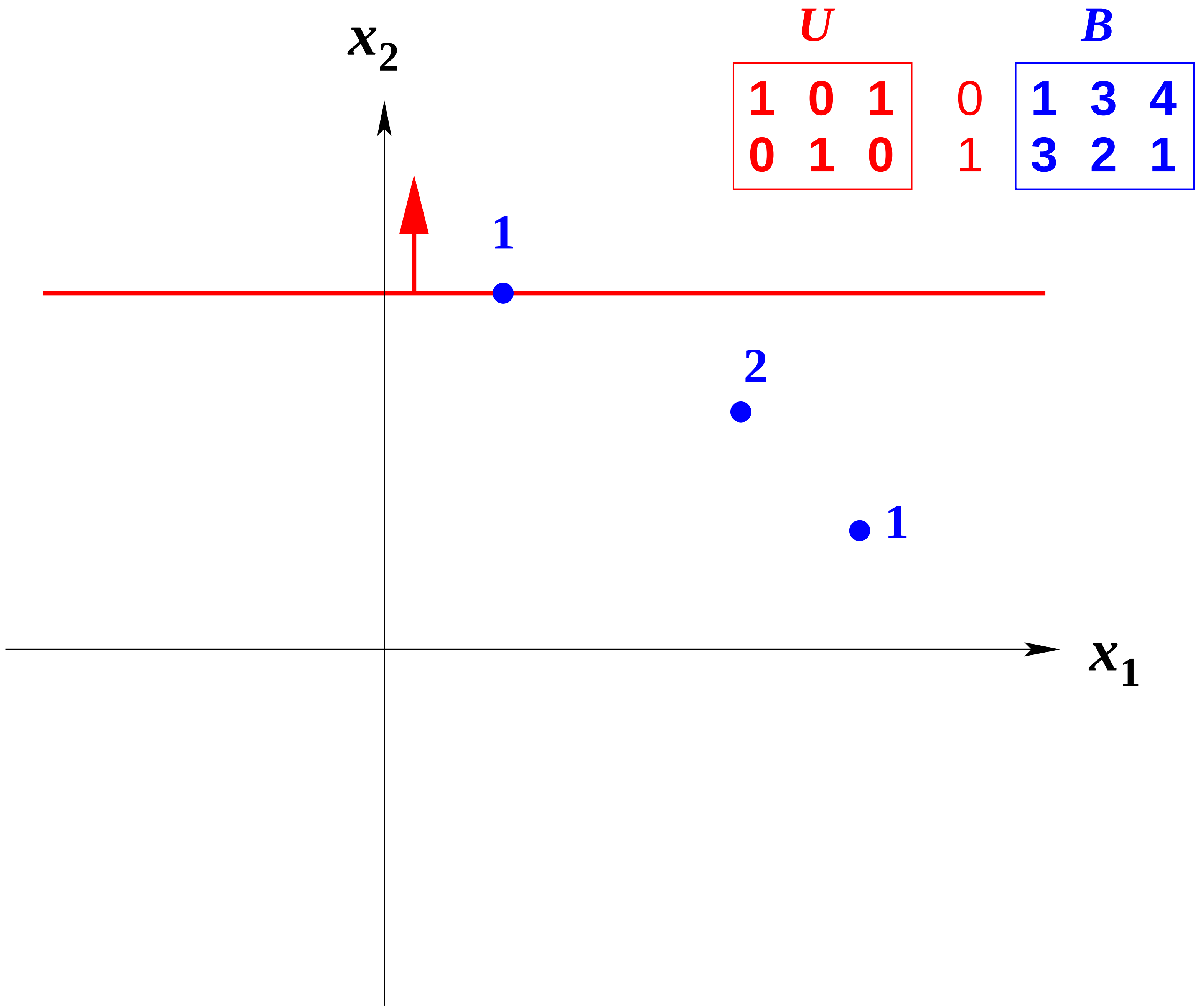
U				*B*		
1	0	1	1/3	1	3	4
0	1	0	2/3	3	2	1



(0, 1) as normal vector



(0, 1) not an equilibrium strategy



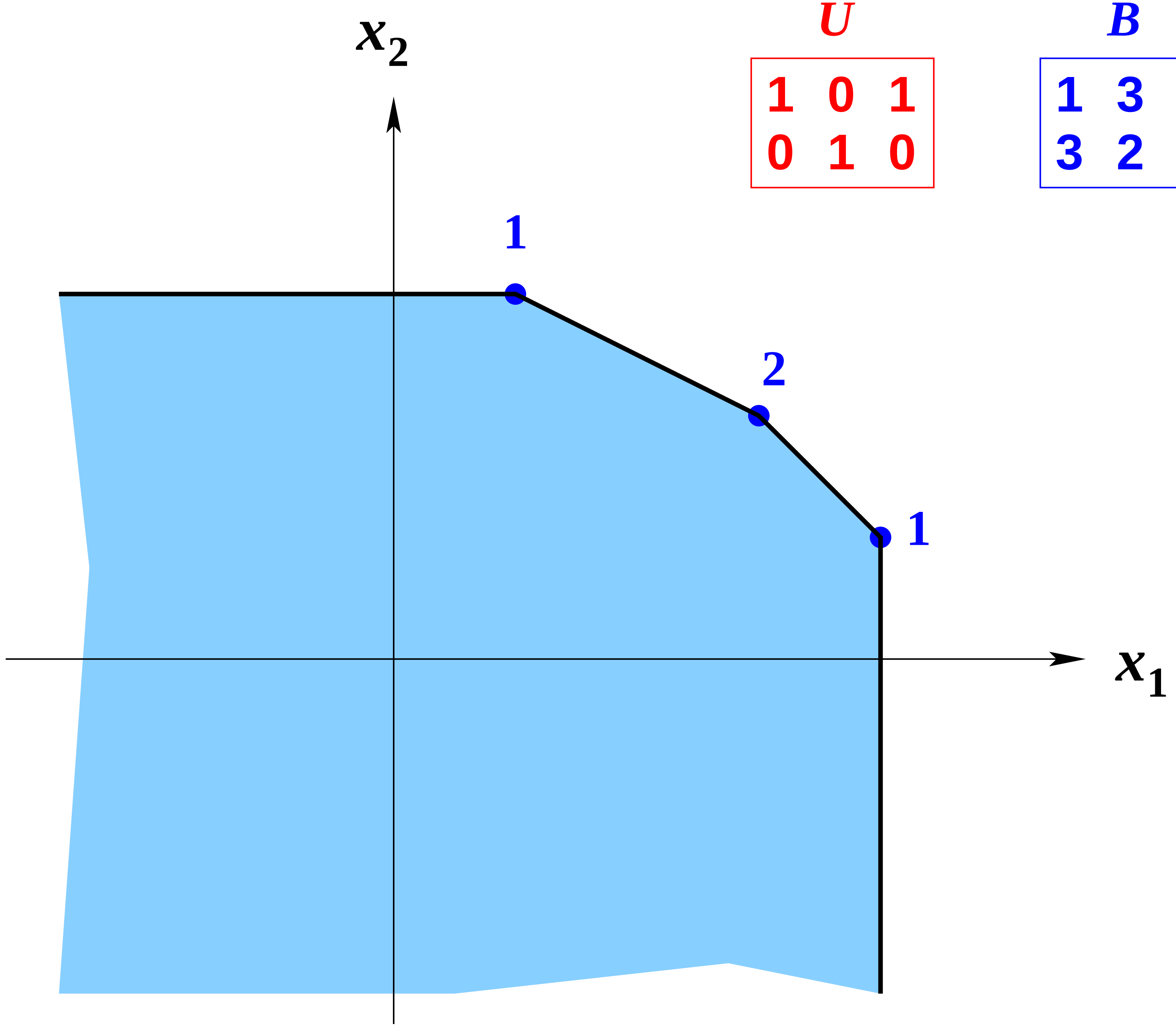
want: nonnegative convex hull

U

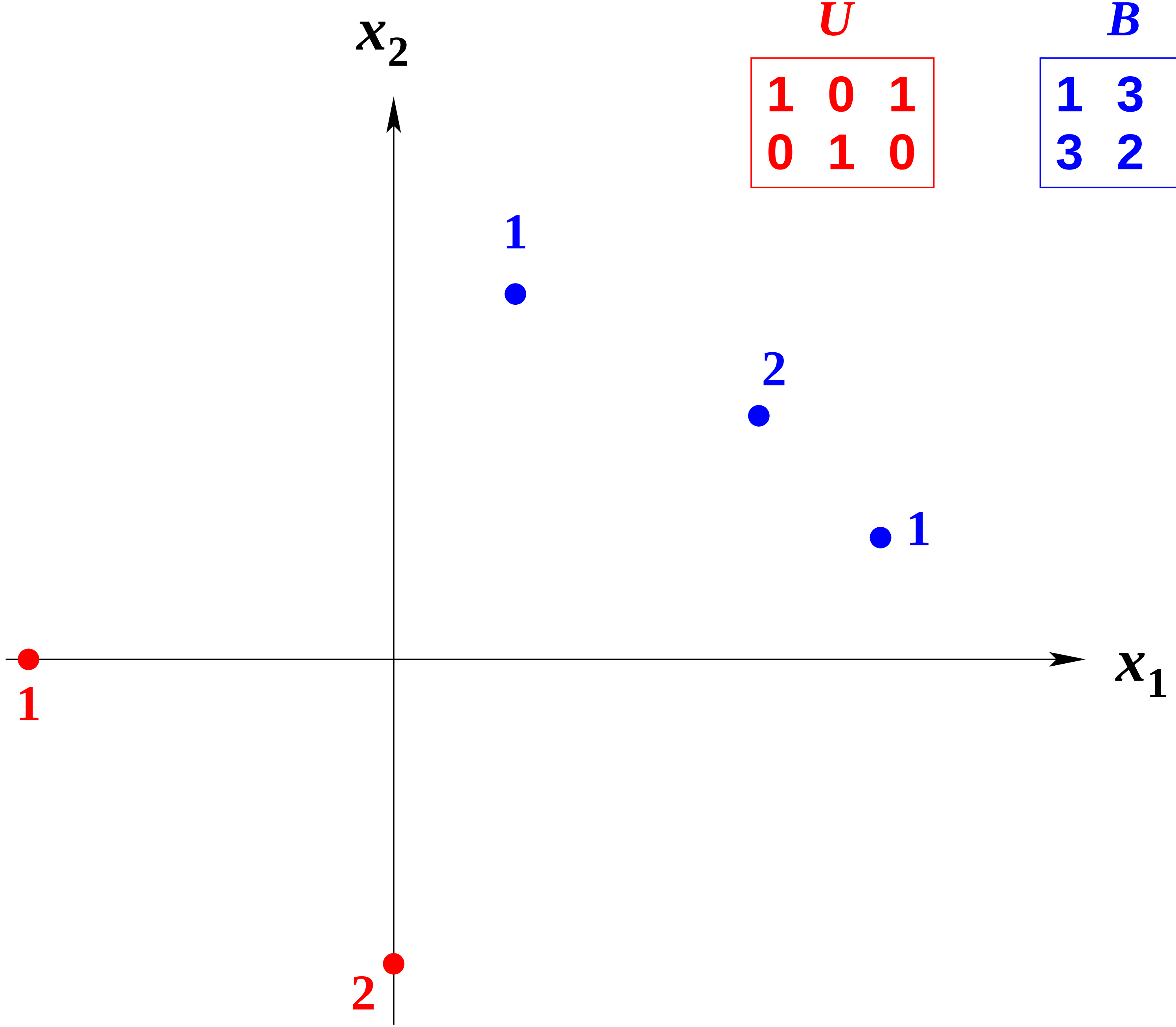
1	0	1
0	1	0

B

1	3	4
3	2	1



use negative unit vectors



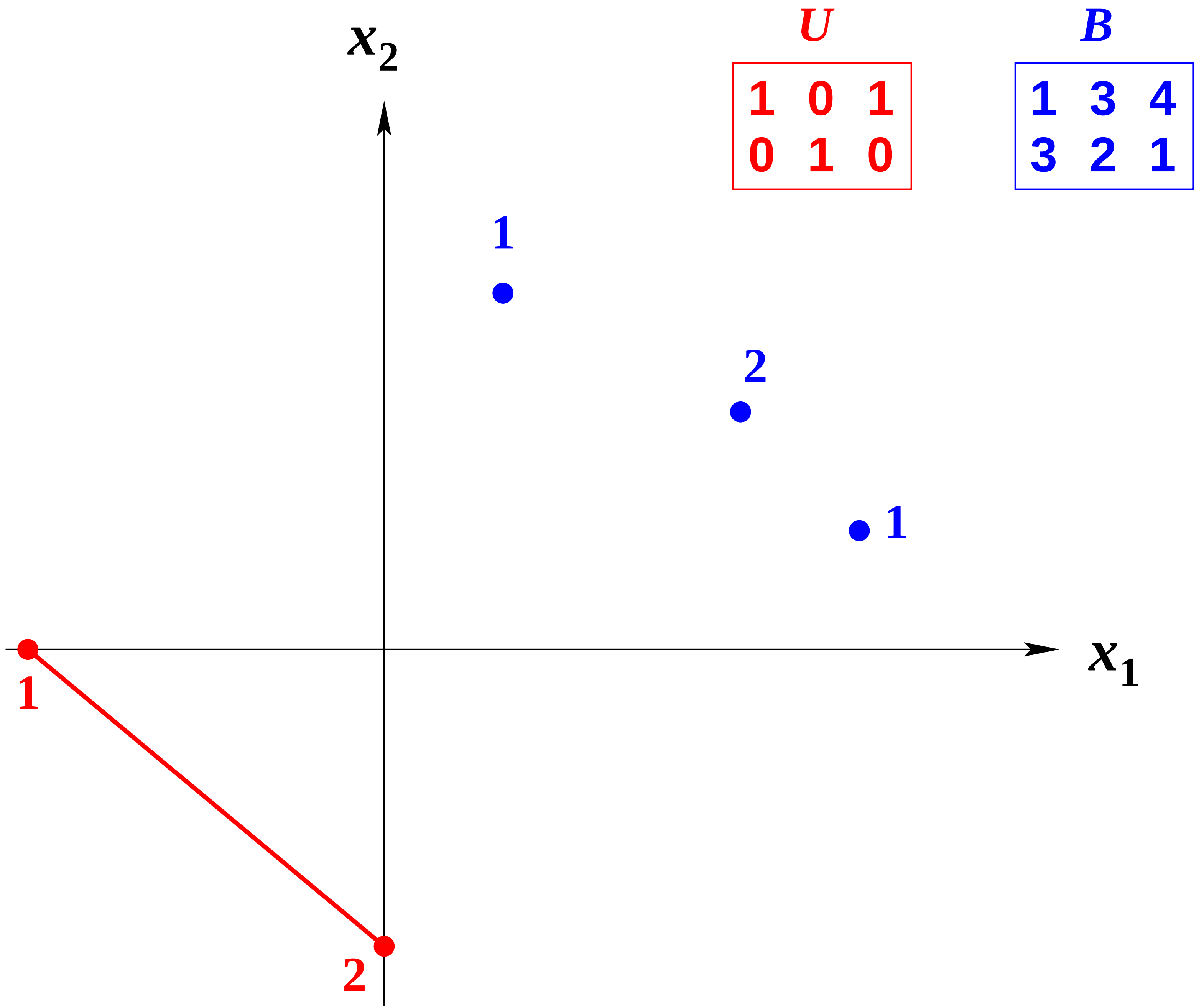
U

1	0	1
0	1	0

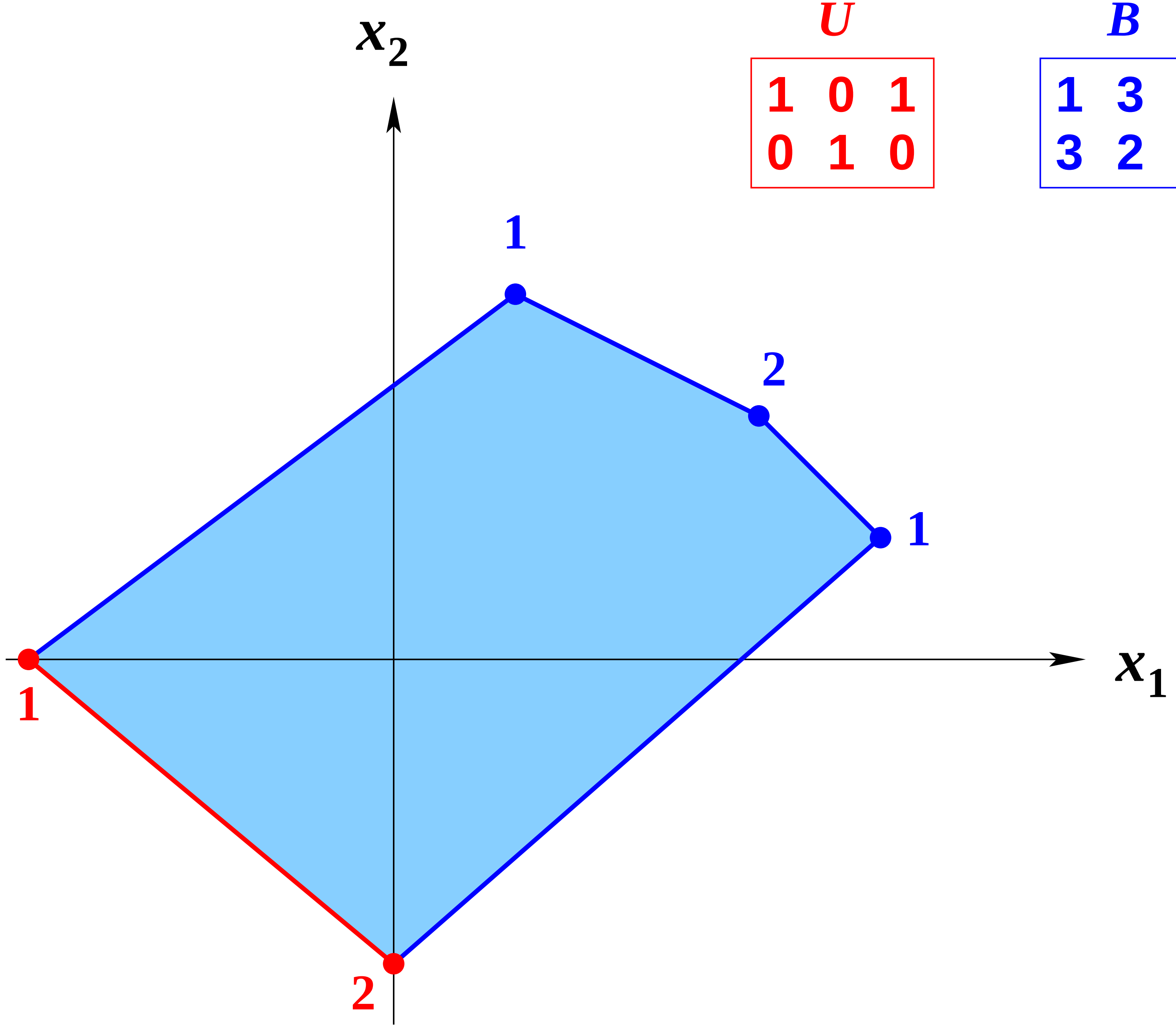
B

1	3	4
3	2	1

(giving trivial panchromatic facet)



otherwise polytope P with equivalent facets



U

1	0	1
0	1	0

B

1	3	4
3	2	1

including equilibria without full support

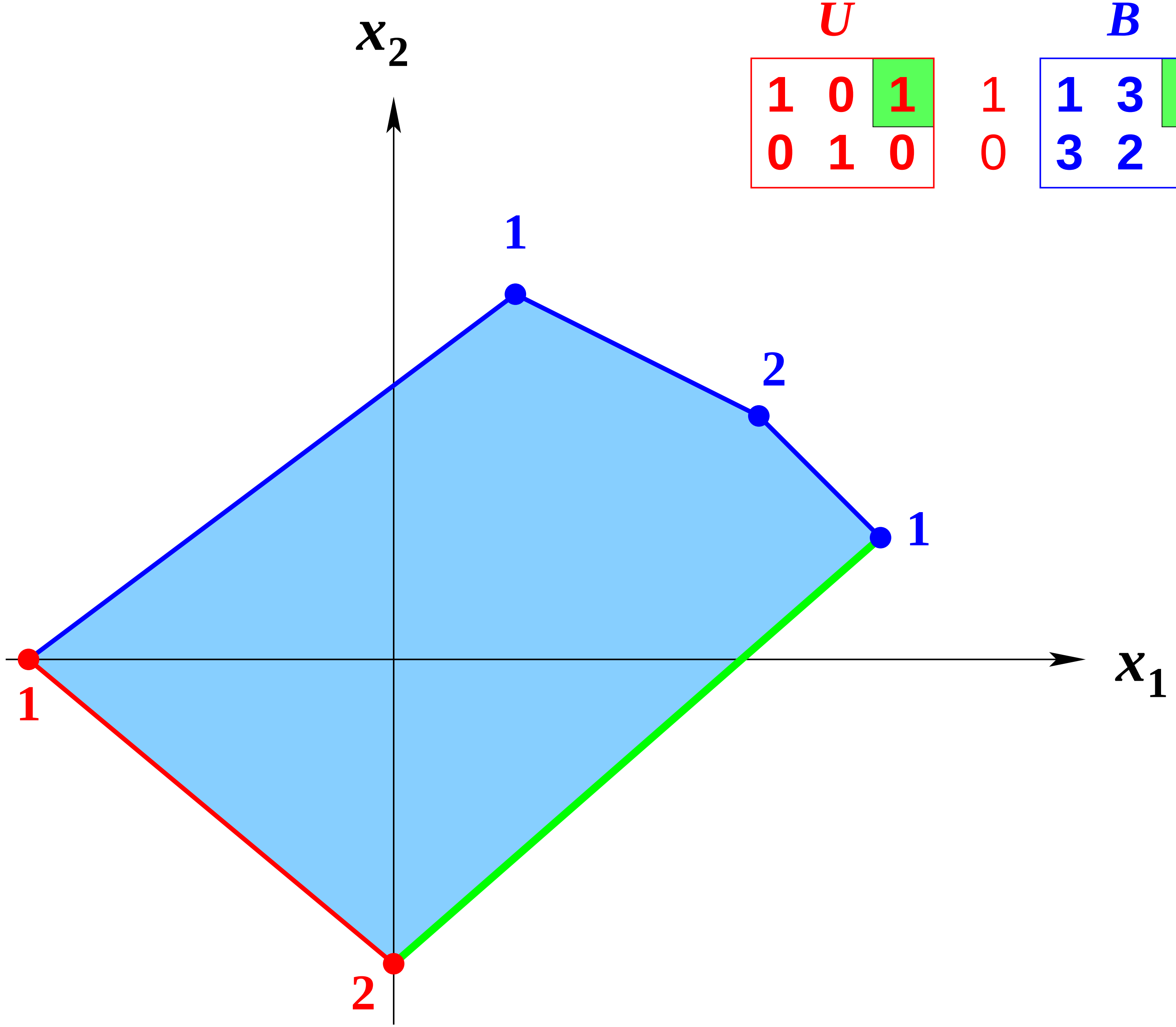
U

1	0	1
0	1	0

1
0

B

1	3	4
3	2	1



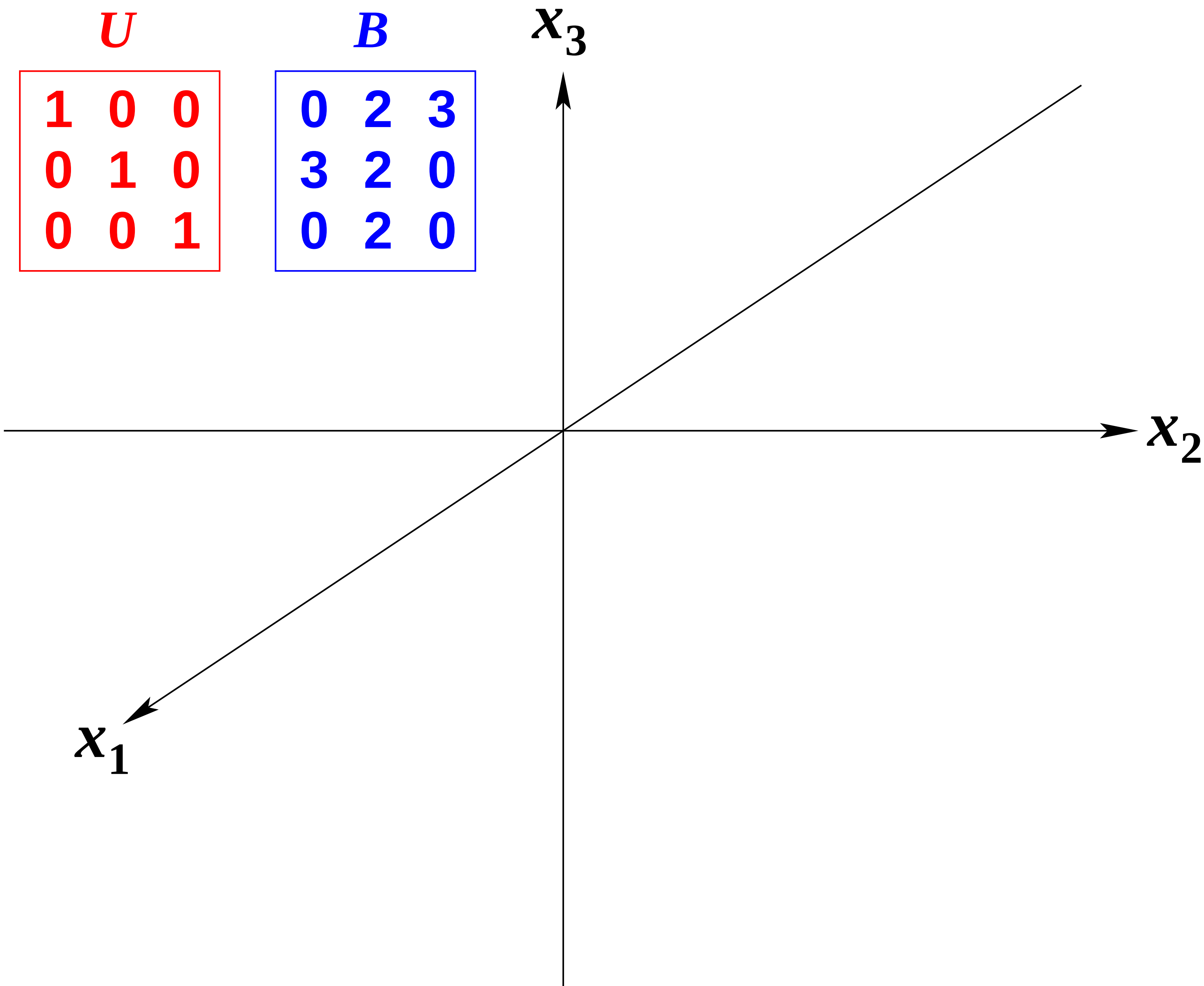
Constructing P for 3 x 3 “imitation game”

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



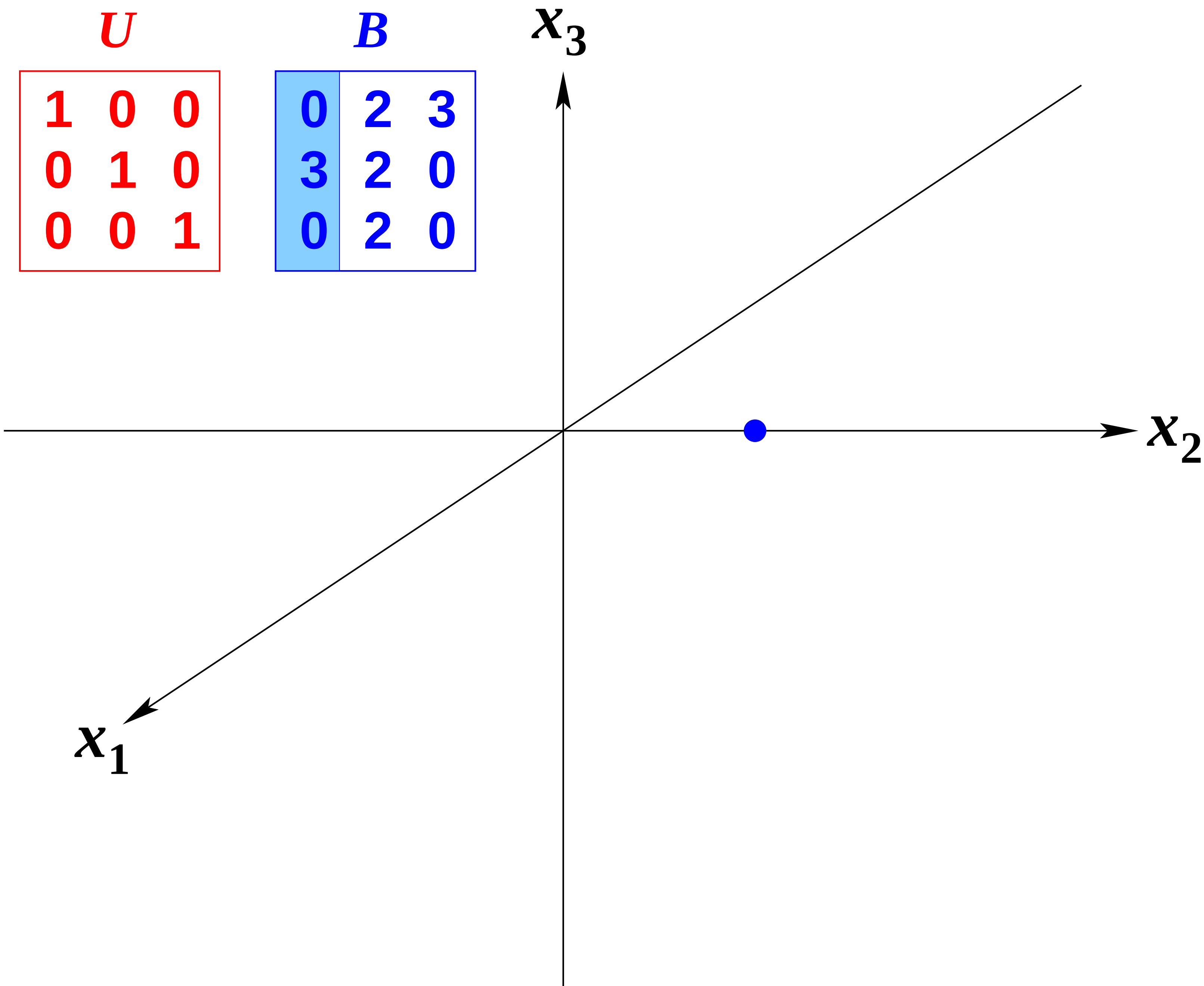
first column of B

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



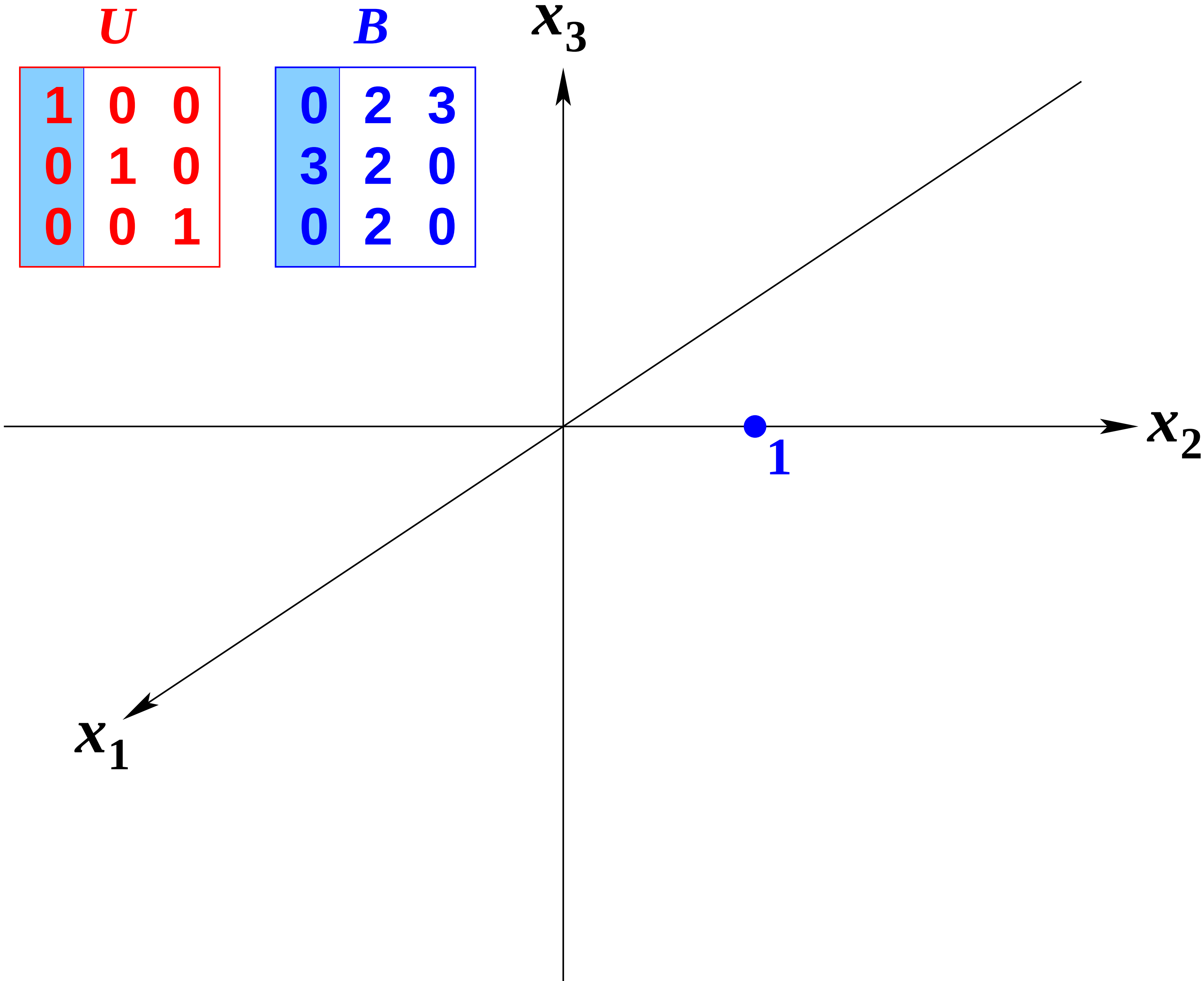
with label 1 if unit vector 1 in U

U

1	0	0
0	1	0
0	0	1

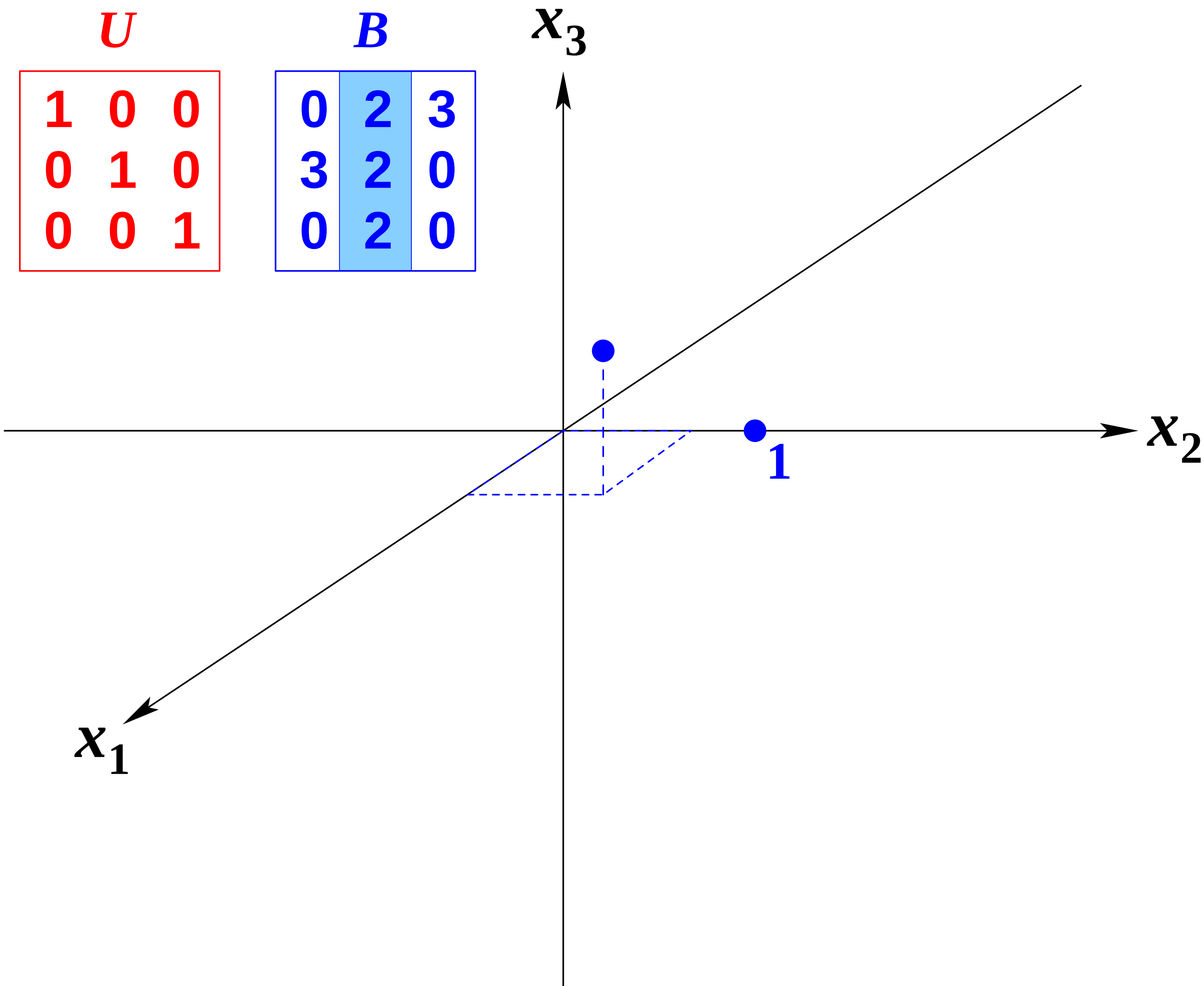
B

0	2	3
3	2	0
0	2	0



second column of B

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 3 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$



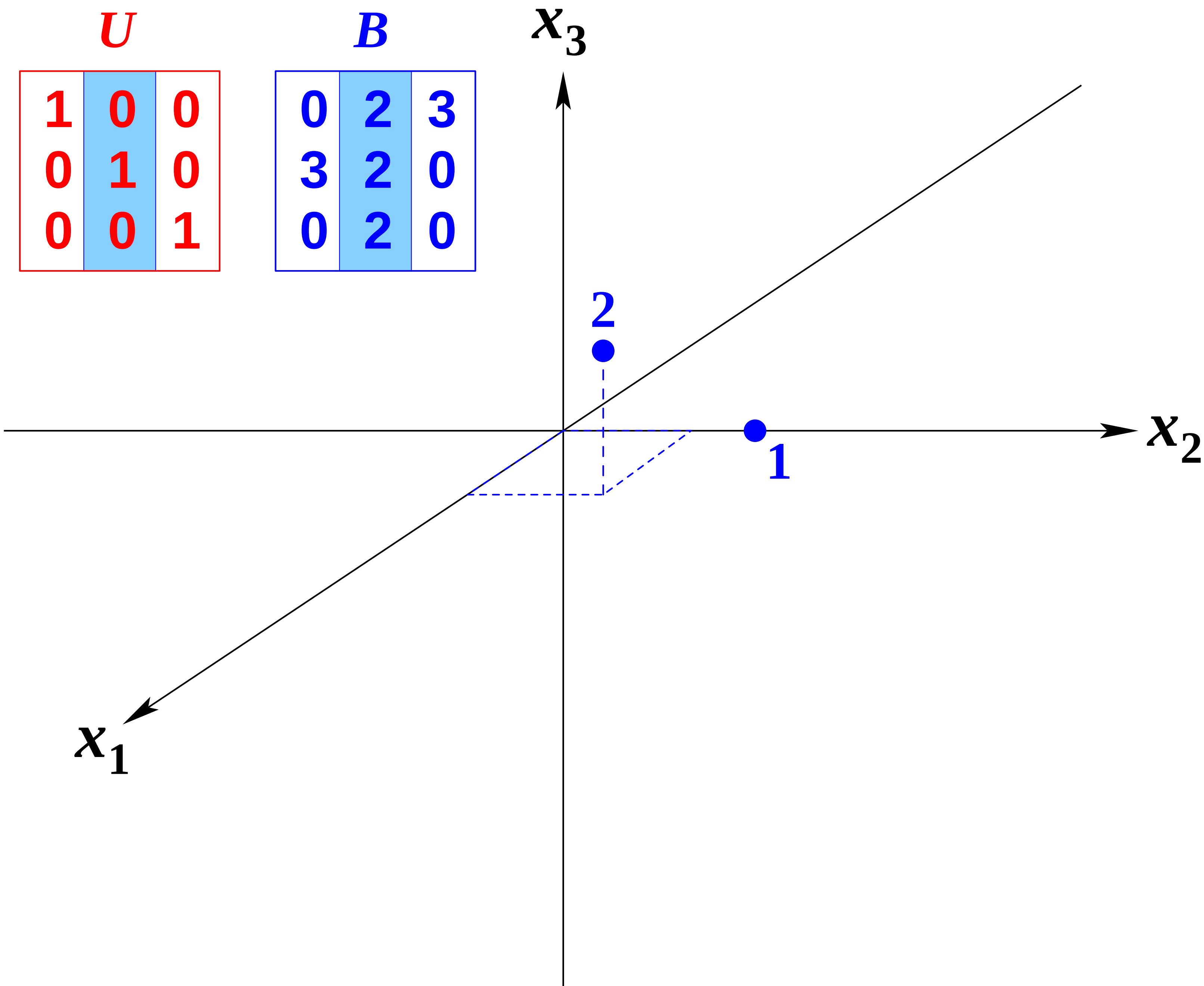
with label 2 if unit vector 2 in U

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



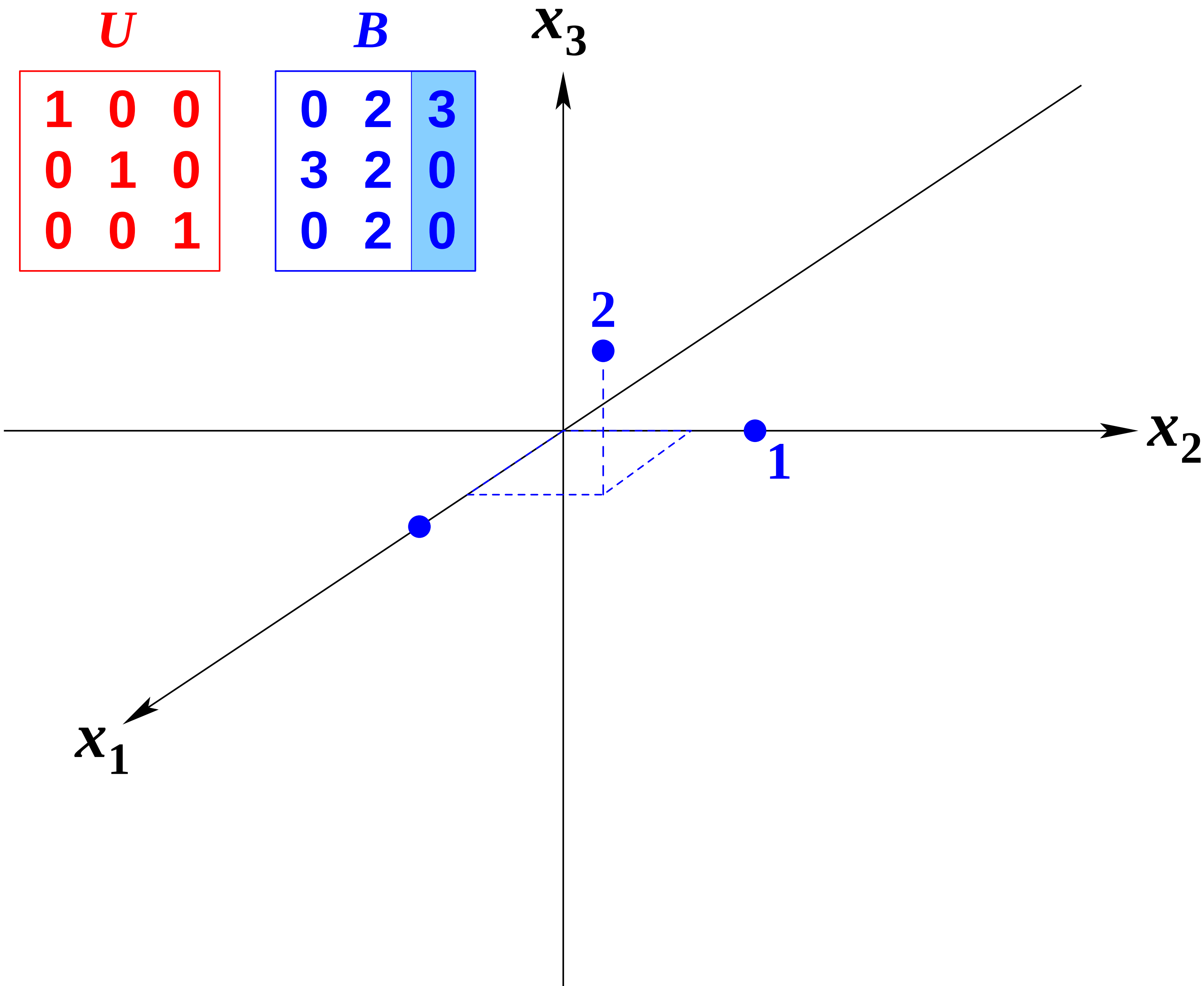
third column of B

U

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

B

$$\begin{pmatrix} 0 & 2 & 3 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$



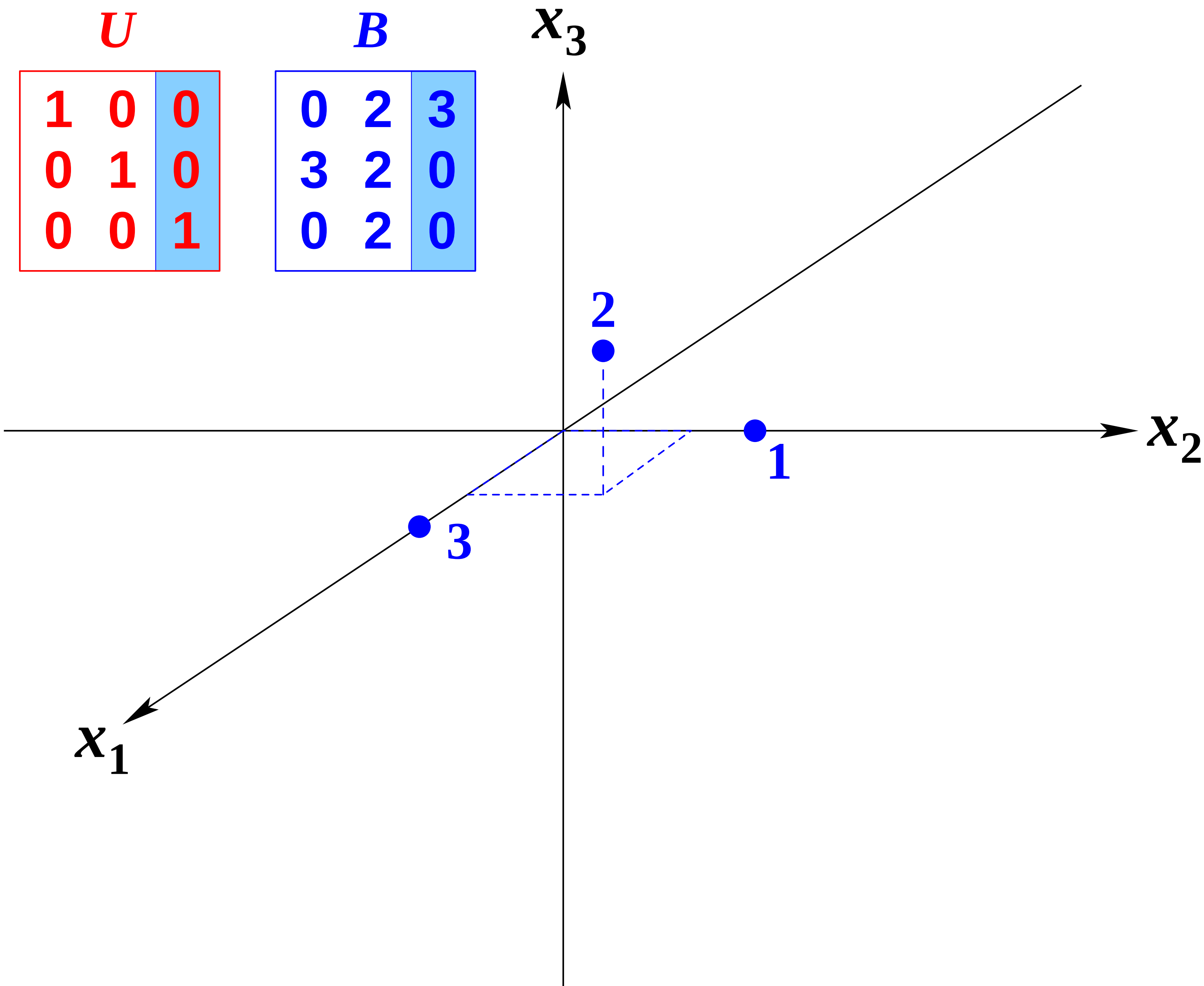
with label 3 if unit vector 3 in U

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



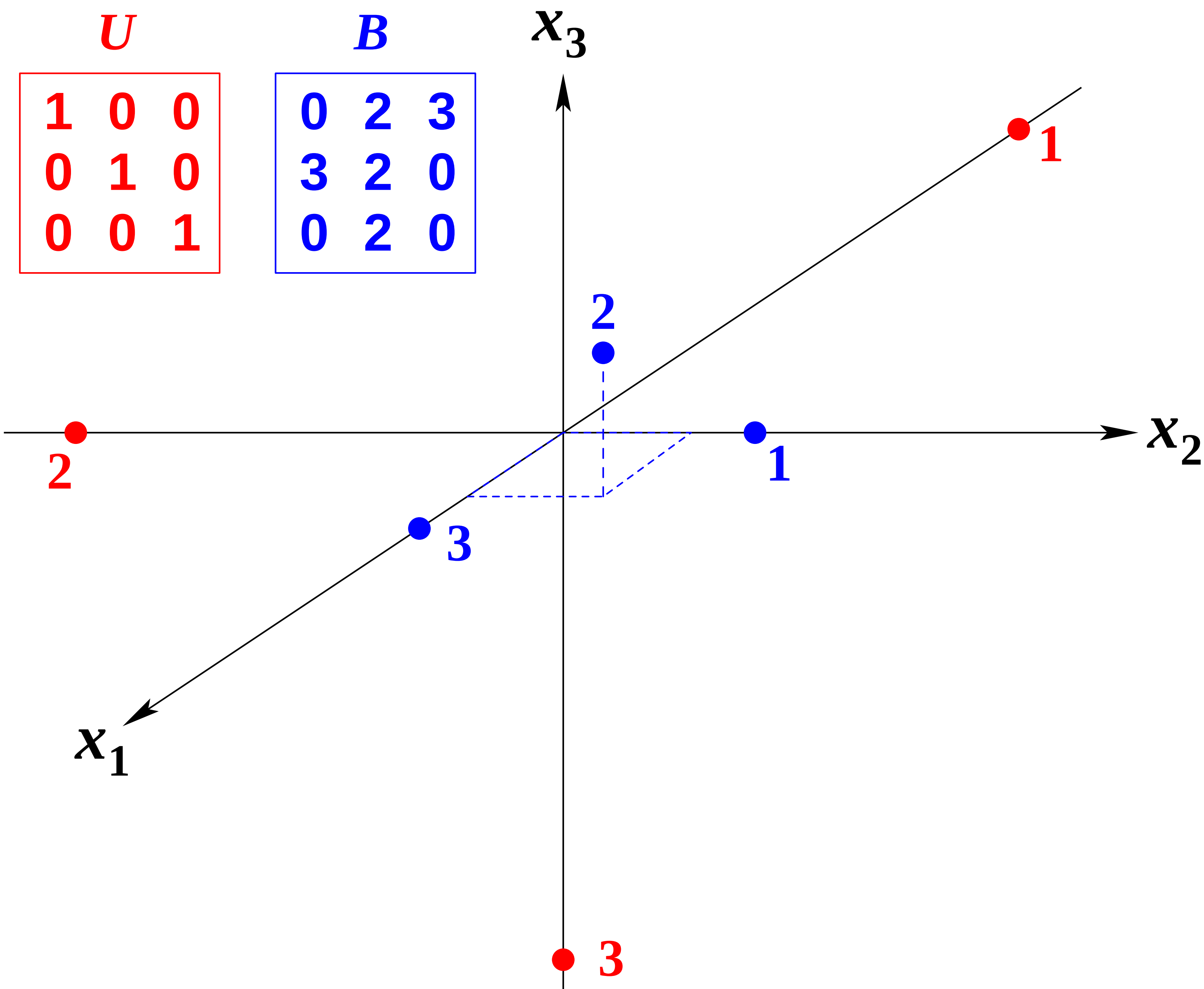
add negative unit vectors

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



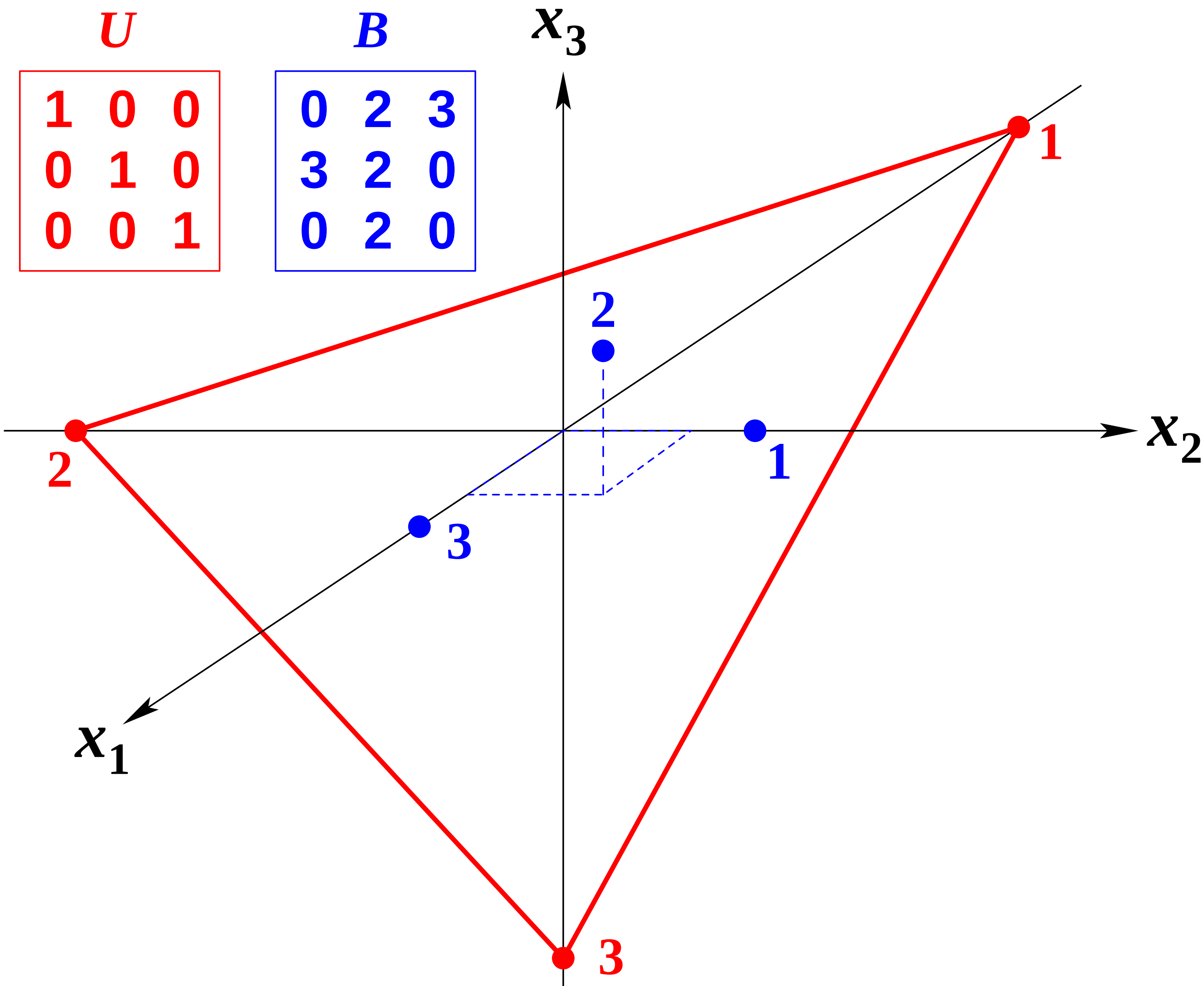
with trivial panchromatic facet

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



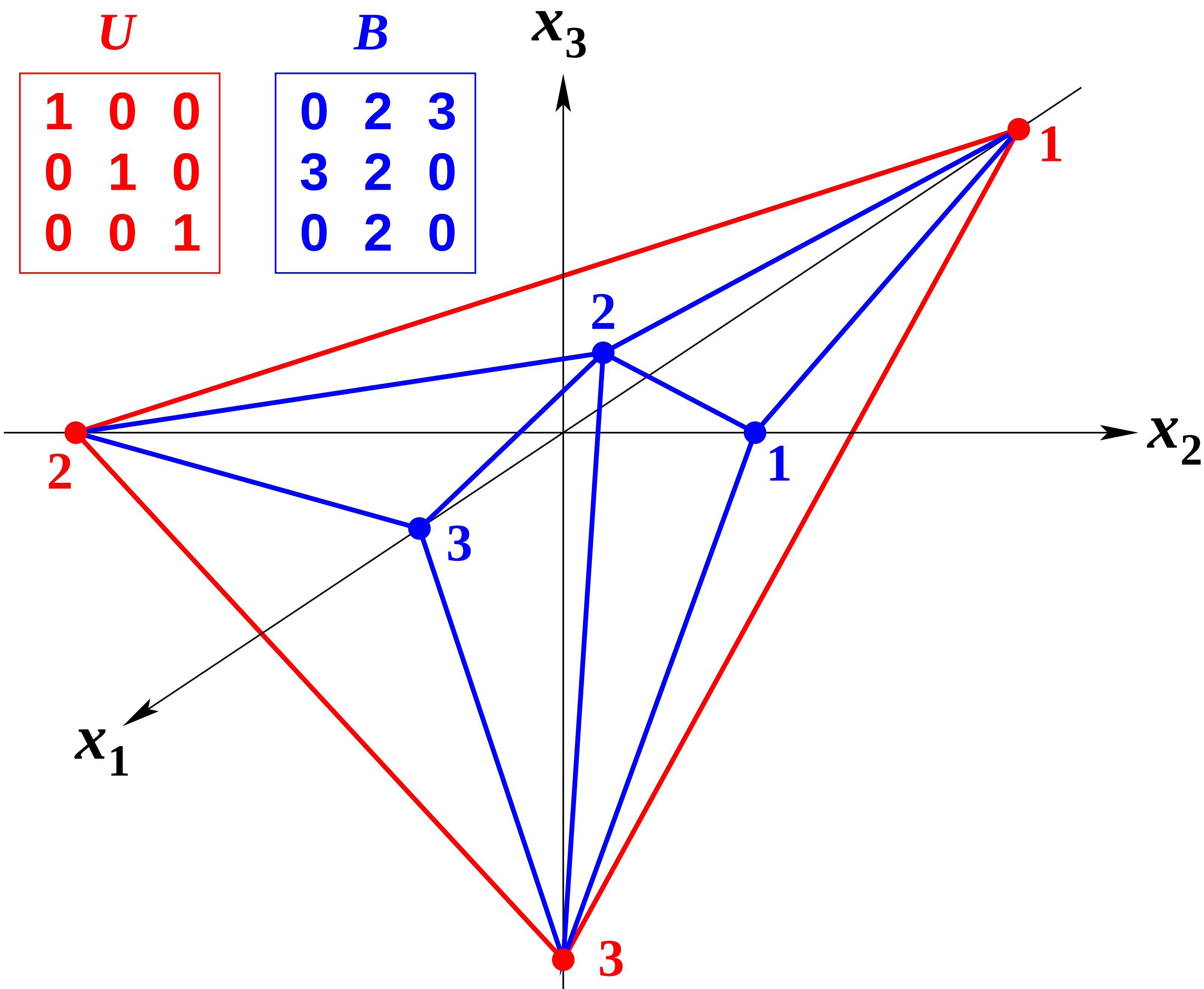
P = convex hull

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



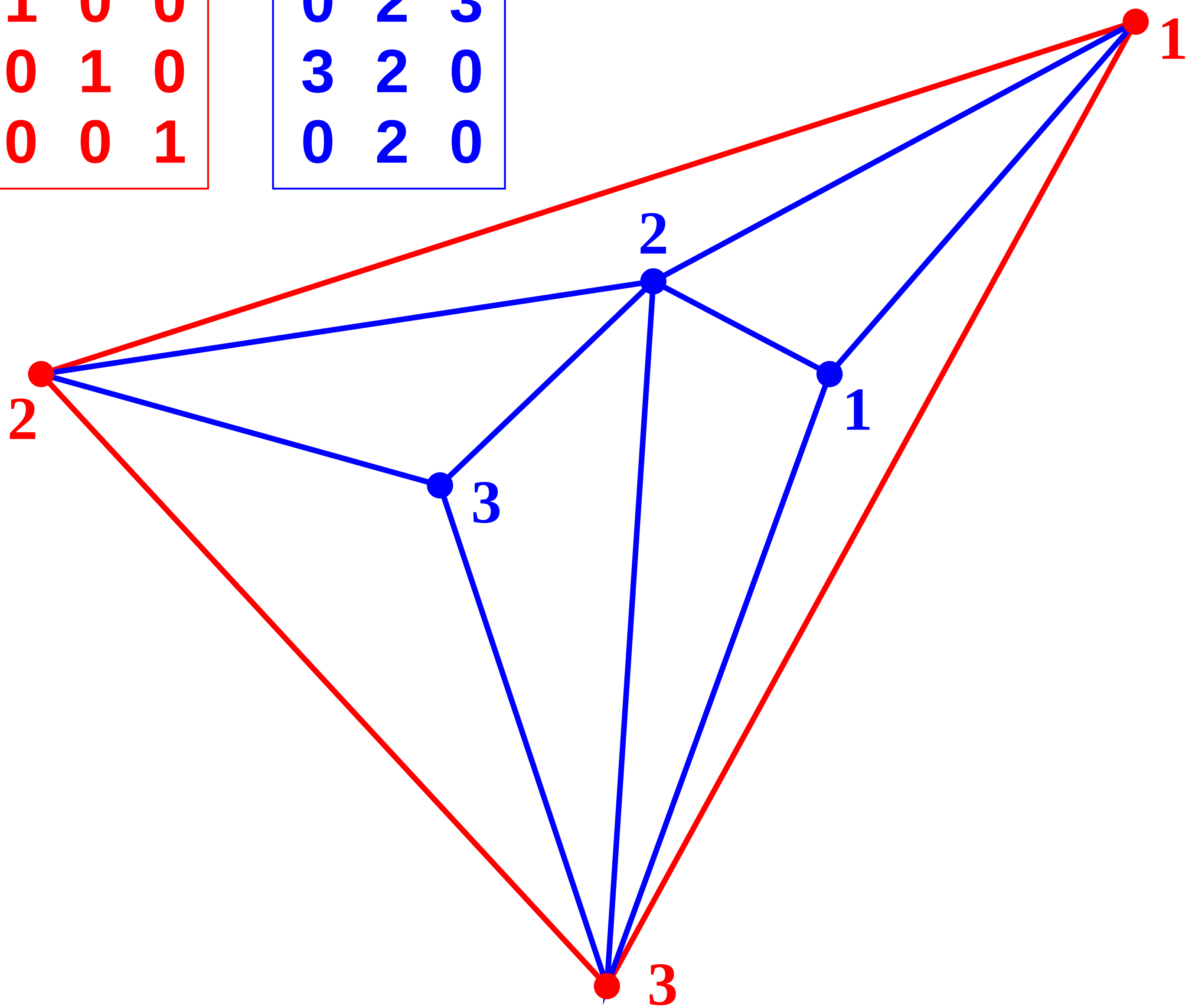
P = convex hull

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



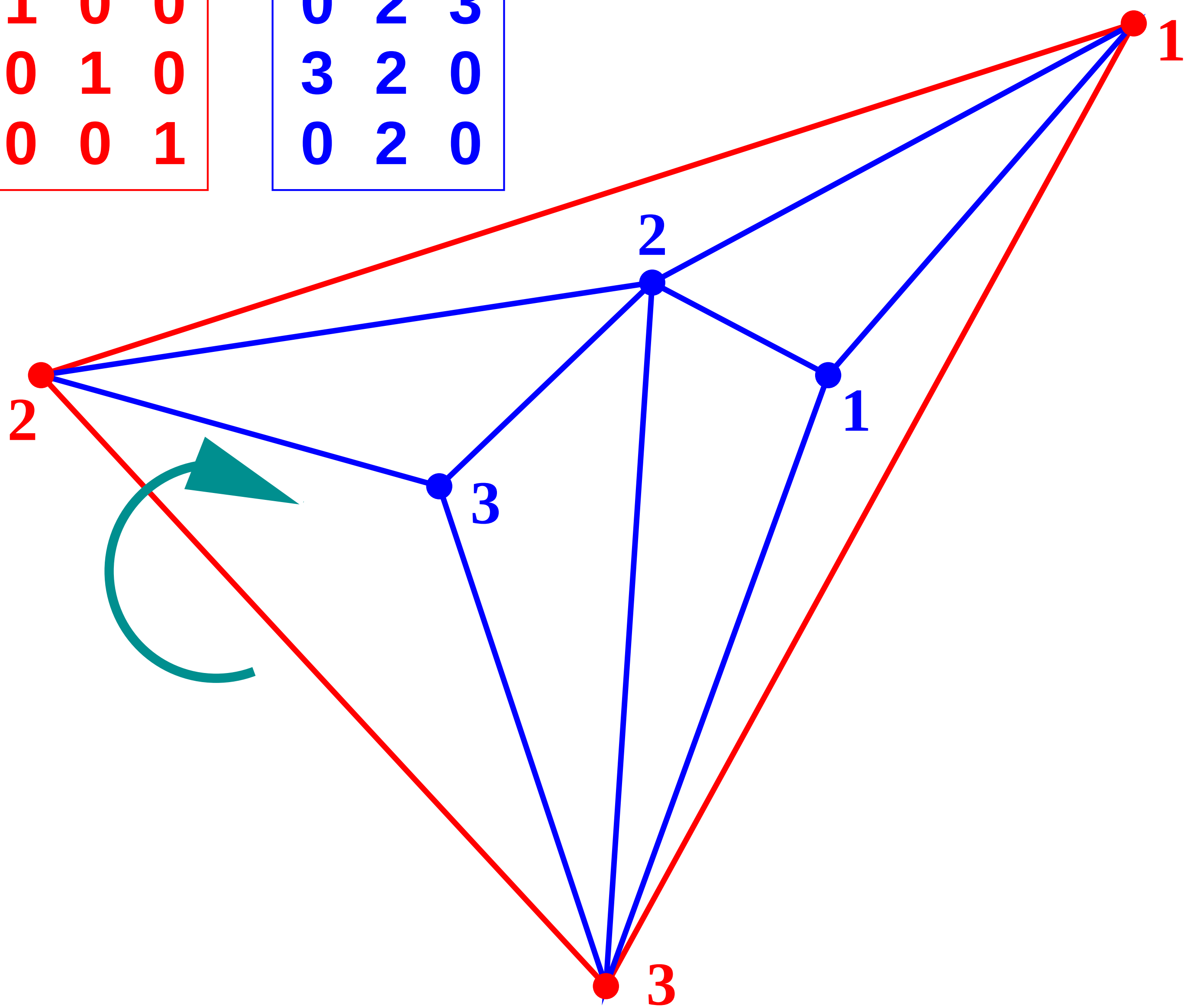
Start path at trivial panchromatic facet

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



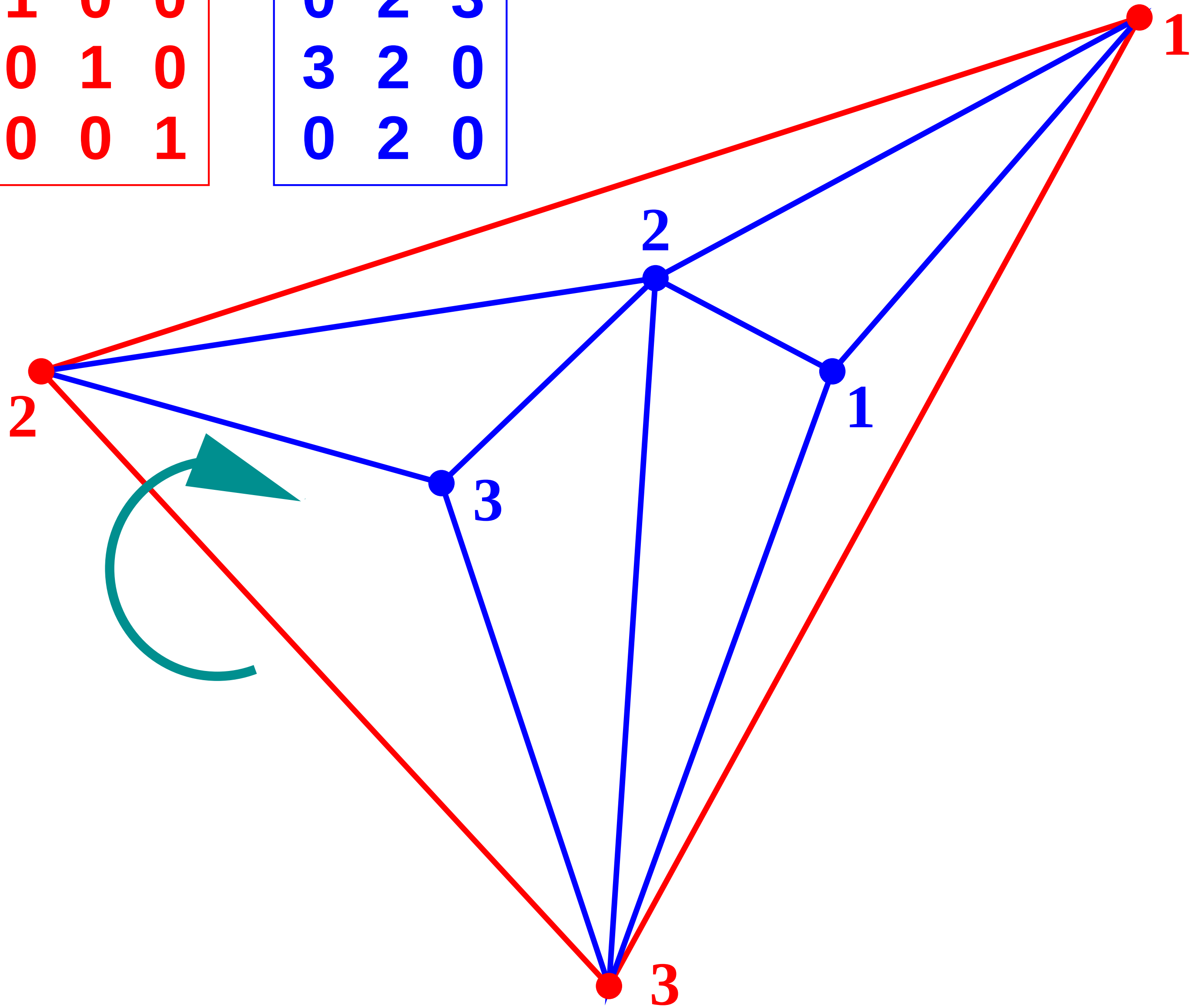
by dropping color 1

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



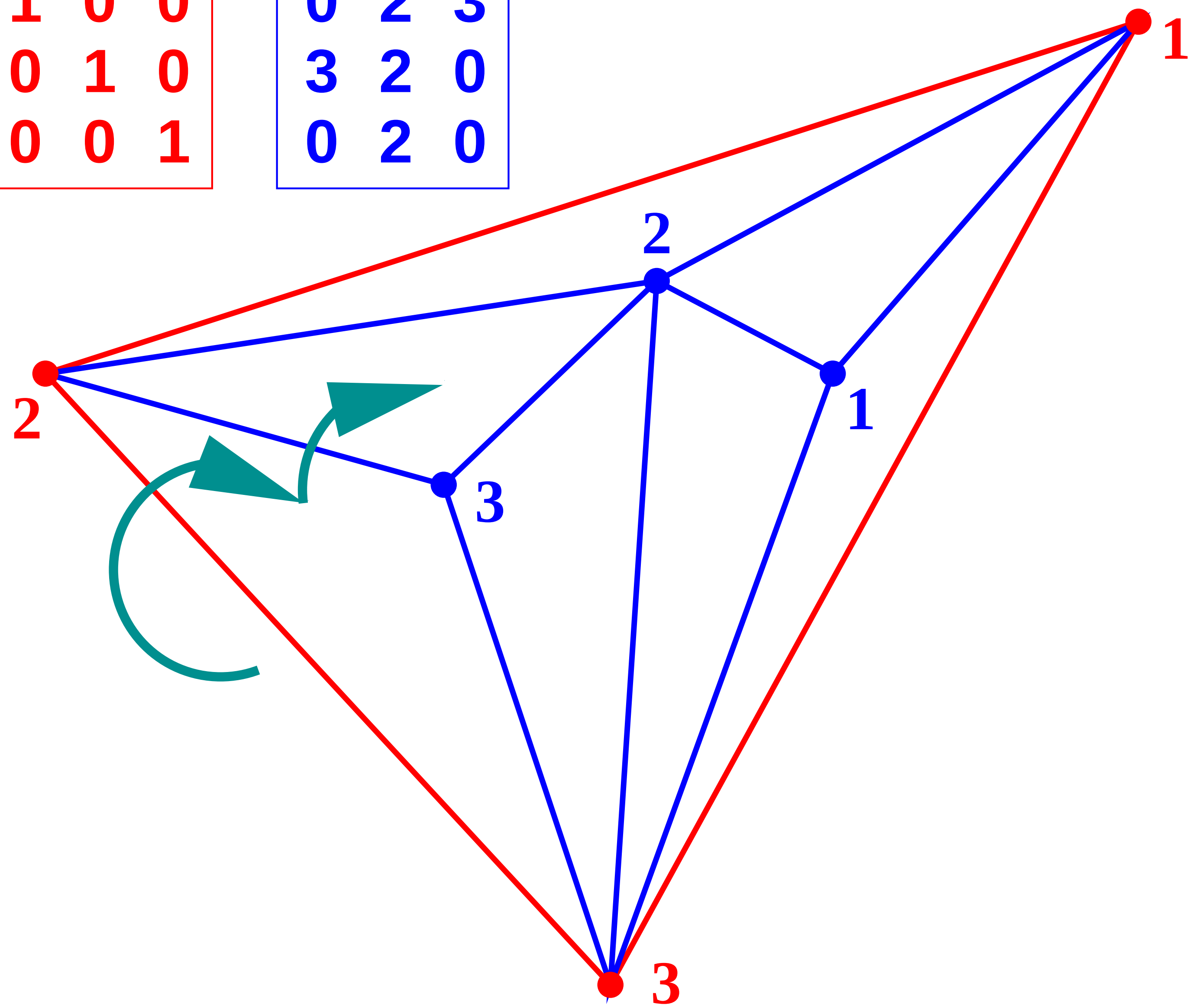
and then dropping duplicate color

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



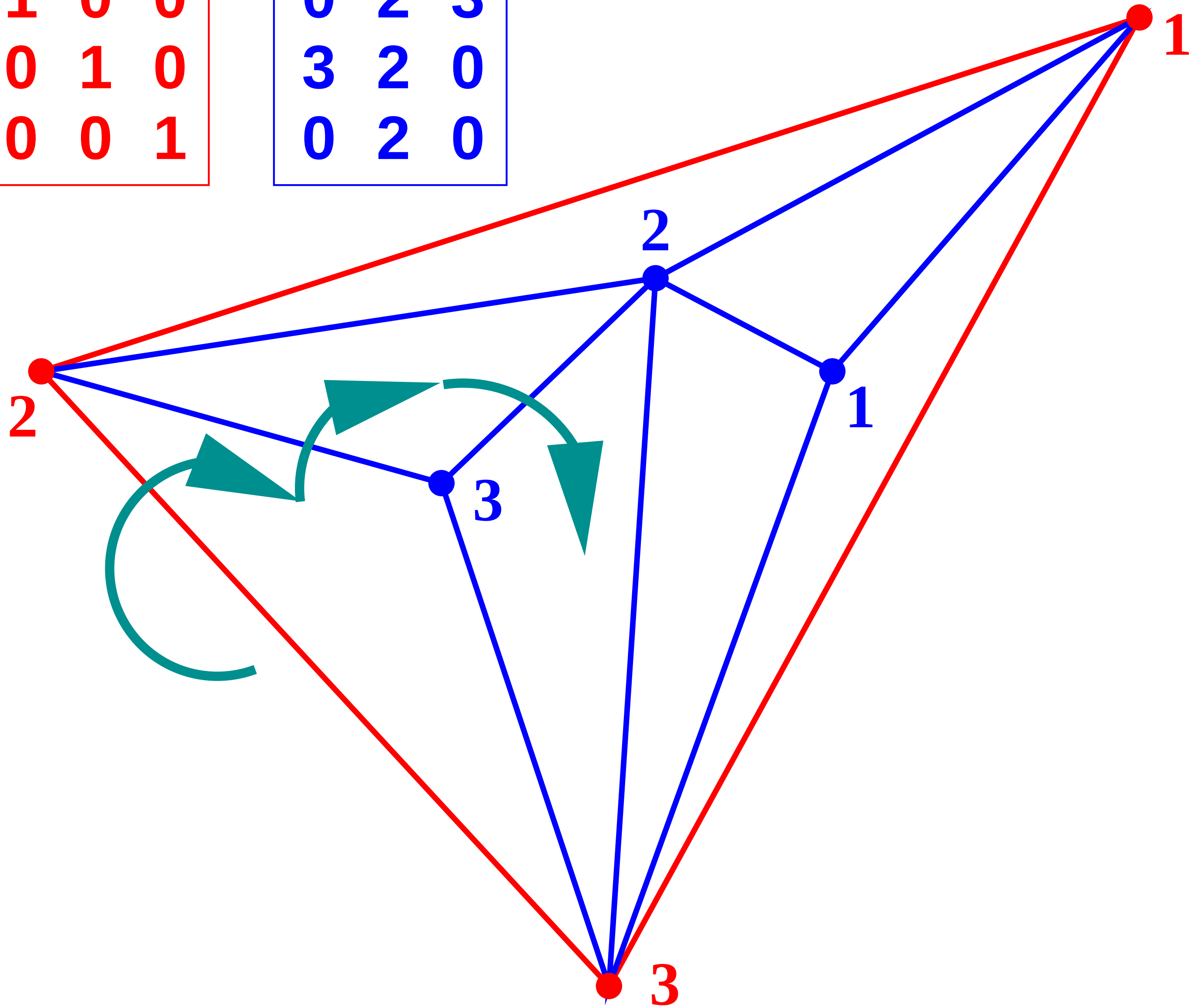
and then dropping duplicate color

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



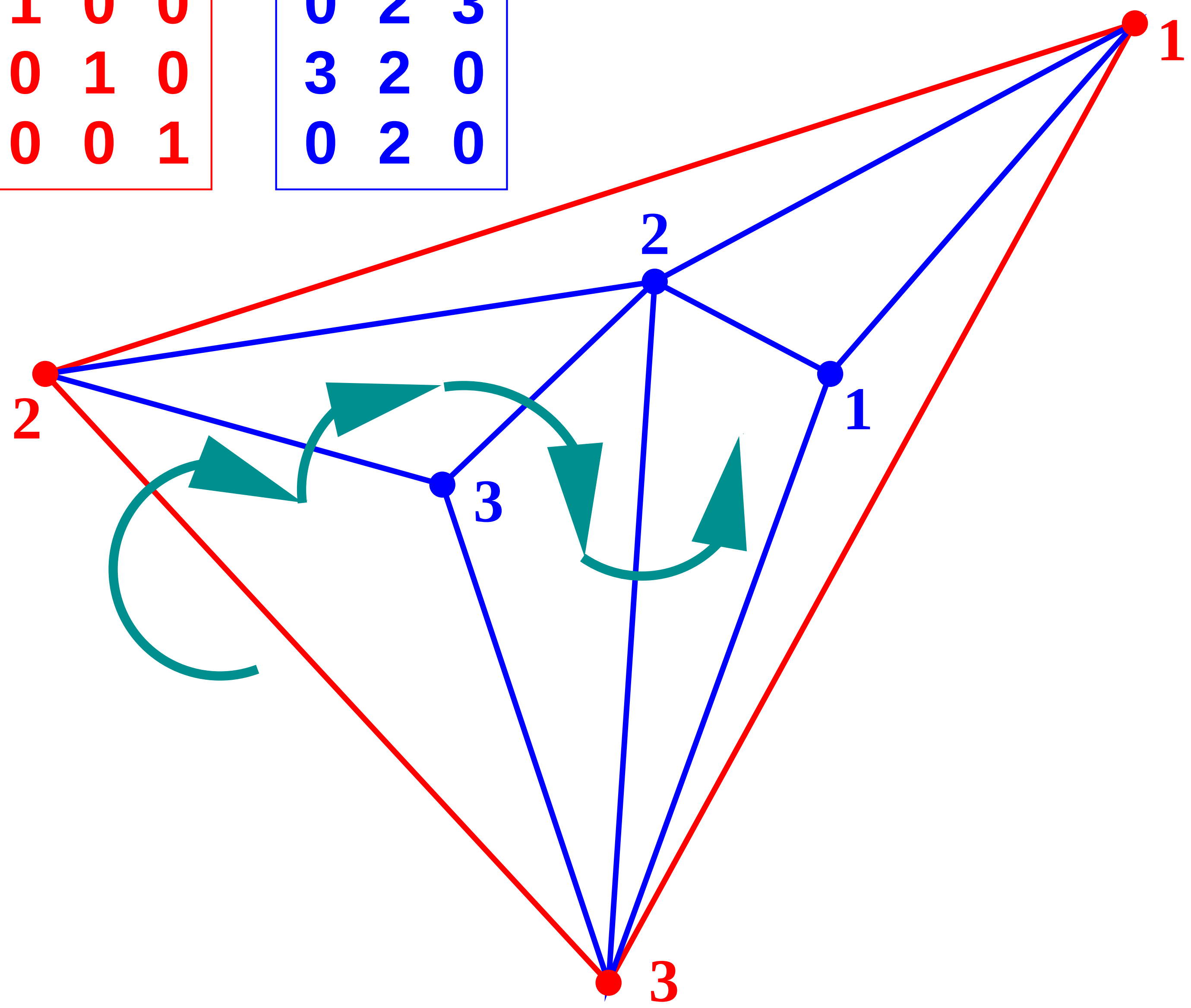
until missing color 1 is found

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



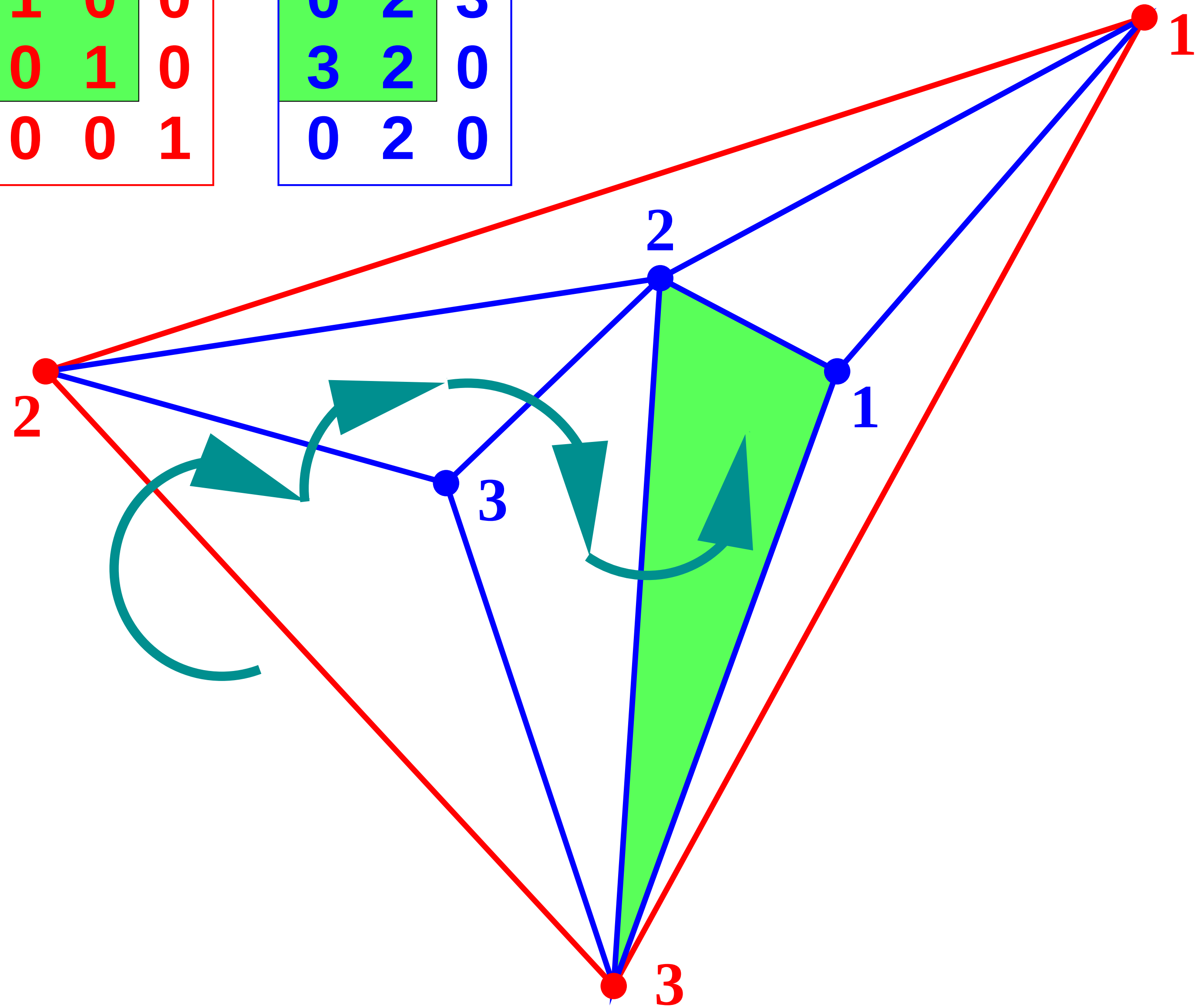
arriving at equilibrium

U

1	0	0
0	1	0
0	0	1

B

0	2	3
3	2	0
0	2	0



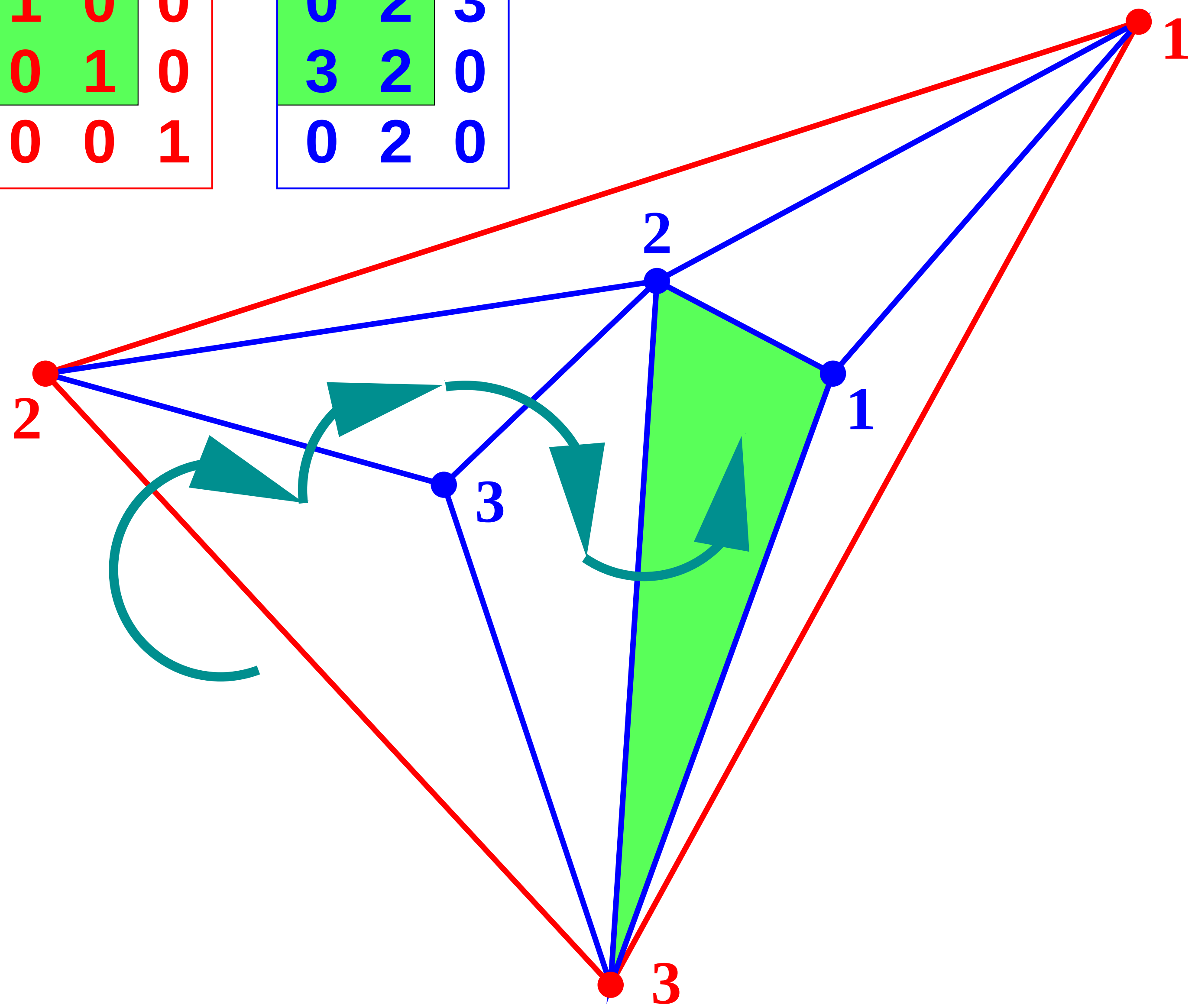
This is the Lemke-Howson algorithm!

U

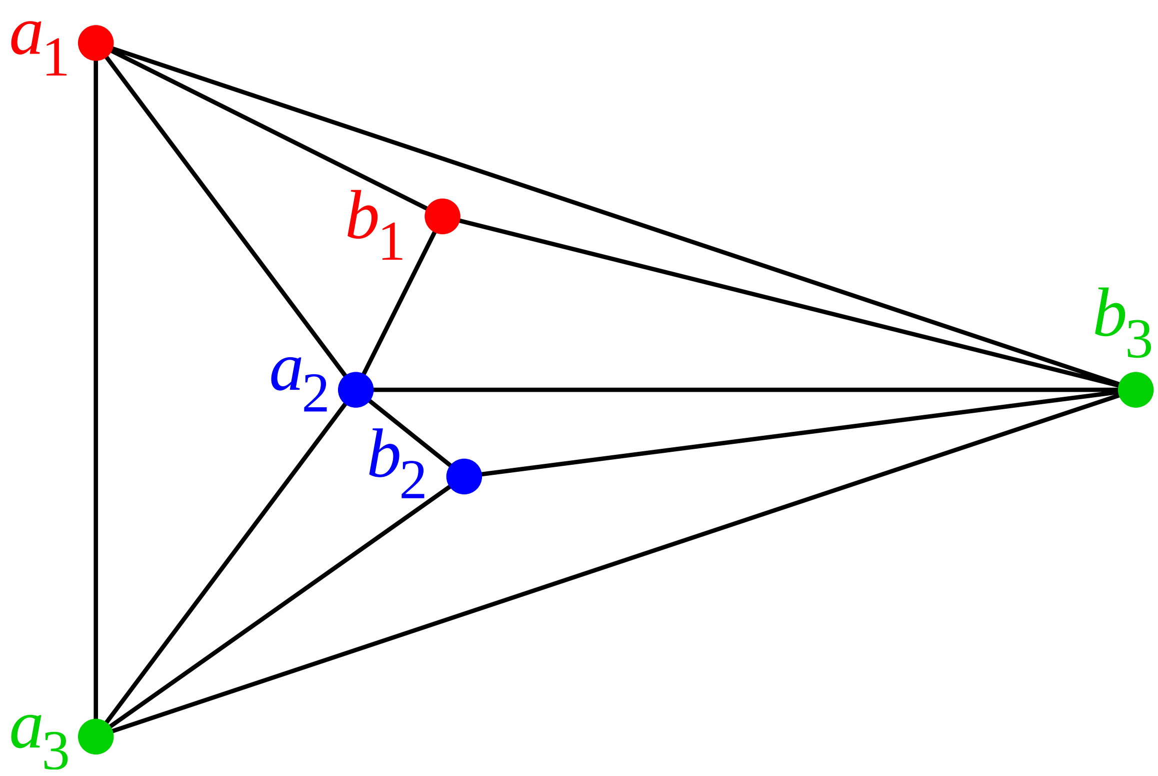
1	0	0
0	1	0
0	0	1

B

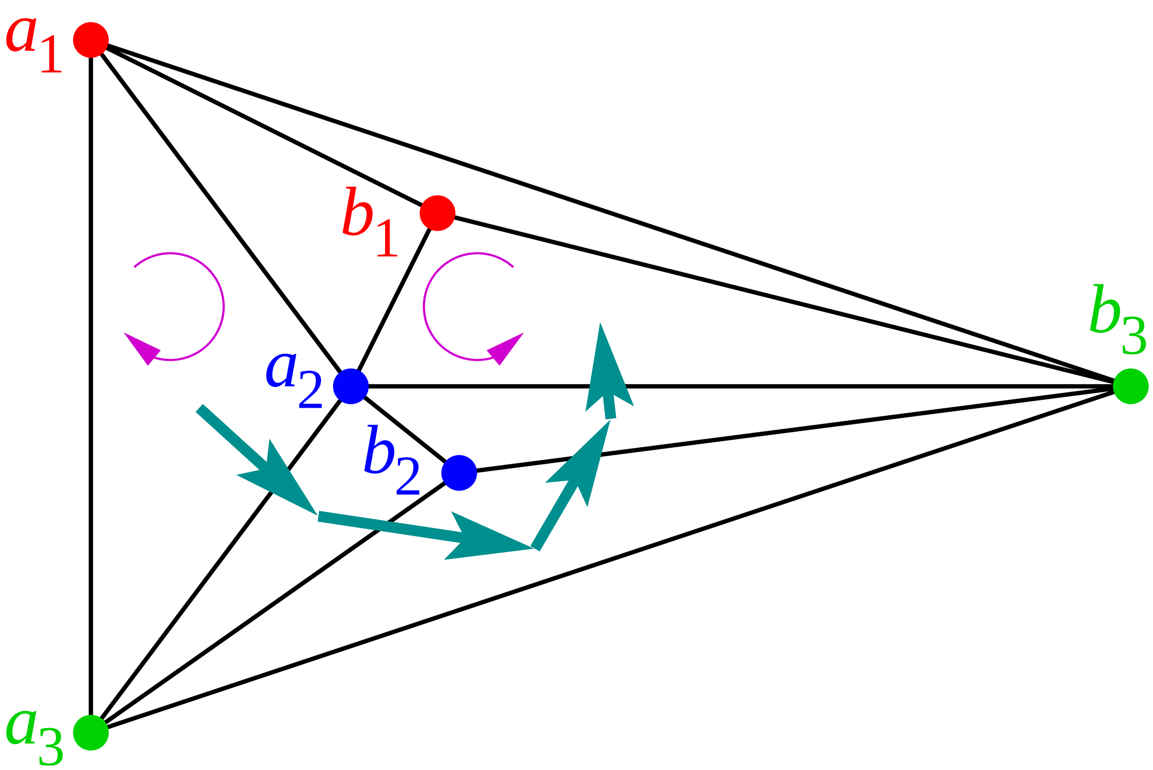
0	2	3
3	2	0
0	2	0



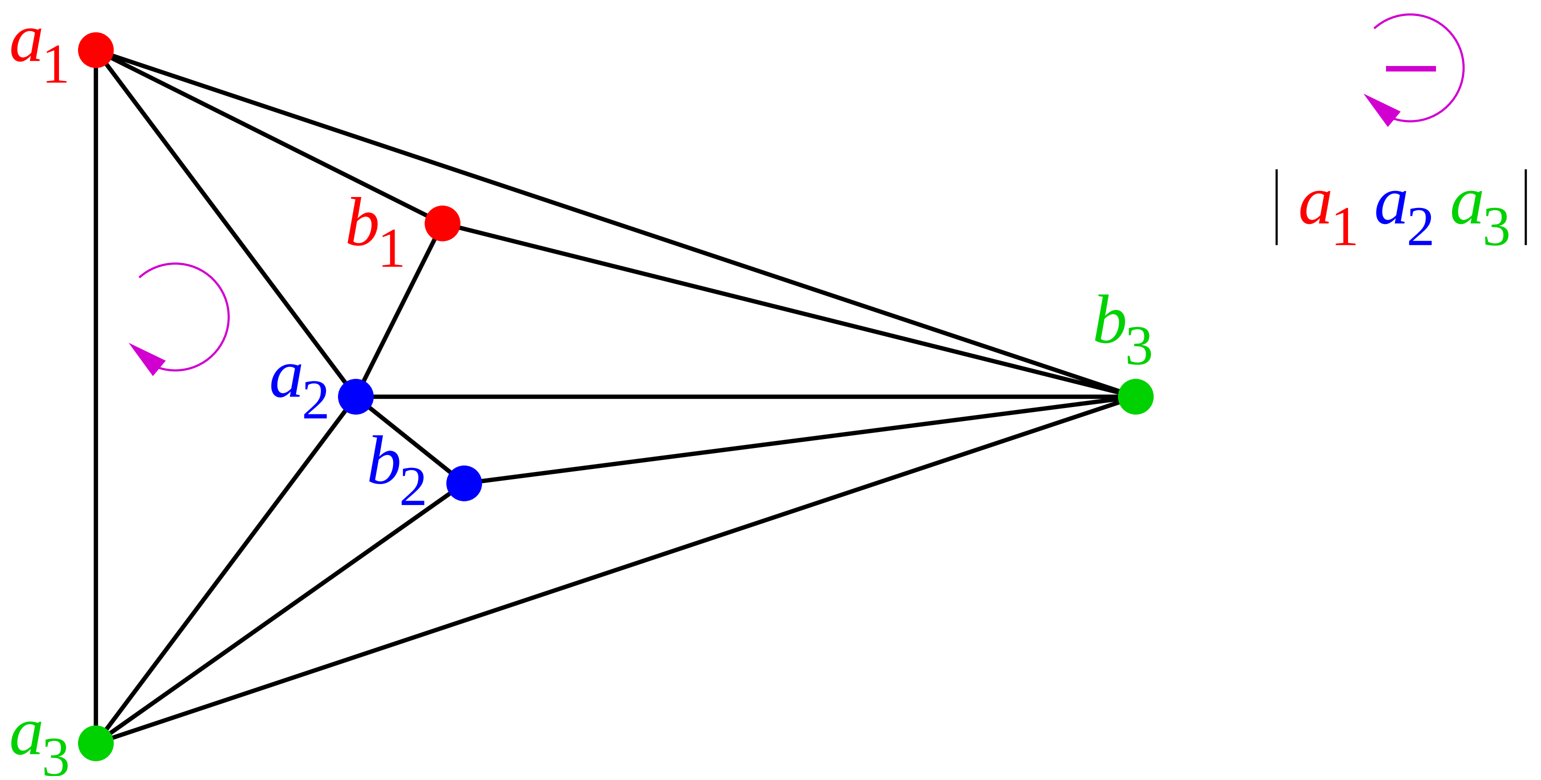
Abstract Sperner



PPAD = opposite orientation of end-rooms



rooms = facets of simplicial polytope

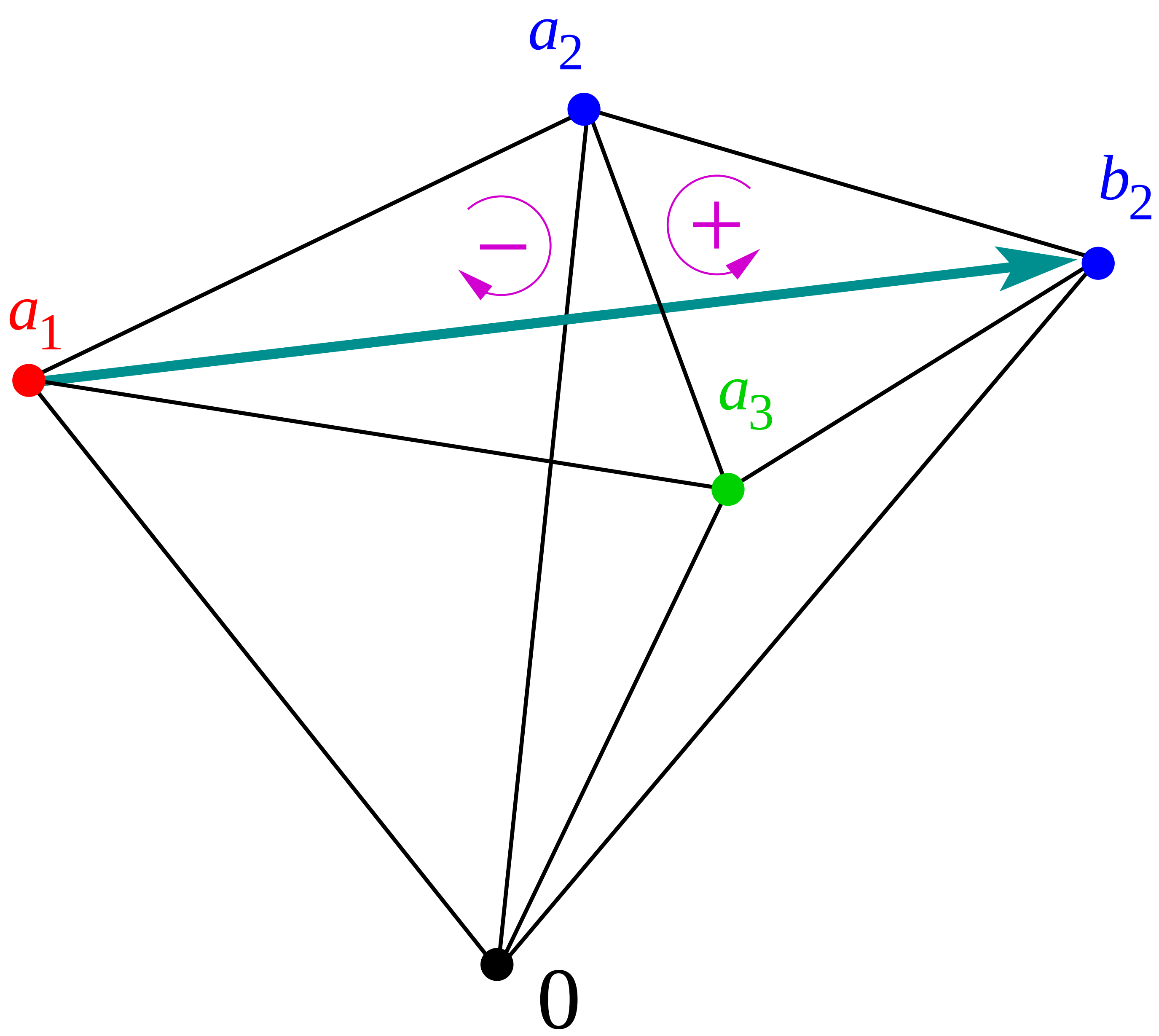


. . . with 0 in interior,

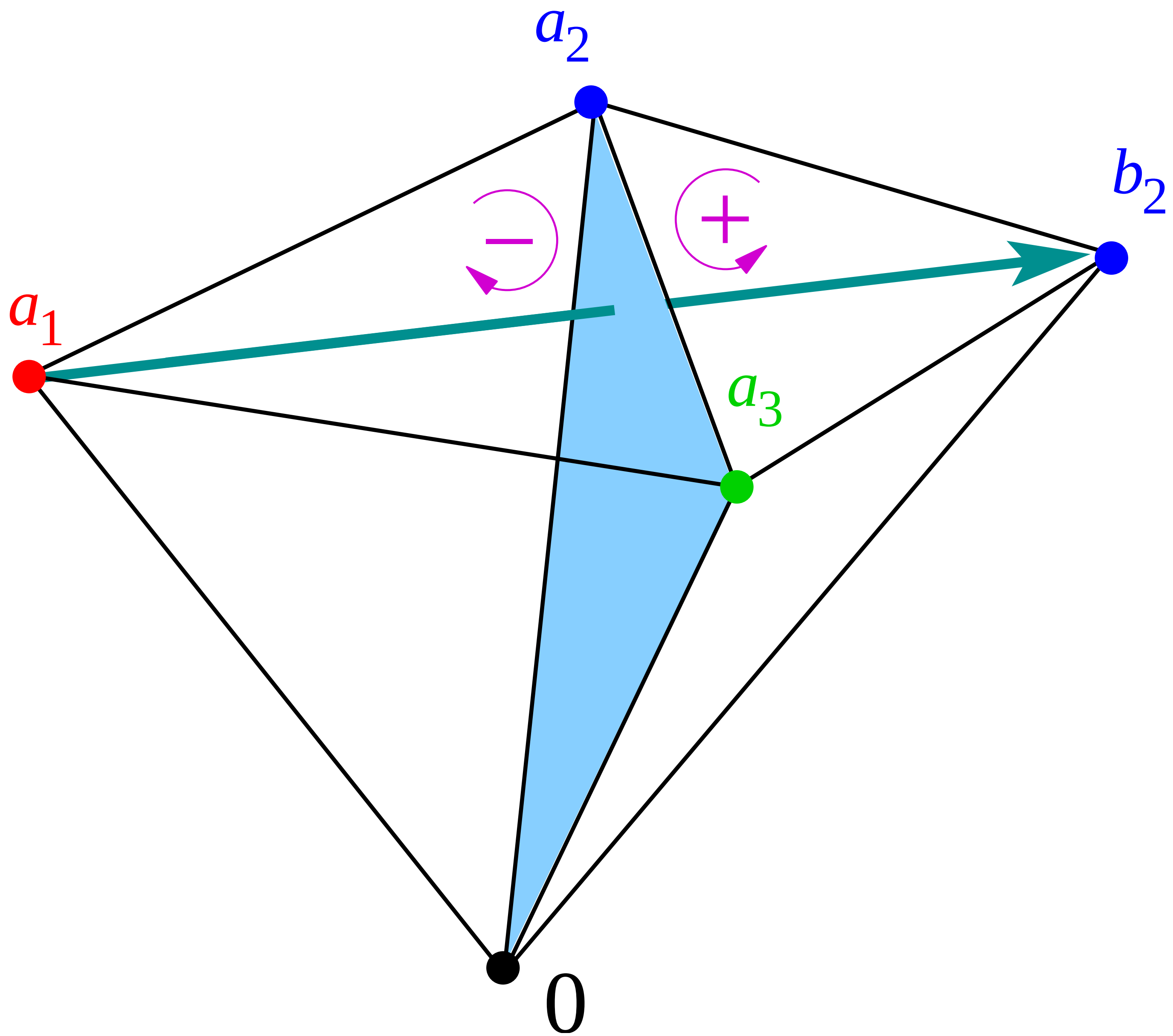
vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

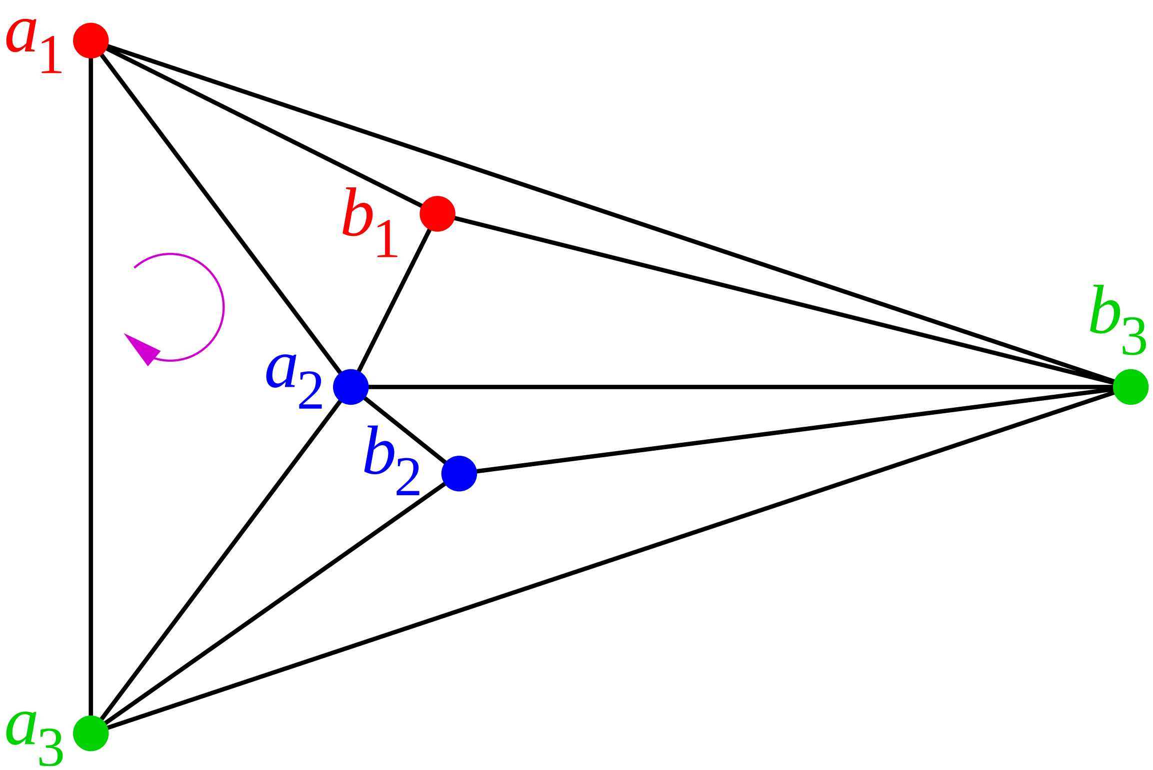
Pivoting changes sign of determinant



Pivoting changes sign of determinant



rooms = facets of simplicial polytope



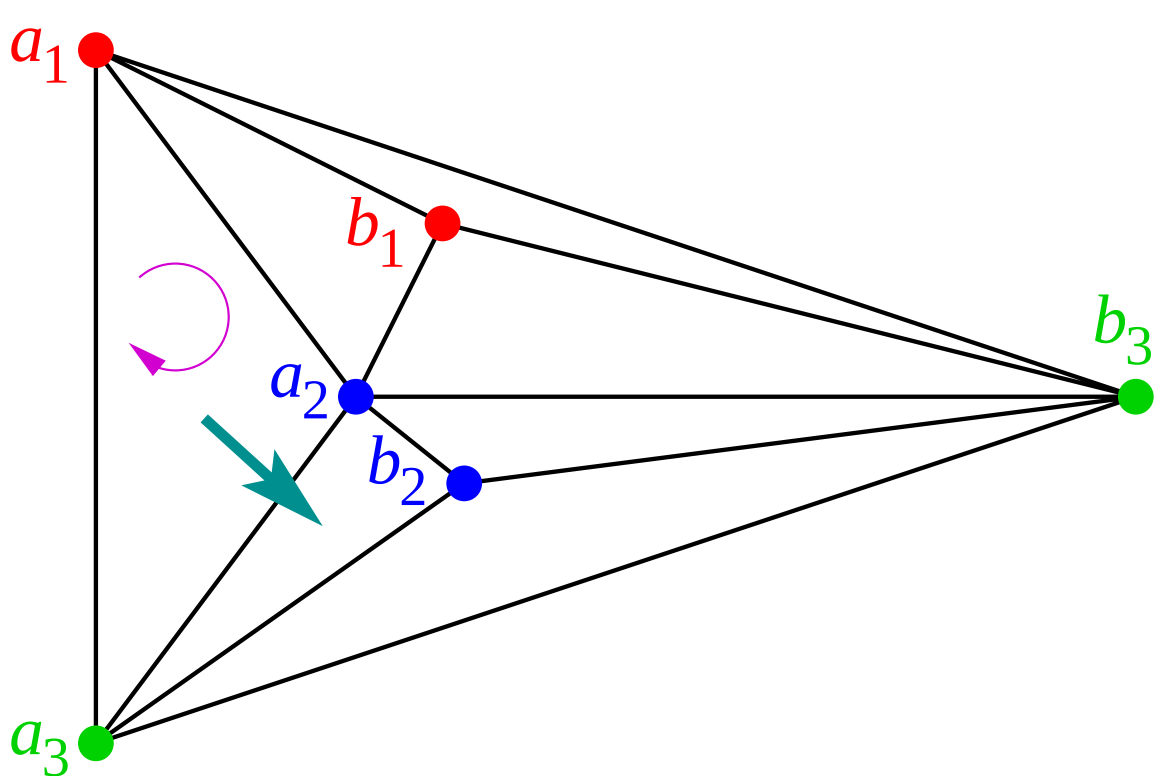
$$\left| a_1 \; a_2 \; a_3 \right|$$

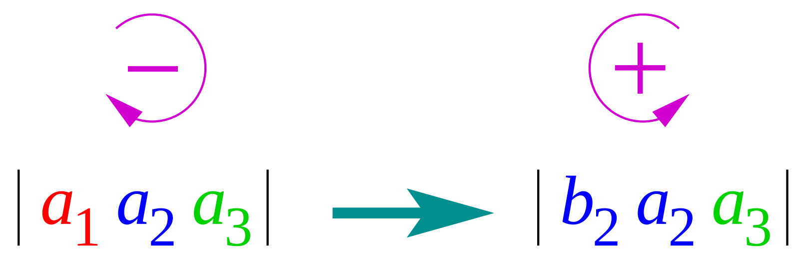
. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope



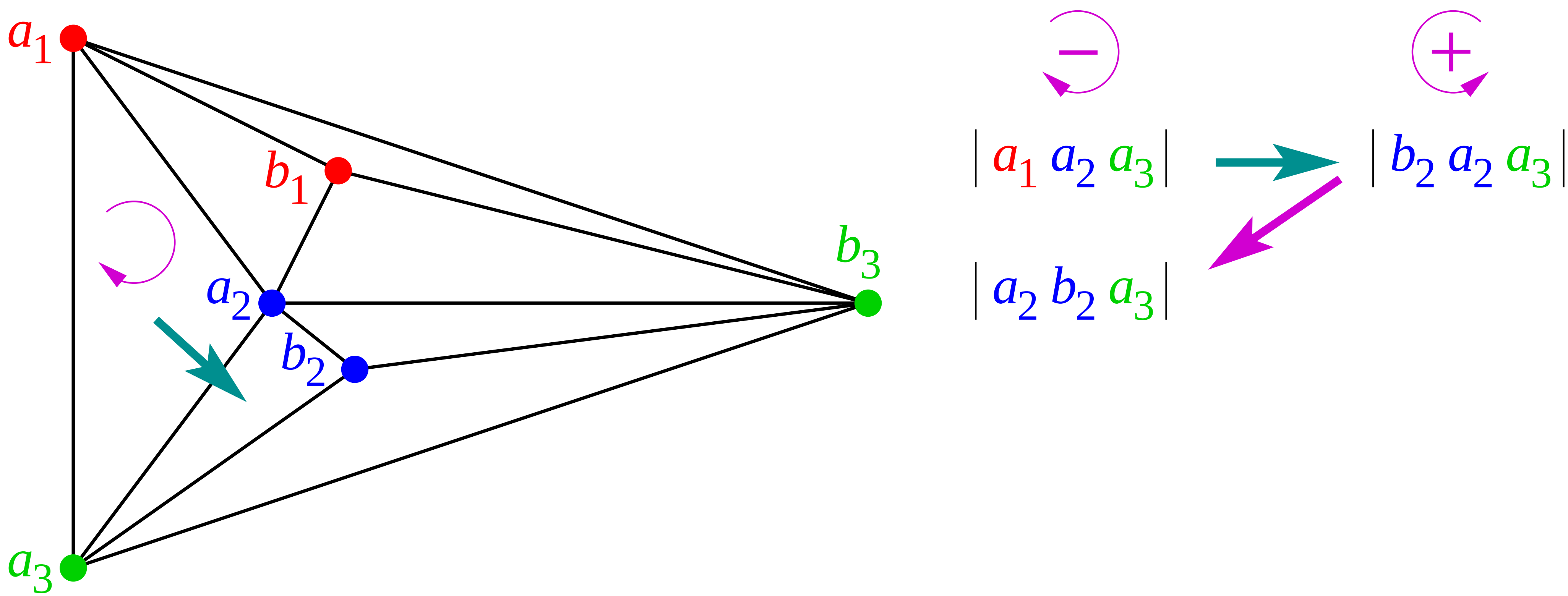


. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope

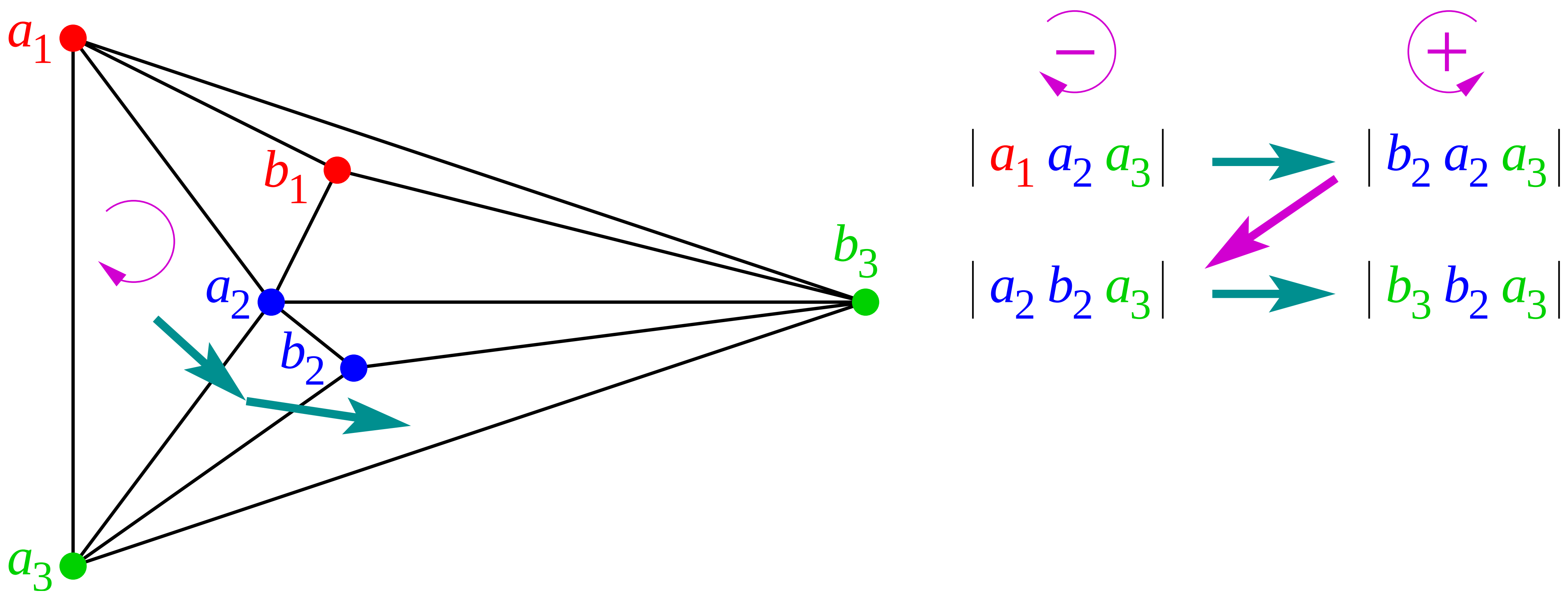


. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope

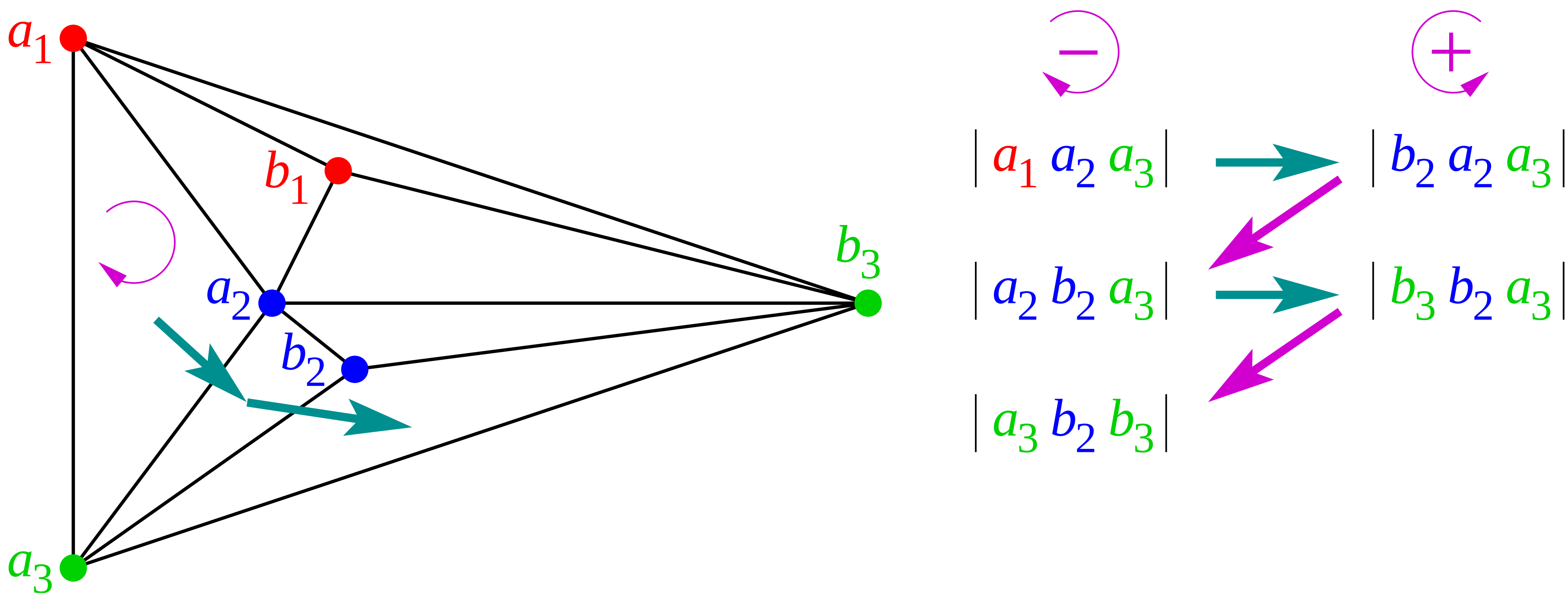


. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope

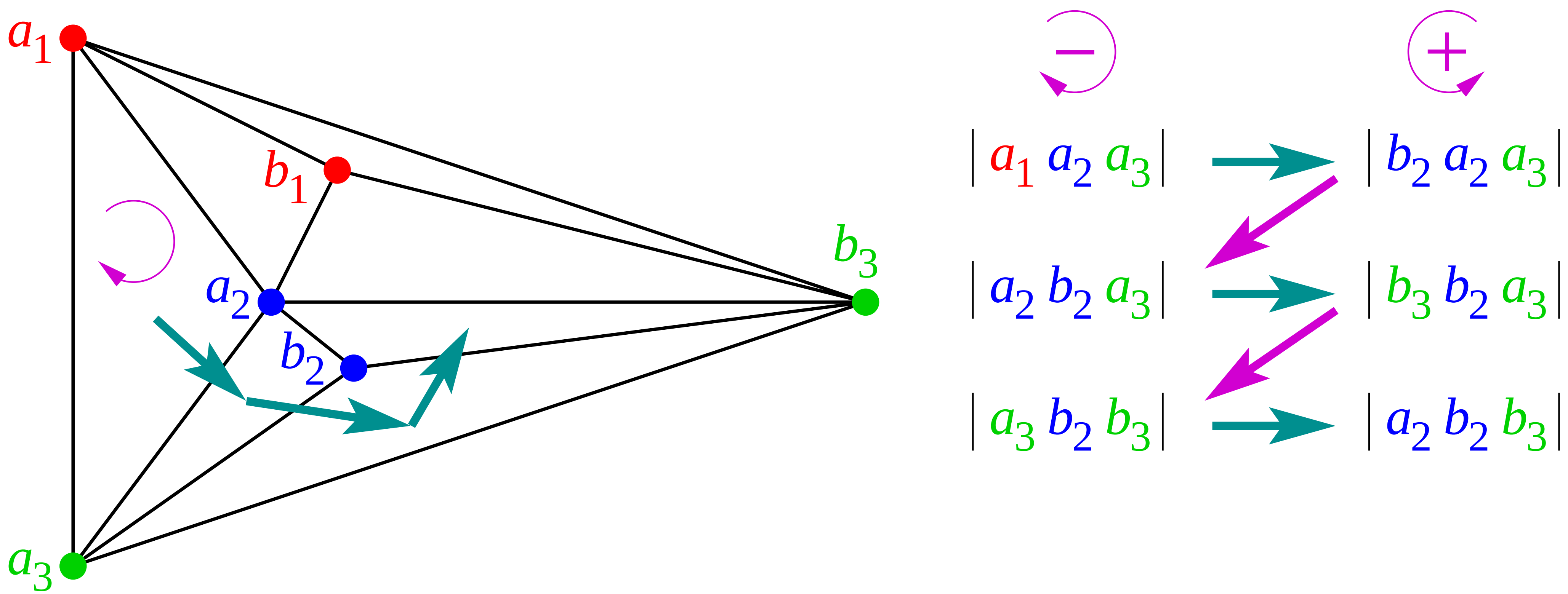


. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope

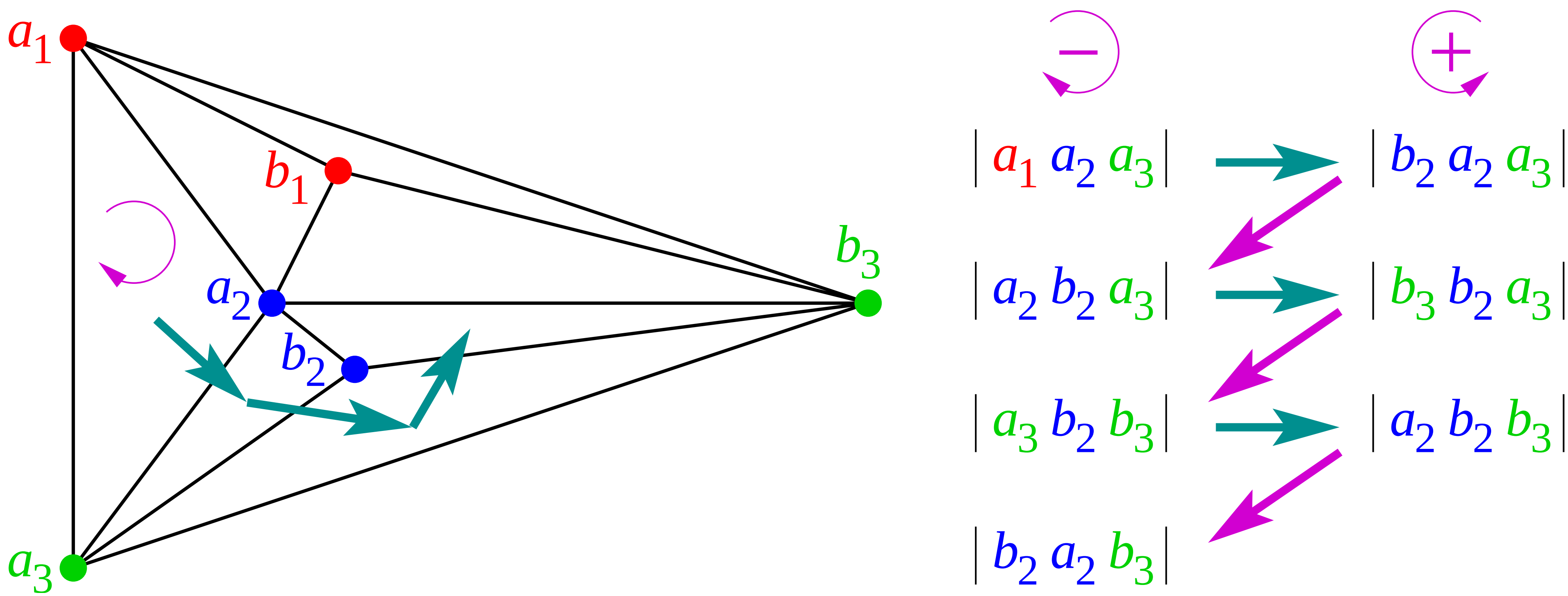


. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope

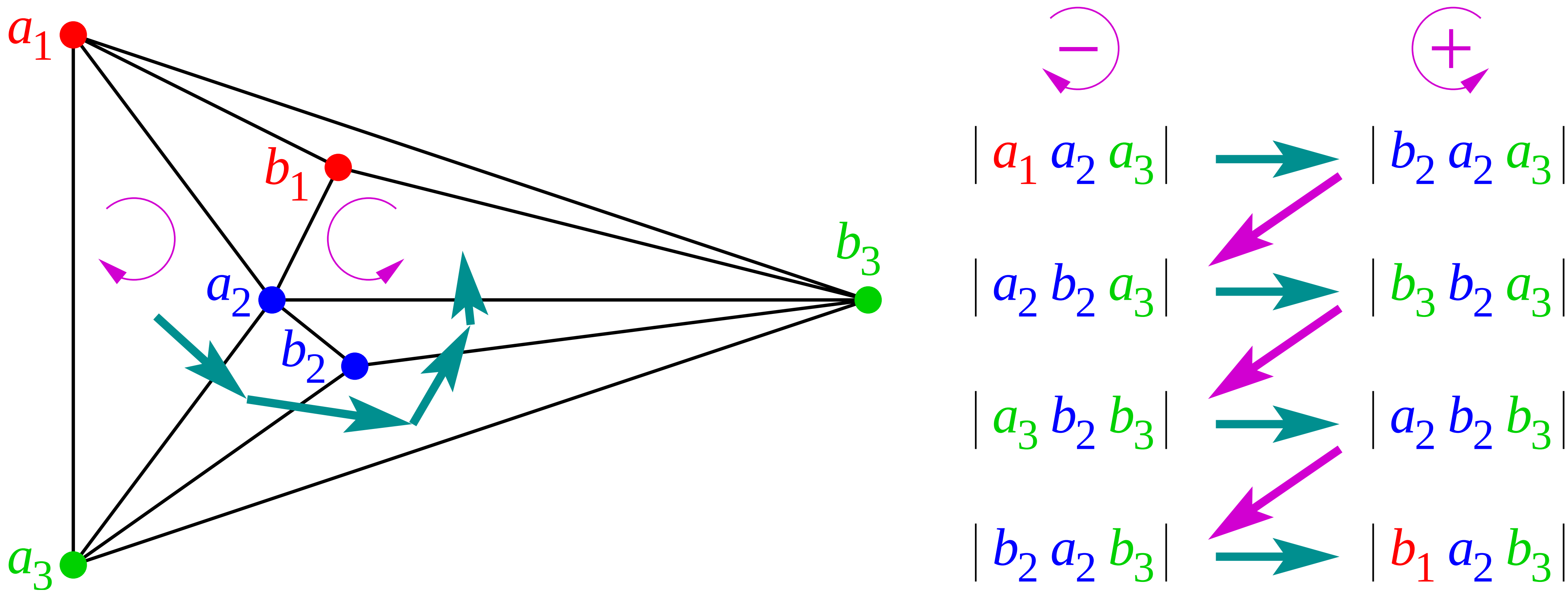


. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

rooms = facets of simplicial polytope



. . . with 0 in interior,

vertices = r-vectors, pivot to next room,

orientation = determinant of vertices in order of labels

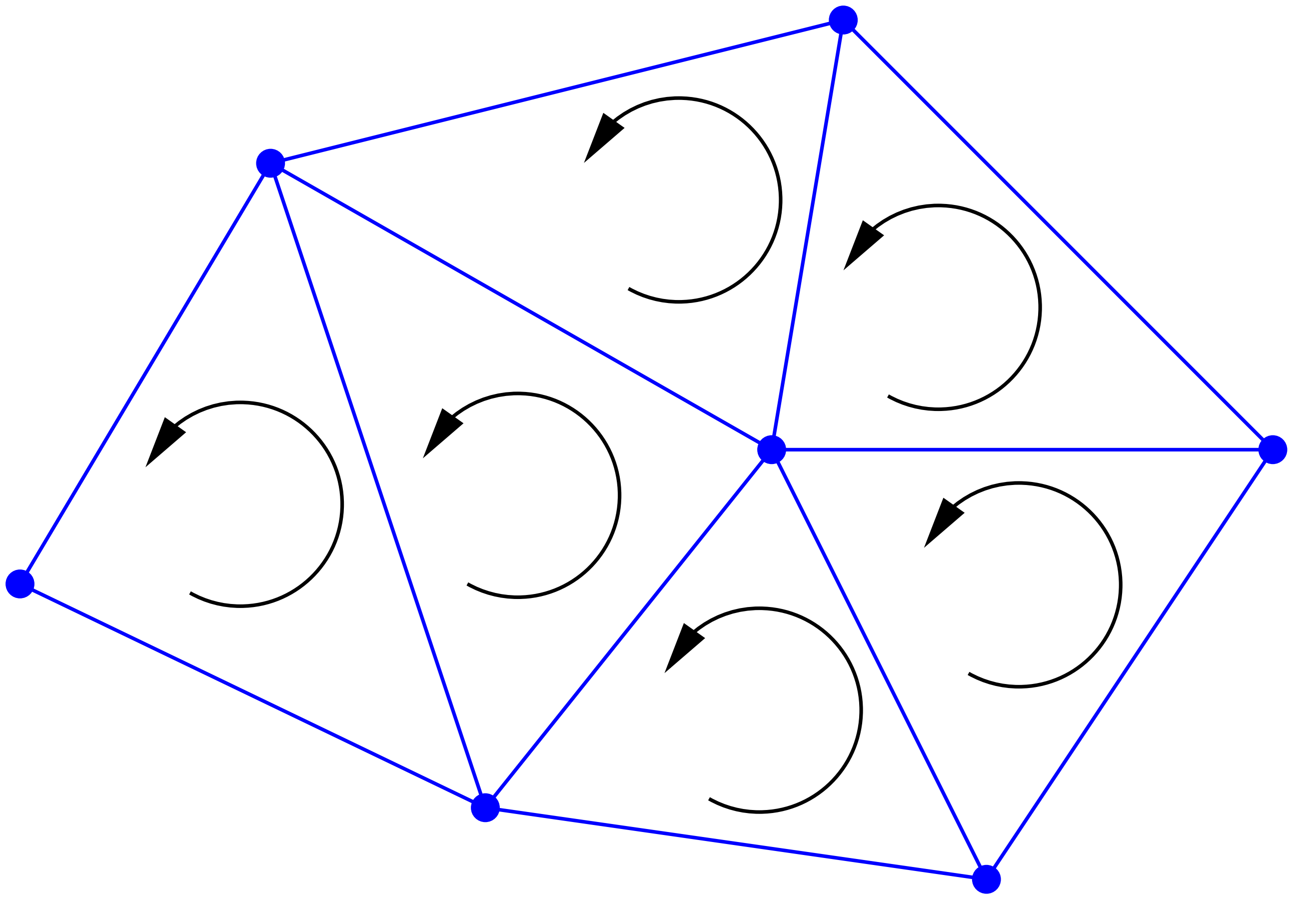
Apply to orientable abstract manifolds

- fix an order of **all** vertices.
- for each room, choose a **sign** (orientation) of the respective order of its vertices
- these signs are **consistent** if the induced orientations on the common wall $\{a_2 \ldots a_r\}$ of two adjacent rooms $\{a_1\ a_2 \ldots a_r\}$, $\{b_1\ a_2 \ldots a_r\}$ are **opposite**:

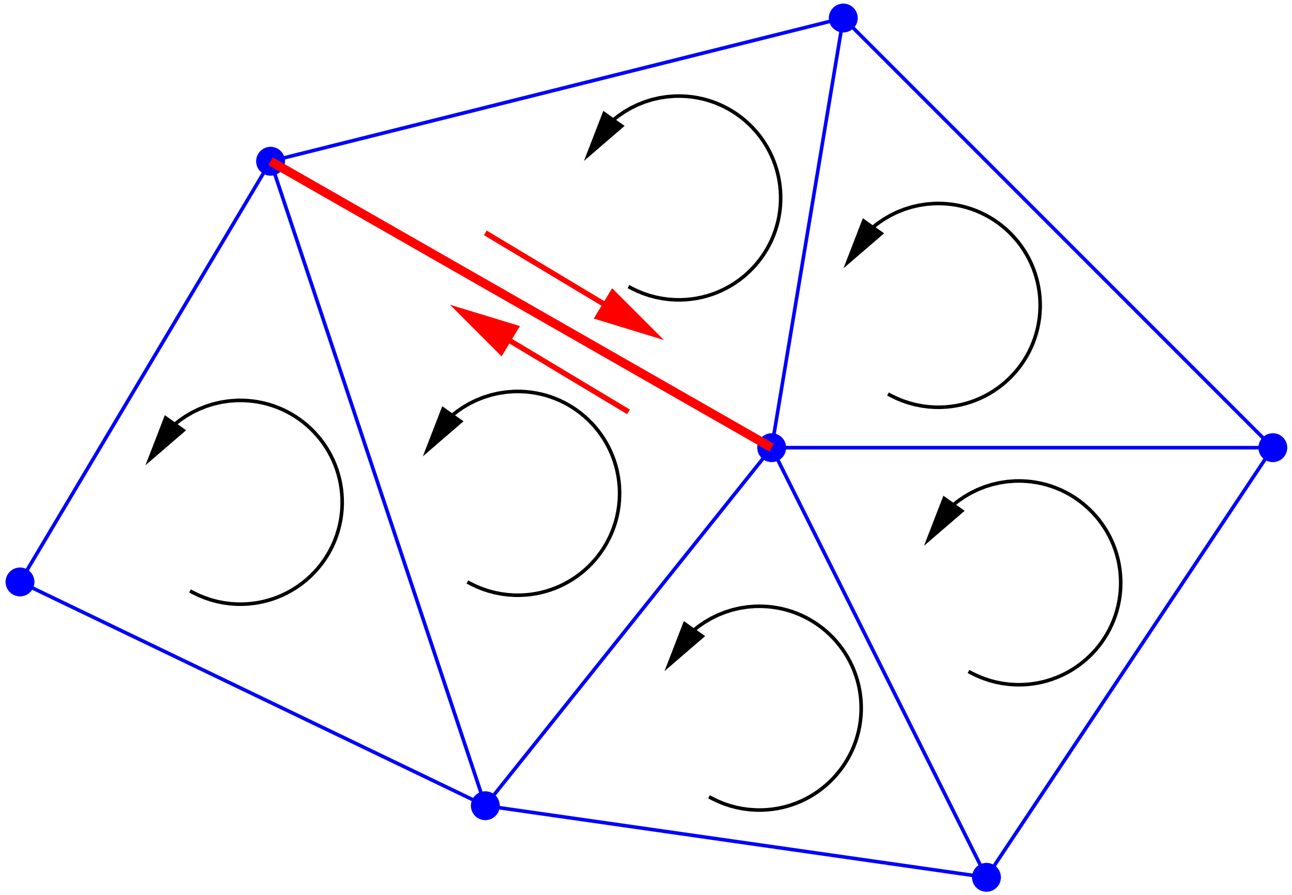
  $\mathrm{sign}(a_1\ a_2 \ldots a_r) = -\,\mathrm{sign}(b_1\ a_2 \ldots a_r)$

- possible for all rooms $\Rightarrow$ oriented manifold

Consistent orientations of rooms



Induced orientations on common wall



Path lengths for Abstract Sperner

- Linear in number of rooms.
- May be **exponential** in number of vertices:

  $*$ if rooms = facets of simplicial polytope,

  room-adjacency via pivoting [Morris 1994]

Room partitionings

Given a manifold M with vertex set V,

room partitioning = partition of V into rooms.

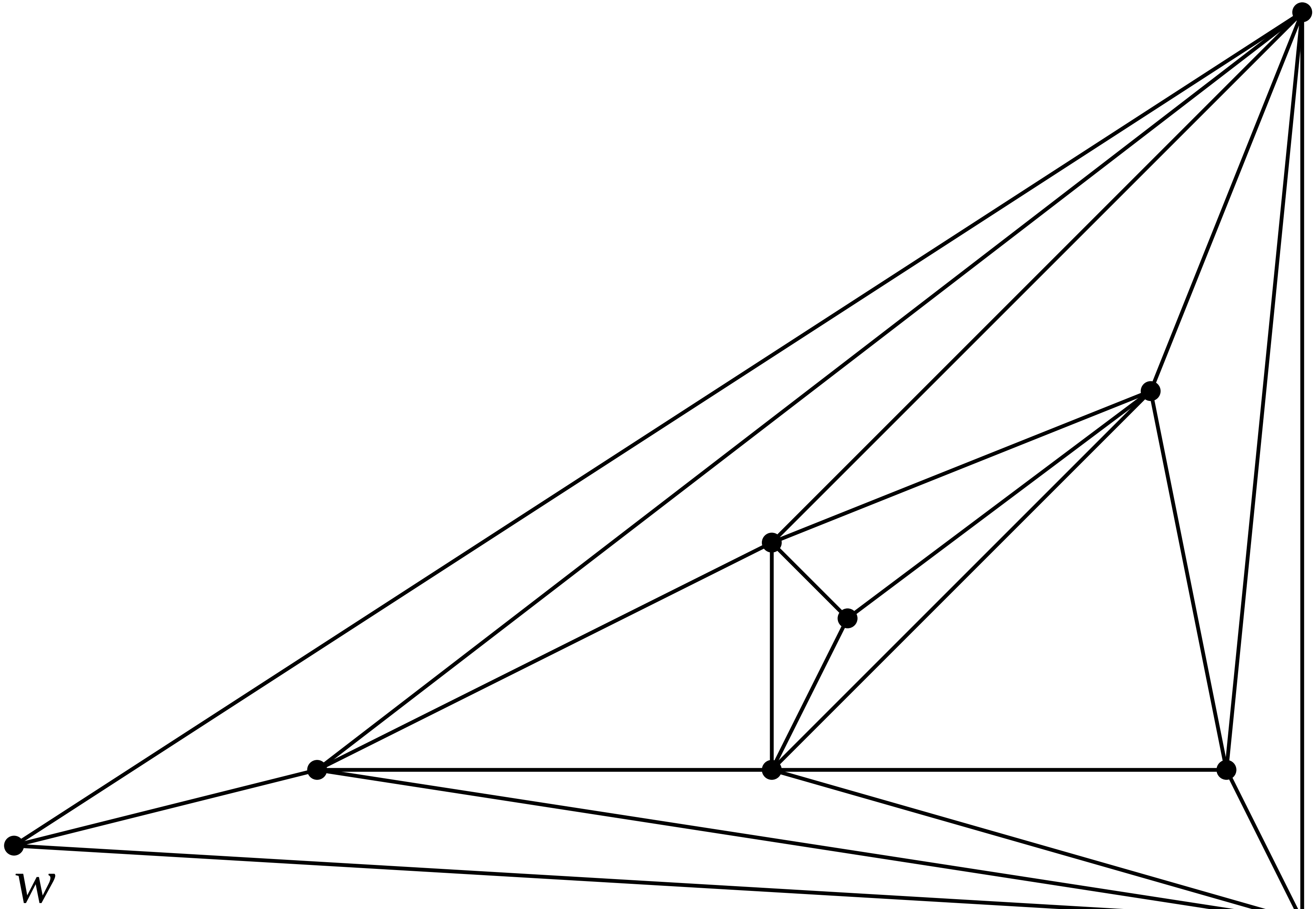
(Then |V| is a multiple of the rank r.)

Theorem.

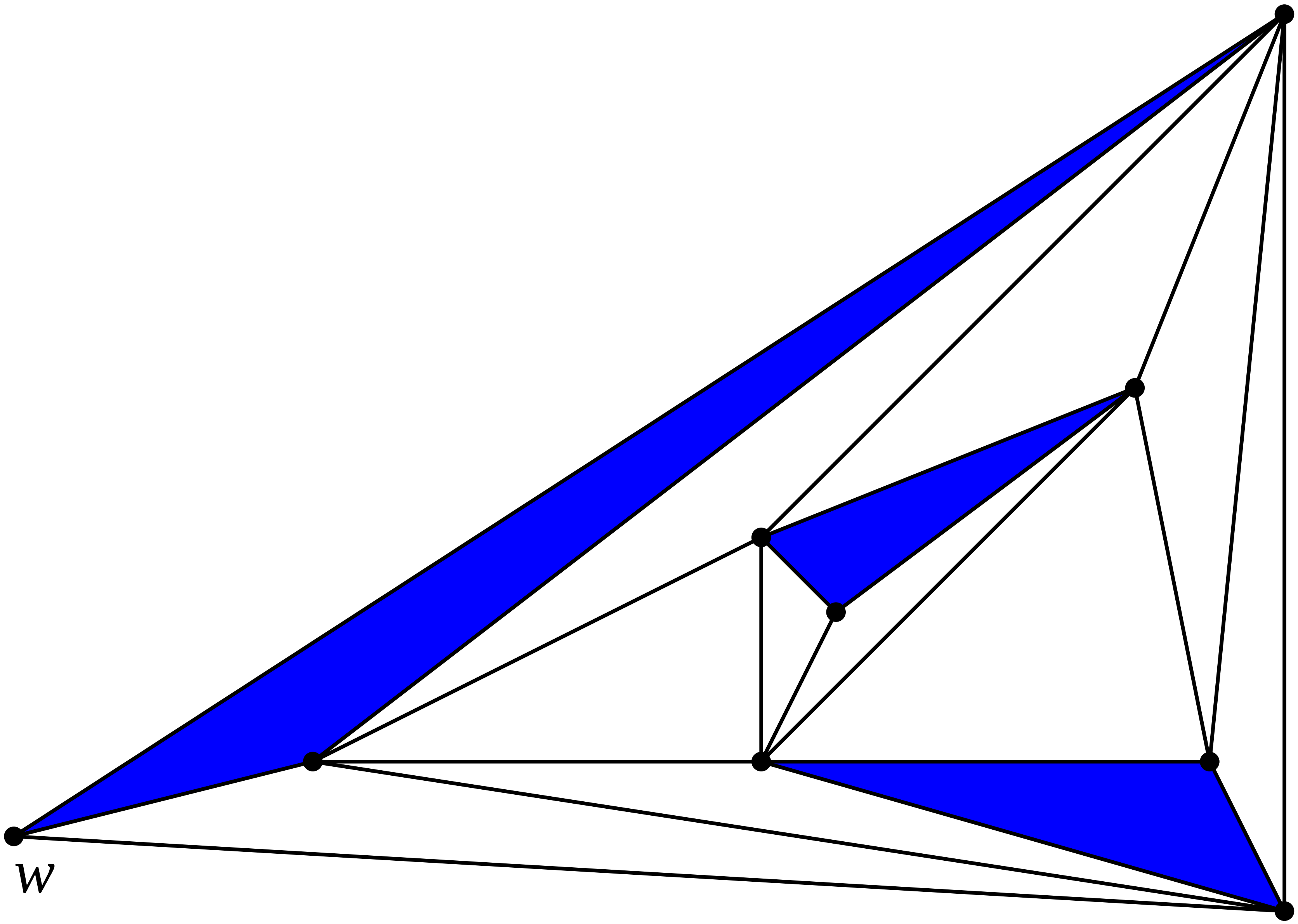
M has an **even** number of room partitionings.

Proof by Parity Argument (PPA).

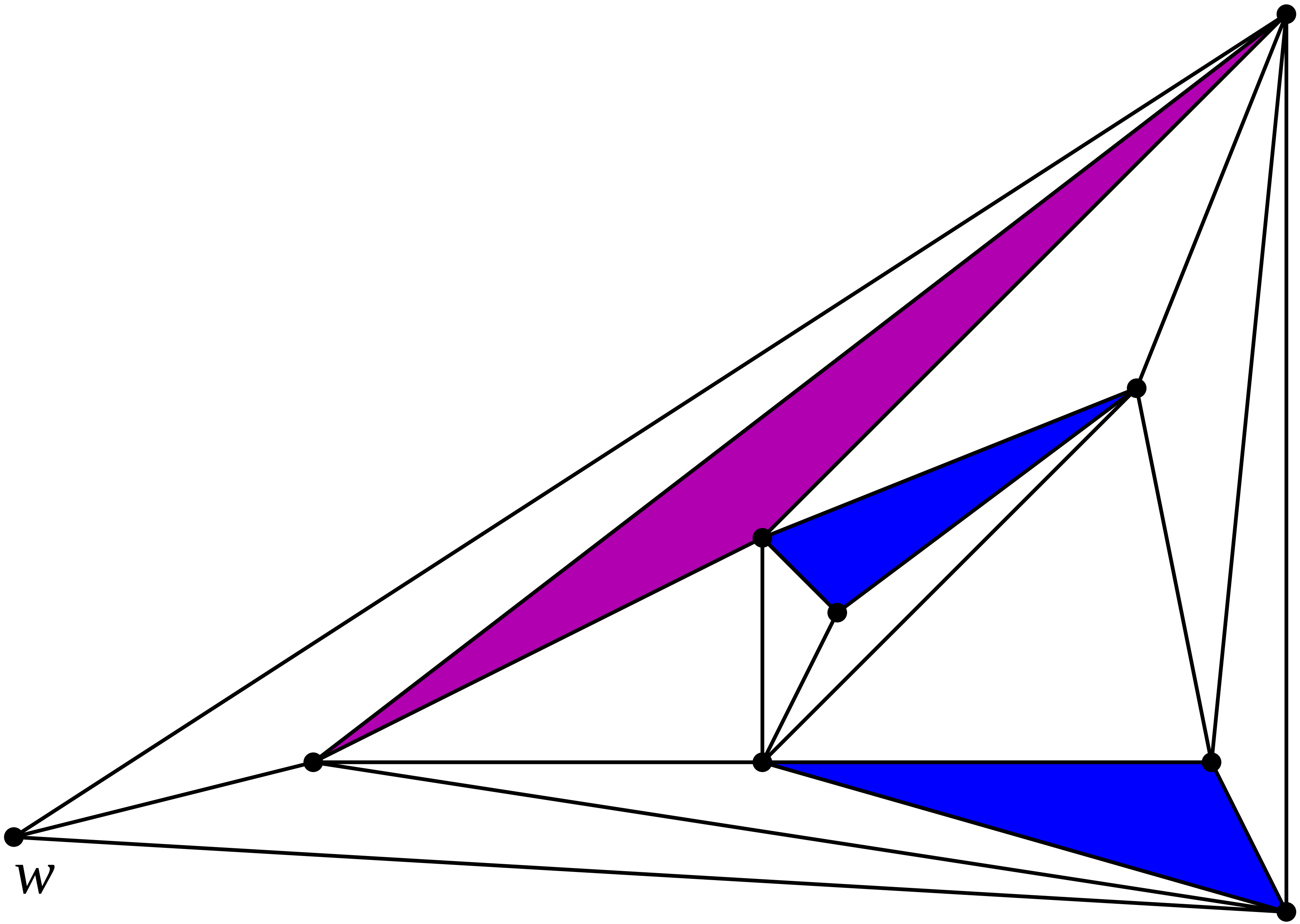
Example: Manifold



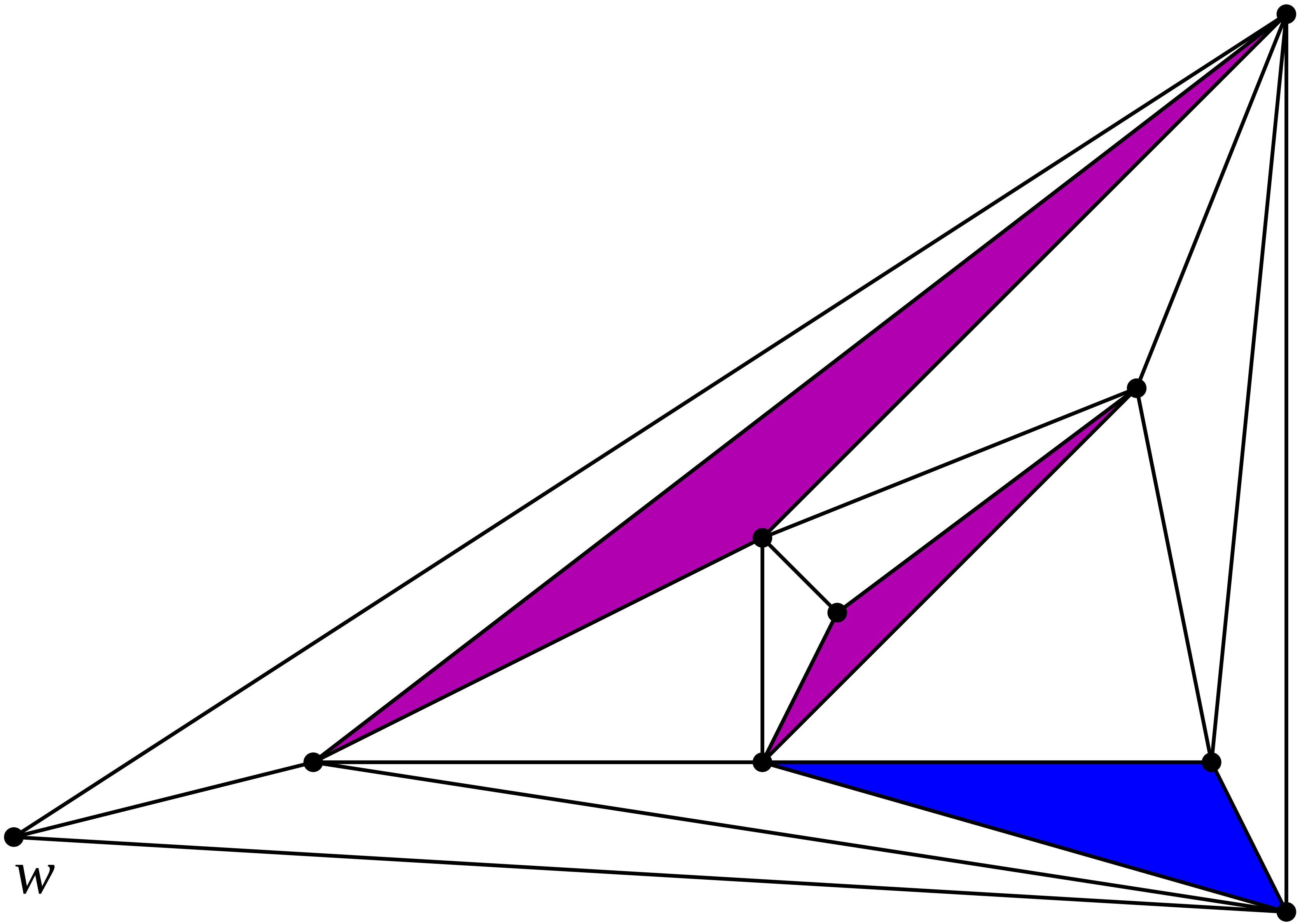
Room partitioning



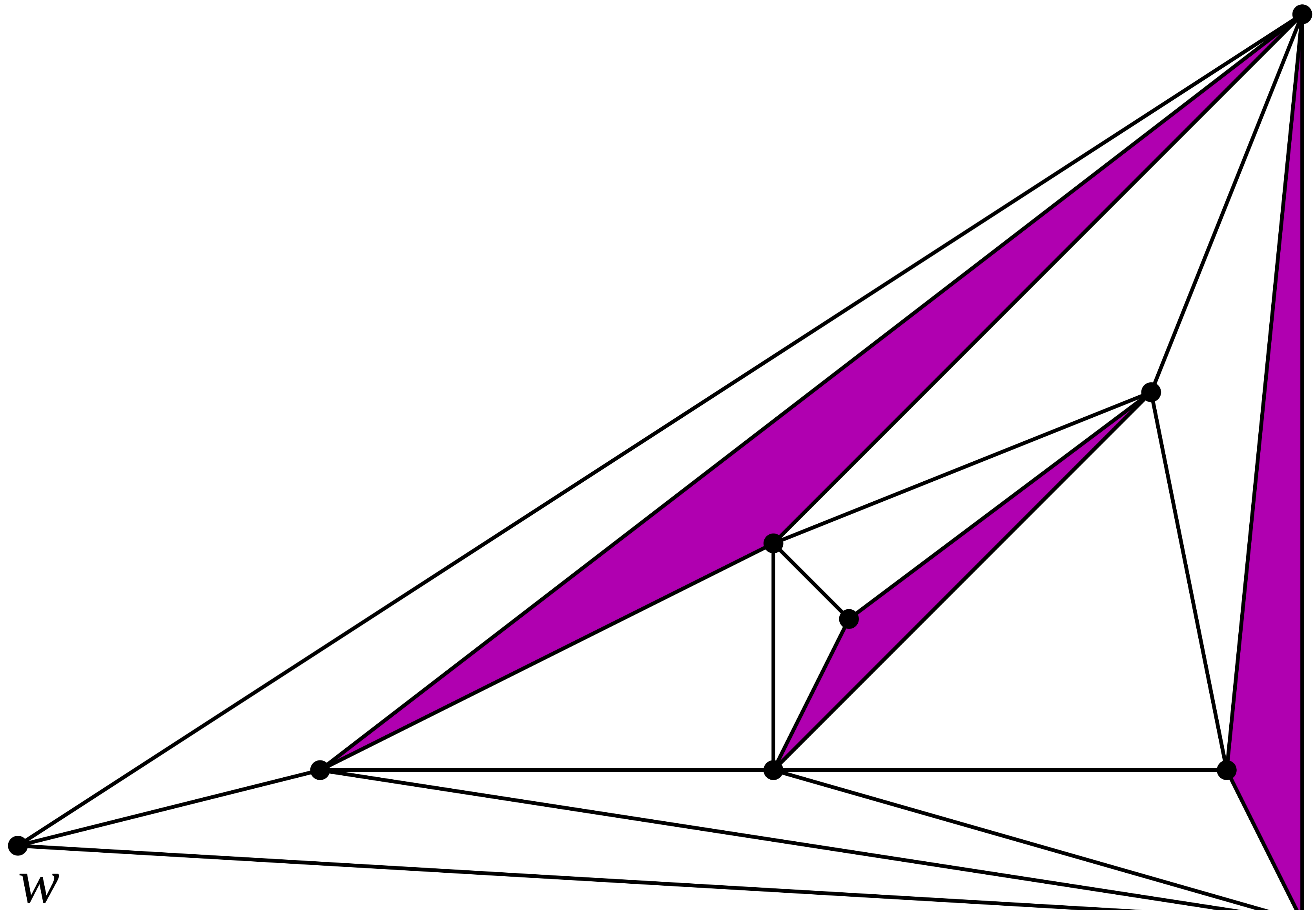
w-almost room partitioning



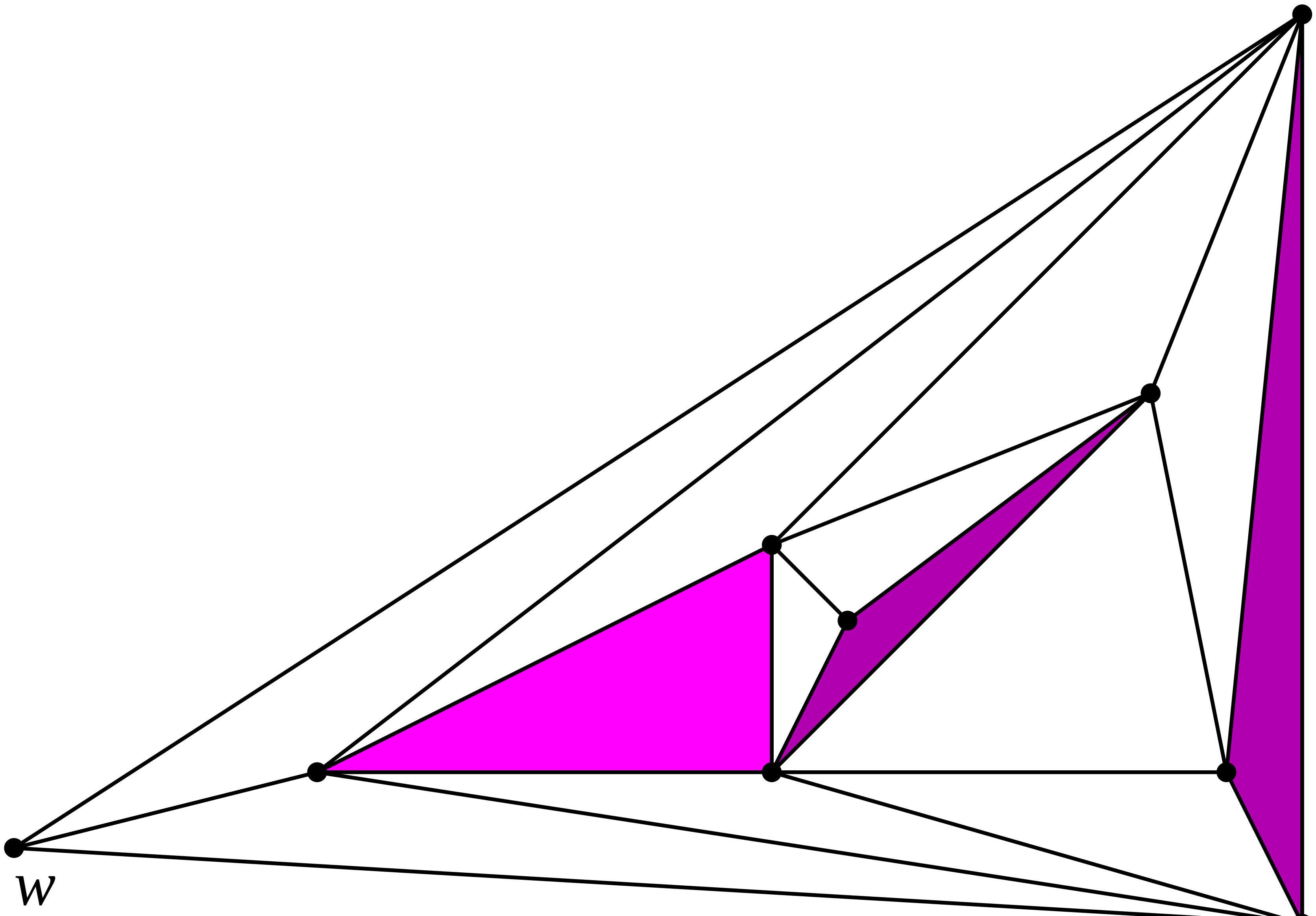
w-almost room partitioning



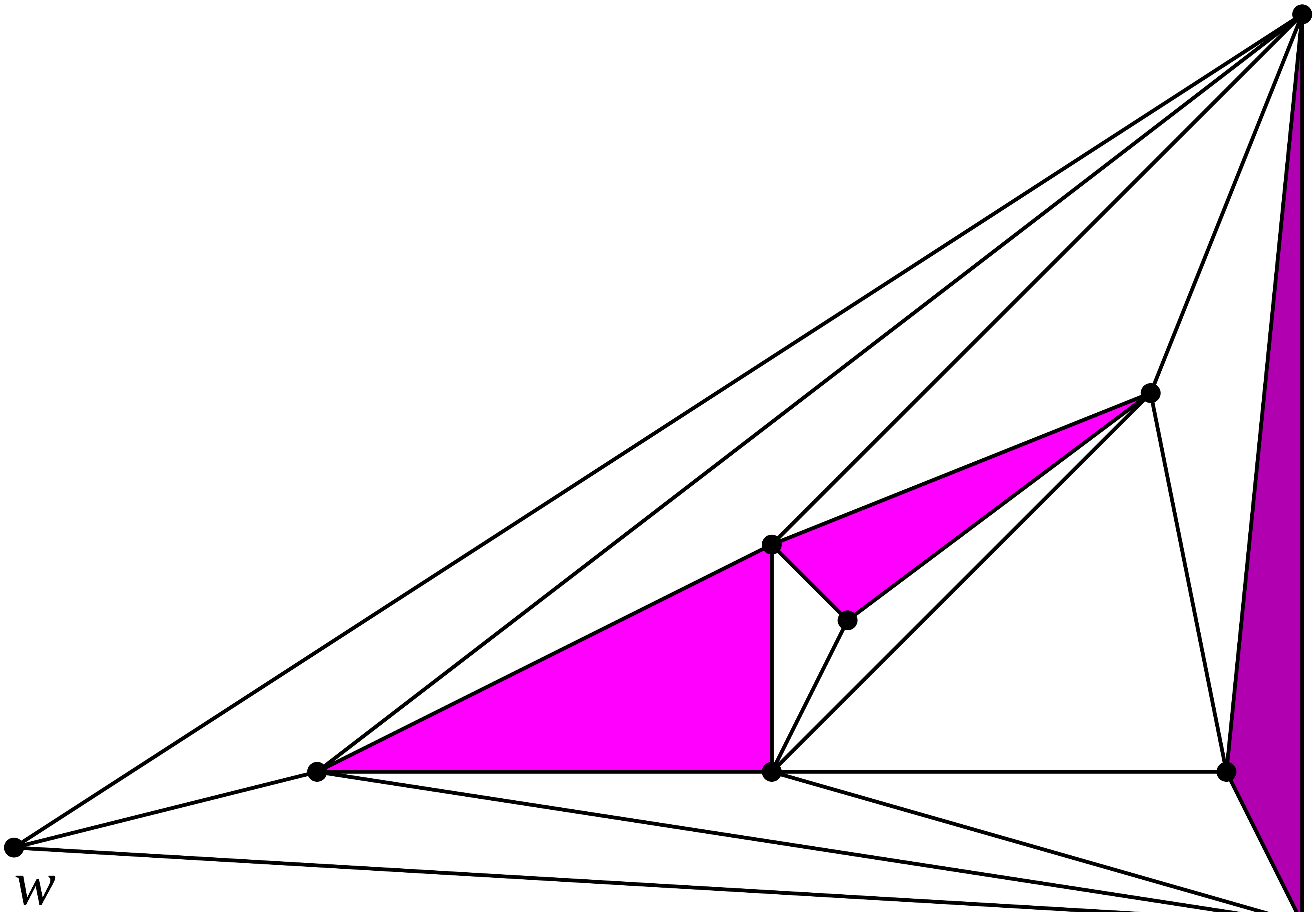
w-almost room partitioning



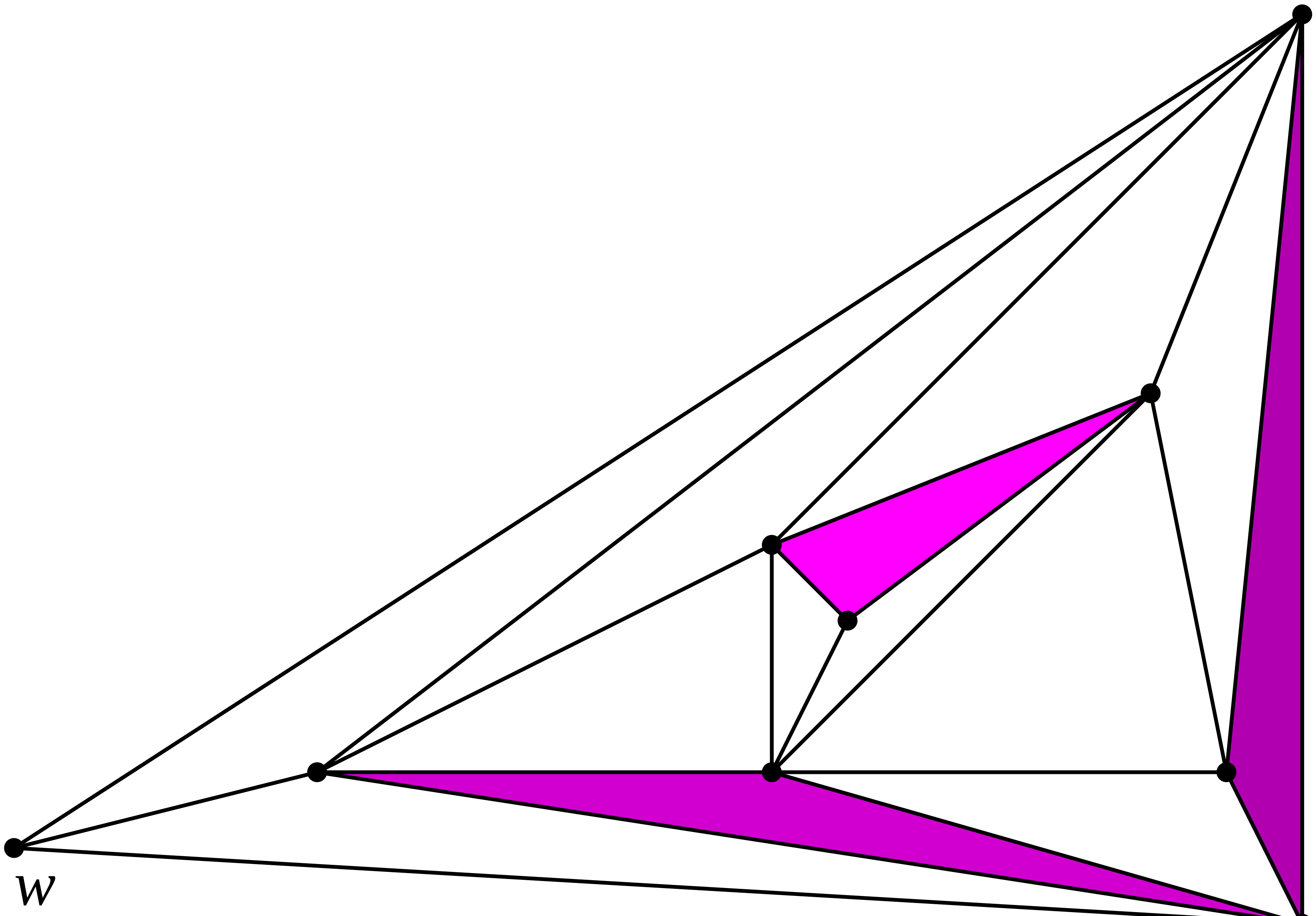
w-almost room partitioning



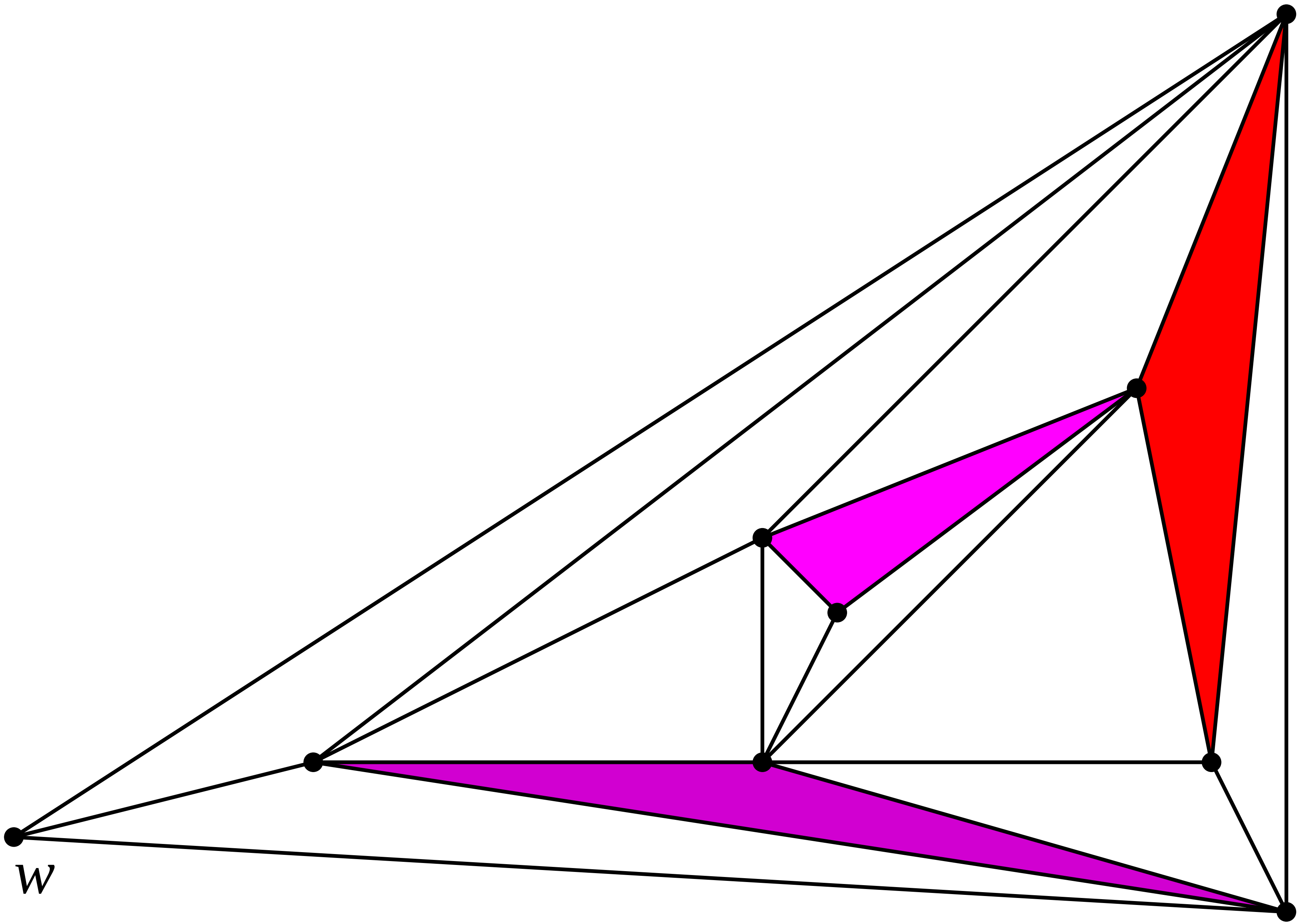
w-almost room partitioning



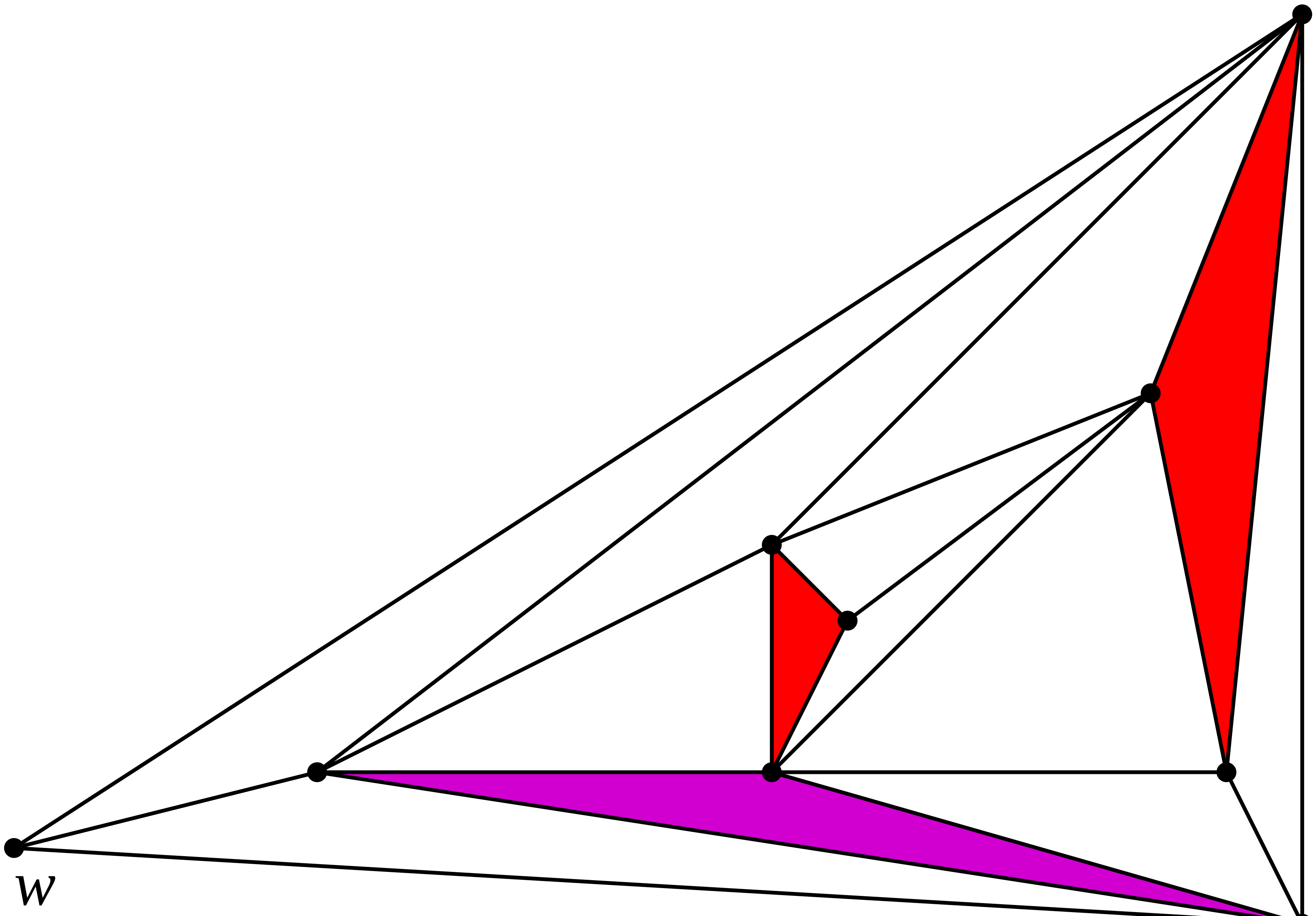
w-almost room partitioning



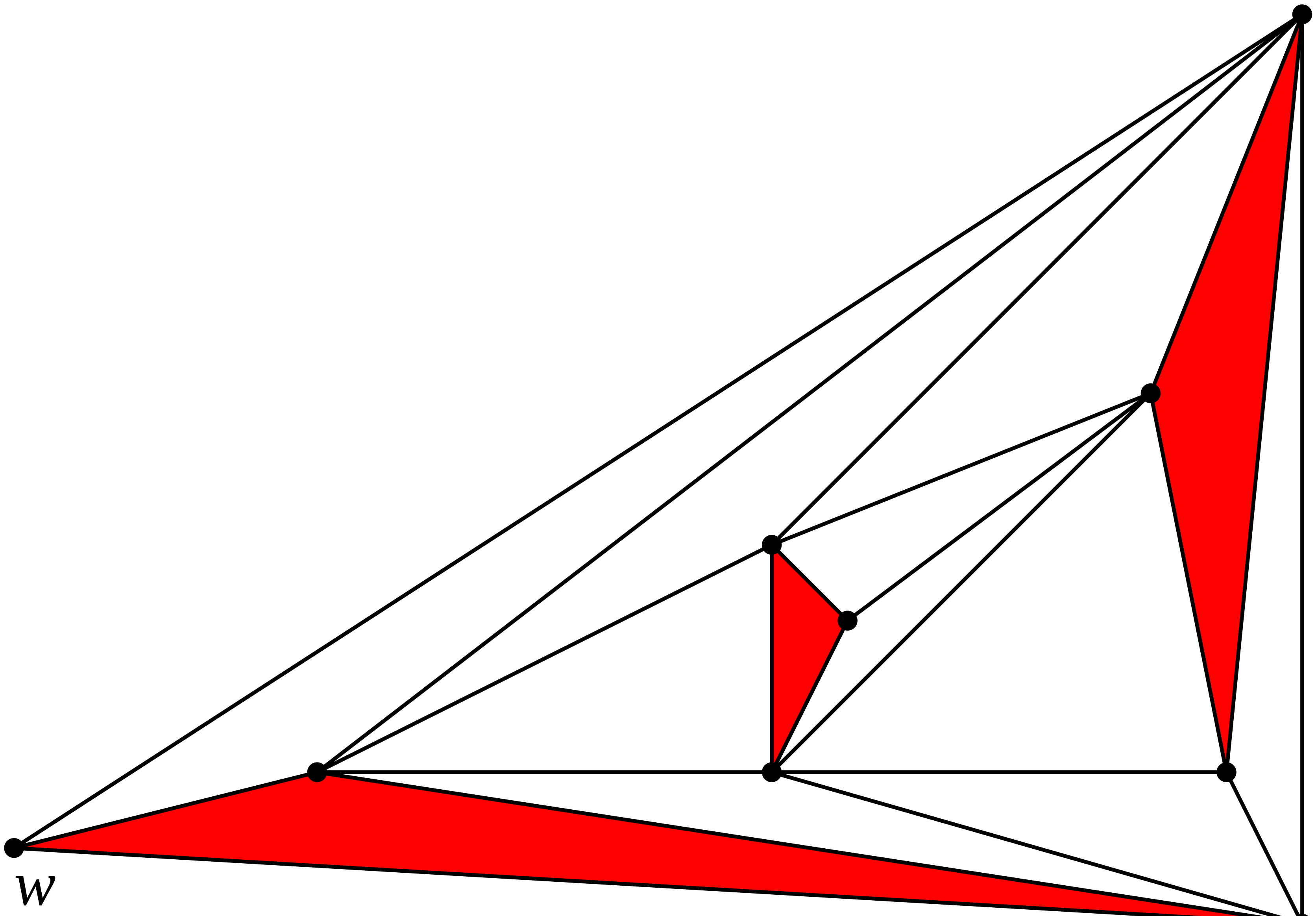
w-almost room partitioning



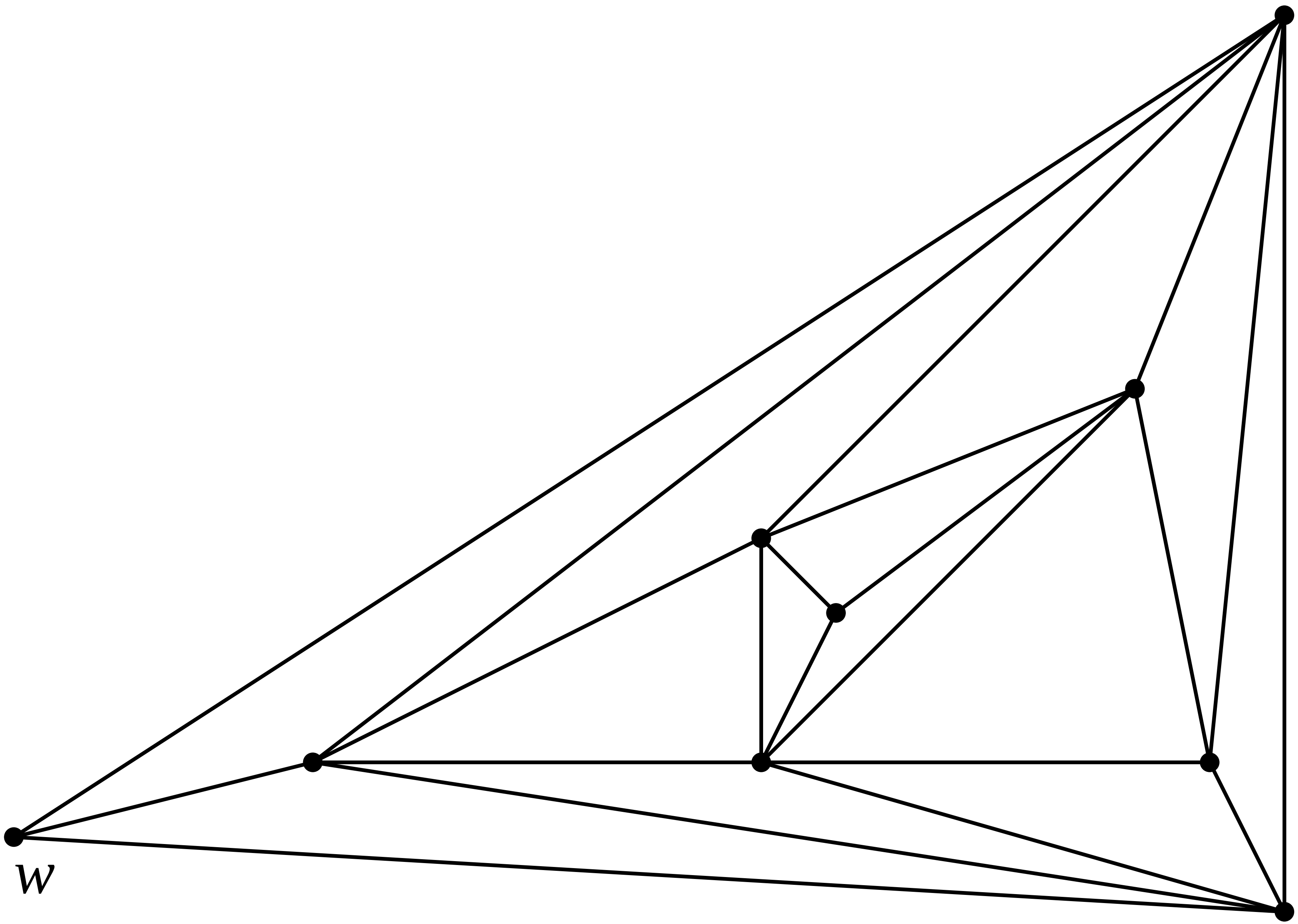
w-almost room partitioning



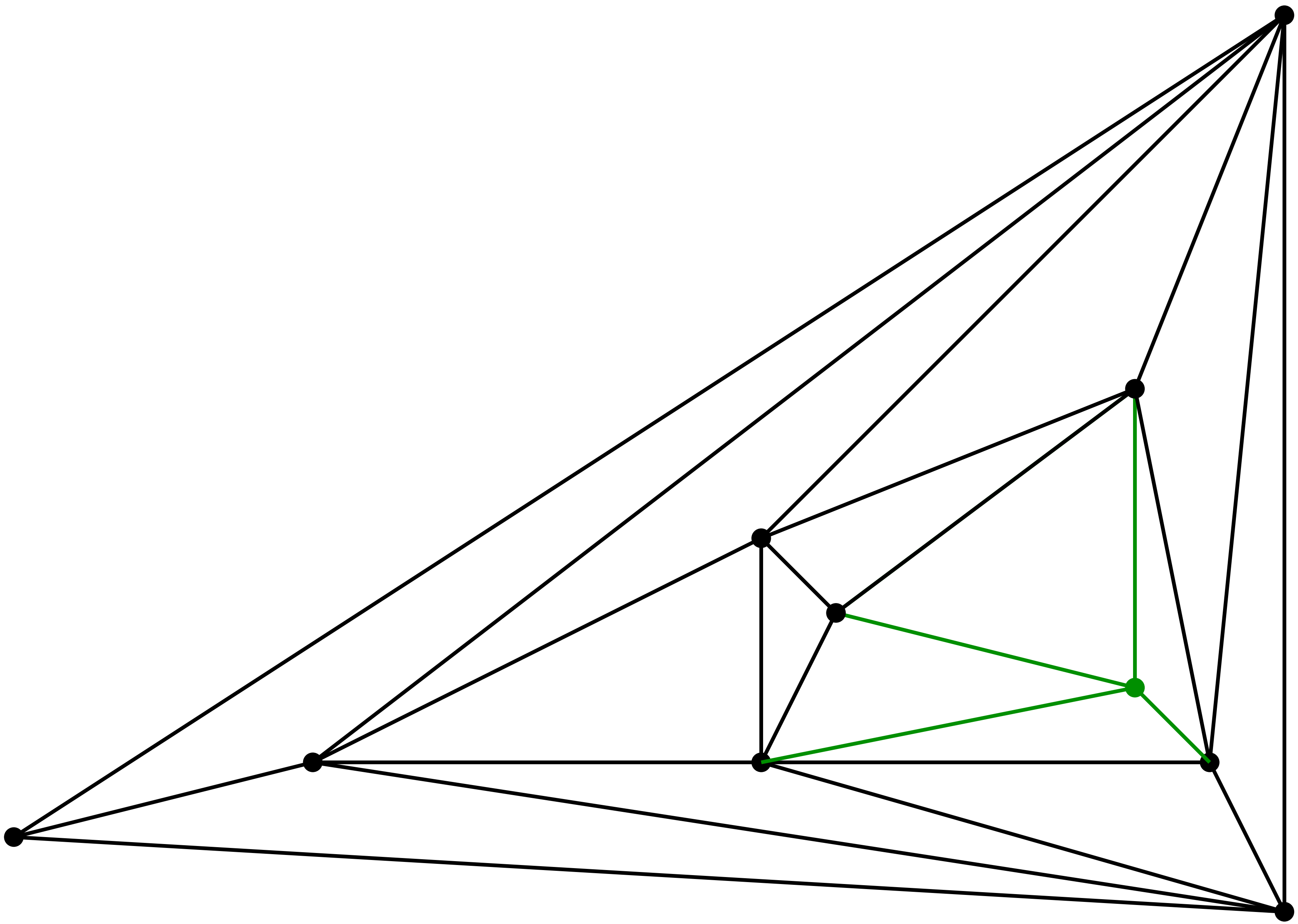
New room partitioning



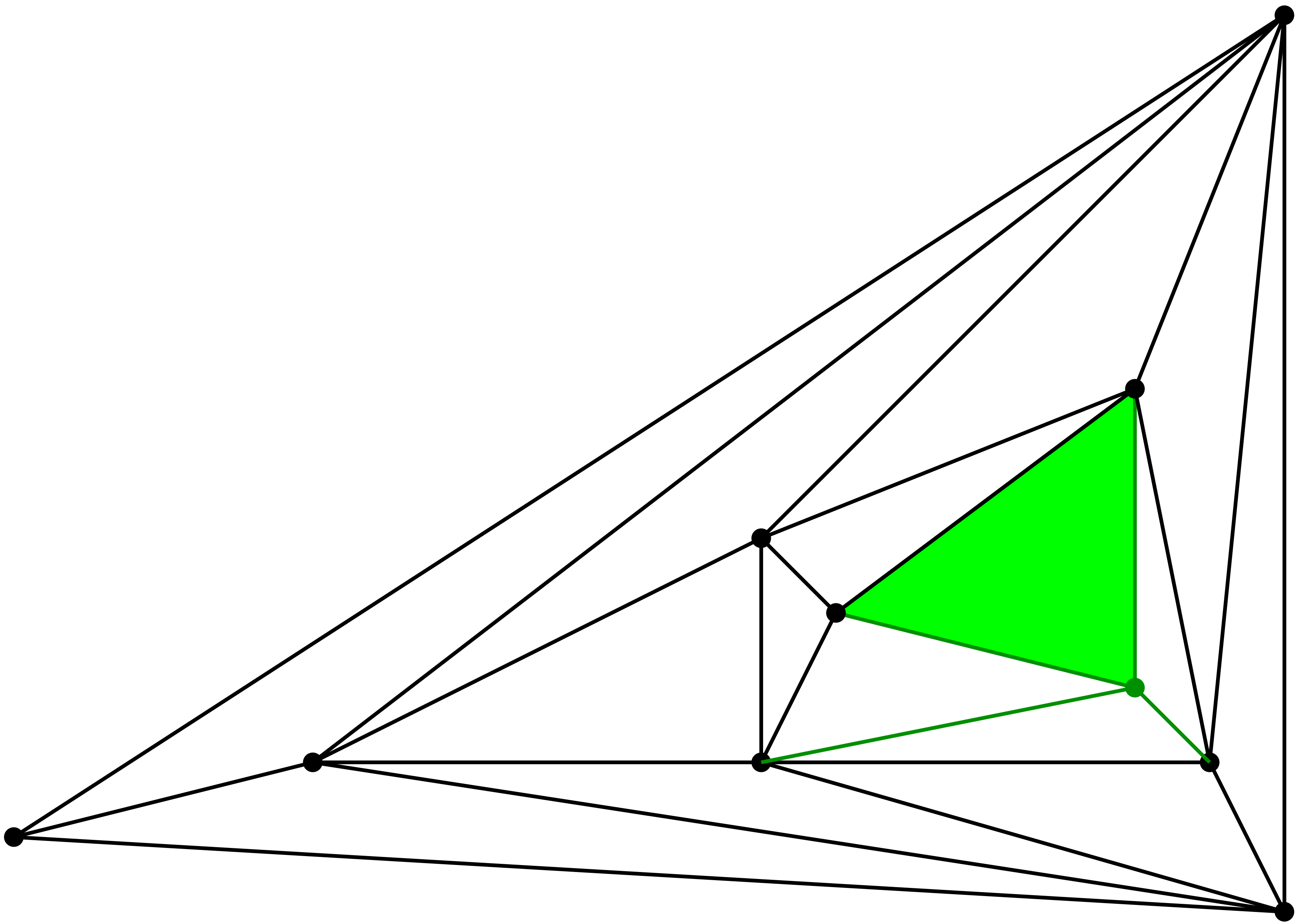
Construct exponential example



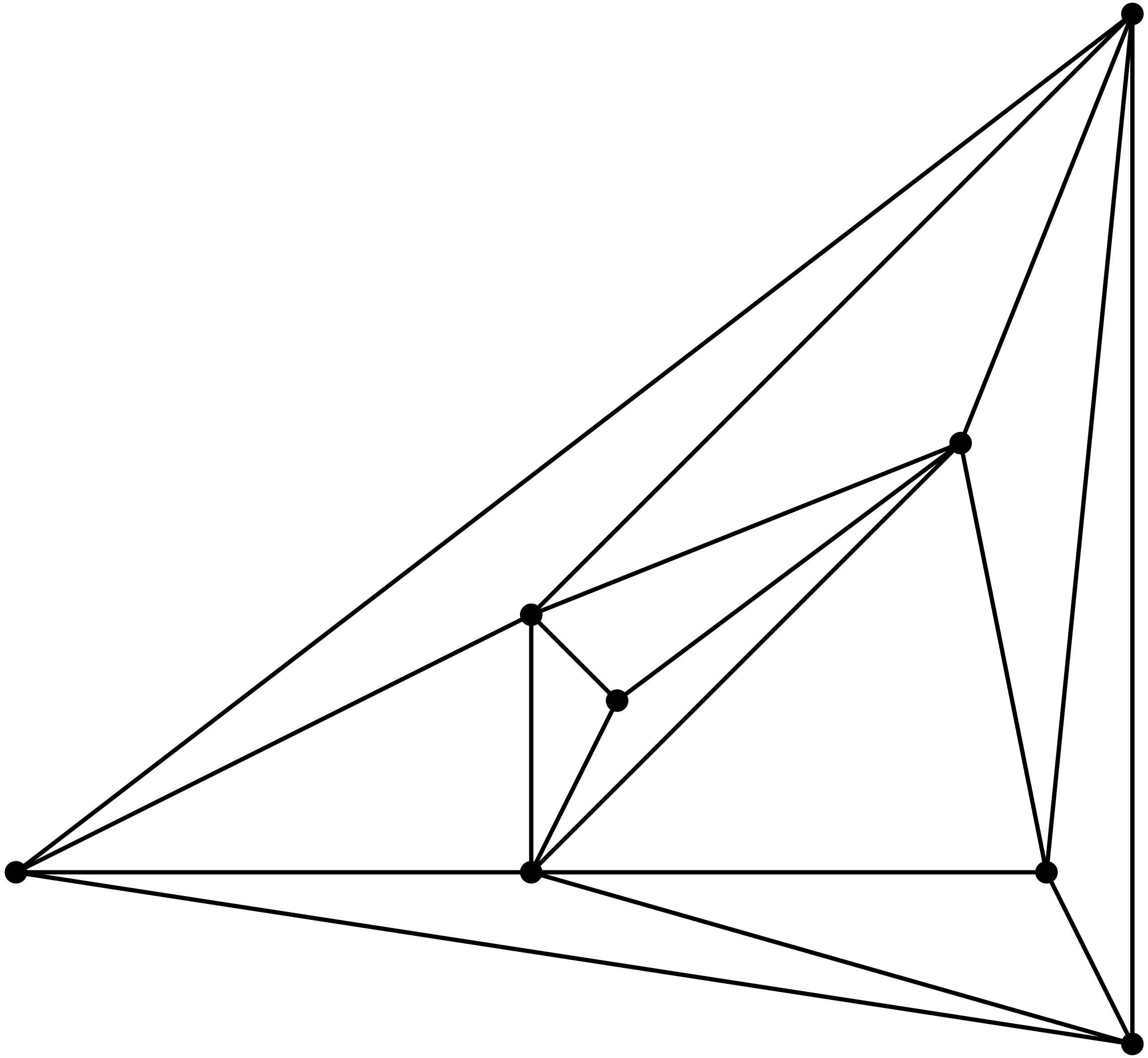
Construct exponential example



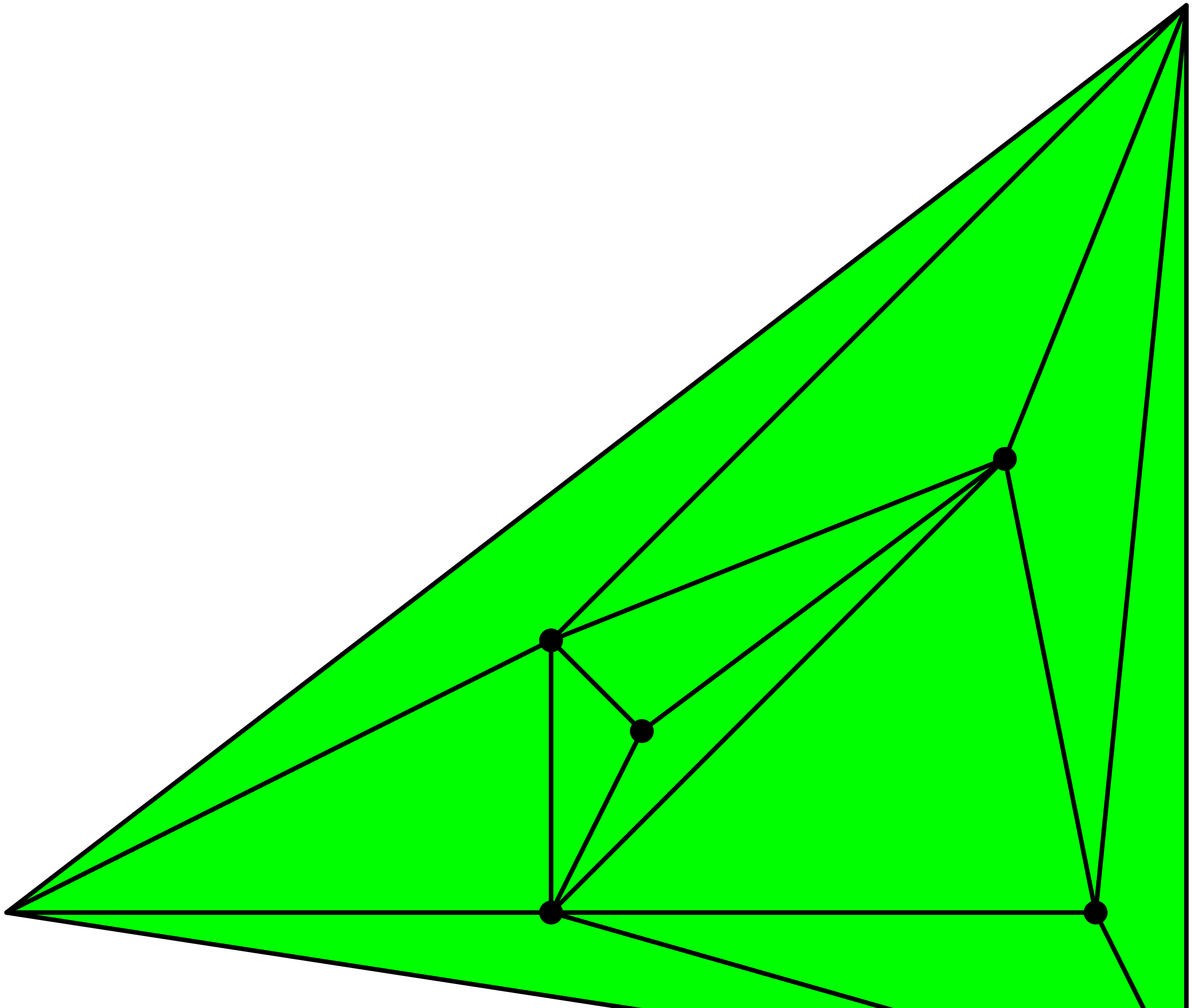
Construct exponential example



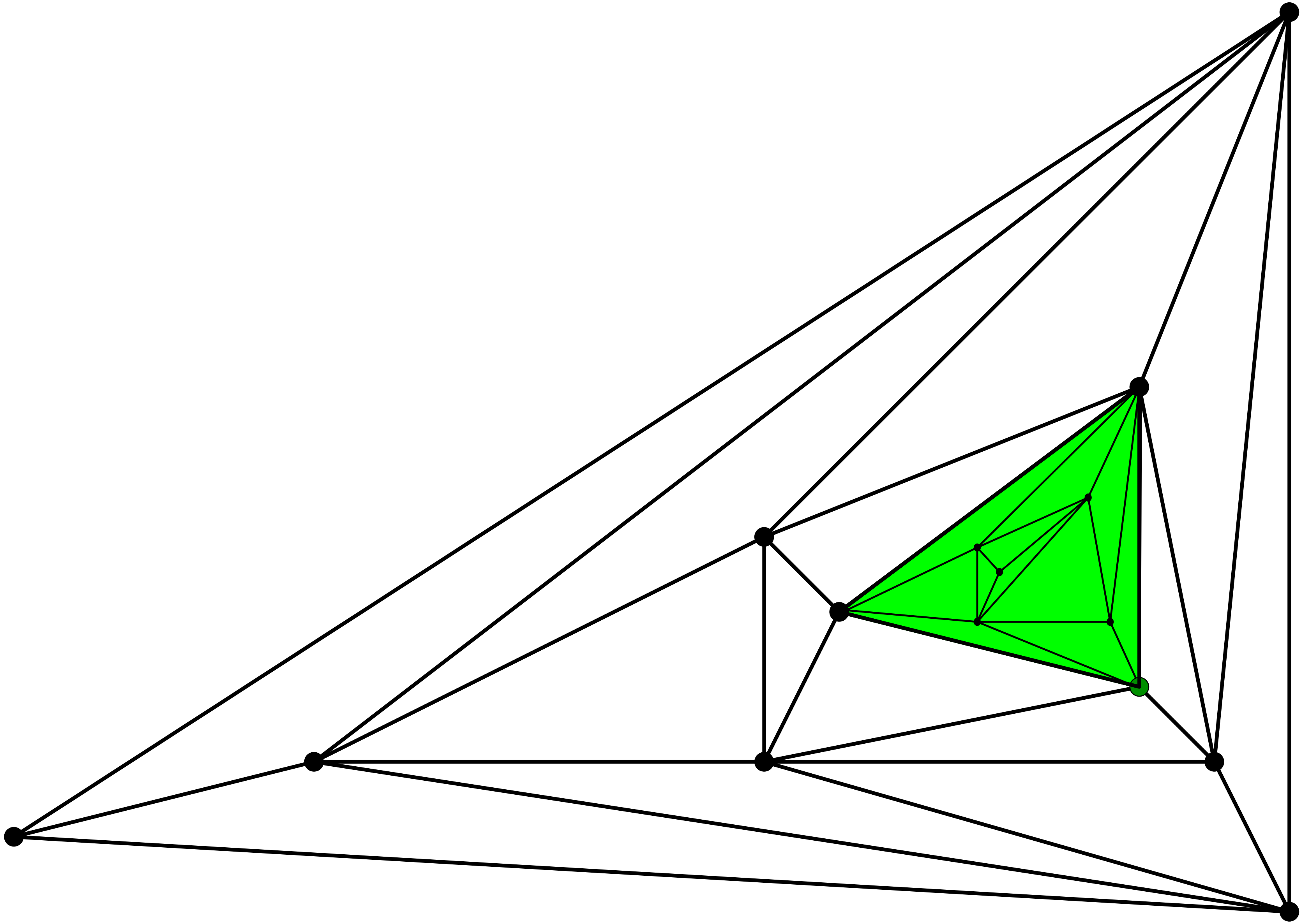
Construct exponential example



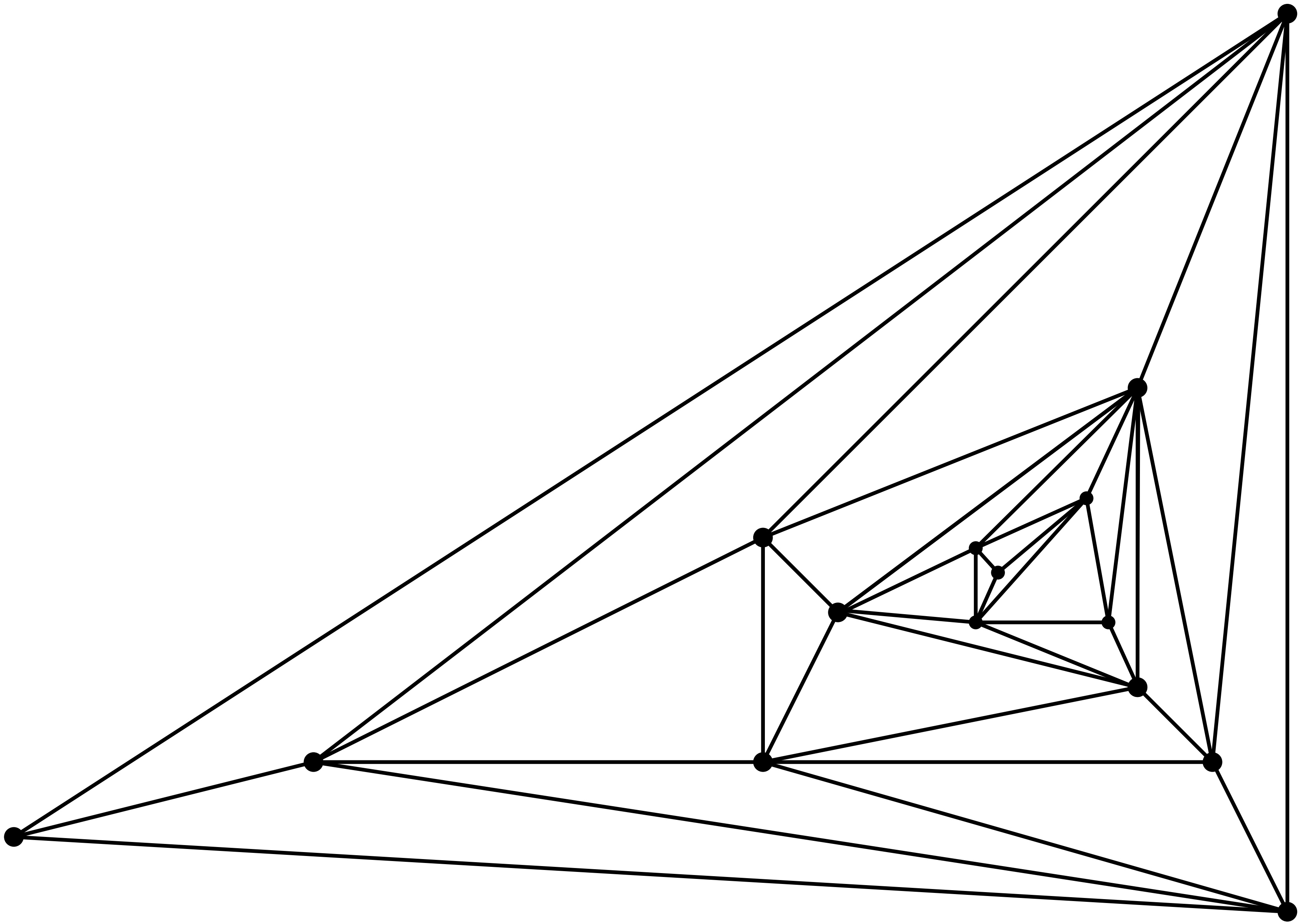
Construct exponential example



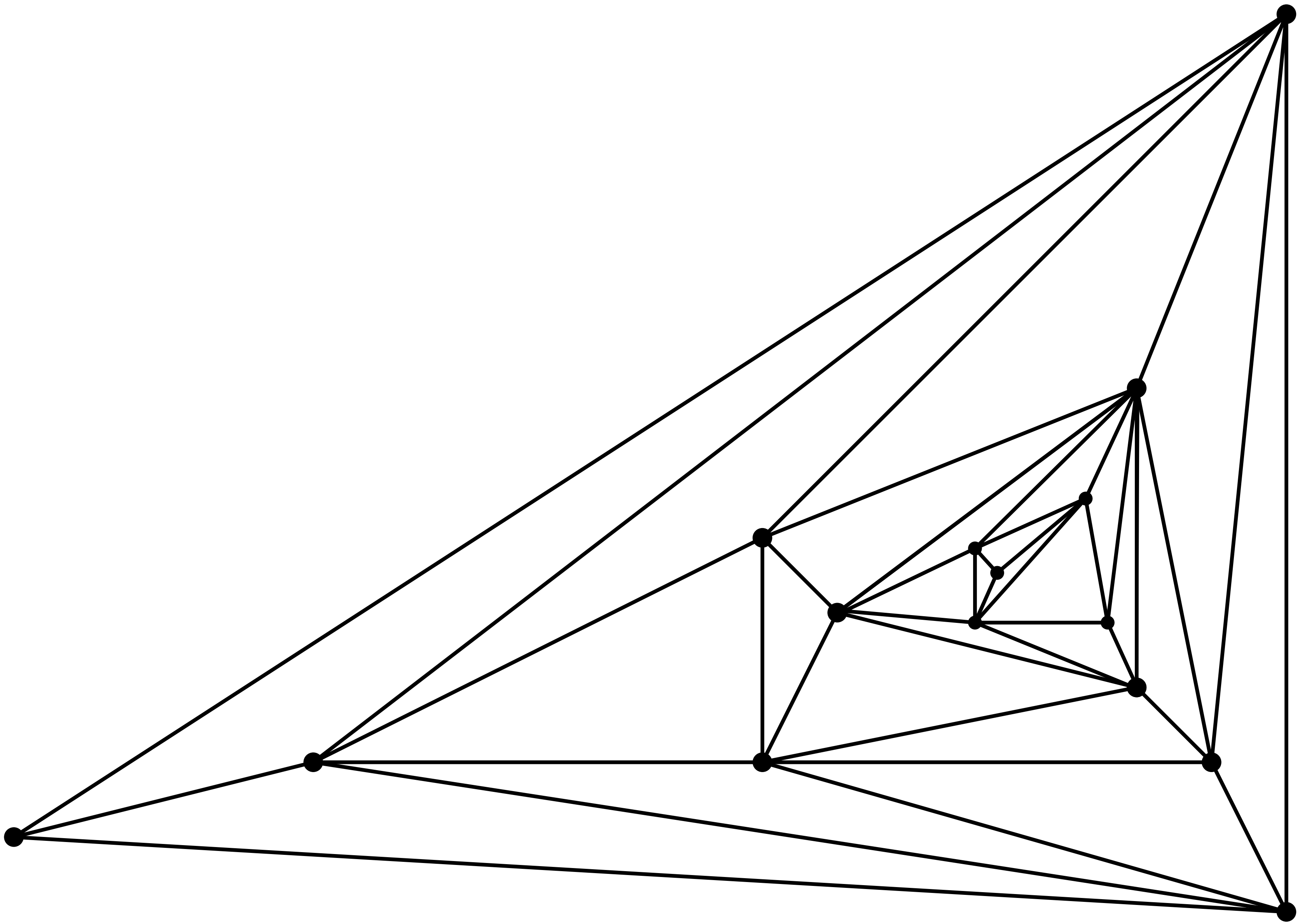
Construct exponential example



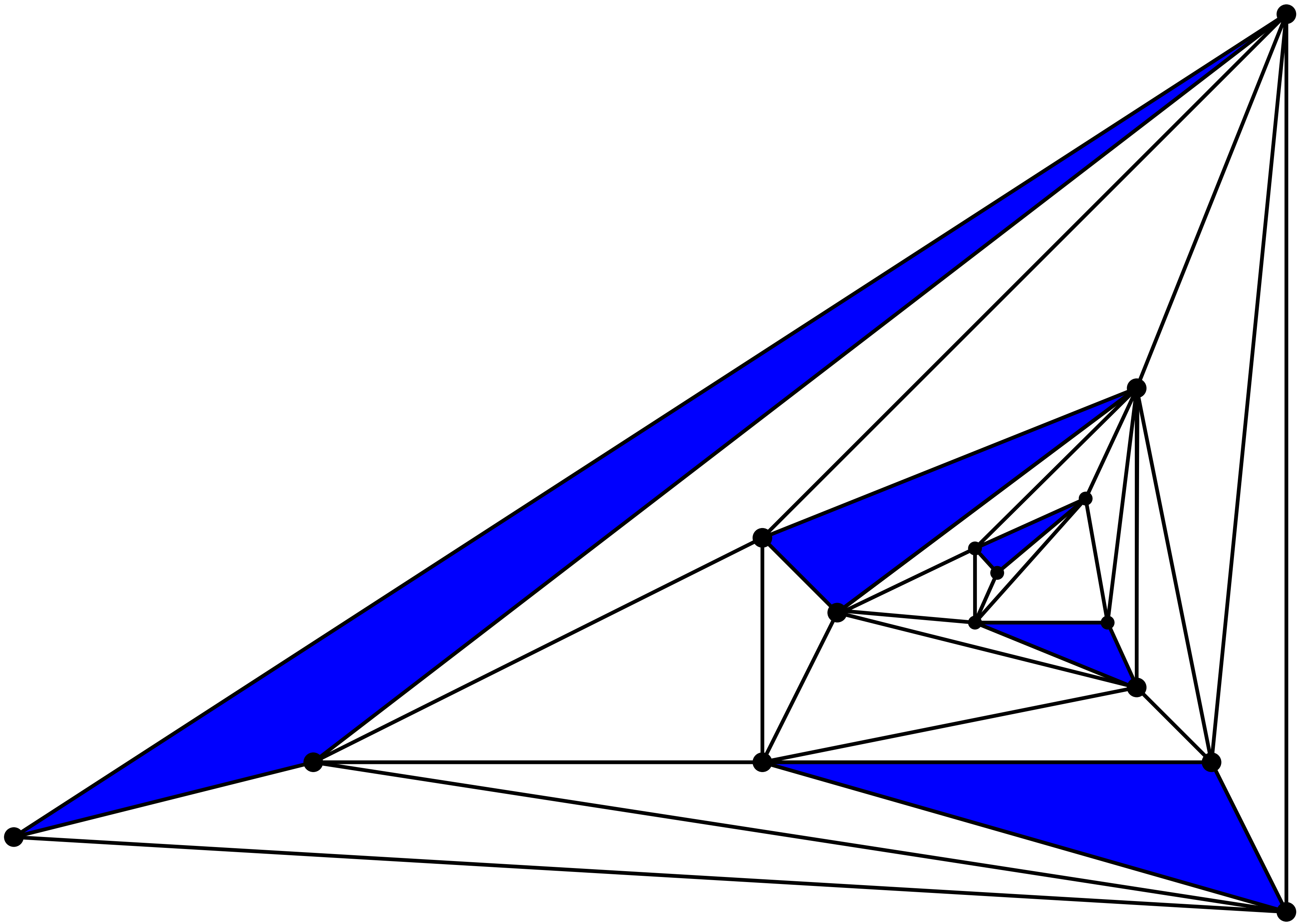
6 extra points, 12 extra rooms



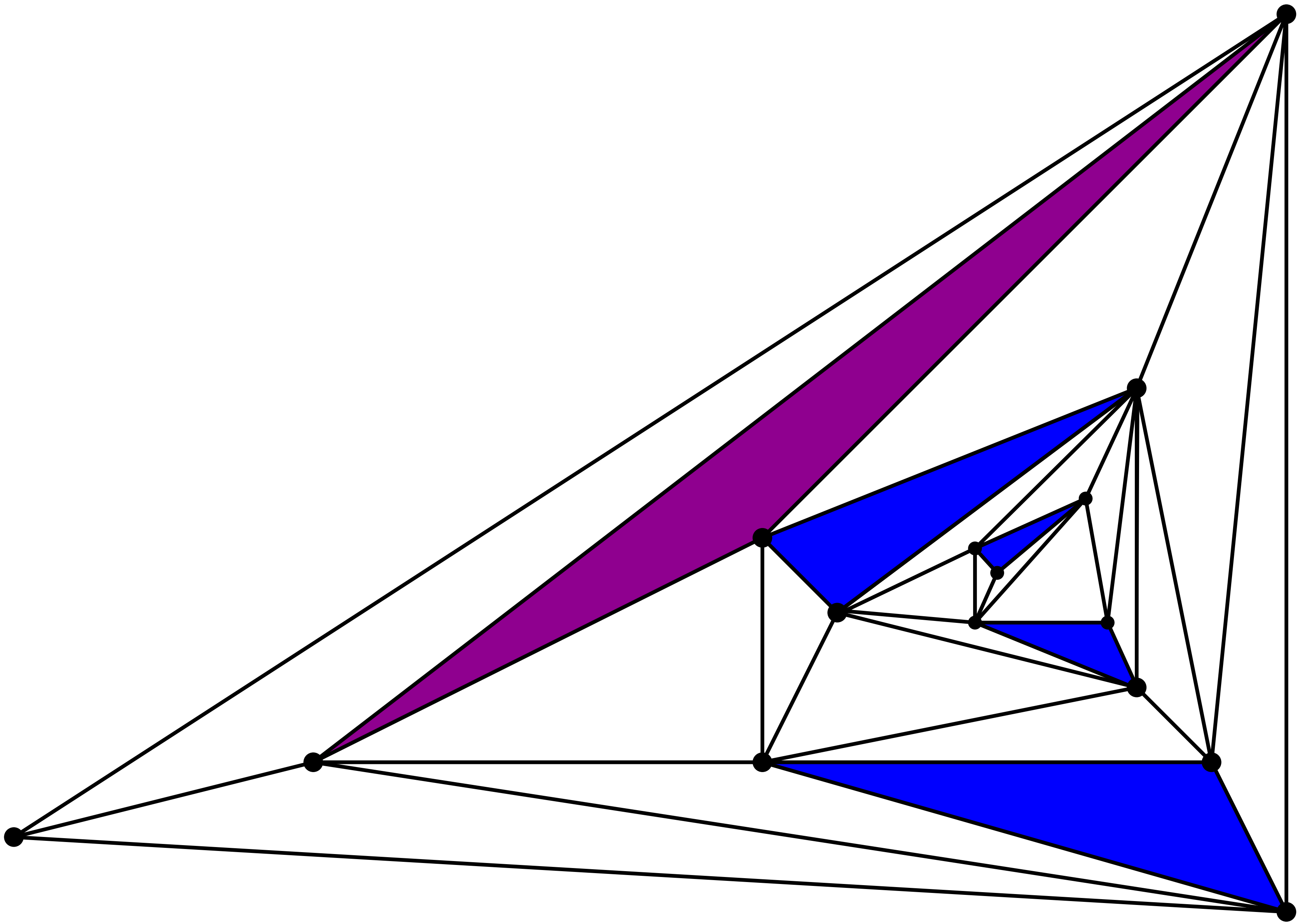
path length more than doubles!



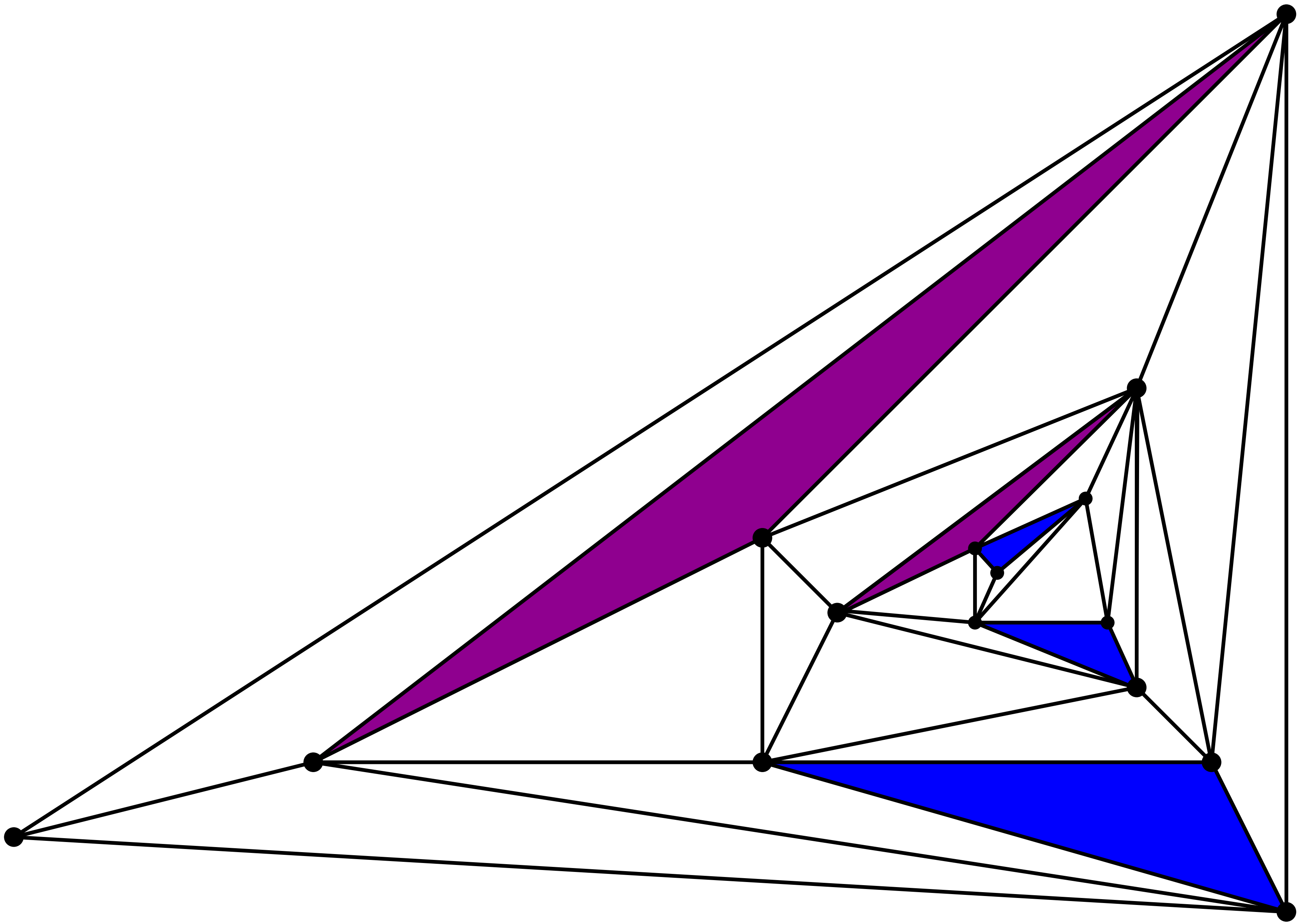
path length more than doubles!



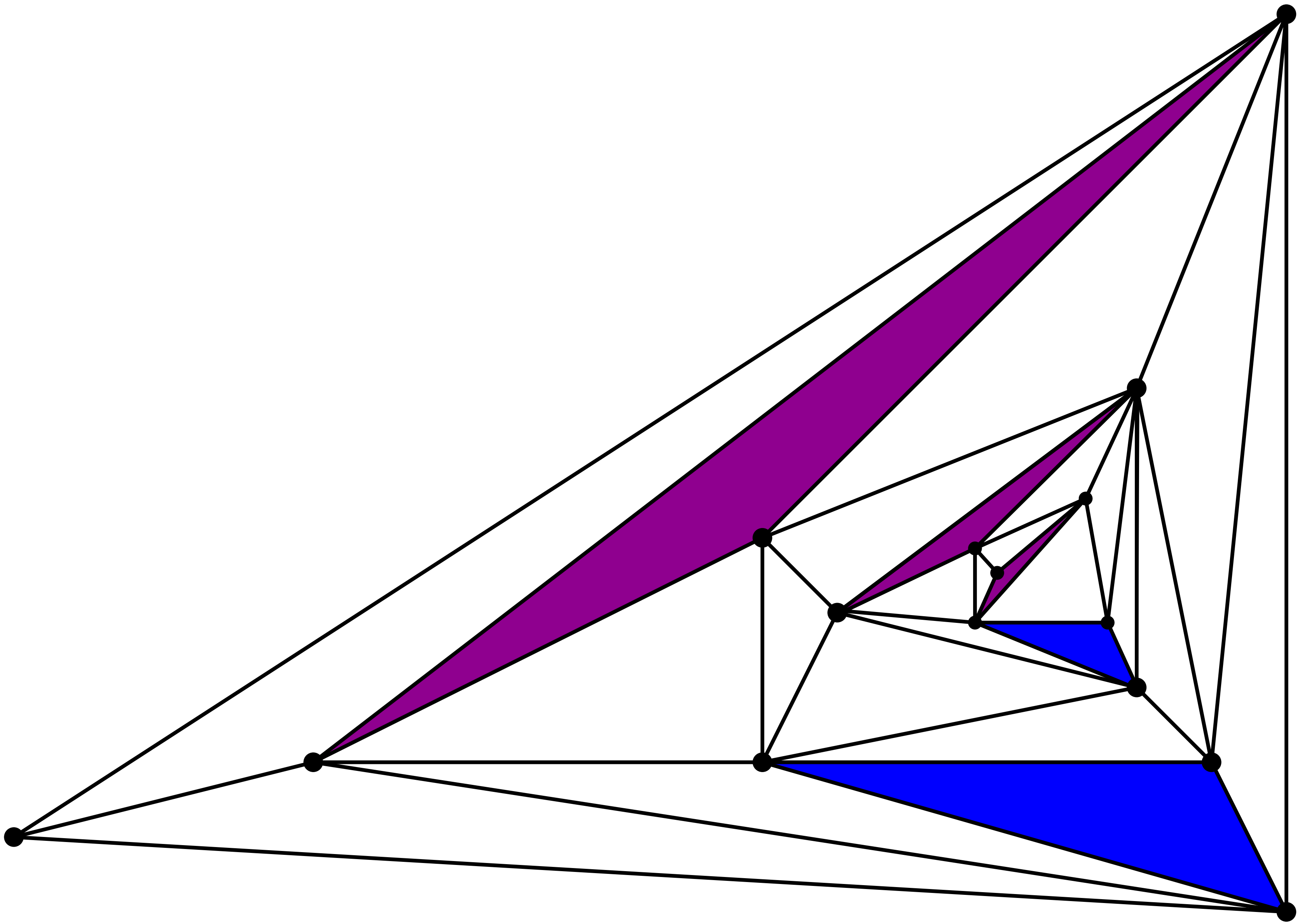
path length more than doubles!



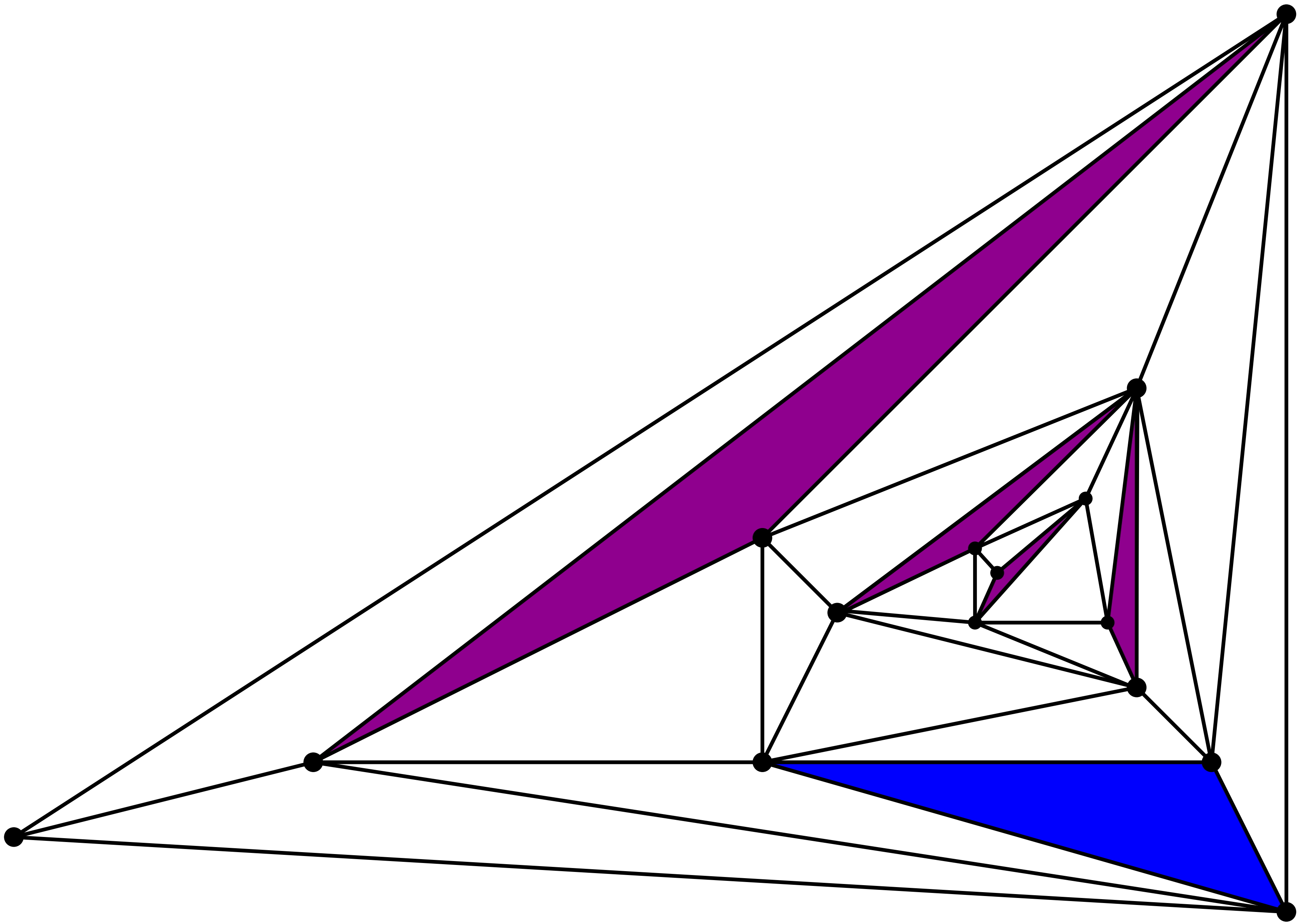
path length more than doubles!



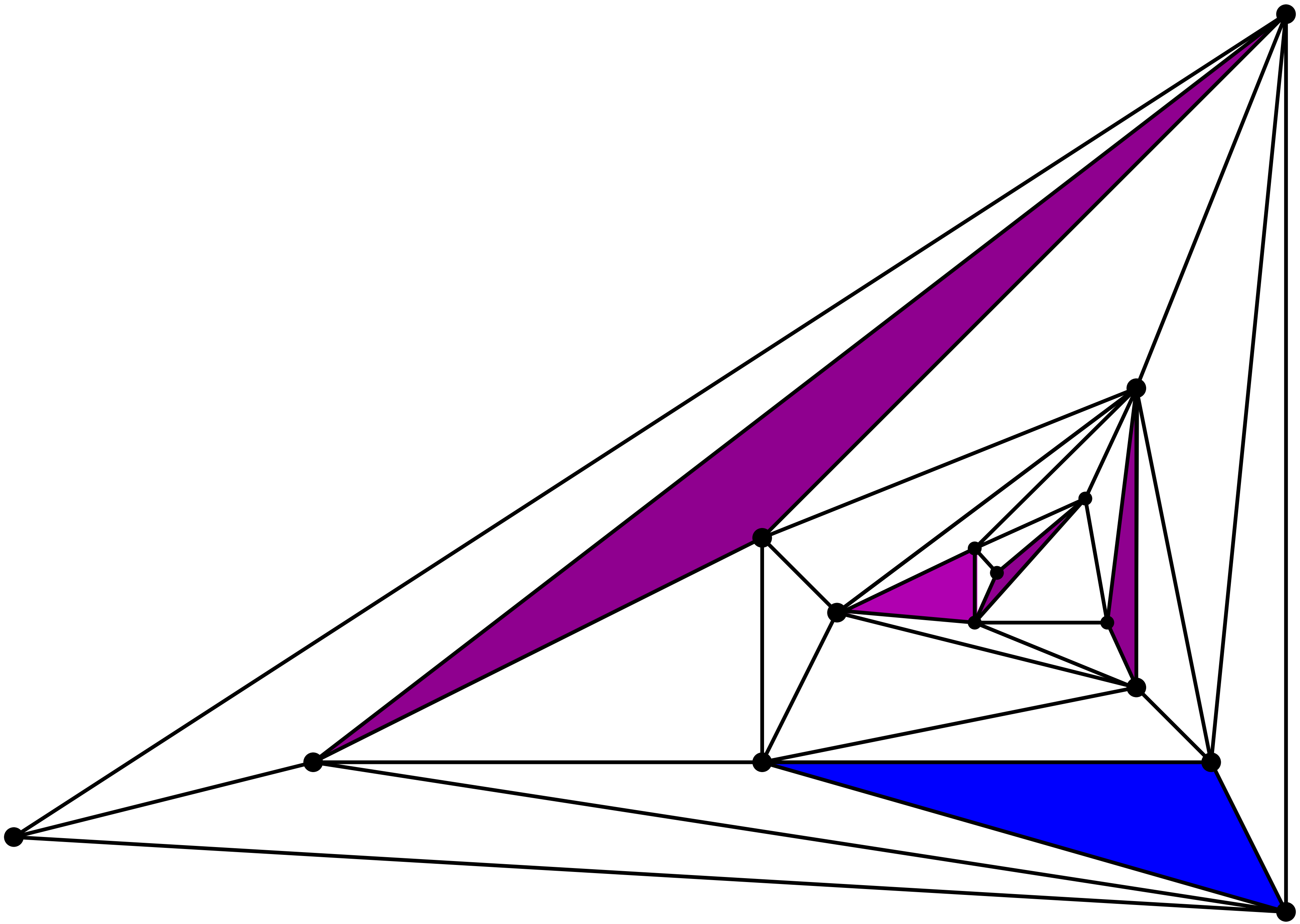
path length more than doubles!



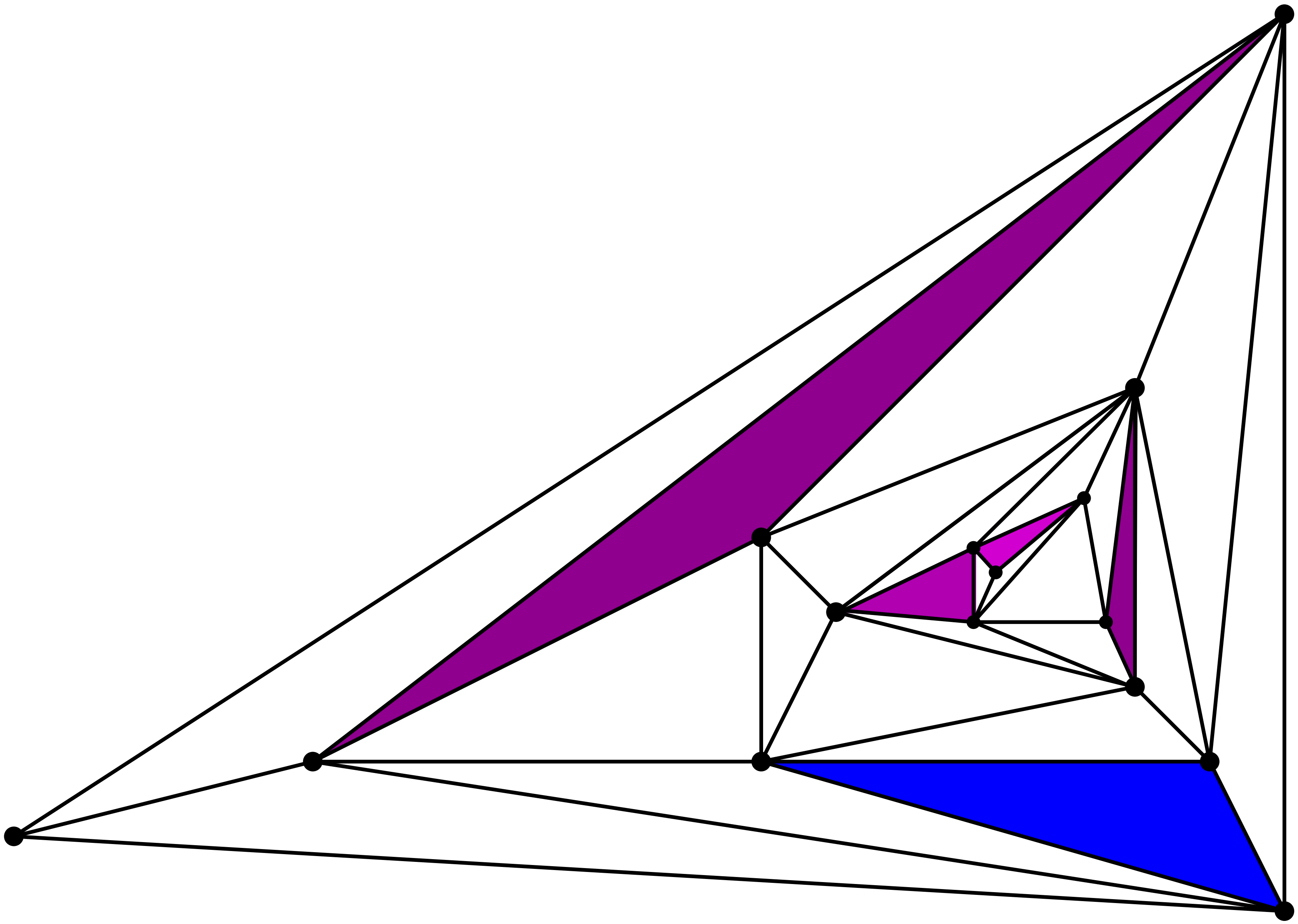
path length more than doubles!



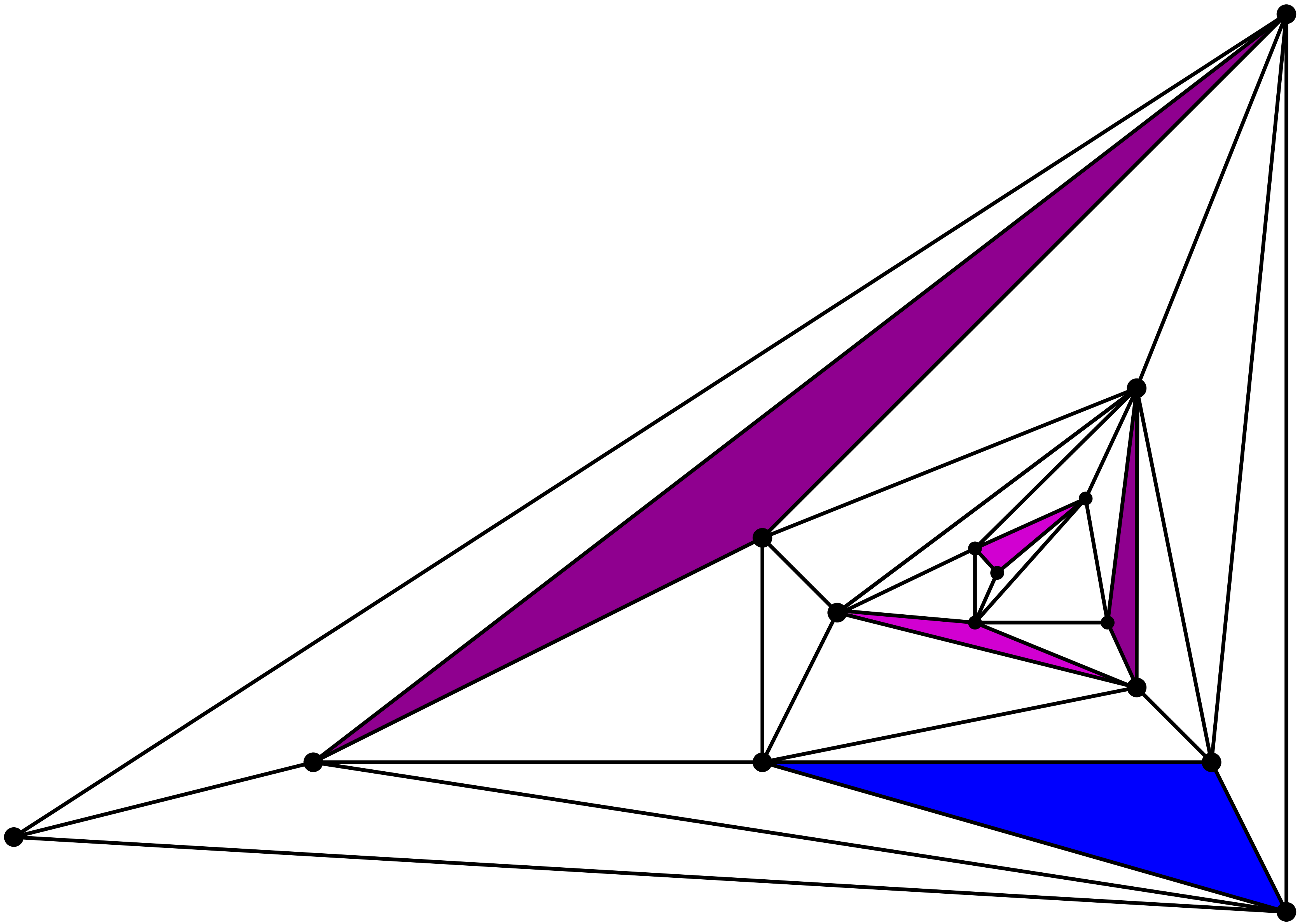
path length more than doubles!



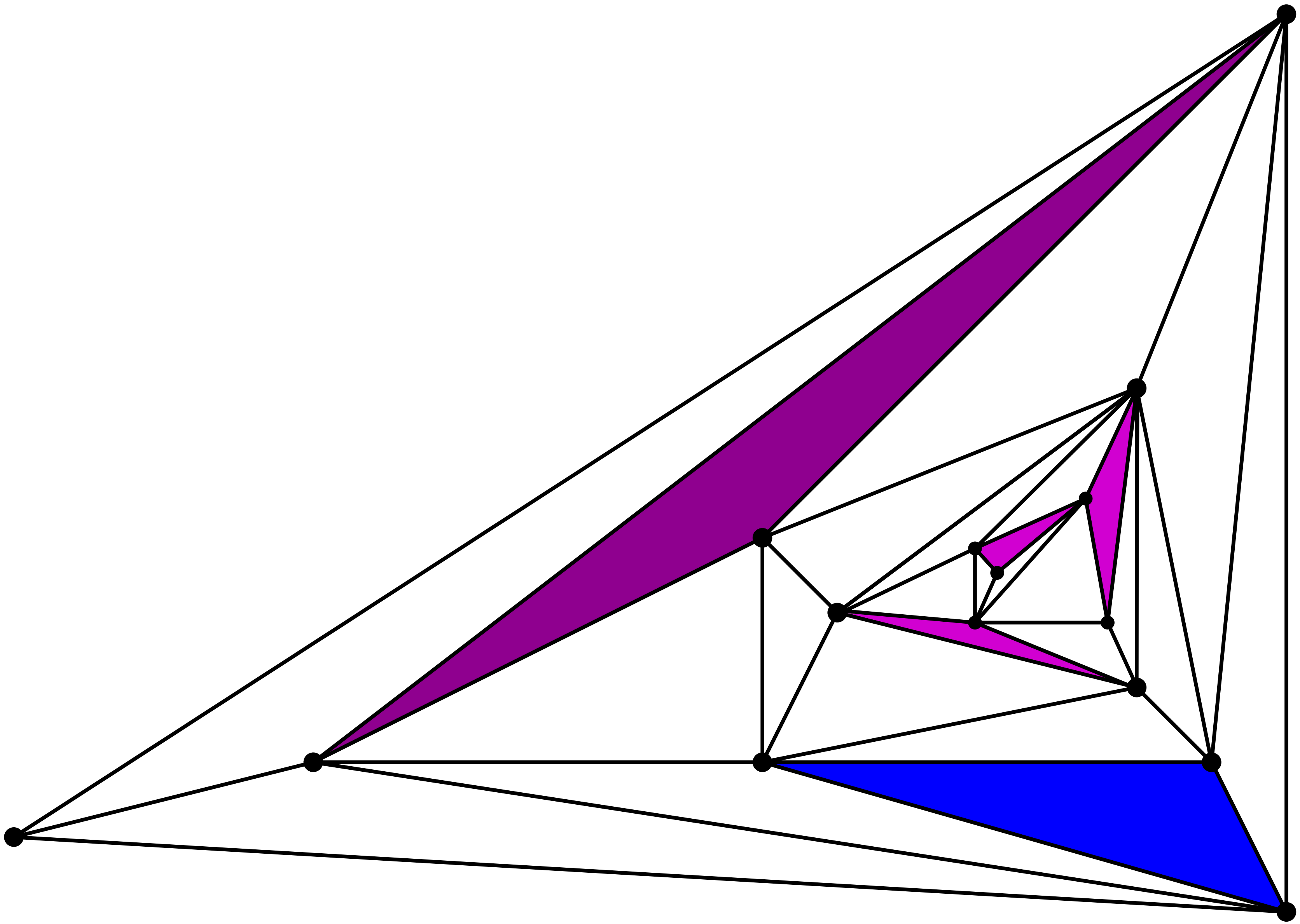
path length more than doubles!



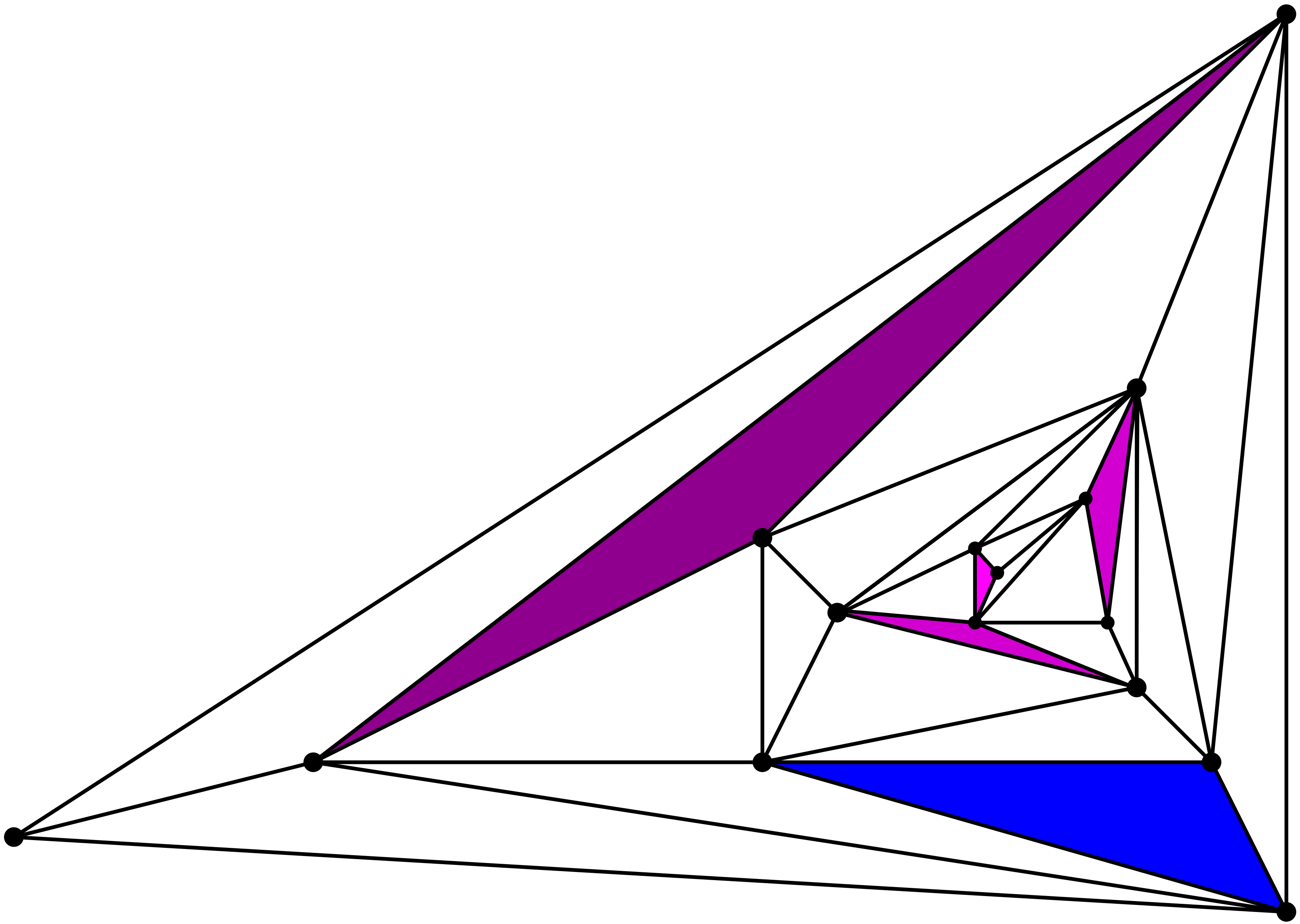
path length more than doubles!



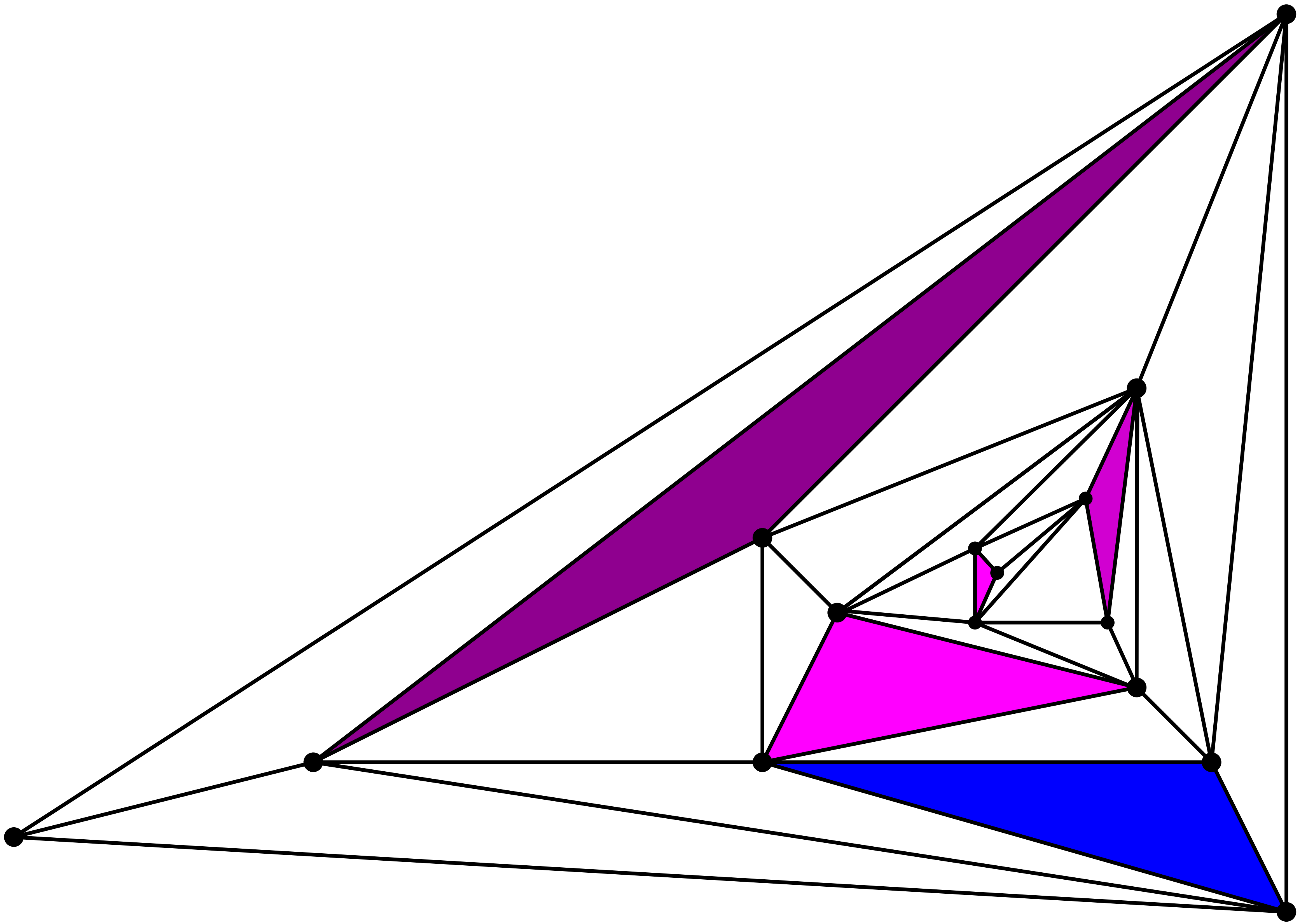
path length more than doubles!



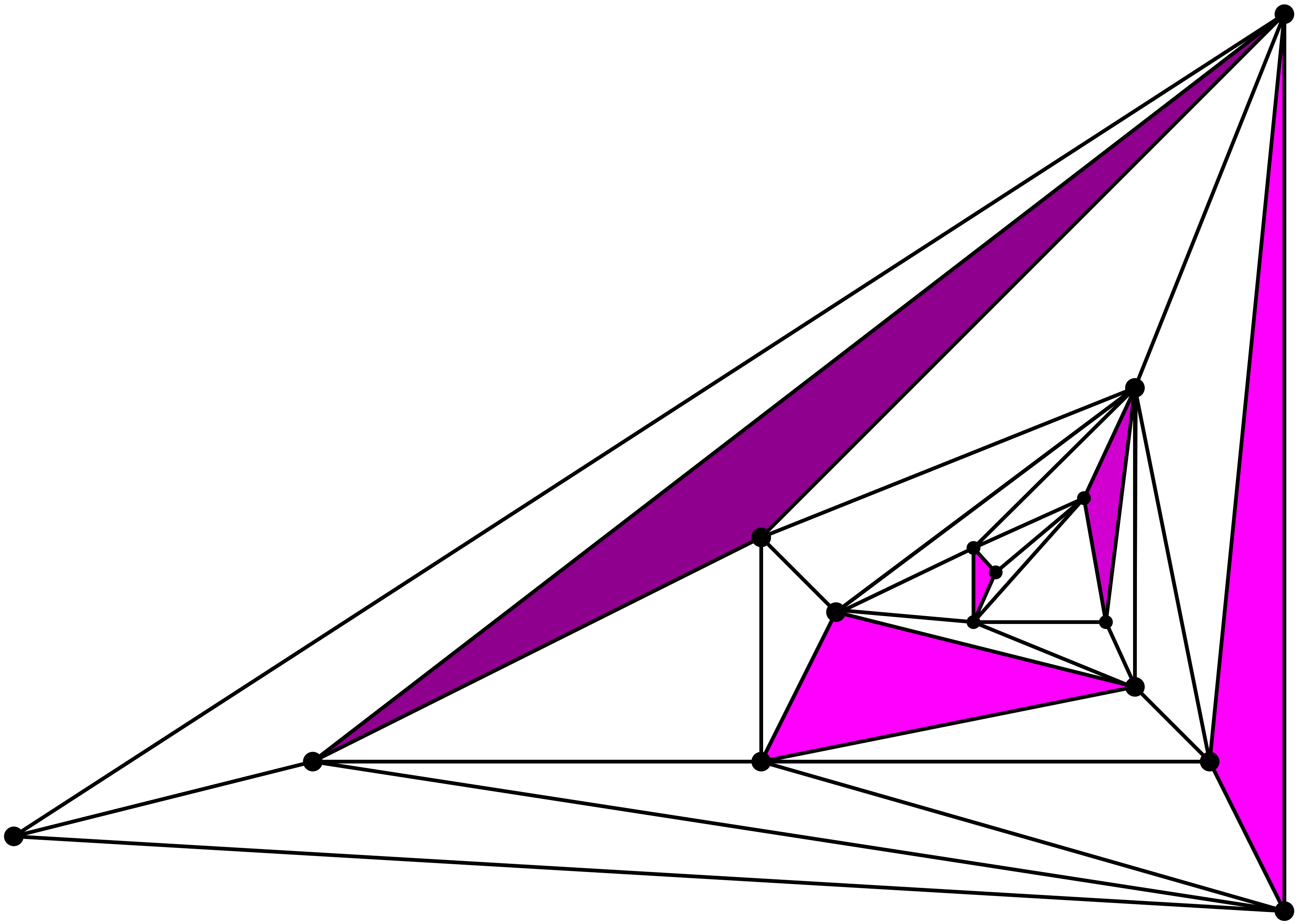
path length more than doubles!



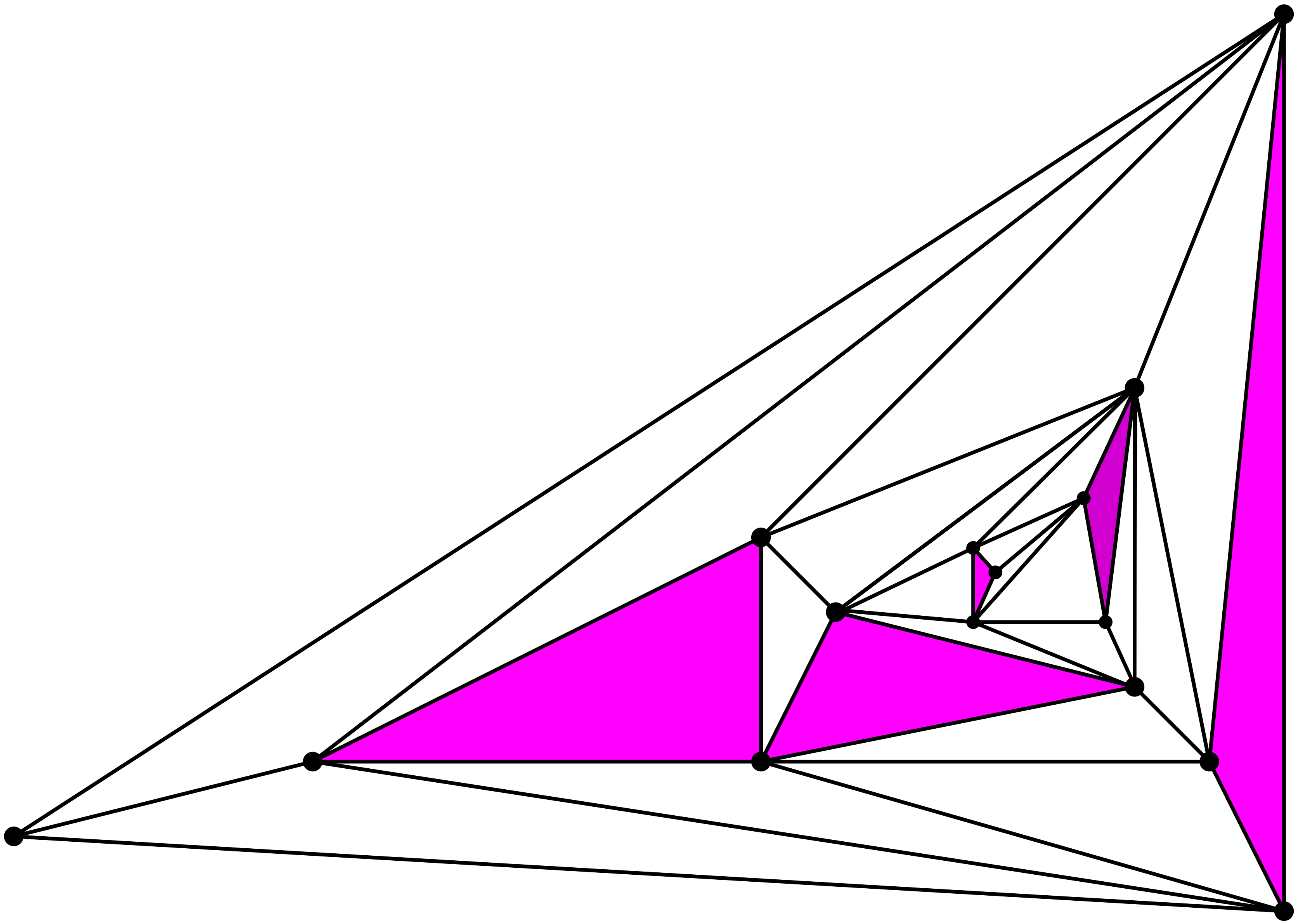
path length more than doubles!



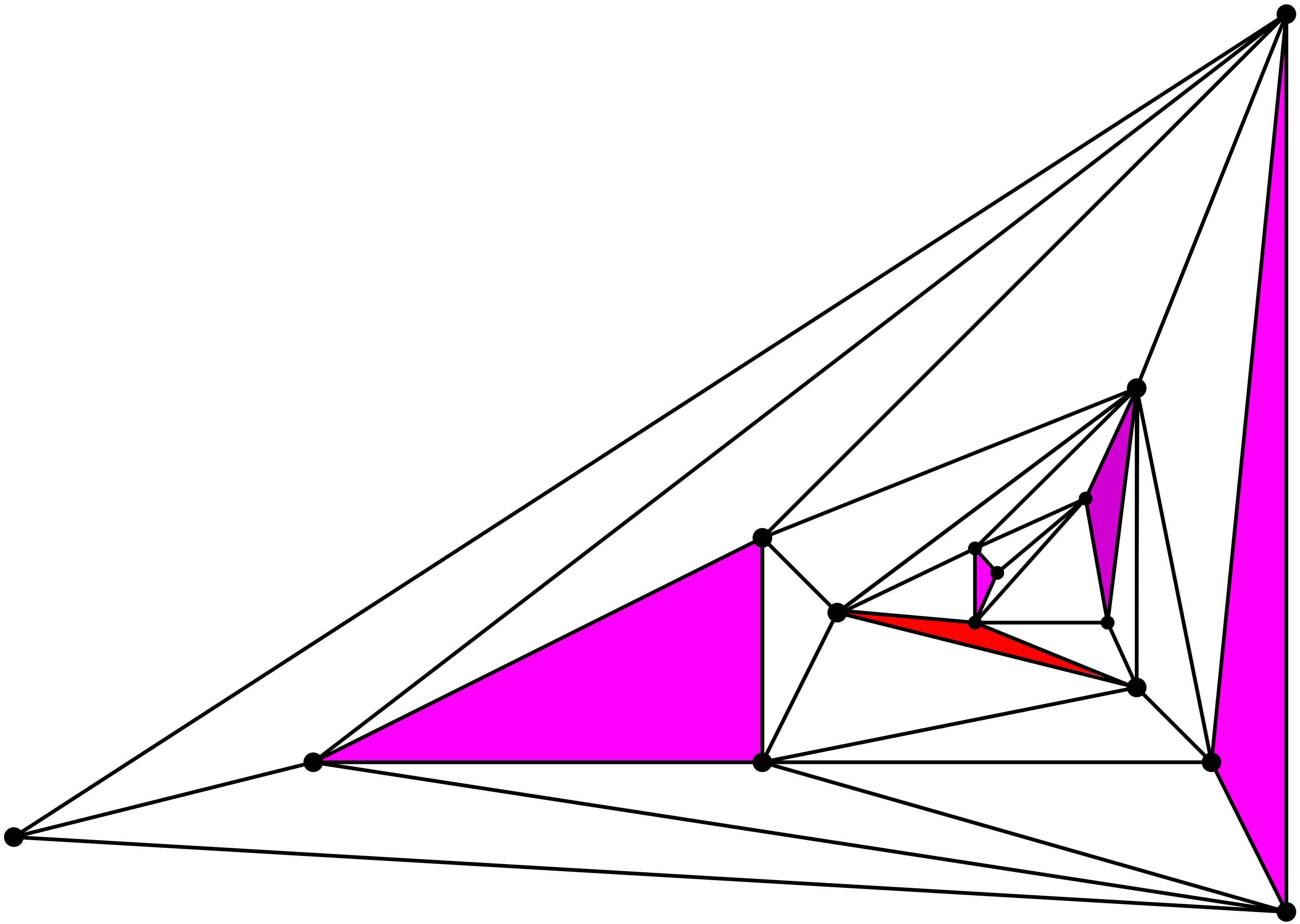
path length more than doubles!



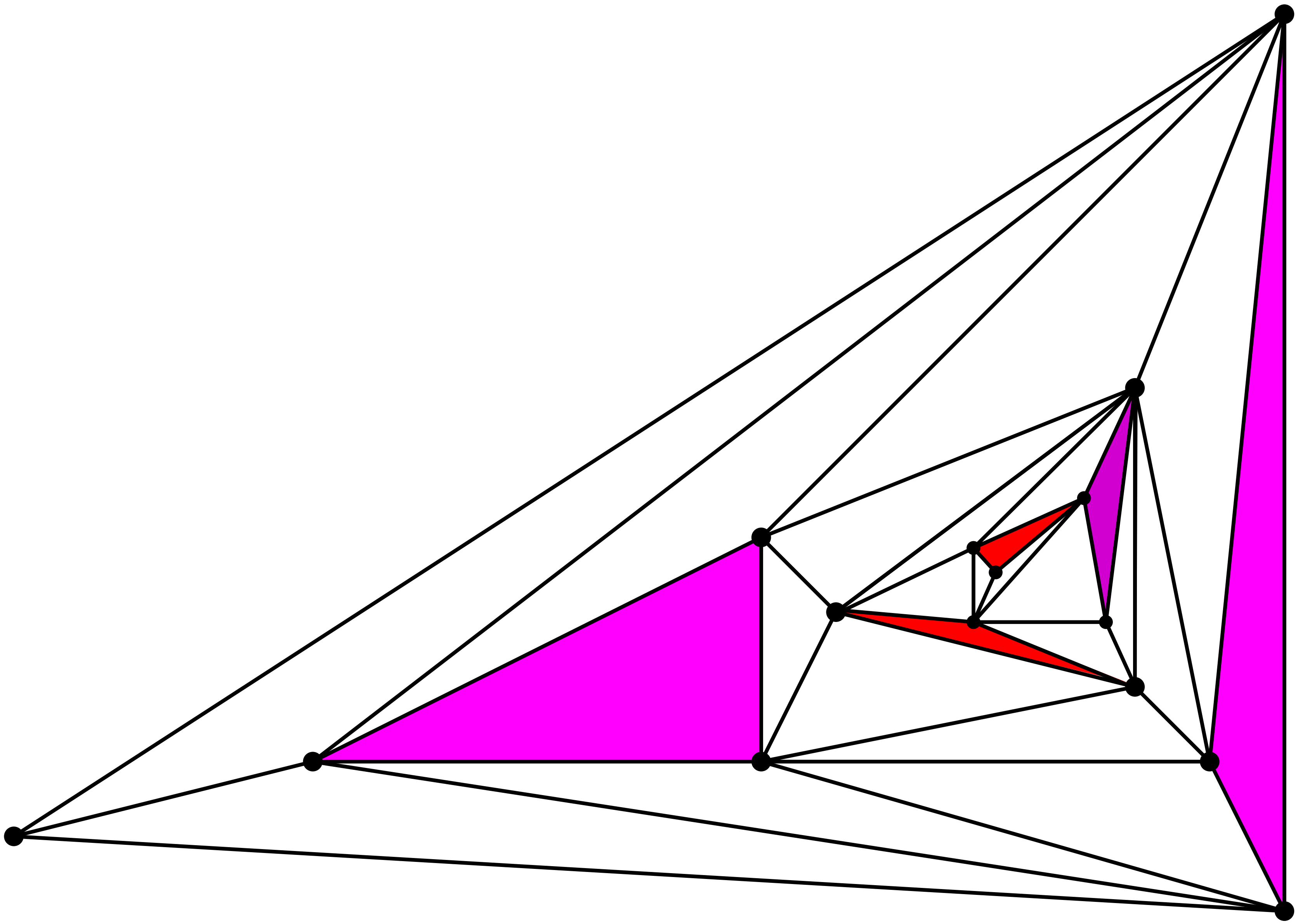
path length more than doubles!



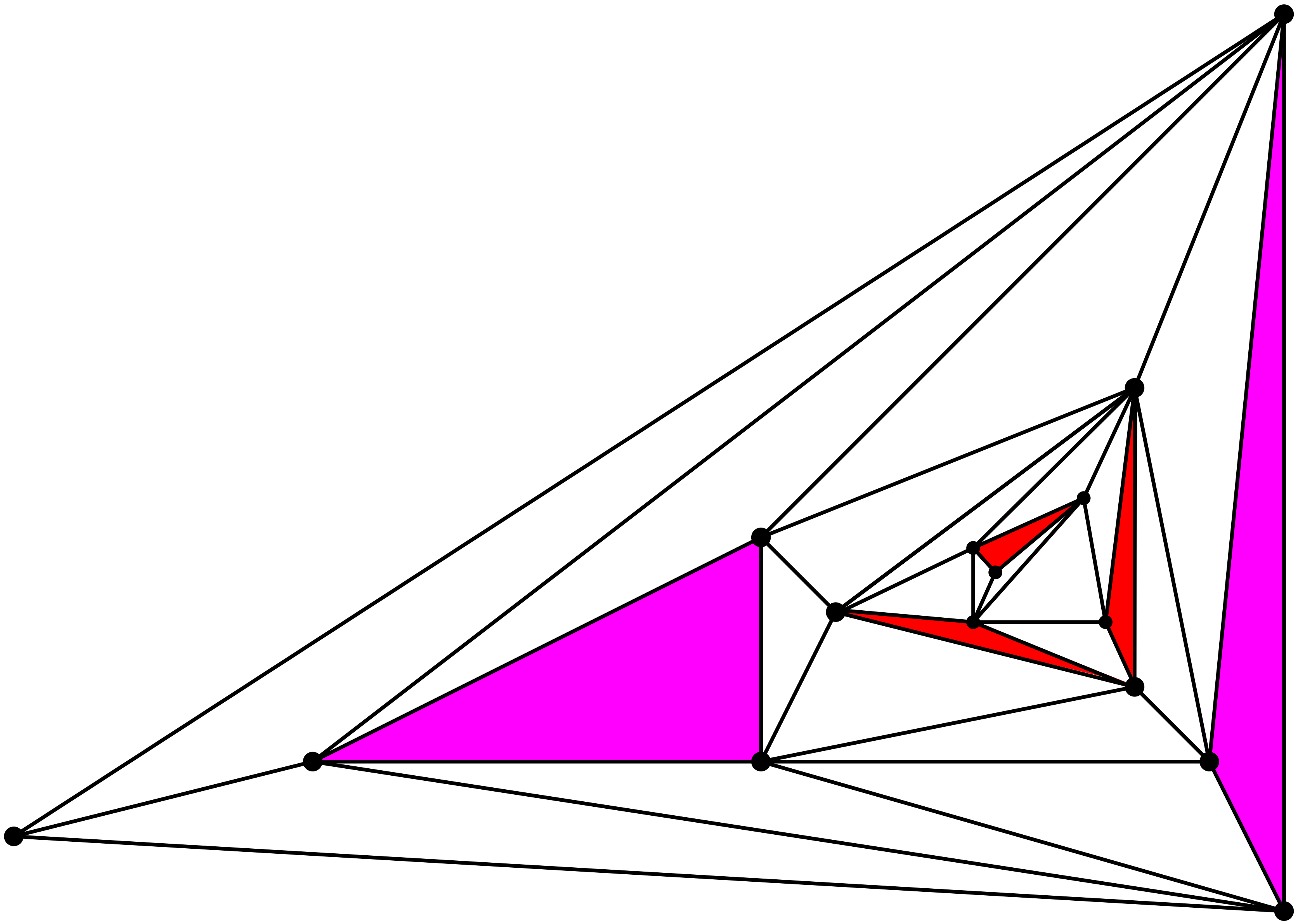
path length more than doubles!



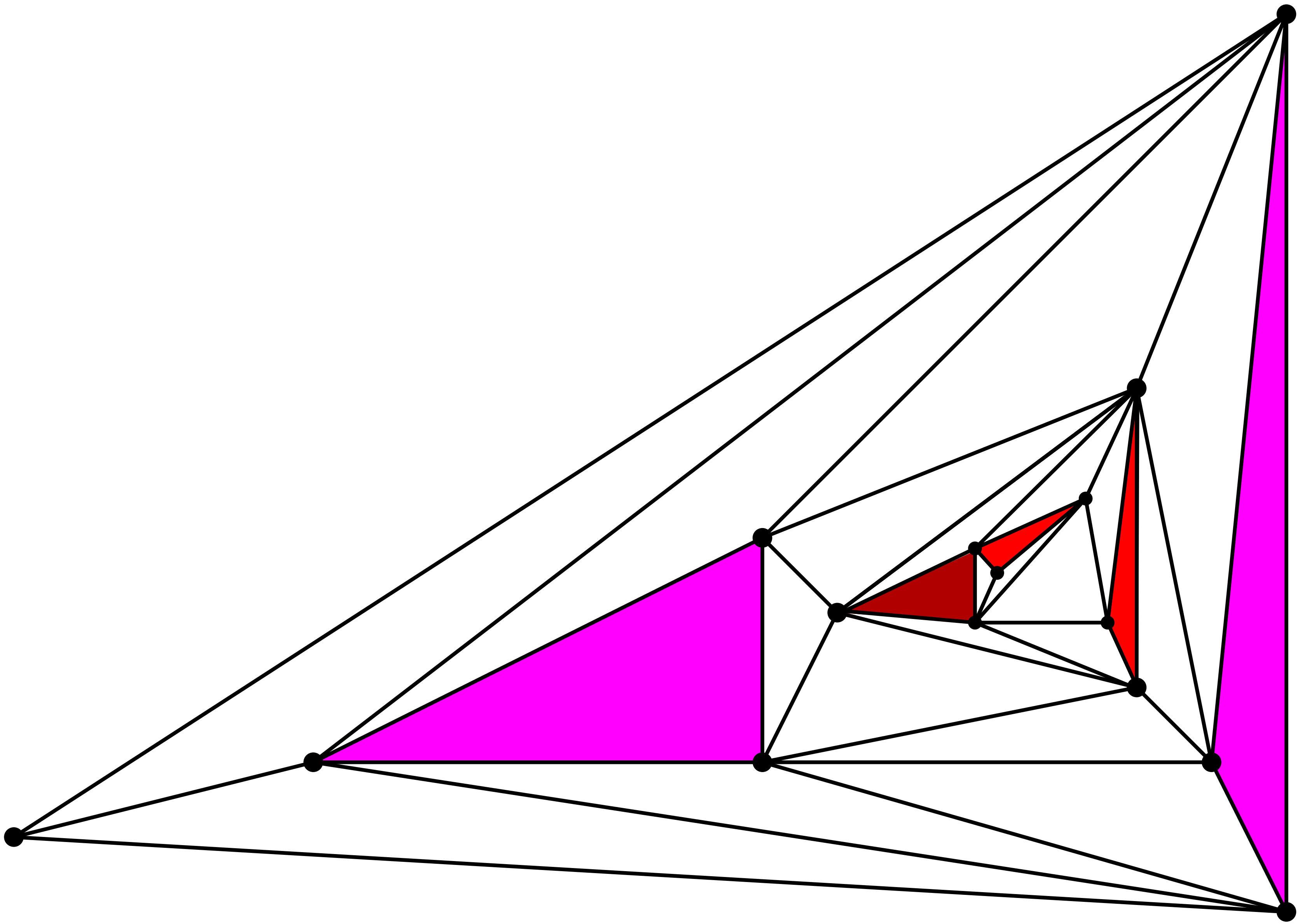
path length more than doubles!



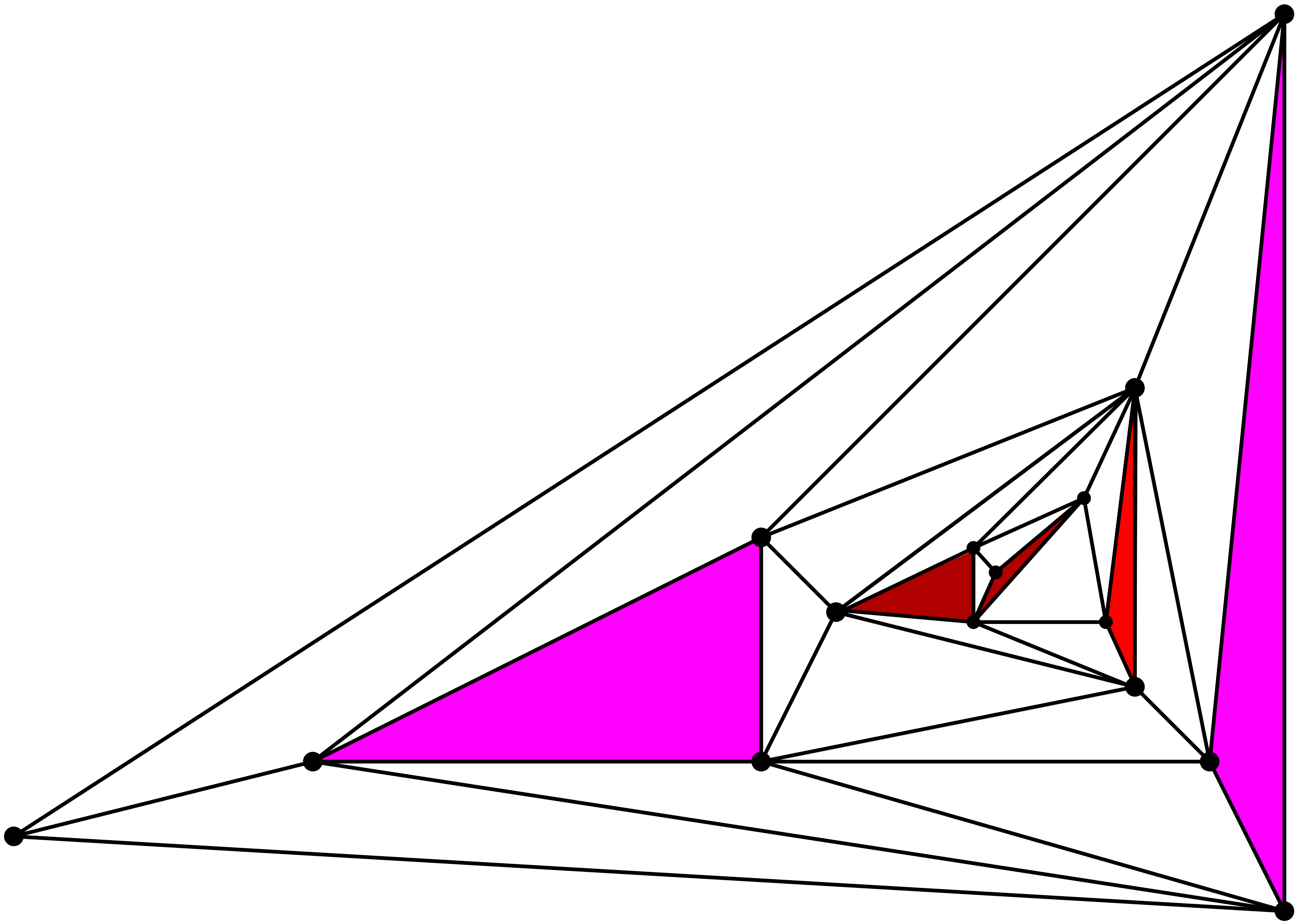
path length more than doubles!



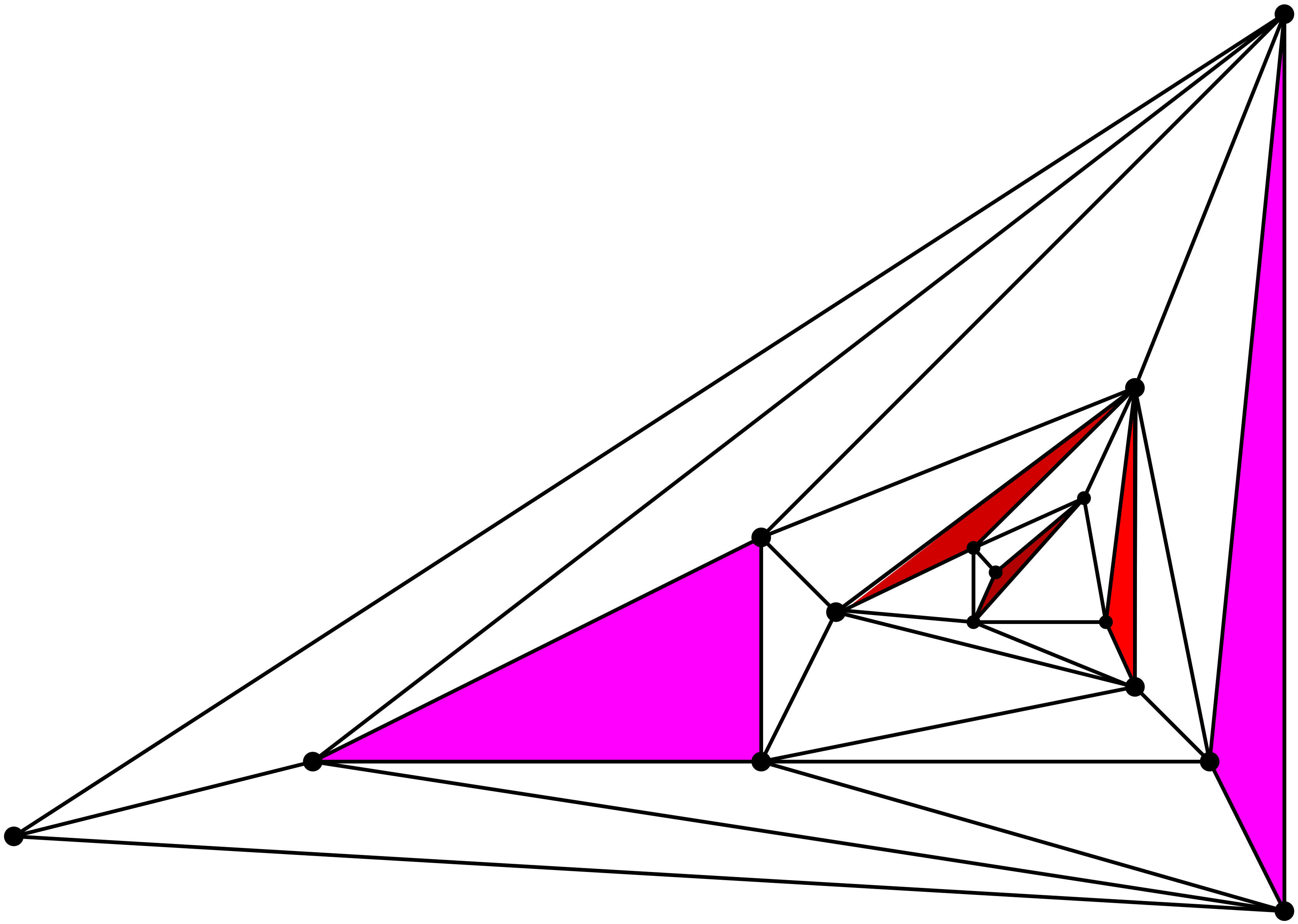
path length more than doubles!



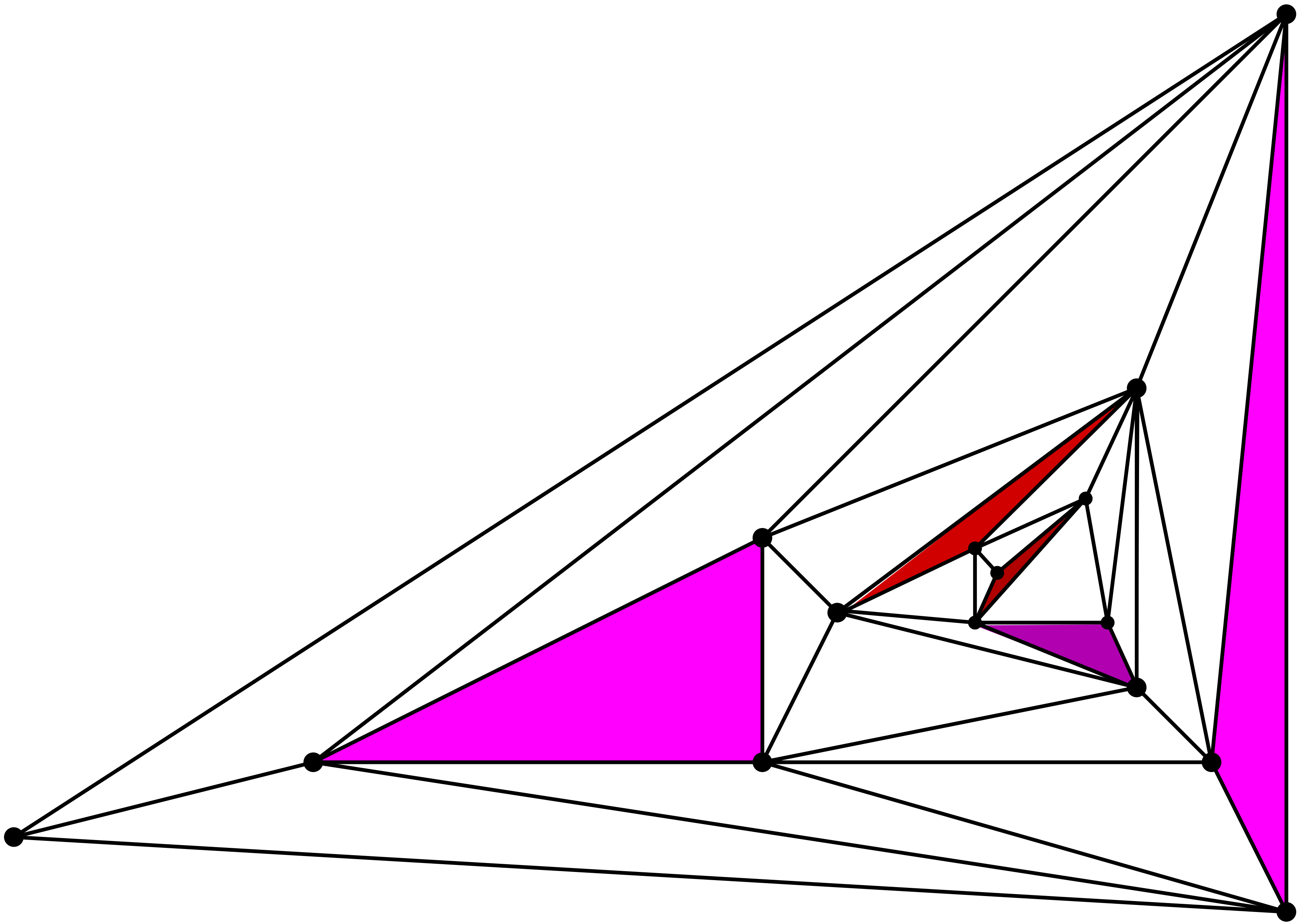
path length more than doubles!



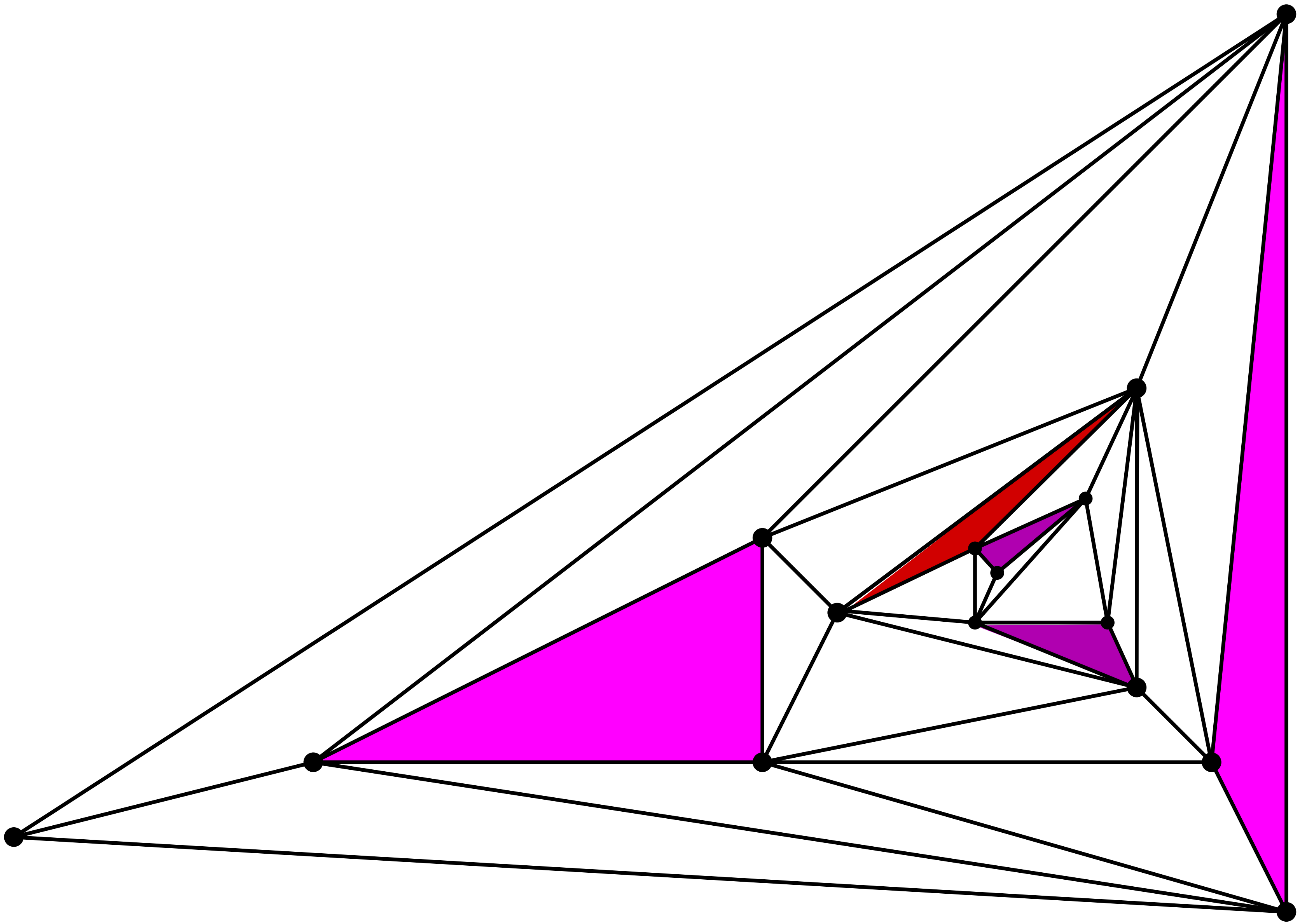
path length more than doubles!



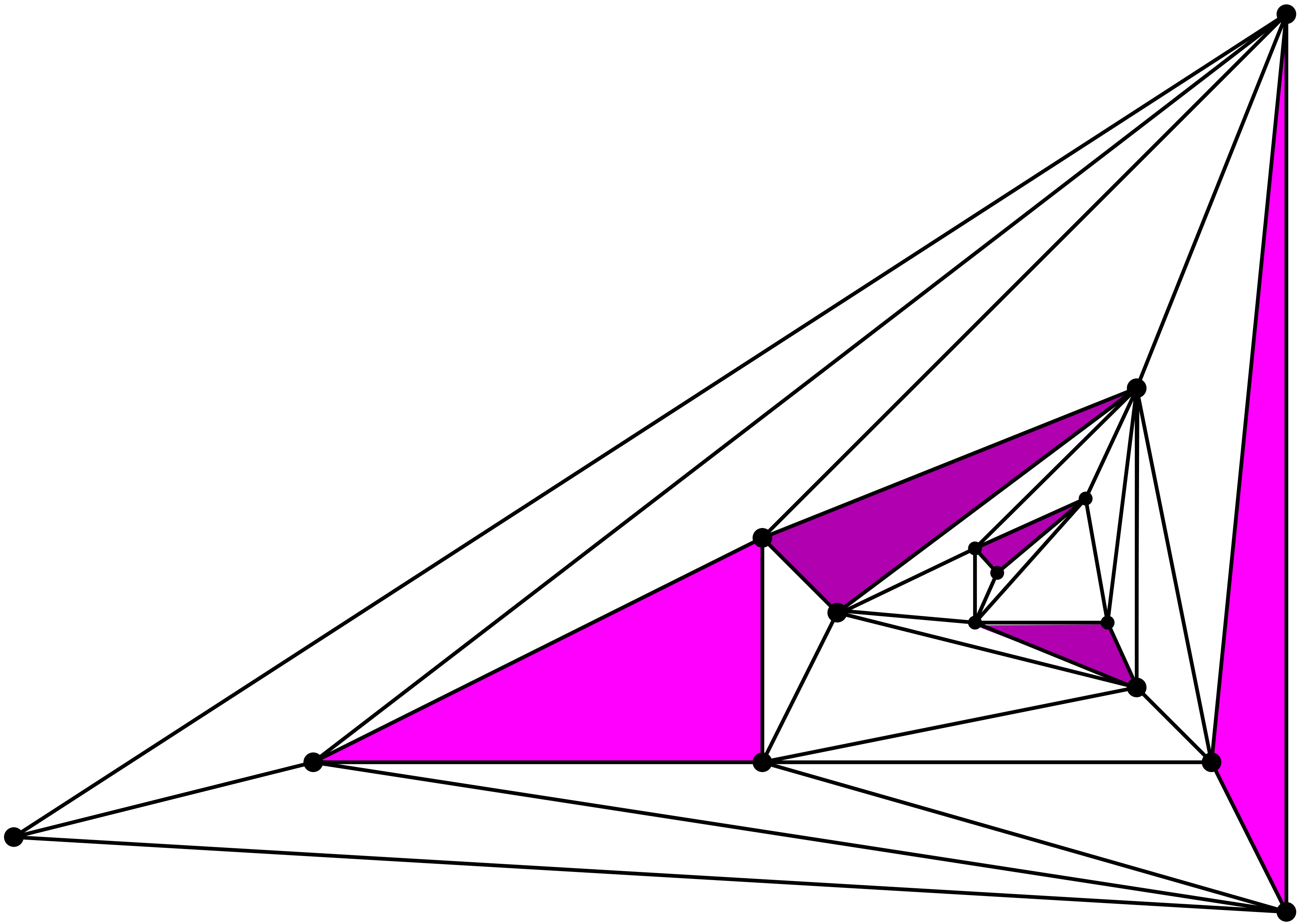
path length more than doubles!



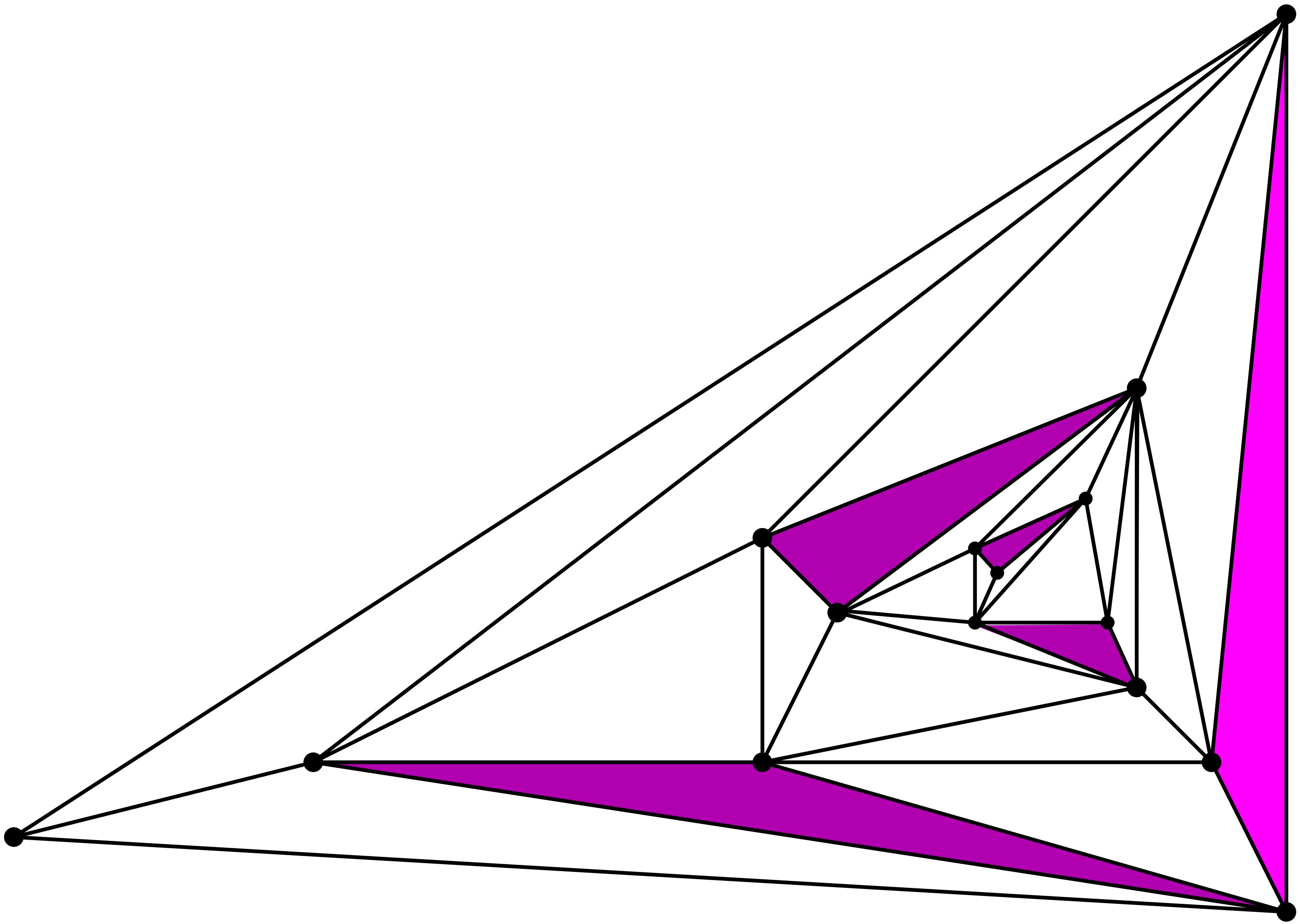
path length more than doubles!



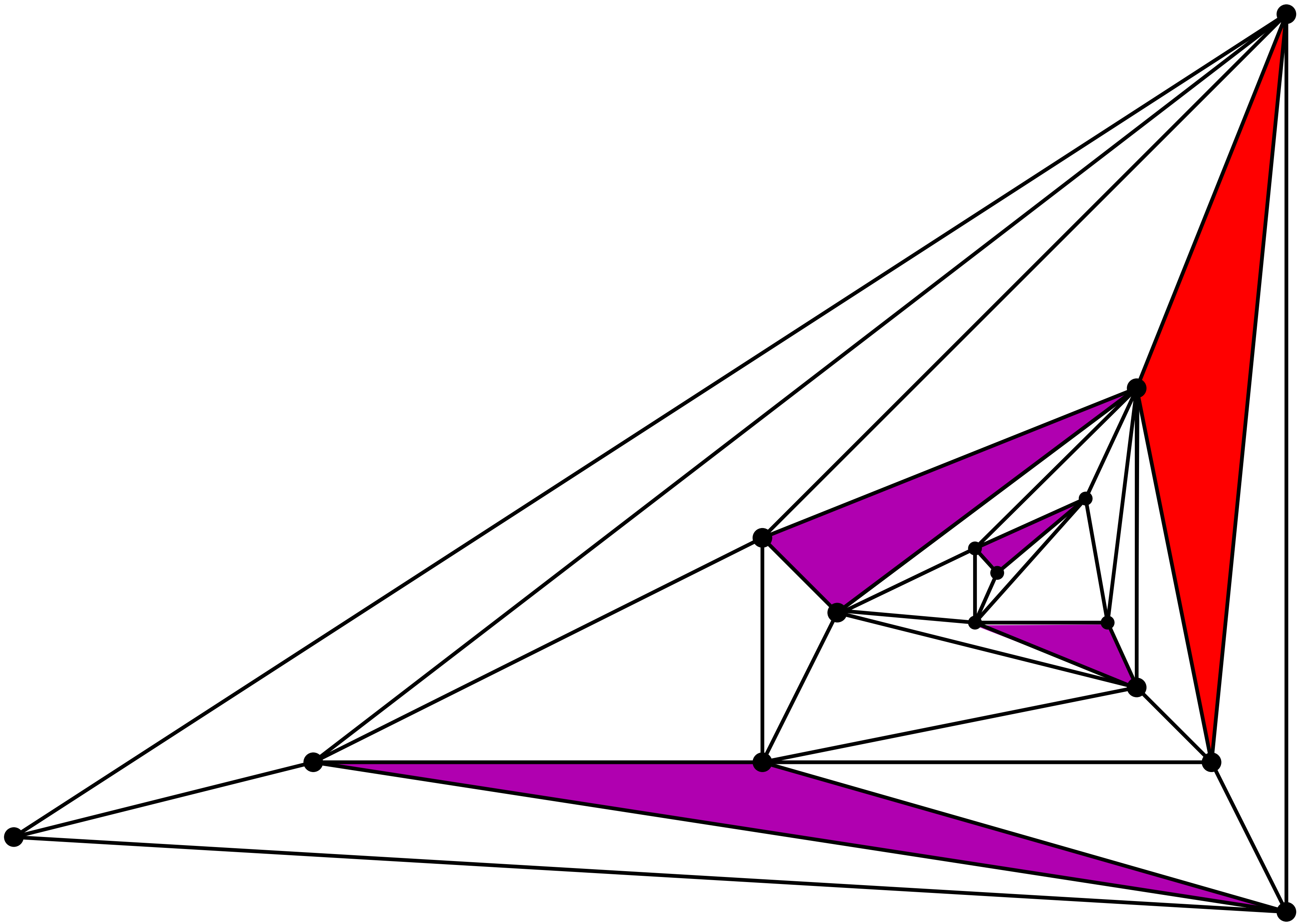
path length more than doubles!



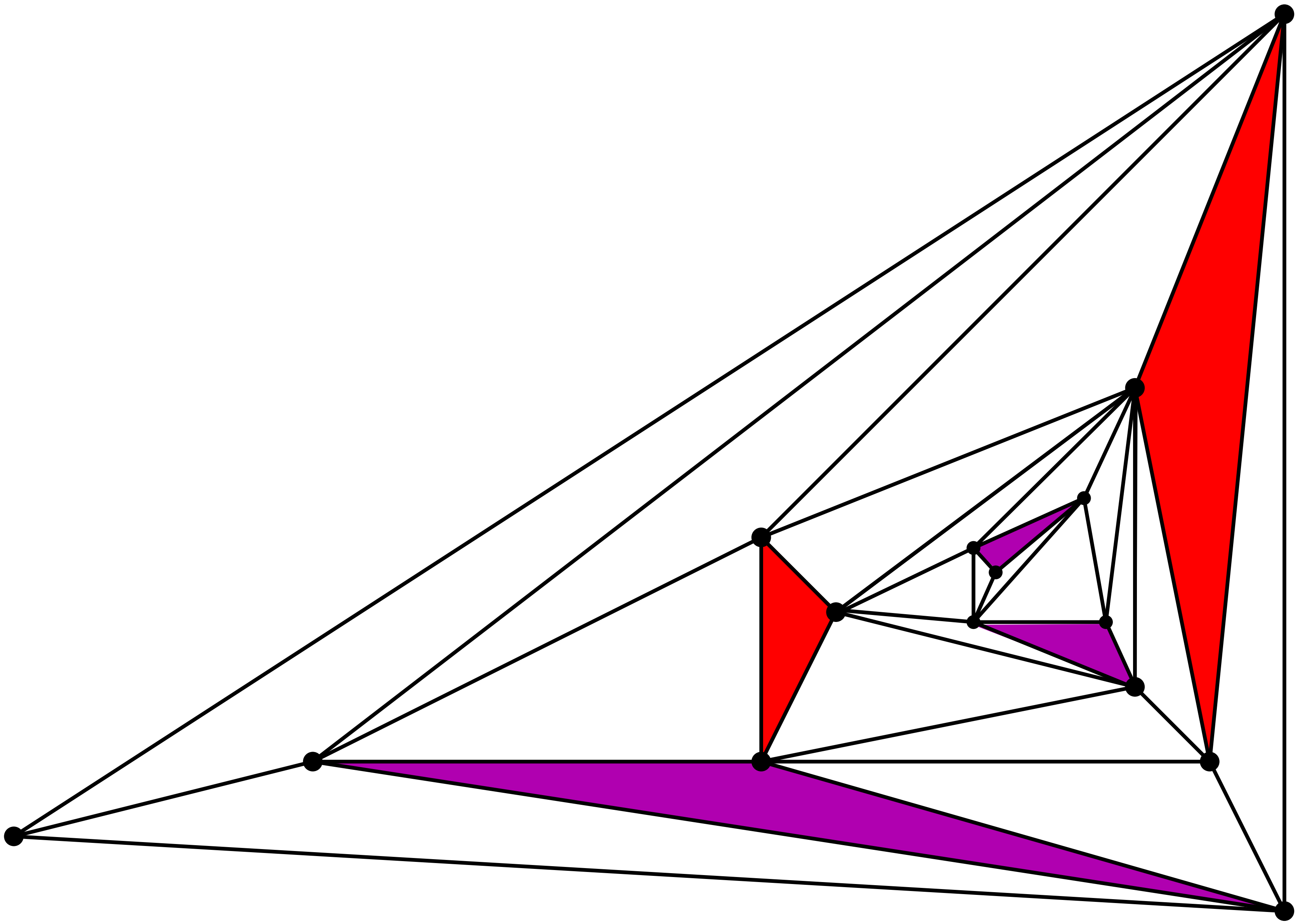
path length more than doubles!



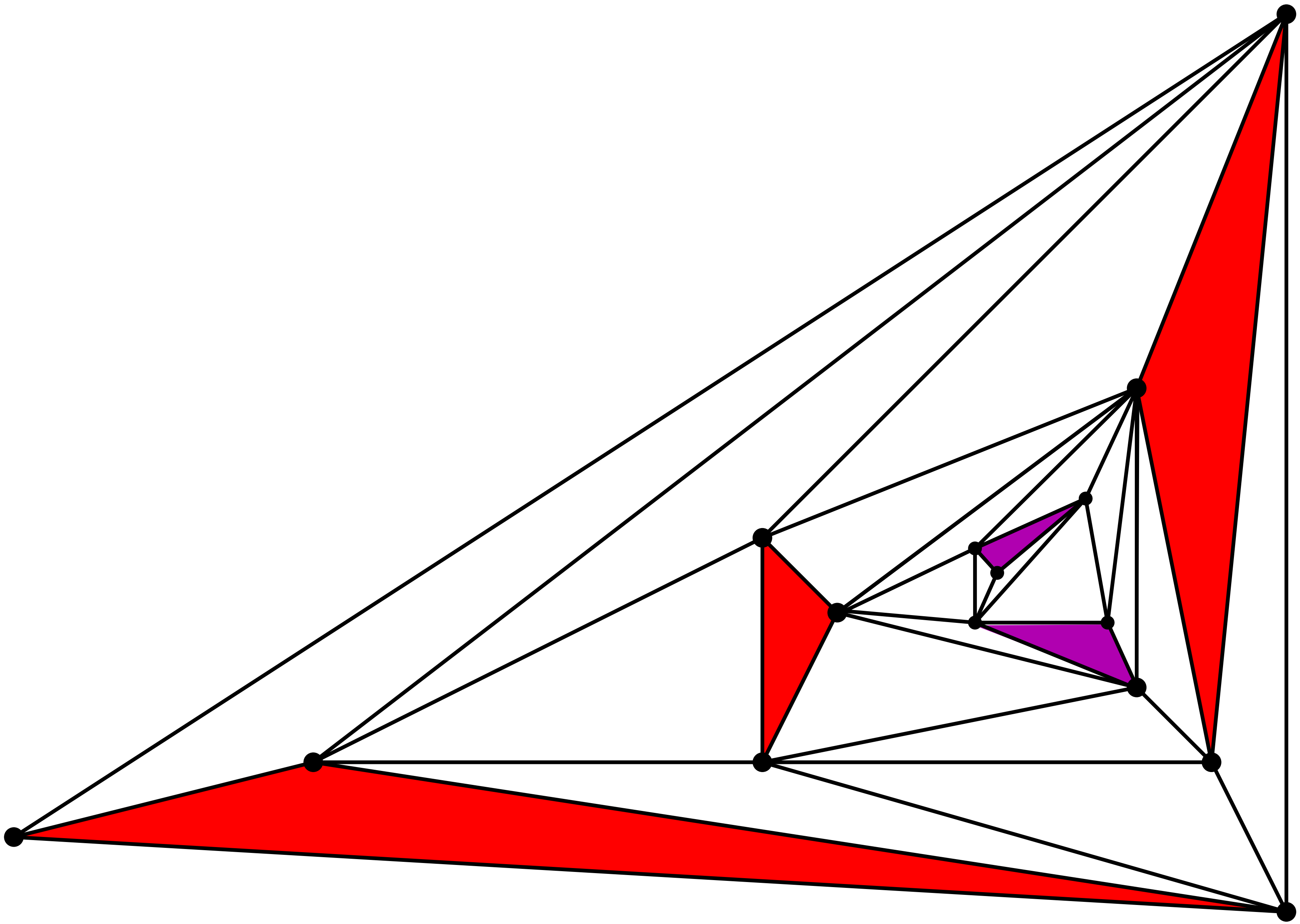
path length more than doubles!



path length more than doubles!



path length more than doubles!



Exponential length of paths

Note:

The path length is exponential in the number of **rooms**.

(The manifold is explicitly given as a list of rooms!)

Open problems

- **Wanted**: a painless path-preserving PPAD-completeness proof for 2-NASH.

- Is "**Find a second room partitioning**" in PPAD?
  [So far: direction only for ordered partitions.]

- ... **PPA-complete**?
  [No such problem known.]