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The 3-colored Ramsey number of odd cycles

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Abstract

For graphs $L_1, \ldots, L_k$, the Ramsey number $R(L_1, \ldots, L_k)$ is the minimum integer N satisfying that for any coloring of the edges of the complete graph K_N on N vertices by k colors there exists a color i for which the corresponding color class contains L_i as a subgraph.

In 1973, Bondy and Erdős conjectured that if n is odd and C_n denotes the cycle on n vertices, then $R(C_n, C_n, C_n) = 4n - 3$. In 1999, Łuczak proved that $R(C_n, C_n, C_n) = 4n + o(n)$, where $o(n)/n \to 0$ as $n \to \infty$. In this paper we strengthen Łuczak's result and verify this conjecture for n sufficiently large.

Keywords: Ramsey number, regularity lemma, stability

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1. Introduction

For graphs $L_1, \ldots, L_k$, the Ramsey number $R(L_1, \ldots, L_k)$ is the minimum integer N satisfying that for any coloring of the edges of the complete graph K_N by k colors there exists a color i for which the corresponding color class contains L_i as a subgraph. In the early 1970's, the Ramsey number $R(C_n, C_m)$ was studied by several authors [1,3,9] (see also [6]). It is known that, for $n > 4$,

$$R(C_n, C_n) = \begin{cases} 2n-1, & n \text{ is odd,} \\ 3n/2-1, & n \text{ is even.} \end{cases} \tag{1}$$

Around the same time, Bondy and Erdős [1] conjectured (see also [2]) that if n is odd, then

$$R(C_n, C_n, C_n) = 4n - 3. \tag{2}$$

This is sharp if true, as shown by the constructions below. Using the Szemerédi Regularity Lemma [10], Łuczak [8] proved that if n is odd, then

$$R(C_n, C_n, C_n) = 4n + o(n), \tag{3}$$

where $o(n)/n \to 0$ as $n \to \infty$.

We shall prove the following.

Theorem 1.1 (Sharp form) *There exists an integer n_0 such that if $n > n_0$ is odd, then*

$$R(C_n, C_n, C_n) = 4n - 3.$$

Remark 1.2 Recently Figaj and Łuczak [4] and slightly later, independently, Gyárfás, Ruszinkó, G. Sárközy and Szemerédi [5] proved that for even n

$$R(C_n, C_n, C_n) = 2n + o(n).$$

The next two constructions prove the lower bound $R(C_n, C_n, C_n) \geq 4n-3$ in Theorem 1.1. We will use the following notation: For a given graph $G =$

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(V, E) and two disjoint subsets A, B of V, denote by $E(A, B)$ the set of all edges with one endpoint in A and the other endpoint in B.

Construction 1.3 *Split the vertices of K_{4n-4} into 4 groups $V_1, \ldots, V_4$, of $n-1$ vertices each. Color the edges inside each group by GREEN, the edges in $E(V_1, V_3) \cup E(V_2, V_4)$ by RED, $E(V_1, V_2) \cup E(V_3, V_4)$ by BLUE, and the edges in $E(V_1, V_4) \cup E(V_2, V_3)$ arbitrarily, by RED and BLUE.*

The special feature of the next construction is that it contains both BLUE and GREEN complete graphs on $n-1$ vertices.

Construction 1.4 *Take 4 groups of $n-1$ vertices, as in Construction 1.3, color the edges in V_1 and V_2 by GREEN, in V_3 and V_4 by BLUE. Then color the edges in $E(V_3, V_4)$ by GREEN and in $E(V_1, V_2)$ by BLUE. Finally, color the edges in $E((V_1 \cup V_2), (V_3 \cup V_4))$ by RED.*

It is easy to observe that if n is odd, then these constructions do not contain monochromatic copies of C_n.

Besides the two constructions above there are other constructions that prove the sharpness of Theorem 1.1 (for example, in Construction 1.4 change one GREEN edge in $E(V_3, V_4)$ to BLUE). However, we shall actually prove that all the other extremal colorings can be obtained from the ones we listed here by changing the colors of a few edges.

2. Methods

In this section we describe (without proof) the main ingredients in our proof of Theorem 1.1: the Regularity Lemma of Szemerédi, the Decomposition Lemma of Łuczak, and the Stability Lemma for 3-colorings with no monochromatic long odd cycles from [7]. We also include a brief outline of the proof without any technical details.

2.1. Regularity lemma for graphs

The Szemerédi Regularity Lemma [10] asserts that each graph of positive edge-density can be approximated by a union of a bounded number of random-like bipartite graphs.

Before we state the Regularity Lemma, we must introduce the concept of *regular pairs*.

Definition 2.1 Let $G = (V, E)$ be a graph and let δ be a positive real number, $0 < \delta \leq 1$. We say that a pair (A, B) of two disjoint subsets of V is δ-regular (with respect to G) if

$$|d(A', B') - d(A, B)| < \delta$$

for any two subsets $A' \subset A$, $B' \subset B$, $|A'| \geq \delta|A|$, $|B'| \geq \delta|B|$. Here, $d(A, B) = |E(A, B)|/(|A||B|)$ stands for the *density* of the pair (A, B).

This definition states that a regular pair has uniformly distributed edges. The Regularity Lemma of Szemerédi [10] enables us to partition the vertex set $V(G)$ of a graph G into $t+1$ sets $V_0 \cup V_1 \cup \ldots \cup V_t$ in such a way that most of the pairs (V_i, V_j) satisfy Definition 2.1. The precise statement is as follows.

Theorem 2.2 (The Regularity Lemma [10]) *For every $\delta > 0$ and $s, t_0 \in \mathbb{N}$ there exist two integers $N_0 = N_0(\delta, s, t_0)$ and $T_0 = T_0(\delta, s, t_0)$ with the following property: for all graphs $G_1, \ldots, G_s$ with the same vertex set V of size $n \geq N_0$ there is a partition of V into $t+1$ classes*

$$V = V_0 \cup V_1 \cup \ldots \cup V_t$$

such that

(i) $t_0 \leq t \leq T_0$,

(ii) $|V_0| \leq \delta n$, $|V_1| = \ldots = |V_t|$, *and*

(iii) *all but at most $\delta\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are δ-regular with respect to every G_k, $1 \leq k \leq s$.*

2.2. The Łuczak decomposition lemma

If an n-vertex graph G is the union of complete graphs of size at most m and a bipartite graph, then it does not contain odd cycles C_t with $t > m$. The following result of Łuczak (see Claim 7 of [8]) shows that up to some error the converse is also true.

Lemma 2.3 (The Decomposition Lemma [8]) *For every $0 < \delta < 10^{-15}$, $\alpha > 2\delta$ and $n \geq \exp(1/\delta^{16}\alpha)$ the following holds. Any graph on n vertices that contains no odd cycles longer than αn contains subgraphs G' and G'' such that:*

(i) *$V(G') \cup V(G'') = V(G)$, $V(G') \cap V(G'') = \emptyset$ and each of the sets $V(G')$ and $V(G'')$ is either empty or contains at least $\alpha\delta n/2$ vertices;*

(ii) *G' is bipartite;*

(iii) *G'' contains not more that $\alpha n|V(G'')|/2$ edges;*

(iv) *all except at most δn^2 edges of G belong to either G' or G''.*

2.3. Stability of 3-colorings with no long monochromatic odd cycles

The following lemma from [7] describes the structure of 3-colorings without long monochromatic cycles and is crucial for the proof of Theorem 1.1.

Lemma 2.4 (The Stability Lemma [7]) *For every $\epsilon > 0$ there exist δ_1, $\delta_2 > 0$ and n_0 with the following property: for every odd $n > n_0$, for any graph G on $N = (7/2 + \epsilon)n$ vertices and with at least $\binom{N}{2} - \delta_1 N^2$ edges, any 3-coloring of G*

(i) *either contains a monochromatic odd cycle longer than n, or*

(ii) *one can remove $\delta_2 N^2$ edges from G and obtain a 3-coloring that can be embedded into Construction 1.3 or 1.4.*

2.4. Idea of the proof

The following is a brief outline of the proof of Theorem 1.1. We refer the interested reader to the full version [7].

Consider an arbitrary 3-coloring of the edges of $G = K_N$, where $N = 4n-3$. Let G_1, G_2, G_3 be its color classes. We apply the Regularity Lemma and obtain a regular partition of the vertex set $V(G)$ into $t+1$ classes $V(G) = V_0 \cup V_1 \cup \ldots \cup V_t$.

We construct an auxiliary graph Γ with vertex set $\{1, \ldots, t\}$ and the edge set formed by pairs $\{i, j\}$ for which (V_i, V_j) is regular with respect to G_1, G_2, and G_3. We 3-color Γ by the majority color in the pair (V_i, V_j).

Then the graph Γ cannot contain an odd monochromatic cycle longer than $t/4$ because (see [8]) that would imply the existence of a monochromatic copy of C_n in the original 3-coloring of G. Hence, by Lemma 2.4, the 3-coloring of Γ has a special structure which in turn yields 4 disjoint subsets V_1', V_2', V_3', V_4' such that the 3-coloring induced on $V_1' \cup V_2' \cup V_3' \cup V_4'$ has precisely the structure described by Constructions 1.3 and 1.4.

The proof is concluded by showing that the remaining vertices of $V(G)$ are then split into four sets $V_1'', V_2'', V_3'', V_4''$ such that all edges induced on $V_i' \cup V_i''$ are of the same color. Since one of these four unions has size at least n, it must contain a monochromatic copy of C_n.

References

[1] Bondy, J. A. and P. Erdős, *Ramsey numbers for cycles in graphs*, J. Combin. Theory Ser. B **14** (1973), 46–54.

[2] Erdős, P., *On the combinatorial problems which I would most like to see solved*, Combinatorica **1** (1981), 25–42.

[3] Faudree, R. J. and R. H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Math. **8** (1974), 313–329.

[4] Figaj, A. and T. Łuczak, *The Ramsey number for a triple of long even cycles*, submitted.

[5] Gyárfás, A., M. Ruszinkó, G. N. Sárközy and E. Szemerédi, personal communication.

[6] Károlyi, G. and V. Rosta, *Generalized and geometric Ramsey numbers for cycles*, Theoretical Computer Science **263** (2001), 87–98.

[7] Kohayakawa, Y., M. Simonovits and J. Skokan, *On a theorem of Łuczak* (tentative title), in preparation.

[8] Łuczak, T., $R(C_n, C_n, C_n) \leq (4 + o(1))n$, J. Combin. Theory Ser. B **75** (1999), 174–187.

[9] Rosta, V., *On a Ramsey type problem of J.A. Bondy and P. Erdős. I-II*, J. Combin. Theory Ser. B **15** (1973), 94–120.

[10] Szemerédi, E., "Regular partitions of graphs", Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), pp. 399–401, Colloq. Internat. CNRS, 260, CNRS, Paris, 1978.