U.S.S.R. Comput.Maths.Math.Phys.,Vol.28,No.5,pp.85-91,1988
Printed in Great Britain

0041-5553/88 $10.00+0.00

CYCLIC GAMES AND AN ALGORITHM TO FIND MINIMAX CYCLE MEANS IN DIRECTED GRAPHS*

V.A. GURVICH, A.V. KARZANOV and L.G. KHACHIVAN

An algorithm is described that finds optimal stationary strategies in dynamic two-person conflicts with perfect information, deterministic transitions, finite sets of positions, and time-averaged limiting integral payoff.

1. Introduction.

We describe an algorithm that finds optimal stationary strategies in dynamic two-person conflicts with perfect information, antagonistic interests, and time-averaged limiting integral payoff. We call the problem described in this paper a cyclic game, and it includes as a special case the standard combinatorial optimization problem which finds a cycle of maximum average edge cost in a diagraph. Other special cases of cyclic games are ergodic extensions of matrix games and finite position games with perfect information. Cyclic games themselves are a special case of stochastic games with perfect information.

A cyclic game may be given the following interpretation. A dynamic system with a finite state set V occupies one of the states $v(t)\in V$ at each instant of time $t=0, 1, \ldots$. The system dynamics is described by the transition digraph $G=(V,E)$, where the edge $e=vu\in E$ signifies that a transition is allowed from state $v=v(t)$ to state $u=v(t+1)$ at the instant t. We assume that there are no multiple edges in the graph G, but loops $e=vv$ are allowed. The graph G is leafless, i.e., for each vertex $v\in V$ the set $E(v)$ of outgoing edges is non-empty. A payoff function $c: E\to\mathbb{R}$ is defined on the edges of the graph, and the control problem is to maximize this payoff function in the mean

$$\lim_{t\to\infty}\Big[\sum_{\tau=0}^{t-1} c(v(\tau)v(\tau+1))\Big]t^{-1}\to\max$$

along the path $\{v(0)\ldots v(t)\ldots\}$. Denote by $V(v)$ the set of states reachable by the system in one step from the state v i.e., the set of end vertices of the edges in $E(v)$. If at each instant the choice of a new state $v(t+1)\in V(v(t))$ is in our power, i.e., the system is completely controllable in any state, then the problem obviously reduces to finding a maximum average cost cycle reachable from the initial state $v(0)$. (Effective strongly polynomial algorithms are available for finding such cycles /1, 2/). For each state $v\in V$, there exists a choice s of a transition $u\in V(v)$, independent of time and the initial state $v(0)$, which optimizes the path payoff as $t\to\infty$. The mapping $s: v\to V(v)$ is naturally interpreted as an optimal stationary control strategy of the system over an infinite time interval.

Now let the state set V of the system of partitioned into two non-intersecting subsets A and B ($A\cup B=V, A\cap B=\varnothing$), so that the choice of the transition $u\in V(v)$ is in our power only when the state $v(t)$ is contained in the set A of controllable states. Following the guaranteed payoff concept, we assume that in the states $v(t)\in B$ the choice of the transition $u\in V(v)$ is left to the opponent; the states $v\in A$ and $v\in B$ are respectively called White and Black *positions* (in a White position, White makes the next move, and in a Black position, Black makes the next move). Consider the pair of mappings

$$s_A: v\to V(v) \text{ for } v\in A,\qquad s_B: v\to V(v) \text{ for } v\in B,$$

which are called *stationary strategies* of White and Black. If we fix a pair of stationary strategies s_A, s_B of both players, then for a given initial position $v=v(0)$, the path-average limiting payoff will be equal to the mean payoff $\bar c(s_A,s_B,v)$ on the cycle that the system reaches in the limit. The function $\bar c(s_A,s_B,v)$ defined on the direct product of the finite sets of White and Black stationary strategies specifies a cyclic game in normal form. The same game in extensive form is defined by specifying the game network (G,A,B,c) and the initial position v.

We will see from what follows that, for any initial position $v\in V$, we have the discrete minimax identity

$$\max_{s_A}\min_{s_B}\bar c(s_A,s_B,v)=\min_{s_B}\max_{s_A}\bar c(s_A,s_B,v)=p(v), \tag{1}$$

which indicates the existence of optimal White and Black stationary strategies in the cyclic game. As in the case of finding the maximum cycle mean $V=A$, the optimal stationary strategies of both players may be chosen as uniform strategies, i.e., strategies independent

*Zh.vychisl.Mat.mat.Fiz.,28,9,1406-1417,1988 of the initial position v (although naturally different initial positions may have a different price $p(v)$). We can thus speak of optimal stationary strategies for the game network (G,A,B,c), which for brevity we sometimes call simply a network. Now, if we factorize the set of positions V by all possible values $p_1<\ldots<p_m$ of the price $p(v)$, then the resulting partition $(V_1,\ldots,V_m)$ of the transition graph vertices has the following properties:

1) for each i, the subgraph in G generated by V_i is leafless;

2) for any $1\leqslant i<j\leqslant m$, the sets of edges $E(V_i\cap A, V_j)$ and $E(V_j\cap B, V_i)$ are empty, i.e., among the edges of G, White has no transitions from the classes V_i with lower indices to the classes V_j with higher indices, while Black conversely has no transitions from the classes V_j with higher indices to the classes V_i with lower indices.

A vertex partitioning of a two-colour leafless graph (G,A,B) having these properties will be called ergodic (and non-trivial for $m\geqslant 2$). Thus, price factorization of positions produces an ergodic partitioning with each ergodic class V_i characterized by one price for all positions, and the optimal behaviour of White and Black is such that, starting in a class V_i, the player will never leave this class. If this partitioning is trivial, i.e., the prices of all the positions $v\in V$ are equal, the game network (G,A,B,c) is called ergodic. Finally note that if White follows an optimal stationary strategy, then Black cannot reduce his average limiting loss per step below $p(v)$ even by switching to non-stationary strategies, and conversely, so that the equilibrium (1) is maintained with non-stationary strategies also.

In order to prove the assertions stated above and to describe an algorithm that finds optimal stationary strategies in game networks, let us consider potential transformations $c'(vu)=c(vu)+\varepsilon(v)-\varepsilon(u)$ of payoffs on network edges, where $\varepsilon: V\to\mathbb{R}$ is an arbitrary real function defined on the network vertices, called the potential. Since the potential transformations $c\xrightarrow{\varepsilon}c'$ do not change the (average) cycle cost, they preserve the normal form of the cyclic game, and therefore the prices and the optimal strategy sets are the same in the networks (G,A,B,c) and (G,A,B,c'). Therefore, the payoff functions c' obtained by potential transformations will be called ***equivalent*** to the functions c.

There exists a function c' equivalent to c for which price determination and the search for optimal strategies in the network are trivial: in each move, White (Black) may choose a maximum (minimum) cost edge among the edge leaving the current position, and this cost will be preserved along the path. For the function $c: E\to\mathbb{R}$ and the vertex $v\in A\cup B$, define

$$\operatorname{ext}(c,v)=\begin{cases}\max\{c(vu): u\in V(v)\}, & \text{if } v\in A,\\ \min\{c(vu): u\in V(v)\}, & \text{if } v\in B,\end{cases}$$

and the vertex set

$$\mathrm{VEXT}(c,v)=\{u\in V(v): c(vu)=\operatorname{ext}(c,v)\}.$$

The edge vu such that $c(vu)=\operatorname{ext}(c,v)$ is called extremal (relative to c). We denote by $\|\cdot\|_\infty$ the componentwise maximum absolute value norm.

Theorem. Let (G,A,B,c) be an arbitrary game network.

1. There exist numbers $p(v)$, $v\in V$, and a function c' equivalent to c such that the following holds:

a) $p(v)=\operatorname{ext}(c',v)$ for all $v\in V$,

b) $p(v)=p(u)$ for $v\in A\cup B$ and $u\in \mathrm{VEXT}(c',v)$,

c) $p(v)\geqslant p(u)$ for $v\in A$ and $u\in V(v)$,

d) $p(v)\leqslant p(u)$ for $v\in B$ and $u\in V(v)$,

e) $\|c'\|_\infty\leqslant 2n\|c\|_\infty$, where $n=|V|$.

2. The numbers $p(v)$ satisfying the conditions a)-d) are uniquely defined for a given network.

The function c' and the network (G,A,B,c') satisfying conditions a)-d) of the theorem will be called canonical. Since all the assertions preceding the theorem are obviously true for a canonical network, part 2 of the theorem does not require a separate proof. We also clearly have the following corollary.

Corollary. Any mappings $s_A: A\to V$ and $s_B: B\to V$ such that $s_A(v)$ and $s_B(v)$ belong to $\mathrm{VEXT}(c',v)$ are optimal stationary (uniform) strategies of White and Black (possibly not all the optimal stationary strategies are generated in this way).

Note that the existence of prices and optimal stationary strategies in cyclic games can be deducted from the general theorem on stationary optimal strategies in stochastic games /3/, which is proved non-constructively using the fixed-point theorem. A cyclic game for the case of a complete bipartite graph G with parts A and B (ergodic extensions of matrix games) is considered in /4/ and the existence of a potential transformation $c\xrightarrow{\varepsilon}c'$ such that $\operatorname{ext}(c', v)=\mathrm{const}$ is shown. This last constant is called the ergodic price of the matrix game $c(A,B)$. The proof in /4/ also relies on the fixed point theorem.

The purpose of this paper is to describe an algorithm which reduces to canonical form an arbitrary network (G,A,B,c) with an integer-valued payoff function $c: E\to\mathbb{Z}$. We thus obtain a constructive proof of Theorem 1 for the integer case. The general (real-valued) case will follow by an appropriate passage to the limit in the payoffs.

Before describing the algorithm, let us consider three additional points.

1. A two-colour leafless transition graph (G,A,B) will be called ergodic if the game

network (G,A,B,c) is ergodic for any payoffs c on the edges. It is easy to see that a graph is non-ergodic if and only if it admits of a non-trivial ergodic partitioning $(V_1,\ldots,V_m)$, and without loss of generality this partitioning may be regarded as a cut $m=2$. An example of an ergodic graph is the complete bipartite graph with the parts A and B which corresponds to ergodic extensions of matrix games. V.N. Lebedev (graduate thesis, Moscow Physical-Technical Institute, 1987) has shown that the problem of deciding non-ergodicity of arbitrary two-colour leafless graphs is NP-complete.

2. Optimal stationary strategies in a cyclic game which attain the equilibrium (1) are called pure: they are deterministic and the transitions $s_A(v)$ and $s_B(v)$ are chosen without tossing a coin. If the players in the game are not antagonists, then a Nash equilibrium does not necessarily exist in pure stationary strategies s_A and s_B. A minimal example of this kind is supplied by a cyclic game on a complete bipartite graph with three White positions a_1, a_2, a_3 and three Black positions b_1, b_2, b_3, in which the White and Black payoff's on moving along the edge (a_i, b_j) in any direction are given by the matrices

$$c_A=\begin{Vmatrix} 0 & 0 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{Vmatrix}, \qquad c_B=\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\varepsilon & 0 & 1 \end{Vmatrix}$$

for sufficiently small ε, e.g., $\varepsilon=0.1$, and the initial position v is, say, $v=a_1$. White and Black have 27 stationary strategies each, and we can show by direct enumeration that the corresponding (27x27) bimatrix game with the payoff matrices $\bar{c}_A(s_A, s_B, v)$ and $\bar{c}_B(s_A, s_B, v)$ has no Nash equilibria. Thus, ergodic extension of the 3x3 bimatrix game does not necessarily have equilibria in pure stationary strategies. Using /5/, we can show that ergodic extensions of $2\times n$ bimatrix games always have an equilibrium in pure stationary strategies. This example reveals a fundamental difference between cyclic and position games with perfect information. It also shows that the existence of equilibrium in pure stationary strategies is not a direct consequence of perfect information and the integral payoff function averaged over $t\to\infty$.

3. Let us return to antagonistic (zero-sum) payoffs and assume that the game network, in addition to white positions $v\in A$ and black positions $v\in B$, also has change positions $v\in C$, each assigned its own probability distribution $\pi(vu)$ on the set $V(v)$ of allowed transitions:

$$\pi(vu)>0,\quad u\in V(v),\qquad \Sigma\{\pi(vu) : u\in V(v)\}=1.$$

Special cases of this model of a stochastic game with perfect information are Markov chains with reward $V=C$, controllable Markov chains $V=A\cup C$, cyclic games $V=A\cup B$, and the maximum cycle mean problem $V=A$. Consider a pair of pure stationary strategies s_A and s_B of White and Black players and assume that the prescribed moves are made with probability 1. As a result, we obtain a Markov chain with reward, in which, as we know, for any initial position $v\in V=A\cup B\cup C$ there exists a limit, as $t\to\infty$, of the expected payoff per move. This limit is preserved under potential transformations and we denote it as before by $\bar{c}(s_A, s_B, v)$. The discrete minimax identity (1) remains valid for the function $\bar{c}(s_A, s_B, v)$ because the theorem on the reduction of a network to canonical form is extended to stochastic games in the following way: in chance positions $v\in C$, the value of

$$\mathrm{ext}(c, v)=\Sigma\{\pi(vu)c(vu) : u\in V(v)\}$$

is defined as the mean payoff over all allowed one-step transitions according to the assigned probability distributions ("expected direct reward" /6/), condition b) of the theorem is supplemented by the requirement

$$p(v)=\Sigma\{\pi(vu)p(u) : u\in V(v)\},\qquad v\in C,$$

of price conservation in the mean in chance positions, and finally the bound e) on the absolute value of payoffs in a canonical network is rewritten in the form $\|c'\|_\infty\leqslant 2n\|c\|_\infty/\gamma$. where γ is the smallest among the non-zero transition probabilities $\pi(vu)$ assigned to chance positions. In particular, the corollary of the theorem remains valid, and given the canonical form of a stochastic game with perfect information, we can find the optimal pure stationary strategies for this game. The "three-colour" version of the theorem can be further generalized to discounted payoffs (see /3/) if we use potential transformations of the form $c'(vu)=c(vu)+\varepsilon(v)-\alpha\varepsilon(u)$, where $\alpha\in[0,1)$ is the discounting factor, so that the average limiting payoff corresponds to discounting as $\alpha\to 1$. However, the combinatorial algorithms described below cannot be extended to these cases, and in what follows we will accordingly consider only cyclic games.

2. An auxiliary algorithm.

The algorithm that reduces the network (G,A,B,c) to canonical form incorporates a procedure using the auxiliary algorithm described below, which is first applied to the original network, and then to some of its subnetworks with transformed edge costs. The auxiliary algorithm is finite for an arbitrary input real payoff function c. We describe it using the same notation as above. The input of the auxiliary algorithm is some network (G,A,B,c). Let

$$M=M(G,A,B,c)=\max\{\mathrm{ext}(c,v) : v\in V\},$$

$$m=m(G,A,B,c)=\min\{\mathrm{ext}(c,v) : v\in V\},$$

and

$$p := (M+m)/2.$$

p keeps the assigned value during the entire execution of the auxiliary algorithm. Note that if $M-m=0$, then the network is canonical (and ergodic), and therefore in what follows we assume that $M-m>0$.

The auxiliary algorithm produces the function c' equivalent to c such that on all the network vertices

$$m \leqslant \mathrm{ext}(c', v) \leqslant M \tag{2}$$

and one of the following three assertions is true:

A) $\mathrm{ext}(c', v) \geqslant p \ \forall v \in V$;

B) $\mathrm{ext}(c', v) \leqslant p \ \forall v \in V$;

C) a regular partitioning (V', V'') of the vertices of the network (G,A,B,c') has been found; this means that (V', V'') is an ergodic partitioning of (G, A, B), $\mathrm{ext}(c', v) \leqslant p$ for $v \in V'$ and $\mathrm{ext}(c', v) \geqslant p$ for $v \in V''$, $\mathrm{VEXT}(c', v) \cap V' \neq \varnothing$ for $v \in V'$ and $\mathrm{VEXT}(c', v) \cap V'' \neq \varnothing$ for $v \in V''$

In cases A) or B), the auxiliary algorithm dichotomizes the range $[m, M]$ of ext on the network. In case C), when a regular partitioning has been found, the network (G,A,B,c') can be reduced to canonical form by reducing to canonical form, independently of each other, the two game subnetworks generated by the ergodic blocks V' and V'', again cutting in each subnetwork the range of ext by one half compared with the initial range. After further recursive application of the auxiliary algorithm, the blocks V' and V'' may in turn again split, and so on, until ergodic classes are obtained.

The auxiliary algorithm runs in iterations, subjecting the function c to potential transformations. Let c be the current function in the execution of the algorithm. The vertex $v \in V$ is called critical, insufficient, or redundant if $\mathrm{ext}(c, v)$ is equal to, less than, or greater than p respectively. The sets of critical, insufficient, and redundant vertices will be denoted by $K(c)$, $S(c)$, and $T(c)$, and the set of white (black) positions among them will be denoted by $K^A(K^B)$, $S^A(S^B)$, $T^A(T^B)$. Similarly, the edge vu is called critical, insufficient, or redundant if $c(vu)$ is equal to, less than, or greater than p respectively. Note that an edge leaving a critical vertex is critical if and only if it is extremal.

The iterations of the algorithm apply potential transformations of a special kind. The function $c': E \to \mathbb{R}$ is called a shift of the function $c: E \to \mathbb{R}$ by an amount δ relative to the vertex set L if

$$c'(vu) = \begin{cases} c(vu) + \delta & \text{for } vu \in E(L, \bar{L}), \\ c(vu) - \delta & \text{for } vu \in E(\bar{L}, L), \\ c(vu) & \text{for other edges,} \end{cases}$$

where $\bar{L}$ denotes the set $V \backslash L$; clearly, the shift transformation is a potential transformation with potential $\varepsilon(v) = \delta$ for $v \in L$ and $\varepsilon(v) = 0$ for $v \in \bar{L}$.

The algorithm preserves the property of monotonicity: for each vertex v, $|\mathrm{ext}(c, v) - p|$ is non-increasing and the sign of $\mathrm{ext}(c, v) - p$ is conserved. In particular, if a vertex becomes critical at some instant of time, it will remain critical until the algorithm stops (although the set of outgoing critical edges may change). Clearly, the monotonicity property ensures that inequality (2) holds. From the monotonicity property it also follows that the sets of insufficient vertices S and redundant vertices T are non-increasing by inclusion from iteration to iteration. If one of them becomes empty, then we have case A) or B) and the auxiliary algorithm stops. Therefore, in describing an iteration of the auxiliary algorithm, we assume that for the current function c, the sets $S=S(c)$ and $T=T(c)$ are non-empty.

Step 1. Construct a labelled vertex set L such that:

1) $S \subseteq L \subseteq V - T$;

2) every edge leading from $L^A = L \cap A$ to $\bar{L}$ is insufficient and every edge leading from $\bar{L}^B = \bar{L} \cap B$ to L is redundant, which is equivalent to the conditions

$$\mathrm{VEXT}(c, v) \subseteq L \quad \forall v \in K^A \cap L, \qquad \mathrm{VEXT}(c, v) \subseteq \bar{L} \quad \forall v \in K^B \cap \bar{L};$$

3) for every labelled critical Black vertex, there is at least one outgoing critical edge that leads to a labelled vertex, and similarly for each unlabelled critical White vertex there is at least one outgoing critical edge leading to an unlabelled vertex:

$$\mathrm{VEXT}(c, v) \cap L \neq \varnothing \qquad \forall v \in K^B \cap L,$$
$$\mathrm{VEXT}(c, v) \cap \bar{L} \neq \varnothing \qquad \forall v \in K^A \cap \bar{L}.$$

The set L is constructed as follows. Put $L_0 = S$ and successively find sets of white and black critical vertices $X_1, Y_1, X_2, Y_2, \ldots$ by the following rule: assume that all these sets have already been determined up to X_i and Y_i inclusive and let $L_i = S \cup X_1 \cup Y_1 \cup \ldots \cup X_i \cup Y_i$; then

$$X_{i+1} = \{v \in K^A - L_i : \mathrm{VEXT}(c, v) \subseteq L_i\},$$
$$Y_{i+1} = \{v \in K^B - L_i : \mathrm{VEXT}(c, v) \cap L_i \neq \varnothing\}.$$

Continue the process until the next sets X_{r+1} and Y_{r+1} are both empty. Finally, put $L=L_r$.

Remark. If we assume that all the non-critical edges originating from critical vertices have been removed from the network, then the set L is obtained by adjoining to S those critical vertices from which Black forces a move to S.

Step 2. Check whether the partition $(L, \bar{L})$ is regular for the network (G,A,B,c) with the current function c. To this end, identify the sets of edges

$$E_1=E(L^A, \bar{L}) \quad \text{and} \quad E_2=E(\bar{L}^B, L),$$

and also the vertex sets

$$\hat{S}=\{v\in S^B : \mathrm{VEXT}(c, v)\subseteq\bar{L}\}, \qquad \hat{T}=\{v\in T^A : \mathrm{VEXT}(c, v)\subseteq L\}.$$

If all these four sets are empty, the auxiliary algorithm stops on case B) and $(L, \bar{L})$ is the sought regular partitioning of the network vertices.

Step 3. Shift the current function c relative to the labelled set L by a positive δ. Choose the maximum shift δ without violating the monotonicity property. It is defined as $\delta=\min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, where

$$\delta_1=\min\{p-c(vu) : vu\in E_1\}, \qquad \delta_2=\min\{c(vu)-p : vu\in E_2\},$$

$$\delta_3=\min\{p-\mathrm{ext}(c, v) : v\in\hat{S}\}, \qquad \delta_4=\min\{\mathrm{ext}(c, v)-p : v\in\hat{T}\},$$

and, as usual, the minimum over the empty set is taken equal to $+\infty$. Note that properties A)-C) of the labelled set guarantee strict positivity of the shift δ.

This ends the iteration. If for the new function we have $m(G, A, B, c')\geqslant p$ and $M(G, A, B, c')\leqslant p$ (by monotonicity, these inequalities may be satisfied only as equalities), the auxiliary algorithm stops on cases A) or B). Otherwise go to the next iteration, and so on. When the algorithm stops, the current function c is the c' delivered on the output of the auxiliary algorithm.

Let us prove that the number of iterations of the auxiliary algorithm is finite. Suppose that in iteration j we had the sets S and T of insufficient and redundant vertices and constructed in Step 1 the sets of critical vertices $X_1, Y_1, \ldots, X_r, Y_r$, where r is the smallest index for which both sets X_{r+1} and Y_{r+1}, are empty. With this iteration we associate a sequence of integers $q(j)=\gamma, \alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$, where $\gamma=|S|+|T|$ and $\alpha_i=|X_i|$, $\beta_i=-|Y_i|$ for $i=1, 2, \ldots, r$. The finiteness of the number of iterations follows from the next lemma.

Lemma. Let $q(j)=\gamma, \alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$ and $q(j+1)=\gamma', \alpha_1', \beta_1', \ldots, \alpha_{r'}', \beta_{r'}'$ be the sequence associated with two successive iterations j and $j+1$. Then $q(j+1)$ is lexicographically less than $q(j)$.

Proof. Objects in the $(j+1)$-th iteration will be denoted by primes, and objects in the j-th iteration will be unprimed. Consider the ordered families of sets $\mathcal{F}=(S\cup T, X_1, Y_1, \ldots, X_r, Y_r)$ and $\mathcal{F}'=(S'\cup T', X_1', Y_1', \ldots, X_{r'}', Y_{r'}')$ in iterations j and $(j+1)$.

1. We will show that $\mathcal{F}'$ is not identical with $\mathcal{F}$. Assume that this is not so. For the function c' in iteration $(j+1)$, from the definition of the shift δ, we have at least one of the following cases:

a) $c'(vu)=p$ for some $v\in L^A$ and $u\in\bar{L}$,

b) $c'(vu)=p$ for some $v\in\bar{L}^B$ and $u\in L$,

c) $\mathrm{ext}(c', v)=p$ for some $v\in S'$,

d) $\mathrm{ext}(c', v)=p$ for some $v\in T'$.

This leads to a contradiction with $\mathcal{F}=\mathcal{F}'$. Indeed, from $\mathcal{F}=\mathcal{F}'$ follows $L=L'$, but then in cases a) and c) the vertex v may not be in L', and in cases b) and d) it may not be in $\bar{L}'$.

2. Take the set Z in $\mathcal{F}$ closest to the beginning which is different from the set Z' with the same serial number in $\mathcal{F}'$ (if $\mathcal{F}$ and $\mathcal{F}'$ contain a different number of sets, pad the smaller of the two with $||\mathcal{F}|-|\mathcal{F}'||$ empty sets). Three cases are possible.

Case 1. $Z=S\cup T$. By the monotonicity property, the sets S and T may only decrease as we pass to the next iteration. Therefore, $\gamma'<\gamma$ and the lemma is true.

Case 2. $Z=X_i$, $i\geqslant 1$. We will show that in this case we have strong inclusion $X_i'\subset X_i$, whence follows the assertion of the lemma $\alpha_i'<\alpha_i$. Assume the contrary, i.e., there is a vertex $v\in X_i'-X_i$. First, $v\notin S\cup T$, since otherwise we would have $S'\cup T'\neq S\cup T$, contrary to the choice of X_i. Thus, $v\in K^A$. Let $L_{i-1}=S\cup X_1\cup Y_1\cup\ldots\cup X_{i-1}\cup Y_{i-1}$, and let L_{i-1}' be the corresponding set in the $(j+1)$-th iteration; by the choice of X_i, we have $L_{i-1}=L_{i-1}'$. From $v\notin X_i\cup L_{i-1}$ and $v\in X_i'$ it follows that $\mathrm{VEXT}(c, v)\not\subseteq L_{i-1}$ and $\mathrm{VEXT}(c', v)\subseteq L_{i-1}$. Then there should be an edge vu which is critical for the function c and non-critical for c':

$$c(vu)=\mathrm{ext}(c, v)=p \quad \text{and} \quad c'(vu)<\mathrm{ext}(c', v)=p.$$

Since c' is a shift of c relative to L by a positive δ, we have from the last conditions $v\in\bar{L}$ and $u\in L$. By property 3) (see Step 1), membership of v in the set $K^A\cap\bar{L}$ implies the existence of a critical edge vu' with an end u' in $\bar{L}$. But if $v\in\bar{L}$ and $u'\in\bar{L}$, then $c'(vu')=c(vu')=p$ and so $u'\in\mathrm{VEXT}(c', v)$ which contradicts $\mathrm{VEXT}(c', v)\in L_{i-1}\in L$.

Case 3. $Z=Y_i$, $i\geqslant 1$. We will show that $Y_i\subset Y_i'$ and $\beta_i>\beta_i'$. Indeed, from $v\in Y_i$ it follows that $\mathrm{VEXT}(c, v)\cap L_{i-2}=\varnothing$ and $\mathrm{VEXT}(c, v)\cap(X_{i-1}\cup Y_{i-1})\neq\varnothing$. By the choice of Y_i we have $L_{i-2}=L_{i-2}'$, and also $X_{i-2}\cup Y_{i-2}=X_{i-2}'\cup Y_{i-2}'$. Since for a critical vertex $v\in Y_i\subseteq K^B\cap L$, the positive shift δ will not convert non-critical edges into critical edges, we obtain that $\mathrm{VEXT}(c', v)\cap L'_{i-2}=\varnothing$. Moreover, since edges with both ends in L do not change their cost under a shift, we have as before $\mathrm{VEXT}(c', v)\cap(X'_{i-1}\cup Y'_{i-1})\neq\varnothing$, which implies that $v\in Y'_i$.

This completes the proof of the lemma, and hence the proof of the finiteness of the number of iterations of the auxiliary algorithm. Note that each iteration of the auxiliary algorithm may be completed in time $O(n^2)$.

3. Proof of the theorem and the algorithm reducing the network to canonical form.

As we have noted above, if the auxiliary algorithm reaches case C), i.e, finds a regular partitioning $(L, \bar{L})=(V', V'')$ of the game network then in order to prove Theorem 1 it suffices to establish its truth separately for each of the ergodic blocks V' and V''. We therefore assume that case C) is not realized. Repeated application of the auxiliary algorithm will then generate a potential transformation $c\xrightarrow{\varepsilon}c'$ of the payoff function of the game network, and the range $[m', M']$ of $\mathrm{ext}(c', v)$ on the network will be made arbitrarily small. We will show that this can always be achieved by a bounded potential transformation

$$\|\varepsilon\|_\infty\leqslant 2(n-1)\|c\|_\infty, \tag{3}$$

where c is the edge cost function in the original network. Indeed, the condition $\mathrm{ext}(c', v)\in[m', M']$ for known vertex sets $\mathrm{VEXT}(c', v)$ may be rewritten as a consistent system of linear inequalities

$$\begin{array}{llll} m'\leqslant c(vu)+\varepsilon(v)-\varepsilon(u)\leqslant M' & \text{for} \quad v\in V & \text{and} & u\in\mathrm{VEXT}(c', v), \\ c(vu)+\varepsilon(v)-\varepsilon(u)\leqslant M' & \text{for} \quad v\in A & \text{and} & u\in V(v), \\ m'\leqslant c(vu)+\varepsilon(v)-\varepsilon(u) & \text{for} \quad v\in B & \text{and} & u\in V(v) \end{array} \tag{4}$$

in $n=|V|$ unknown potentials, one of which may be set equal to zero. This system has a completely unimodular constraint matrix and the right-hand sides do not exceed $2\|c\|_\infty$ in absolute value. Hence it follows that it has a solution $\varepsilon(v)$, $v\in V$, whose components do not exceed $2(n-1)\|c\|_\infty$ in absolute value. Part 1 of the theorem now follows from the passage to the limit $M'-m'\to 0$ and compactness of the cube (3). This completes the proof of the theorem.

If the payoff function c of the original network is integer-valued, then in order to find a potential transformation $\varepsilon: V\to\mathbb{R}$ reducing the network to canonical form, it is sufficient to apply the auxiliary algorithm until the difference $M'-m'$ in the ergodic blocks does not exceed $1/(2n^2)$. Indeed, each of the numbers $p(v)$ is equal to the average cost c over the edges of some simple cycle, and it is therefore a rational number with denominator not greater than n. Therefore, if $M'-m'<1/(2n^2)$, $p(v)$ can be obtained using continued fractions. Now, consistency of the system (4) in each block implies consistency of the system

$$\begin{array}{llll} c(vu)+\varepsilon(v)-\varepsilon(u)=p(v) & \text{for} \quad v\in V & \text{and} & u\in\mathrm{VEXT}(c', v), \\ c(vu)+\varepsilon(v)-\varepsilon(u)\leqslant p(v) & \text{for} \quad v\in A & \text{and} & u\in V(v), \\ p(v)\leqslant c(vu)+\varepsilon(v)-\varepsilon(u) & \text{for} \quad v\in B & \text{and} & u\in V(v), \end{array}$$

and its solution can be obtained in a time n^3 (see, e.g., /7/) from the solution of system (4) generated by the auxiliary algorithm, although it does not necessarily belong to the cube (3). Thus, in the integer-valued case, the time to reduce the game network to canonical form does not exceed in order of magnitude $In^2\log(n\|c\|_\infty)+n^3$, where I is the number of auxiliary algorithm iterations.

An experimental study of the auxiliary algorithm carried out by V.N. Lebedev has shown that in most cases the number of iterations I is not more than a few times greater than the number of network vertices n, so that this algorithm can be successfully applied in game networks with hundreds of positions. This practical behaviour, however, clashes with Lebedev's example of a game network in which the number of iterations of the auxiliary algorithm is $2^{n/2-1}+1$ - a situation which is similar, to a certain extent, to the behaviour of the simplex method. This naturally raises the question of the existence of an algorithm polynomial in n and $\log\|c\|_\infty$ for reducing a game network with integer-valued payoff function to canonical form. Such a polynomial algorithm can indeed be constructed using the dynamic programming solution of "trimmed" games in finite time intervals and the technique of "digit-wise reduction of errors". Its description, however, is outside the scope of this paper.

REFERENCES

1. KARP R.M., A characterisation of the minimum cycle mean in a digraph, Discrete Math., 23, 3, 309-311, 1978.
2. KARZANOV A.V., On minimum mean weight cuts and cycles in a digraph, in: Qualitative and Approximate Methods of Investigating Operator Equations, Izd. YarGU, Yaroslavl. 1985.
3. MOULIN H., Prolongement des jeux a deux joueurs de somme nulle, Bull. Soc. Math. France, Mem.45, 1976.

4. PARTHASARATHY T. and RAGHAVAN T., Some Topics in Two-Person Games, Elsevier, New York, 1971.
5. GURVICH V.A., Solubility of position games in pure strategies, Zh. vychisl. Mat. mat. Fiz., 15, 2, 358-371, 1975.
6. HOWARD R.A., Dynamic Programming and Markov Processes. MIT Press, Cambridge, Mass., 1960.
7. KHACHIYAN L.G., On the exact solution of systems of linear inequalities and linear programming problems. Zh. vychisl. Mat. mat. Fiz., 22, 4, 999-1002, 1982.

Translated by Z.L.

U.S.S.R. Comput.Maths.Math.Phys.,Vol.28,No.5,pp.91-93,1988
Printed in Great Britain

0041-5553/88 $10.00+0.00

SHORT COMMUNICATIONS

THE WELL-POSEDNESS OF POTENTIAL-TYPE FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND FOR THIN WIRES*

V.A. STRIZHKOV

Regularizing spaces are constructed for investigating the integral equations of the thin-wire approximation. Convergence of the Galerkin method is established. Necessary conditions are established for the stability of certain numerical algorithms.

When handling problems of the diffraction and excitation of electromagnetic waves in compound three-dimensional wire structures one often encounters integral equations of the form

$$\mathscr{L}_n[I]\equiv\int_L I(s',\varepsilon)\frac{K(s,s',\varepsilon)}{[(s-s')^2+\varepsilon^2]^{n/2}}ds'=G(s),\qquad s\in L,\quad n=1,2,\ldots,\tag{1}$$

where $I(s,\varepsilon)$ is an unknown function (usually the axial current), $K(s,s',\varepsilon)$, $G(s)$ are known functions, L is a three-dimensional curve representing the structure of the antenna graph, and $\varepsilon\ll1$ is a small parameter (the ratio of the wire radius r to the wavelength). This is the case, for example, for the integral equations of Hallen, Mei, Pocklington etc. (see the bibliography in /1, 2/).

Setting up mathematical models of electrodynamic processes using the integral equations of the thin-wire approximation (1) requires a consideration of the following problems:

1) to check whether problem (1) is well-posed for small ε, i.e., to construct a space of functions in which the problem has a unique solution $I(s,\varepsilon)$ depending continuously on the right-hand side $G(s)$;

2) to estimate the accuracy to which the axial current $I(s,\varepsilon)$ approximates the real surface current $I_P(s,\varepsilon)$ and the accuracy with which the other electrodynamic quantities are determined;

3) to select the most efficient numerical algorithm for solving problem (1), and establish its convergence and its range of applicability.

The most natural solution space for Eq.(1) is the following regularizing class of functions:

$$V_n=\left\{I(s,\varepsilon)=\psi_n(\varepsilon)\sum_{k=0}^{\infty}\psi_n^k(\varepsilon)j_k(s,\varepsilon)\right\},$$

where $\psi_n(\varepsilon)$ are the essentially singular manifolds, determined by the order of singularity of the initial integral operator $\mathscr{L}_n$ as $\varepsilon\to+0$. In the case when $n=1$ we have $\psi_1(\varepsilon)=|\ln\varepsilon|^{-1}$ and $\psi_n(\varepsilon)=\varepsilon^{1-n}$ for any $n\geqslant2$. We have $j_k(s,\varepsilon)\in\mathring{W}_2^1(L)$ uniformly in ε, $\|I\|_{V_n}\equiv\psi_n^{-1}(\varepsilon)\|I\|_{W_2^1}$.

The existence and uniqueness of the solution in class V_n have been investigated for the following generalized problem:

$$(\varphi,\mathscr{L}_n[I])_{\mathscr{L}_2}=(\varphi,G)_{\mathscr{L}_2}\quad\forall\varphi\in\mathring{W}_2^1(L).\tag{2}$$

Theorem 1. Assume that: a) $K(s,s',\varepsilon)\in C^1(L\times L)$ uniformly in ε; b) $G(s)\in\mathscr{L}_2(L)$; c) $|K(s,s,0)|\geqslant\delta>0\ \forall s\in L$.

Then there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0]$ problem (2) has a unique solution in V_n, and moreover $\|I\|_{V_n}\leqslant C\|G\|_{\mathscr{L}_2}$.

---

*Zh.vychisl.Mat.mat.Fiz.,28,9,1418-1420,1988