Combinatorics, Probability and Computing (2003) 00, 000–000.
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

Counting small cliques in 3-uniform hypergraphs

Y. PENG[1]† V. RÖDL[2]‡ and J. SKOKAN[3]§

[1] Department of Mathematics and Computer Science,
Indiana State University, Terre Haute, IN 47809, USA
(mapeng@isugw.indstate.edu)

[2] Department of Mathematics and Computer Science
Emory University, Atlanta, GA 30032, USA
(rodl@mathcs.emory.edu)

[3] Department of Mathematics, MC–382,
University of Illinois at Urbana–Champaign,
1409 W. Green Street, Urbana, IL, 61801, USA
(jozef@member.ams.org)

Received 17 May 2002; revised 5 June 2003

Many applications of Szemerédi's Regularity Lemma for graphs are based on the following counting result. If $\mathcal{G}$ is an s-partite graph with partition $V(\mathcal{G}) = \bigcup_{i=1}^{s} V_i$, $|V_i| = m$ for all $i \in [s]$, and all pairs (V_i, V_j), $1 \leq i < j \leq s$, are ϵ-regular of density d, then $\mathcal{G}$ contains $(1 \pm f(\epsilon))d^{\binom{s}{2}} m^s$ cliques K_s, provided $\epsilon < \epsilon(d)$, where $f(\epsilon)$ tends to 0 as ϵ tends to 0.

Guided by the regularity lemma for 3-uniform hypergraphs established earlier by Frankl and Rödl, Nagle and Rödl proved a corresponding counting lemma. Their proof is rather technical, mostly due to the fact that the 'quasi-random' hypergraph arising after application of Frankl-Rödl's regularity lemma is 'sparse' and consequently is difficult to handle.

If the 'quasi-random' hypergraph is dense, then Kohayakawa, Rödl and Skokan [*J. Combin. Theory Ser. A* **97**, pp. 307–352] found a simpler proof of the counting lemma. Their result applies even to k-uniform hypergraphs for arbitrary k. While the Frankl-Rödl regularity lemma will not render the dense case, in this paper, for $k = 3$, we are able, nevertheless, to reduce the harder, sparse case to the dense case.

Namely, we prove that a 'dense substructure' randomly chosen from the 'sparse δ-regular structure' is δ-regular as well. This allows to count the number of cliques (and

† Correspondence to: Y. Peng
‡ The second author was partially supported by NSF grants DMS-0071261 and INT-0072064.
§ The third author was partially supported by NSF grant INT-0072064.

other subhypergraphs) using Kohayakawa-Rödl-Skokan result and provides an alternative proof of the counting lemma in the sparse case. Since the counting lemma in the dense case applies to k-uniform hypergraphs for arbitrary k, there is a possibility that the approach of this paper can be adopted to the general case as well.

1. Introduction and the Main Result

For a graph $\mathcal{G} = (V, E)$ and two disjoint nonempty sets $A, B \subset V$, denote by $E(A, B)$ the set of edges $\{a, b\} \in E$ with $a \in A$ and $b \in B$. Let $d(A, B) = |E(A, B)|/|A||B|$ be the density of a pair (A, B). For a given $\epsilon > 0$, the pair (A, B) is ϵ-regular if

$$|d(A, B) - d(A', B')| < \epsilon$$

holds whenever $A' \subseteq A, B' \subseteq B$, and $|A'| \geq \epsilon|A|$, $|B'| \geq \epsilon|B|$.

Szemerédi's Regularity Lemma [9] says that all graphs can be decomposed into ϵ-regular, 'random like' pieces.

Theorem 1.1 (Szemerédi's Regularity Lemma). *For every given $\epsilon > 0$ and integer t, there exist integers $T = T(\epsilon, t)$ and $N = N(\epsilon, t)$ such that every graph $\mathcal{G} = (V, E)$ with $\big|V\big| \geq N$ vertices admits a partition $V = \bigcup_{i=1}^{s} V_i$, such that*

(1) $t \leq s \leq T$,
(2) $\big||V_i| - |V_j|\big| \leq 1$ for all pairs $(i, j), 1 \leq i < j \leq s$, and
(3) pairs (V_i, V_j) are ϵ-regular for all but $\epsilon\binom{s}{2}$ pairs (i, j), $1 \leq i < j \leq s$.

A partition described in the above theorem is called an ϵ-regular partition.

Szemerédi's Regularity Lemma is one of the most powerful tools in extremal graph theory. Many applications of Szemerédi's Regularity Lemma are based on a 'counting lemma' which states that the number of cliques K_s, i.e. complete subgraphs with s vertices, in a 'random-like' graph is as expected. For a graph $\mathcal{G}$, let $\mathcal{K}_s(\mathcal{G})$ be the set of all s-element sets that induces a copy of K_s in $\mathcal{G}$. The following fact is easy to prove.

Fact 1.2 (Counting Lemma). *Let $\mathcal{G}$ be an s-partite graph with a vertex partition $V(\mathcal{G}) = \bigcup_{i=1}^{s} V_i$ such that*

(1) $\big|V_i\big| = m$ for all $i \in [s]$, and
(2) all pairs (V_i, V_j), $1 \leq i < j \leq s$, are ϵ-regular with density d.

Then $\mathcal{G}$ contains $(1 \pm f(\epsilon))d^{\binom{s}{2}}m^s$ cliques K_s, provided $\epsilon < \epsilon(d)$, where $f(\epsilon) \to 0$ as $\epsilon \to 0$. In other words,

$$|\mathcal{K}_s(\mathcal{G})| = (1 \pm f(\epsilon))d^{\binom{s}{2}}m^s.$$

A natural question arises whether the Regularity Lemma can be generalized to hypergraphs in a way that allows for a similar counting lemma. It turns out that this is a hard problem. Frankl and Rödl developed such a regularity lemma for 3-uniform hypergraphs in [4]. In a way similar to Theorem 1.1, Frankl-Rödl's regularity lemma allows one to count the number of cliques $K_s^{(3)}$. This was done for $s = 4$ in [4] and later generalized by Nagle and Rödl [7], replacing 4 with an arbitrary integer s. The result of [7], which extends Fact 1.2 to 3-uniform hypergraphs, replaces the concept of ϵ-regularity with (δ, d, r)-regularity (cf. Definition 1.3). While the proof of Fact 1.2 is simple, the proof of counting lemma given in [7] is surprisingly very technical.

Since in this paper we are interested mainly in an extension of Fact 1.2 to 3-uniform hypergraphs, we will not state the Frankl-Rödl regularity lemma here. We need however their concept of regularity for 3-uniform hypergraphs (cf. [4]).

Definition 1.3. *Let δ, d be positive real numbers and r be a positive integer. Let $\mathcal{G}$ be a 3-partite graph and $\mathcal{H}$ be a 3-partite 3-uniform hypergraph with the same vertex partition. We say that $\mathcal{H}$ is (δ, d, r)-regular with respect to $\mathcal{G}$ if the following property is satisfied.*

Whenever $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$ are subgraphs of $\mathcal{G}$ such that

$$\Big|\bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{Q}_j)\Big| \geq \delta\big|\mathcal{K}_3(\mathcal{G})\big|,$$

then

$$\Big|\mathcal{H} \cap \bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{Q}_j)\Big| = (1 \pm \delta) d \Big|\bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{Q}_j)\Big|. \tag{1.1}$$

Here $\mathcal{K}_3(\mathcal{G})$ stands for the number of triangles in a graph $\mathcal{G}$ and we refer the reader to Section 2 for more notation used through this paper.

Remark 1.4. *(δ, d, r)-regularity implies (δ', d, r')-regularity when $r' \leq r$ and $\delta' \geq \delta$.*

Definition 1.5. *Let $\mathcal{G}$ be an s-partite graph and $\mathcal{H}$ be an s-partite 3-uniform hypergraph with the same partition $\bigcup_{i=1}^{s} V_i$. We say that $\mathcal{H}$ is (δ, d, r)-regular with respect to $\mathcal{G}$ if $\mathcal{H}[V_i \cup V_j \cup V_k]$ is (δ, d, r)-regular with respect to $\mathcal{G}[V_i \cup V_j \cup V_k]$ for every $\{i, j, k\} \in [s]^3$.*

Definition 1.6. *Let $\mathcal{G}$ be a graph and $\mathcal{H}$ be a 3-uniform hypergraph. We say that $\mathcal{G}$ underlies $\mathcal{H}$ if all edges of $\mathcal{H}$ are triangles of $\mathcal{G}$, in other words, $\mathcal{H} \subset \mathcal{K}_3(\mathcal{G})$.*

We found it convenient to work with the following alternative (but equivalent) definition of ϵ-regularity.

Definition 1.7. *Let $0 < \epsilon, d \leq 1$ and (V_1, V_2) be a disjoint pair in graph $\mathcal{G}$. The pair (V_1, V_2) is called (ϵ, d)-regular if*

$$d(W_1, W_2) = (1 \pm \epsilon) d$$

holds for every $W_1 \subset V_1$, $W_2 \subset V_2$ with $|W_1||W_2| \geq \epsilon |V_1||V_2|$.

Definition 1.8. *Let $\mathcal{G} = (V, E)$ be an s-partite graph with partition $V = \bigcup_{i=1}^{s} V_i$. Then $\mathcal{G}$ is called (ϵ, d)-regular if all pairs (V_i, V_j), $1 \leq i < j \leq s$, are (ϵ, d)-regular.*

Now we are ready to state the counting lemma for 3-uniform hypergraphs.

Theorem 1.9 (Counting lemma for 3-uniform hypergraphs [7]). *Let $s \geq 3$ be an integer. For every $\mu > 0$ and $d_3 \in (0, 1]$ there exists $\delta > 0$ such that for every $d_2 \in (0, 1]$ there exist $\epsilon > 0$ and integers r and m_0 such that the following assertion holds.*

If $\mathcal{G}$ is an s-partite graph with partition $V = \bigcup_{i=1}^{s} V_i$, where $|V_i| = m > m_0$ for $1 \leq i \leq s$, and $\mathcal{H}$ is an s-partite 3-uniform hypergraph with the same partition such that

(1) $\mathcal{G}$ is (ϵ, d_2)-regular, and

(2) $\mathcal{G}$ underlies $\mathcal{H}$, and $\mathcal{H}$ is (δ, d_3, r)-regular with respect to $\mathcal{G}$,

then $\mathcal{H}$ contains $(1 \pm \mu) d_2^{\binom{s}{2}} d_3^{\binom{s}{3}} m^s$ copies of $K_s^{(3)}$.

Note that due to the quantification of Theorem 1.9 ($\forall \mu\ \forall d_3\ \exists \delta\ \forall d_2\ \exists \epsilon\ \exists r\ \exists m_0\ \forall m$), one must prove this theorem for every d_2, and consequently for $d_2 \ll \delta$. In this case, the underlying graph $\mathcal{G}$ is sparse which is the main reason why all known proofs became very technical. Unfortunately, the situation $d_2 \ll \delta$ cannot be avoided after the application of the regularity lemma from [4].

If the underlying graph $\mathcal{G}$ is dense, then it is relatively simple to count the number of cliques. Recently, Kohayakawa, Rödl and Skokan [6] proved a counting theorem for k-uniform hypergraphs, which for $k = 3$ reduces to a special case of Theorem 1.9. Specifically, they showed that Theorem 1.9 is true when $\mathcal{G}$ is a *complete* s-partite graph and $r = 1$.

Lemma 1.10. *[6] Let $s \geq 3$ be an integer. For every $\mu > 0$ and every $d \in (0, 1]$, there exist $\delta_0 > 0$ and $m_0 > 0$ such that the following holds. If*

(1) $\mathcal{G}$ is a complete s-partite graph with partition $V = \bigcup_{i=1}^{s} V_i$, where $|V_i| = m \geq m_0$ for $1 \leq i \leq s$, and

(2) $\mathcal{H}$ is an s-partite 3-uniform hypergraph with the same partition $V = \bigcup_{i=1}^{s} V_i$ and $\mathcal{H}$ is $(\delta, d, 1)$-regular with respect to $\mathcal{G}$, where $\delta \leq \delta_0$,

then $\mathcal{H}$ contains $(1 \pm \mu) d^{\binom{s}{3}} m^s$ copies of $K_s^{(3)}$.

In this paper, we show how to reduce the harder, sparse case (i.e. when $d_2 \ll \delta$) to the dense case (when $\delta \ll d_2 = 1$). We prove that a 'dense substructure' randomly chosen from the 'sparse δ-regular structure' is δ-regular as well. In order to state our result, we need to describe the environment (or our 'sparse δ-regular structure') in which we will work.

Through the remaining part of this paper, we will work within the following setup. Due to the quantification of Theorem 1.9:

$$\forall \mu\ \forall d_3\ \exists \delta\ \forall d_2\ \exists\ \epsilon\ \exists r\ \exists m_0\ \forall m \geq m_0,$$

we may assume the following.

Setup.

(S1)

$$\frac{1}{m} \ll \epsilon \ll \frac{1}{r} \ll d_2, \delta \text{ and } \delta \ll d_3. \tag{1.2}$$

(S2) Let h be an integer satisfying $\epsilon \ll 1/h \ll \delta$.

(S3) Let $V = \bigcup_{i=1}^{s} V_s$ be a partition of V satisfying $|V_i| = m$ for $i \in [s]$. Suppose that

(i) $\mathcal{G}$ is an (ϵ, d_2)-regular s-partite graph with s-partition $V = \bigcup_{i=1}^{s} V_s$,

(ii) $\mathcal{H}$ is an s-partite 3-uniform hypergraph with the same partition as $\mathcal{G}$, and

(iii) $\mathcal{G}$ underlies $\mathcal{H}$ and $\mathcal{H}$ is (δ, d_3, r)-regular with respect to $\mathcal{G}$.

We also need the following definition.

Definition 1.11. *For $2 \le t \le s$, we call a t-tuple of h-subsets $(L_1, \dots, L_t)$, where $L_1 \subset V_1, \dots, L_t \subset V_t$, complete if $\mathcal{G}[L_1 \cup \ldots \cup L_t]$ is a complete t-partite graph. For $t = 1$, every h-subset $L_1 \subset V_1$ is called complete.*

The following remark estimates the number of complete s-tuples of h-subsets. Note that it is also a generalized version of the Counting Lemma for the graph case given in Fact 1.2.

Remark 1.12. *For $1 \le i \le s$, set $M_i = d_2^{(i-1)h} m$. Then the number of complete s-tuples of h-subsets $(L_1, \dots, L_s)$ is (cf. Fact* A.7*(1))*

$$(1 \pm \epsilon^{1/2^{s+1}}) \prod_{i=1}^{s} \binom{M_i}{h}.$$

Thus the quantity $\prod_{i=1}^{s} \binom{M_i}{h}$ counts asymptotically the number of complete s-tuples of h-subsets in $\mathcal{G}$. Consequently, the quantification

“For all but $f(\delta) \displaystyle\prod_{i=1}^{s} \binom{M_i}{h}$ complete s-tuples of h-subsets, where $f(\delta) \to 0$ as $\delta \to 0$”

means

“For almost all complete s-tuples of h-subsets”.

Now we present our main result.

Theorem 1.13 (Main theorem). *There exists a positive function $f(\delta)$ with the property $f(\delta) \to 0$ as $\delta \to 0$ such that the following holds.*

For all but at most $f(\delta) \displaystyle\prod_{i=1}^{s} \binom{M_i}{h}$ complete s-tuples of h-subsets $(L_1, L_2, \dots, L_s)$, the induced subhypergraph $\mathcal{H}[\cup_{i=1}^{s} L_i]$ is $(f(\delta), d_3, 1)$-regular with respect to $\mathcal{G}[\cup_{i=1}^{s} L_i]$.

Consequently, all but at most $f(\delta) \displaystyle\prod_{i=1}^{s} \binom{M_i}{h}$ complete s-tuples $(L_1, L_2, \dots, L_s)$ of h-subsets satisfy assumptions of Lemma 1.10, and thus enable to count the number of cliques $K_s^{(3)}$ in $\mathcal{H}[\cup_{i=1}^{s} L_i]$. The following result is a direct consequence of Theorem 1.13 and Lemma 1.10.

Corollary 1.14. *There exists a positive function $f(\delta)$ with the property $f(\delta) \to 0$ as $\delta \to 0$ such that the following holds.*

For all but at most $f(\delta)\prod_{i=1}^{s}\binom{M_i}{h}$ complete s-tuples of h-subsets $(L_1, L_2, \ldots, L_s)$, the hypergraph $\mathcal{H}[\cup_{i=1}^{s} L_i]$ contains $(1 \pm f(\delta))d_3^{\binom{s}{3}} h^s$ copies of $K_s^{(3)}$.

Applying Corollary 1.14 and a double counting argument, we can easily enumerate the number of cliques $K_s^{(3)}$ in $\mathcal{H}$. This gives an alternative proof of Theorem 1.9. Details are given in Section 3.

2. Notation

$[W]^k$ denotes all k element subsets of a set W.

$V(\mathcal{G})$ is the vertex set of a graph or a hypergraph $\mathcal{G}$.

$E(\mathcal{G})$ is the edge set of a graph or a hypergraph $\mathcal{G}$.

$\mathcal{G}[W]$ stands for the subhypergraph of $\mathcal{G}$ induced on a set W.

$\mathcal{G}(x)$ denotes the neighborhood of a vertex x in a graph $\mathcal{G}$.

$\mathcal{G}(W) = \bigcap_{x \in W} \mathcal{G}(x)$ is called the joint neighborhood of a set W in a graph $\mathcal{G}$.

$\mathcal{G}(x_1, \ldots, x_k)$ is an abbreviated form of $\mathcal{G}(\{x_1, \ldots, x_k\})$.

K_j is a clique of size j in a graph.

$\mathcal{K}_3(\mathcal{G})$ denotes the set of all triangles in a graph $\mathcal{G}$.

$K_j^{(3)}$ is a clique of size j in a 3-uniform hypergraph.

$\mathcal{K}_j(\mathcal{H})$ denotes the set of all cliques of size j in a 3-uniform hypergraph $\mathcal{H}$.

$K_{2,2,2}^{(3)}$ is a complete 3-partite 3-uniform hypergraph whose every partite set contains precisely two vertices.

$\mathcal{H}(x) = \{e \setminus \{x\} : e \in E(\mathcal{H}),\ x \in e\}$ is the link of a vertex x in a 3-uniform hypergraph $\mathcal{H}$.

$\mathcal{H}(W) = \bigcap_{x \in W} \mathcal{H}(x)$ is called the joint link of a set W in a 3-uniform hypergraph $\mathcal{H}$.

$\mathcal{H}(x_1, \ldots, x_k)$ is an abbreviated form of $\mathcal{H}(\{x_1, \ldots, x_k\})$.

$\mathcal{H}(f) = \{e \setminus f : e \in E(H),\ f \subset e\}$ is the link of a 2-subset $f \in [V(\mathcal{H})]^2$ in a 3-uniform hypergraph $\mathcal{H}$.

$\mathcal{H}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \mathcal{H}(f)$ is the link of a family $\mathcal{F} \subset [V(\mathcal{H})]^2$ in a 3-uniform hypergraph $\mathcal{H}$.

For three numbers a, b and $\delta > 0$, $b = a \pm \delta$ means that $b \in (a - \delta, a + \delta)$.

3. Proof of Theorem 1.9

At this point, we prove Theorem 1.9 by applying Corollary 1.14 and a double counting argument. We will frequently use some easy facts regarding ϵ-regularity of graphs. These facts are summarized in the Appendix A as Facts A.1-A.10.

Denote by $\mathcal{C}_{h,t}$ the set of all complete t-tuples of h-sets in $\mathcal{G}$, that is

$$\mathcal{C}_{h,t}(\mathcal{G}) = \left\{\text{complete}(L_1, \dots, L_t) \in [V_1]^h \times \dots \times [V_t]^h\right\}.$$

Definition 3.1. *We say that an s-tuple $(a_1, \dots, a_s) \in V_1 \times \dots \times V_s$ is complete if the induced subgraph $\mathcal{G}[\{a_1, \dots, a_s\}]$ of $\mathcal{G}$ is a complete graph.*

Now we shall outline the proof of Theorem 1.9. We first apply Corollary 1.14 to estimate

$$A = \sum_{\mathcal{C}} \left|\mathcal{K}_s(\mathcal{H}[L_1 \cup \dots \cup L_s])\right|,$$

where $\sum_{\mathcal{C}}$ stands for the summation over all s-tuples $(L_1, \dots, L_s) \in \mathcal{C}_{h,s}(\mathcal{G})$. This is done in Claim 3.2. Then, for a fixed complete s-tuple $(a_1, \dots, a_s) \in V_1 \times \dots \times V_s$, we estimate the number of s-tuples $(L_1, \dots, L_s) \in \mathcal{C}_{h,s}(\mathcal{G})$ such that $(a_1, \dots, a_s) \in L_1 \times \dots \times L_s$ (cf. Claim 3.3). Since this quantity is essentially the same (say, equal to B) for almost all $(a_1, \dots, a_s) \in V_1 \times \dots \times V_s$, we can conclude that the number of copies of $K_s^{(3)}$ is about A/B.

Claim 3.2.

$$\sum_{\mathcal{C}} \left|\mathcal{K}_s(\mathcal{H}[L_1 \cup \dots \cup L_s])\right| = (1 \pm h(\delta))\, d_2^{h^2\binom{s}{2}} d_3^{\binom{s}{3}} h^s m^{hs}/(h!)^s,$$

where $h(\delta)$ is a positive function with the property $h(\delta) \to 0$ as $\delta \to 0$.

Proof of Claim 3.2. Corollary 1.14, Remark 1.12, and $1/m \ll \epsilon \ll 1/h \ll \delta$ imply that

$$\begin{aligned}
&\sum_{\mathcal{C}} \left|\mathcal{K}_s(\mathcal{H}[L_1 \cup \dots \cup L_s])\right| \\
&\quad \geq (1 - f(\delta)) d_3^{\binom{s}{3}} h^s \times \left(\left(1 - \epsilon^{1/2^{s+1}}\right) \prod_{i=1}^{s} \binom{d_2^{(i-1)h} m}{h} - f(\delta) \prod_{i=1}^{s} \binom{d_2^{(i-1)h} m}{h} \right) \\
&\quad \geq (1 - h(\delta)) d_2^{\binom{s}{2}h^2} d_3^{\binom{s}{3}} h^s m^{hs}/(h!)^s, \qquad (3.1)
\end{aligned}$$

where $f(\delta)$ is the function from Corollary 1.14 and $h(\delta)$ is a positive function with the property that $h(\delta) \to 0$ as $\delta \to 0$. Corollary 1.14 and Fact A.7(3) also imply that

$$\begin{aligned}
&\sum_{\mathcal{C}} \left|\mathcal{K}_s(\mathcal{H}[L_1 \cup \dots \cup L_s])\right| \\
&\quad \leq (1 + f(\delta)) d_3^{\binom{s}{3}} h^s \times \left(1 + \epsilon^{1/2^{s+1}}\right) \prod_{i=1}^{s} \binom{d_2^{(i-1)h} m}{h} + h^s \times f(\delta) \prod_{i=1}^{s} \binom{d_2^{(i-1)h} m}{h} \\
&\quad \leq (1 + h(\delta)) d_2^{\binom{s}{2}h^2} d_3^{\binom{s}{3}} h^s m^{hs}/(h!)^s \qquad (3.2)
\end{aligned}$$

since $\epsilon \ll \delta \ll d_3$. □

Let $(a_1, \dots, a_s) \in V_1 \times \cdots \times V_s$ be a complete s-tuple. Now we estimate the number of s-tuples $(L_1, \dots, L_s) \in \mathcal{C}_{h,s}(\mathcal{G})$ for which $(a_1, \dots, a_s) \in L_1 \times \cdots \times L_s$.

Claim 3.3. *All but at most $\epsilon^{1/8}m^s$ complete s-tuples $(a_1, \dots, a_s) \in V_1 \times \cdots \times V_s$ satisfy*

$$\left|\left\{(L_1, \dots, L_s) \in \mathcal{C}_{h,s}(\mathcal{G}) : (a_1, \dots, a_s) \in L_1 \times \cdots \times L_s\right\}\right| \\ = \left(1 \pm \epsilon^{1/2^{s+3}}\right) d_2^{(h^2-1)\binom{s}{2}} m^{(h-1)s}/((h-1)!)^s. \quad (3.3)$$

Proof of Claim 3.3. Let $(a_1, \dots, a_s) \in V_1 \times \cdots \times V_s$ be a complete s-tuple. We need to estimate the number of complete s-tuples $(B_1, \dots, B_s)$ of $(h-1)$-subsets $B_1 \subset V_1, \dots, B_s \subset V_s$ such that

$$B_i \in \left[\mathcal{G}(\{a_1, \dots, a_s\} \setminus \{a_i\})\right]^{h-1}$$

for every $i \in [s]$.

By Fact A.3 and $\epsilon \ll d_2 \le 1$, for each $i \in [s]$, all but at most $2(s-1)\epsilon^{1/2}m^{s-1} \le \epsilon^{1/4}m^{s-1}$ $(s-1)$-element sets $\{a_1, \dots, a_s\} \setminus \{a_i\}$ satisfy

$$\left|\mathcal{G}(\{a_1, \dots, a_s\} \setminus \{a_i\})\right| = \left(1 \pm \epsilon^{1/4}\right)^{s-1} d_2^{s-1} m. \quad (3.4)$$

Since the right hand side of (3.4) is at least $\epsilon^{1/4}m$, by Fact A.1, all but at most $s \times m \times \epsilon^{1/4}m^{s-1} \le \epsilon^{1/8}m^s$ s-tuples $(a_1, \dots, a_s)$ satisfy that the graph

$$\mathcal{G}\left[\bigcup_{i=1}^{s} \mathcal{G}(\{a_1, \dots, a_s\} \setminus \{a_i\})\right] \text{ is } (\epsilon^{1/2}, d_2)\text{-regular}. \quad (3.5)$$

Consequently, by Fact A.7(1), we have

$$\left|\left\{(L_1, \dots, L_s) \in \mathcal{C}_{h,s}(\mathcal{G}) : (a_1, \dots, a_s) \in L_1 \times \cdots \times L_s\right\}\right| \\ = \left(1 \pm \epsilon^{1/2^{s+2}}\right) \prod_{i=1}^{s} \binom{d_2^{(h-1)(i-1)}\left(1 \pm \epsilon^{1/4}\right)^{s-1} d_2^{s-1} m}{h-1} \\ = \left(1 \pm \epsilon^{1/2^{s+3}}\right) d_2^{(h^2-1)\binom{s}{2}} m^{s(h-1)}/((h-1)!)^s.$$

□

Now we use double counting and Claims 3.2 and 3.3 to finish the proof.

Proof of Theorem 1.9. In Claim 3.2, we proved that the summation of the number of cliques in $\mathcal{H}[L_1 \cup \ldots \cup L_s]$ over all complete s-tuples of h-subsets $(L_1, \dots, L_s)$ is $(1 \pm h(\delta))\, d_2^{h^2\binom{s}{2}} d_3^{\binom{s}{3}} h^s m^{hs}/(h!)^s$.

In Claim 3.3, we proved that for all but at most $\epsilon^{1/8}m^s$ complete s-tuples $(a_1, \dots, a_s) \in V_1 \times \cdots \times V_s$, the number of complete s-tuples of h-subsets $(L_1, \dots, L_s)$ such that $a_i \in L_i$ is $\left(1 \pm \epsilon^{1/2^{s+3}}\right) d_2^{(h^2-1)\binom{s}{2}} m^{(h-1)s}/((h-1)!)^s$.

Combining these two claims and the fact that $\epsilon \ll \delta, d_2, d_3$, we obtain

$$\begin{aligned}\left|\mathcal{K}_s(\mathcal{H})\right| &\le \frac{(1+h(\delta))\, d_2^{h^2\binom{s}{2}} d_3^{\binom{s}{3}} h^s m^{hs}/(h!)^s}{\left(1-\epsilon^{1/2^{s+3}}\right) d_2^{(h^2-1)\binom{s}{2}} m^{s(h-1)}/((h-1)!)^s} + \epsilon^{1/8} m^s \\ &\le (1+2h(\delta)) d_2^{\binom{s}{2}} d_3^{\binom{s}{3}} m^s,\end{aligned}$$

and

$$\begin{aligned}\left|K_s(\mathcal{H})\right| &\ge \frac{(1-h(\delta))\, d_2^{h^2\binom{s}{2}} d_3^{\binom{s}{3}} h^s m^{hs}/(h!)^s - \epsilon^{1/8} m^s \binom{m}{h-1}^s}{(1+\epsilon^{1/2^{s+3}}) d_2^{(h^2-1)\binom{s}{2}} m^{s(h-1)}/((h-1)!)^s} \\ &\ge (1-2h(\delta)) d_2^{\binom{s}{2}} d_3^{\binom{s}{3}} m^s.\end{aligned}$$

This completes the proof. □

4. Proof of Theorem 1.13

It follows from Section 3 that the main task is to prove Theorem 1.13. In order to do so, we apply an equivalent condition for $(\delta, d, 1)$-regularity of 3-uniform hypergraphs when the underlying graph is complete. The corresponding equivalence was proved by Y. Kohayakawa, V. Rödl and J. Skokan in [6]. Before stating this result, we introduce a definition.

Definition 4.1. *Let $\mathcal{H}_0$ be a 3-partite 3-uniform hypergraph with partition $U = U_1 \cup U_2 \cup U_3$. We define the density $\boldsymbol{den}(\mathcal{H}_0)$ of $\mathcal{H}_0$ by*

$$\boldsymbol{den}(\mathcal{H}_0) = \frac{|\mathcal{H}_0|}{|U_1||U_2||U_3|},$$

where $|\mathcal{H}_0|$ is the number of edges in $\mathcal{H}_0$. We also denote by $\boldsymbol{com}(\mathcal{H}_0)$ the number of copies of $K^{(3)}_{2,2,2}$ in $\mathcal{H}_0$.

Now we are ready to state the result from [6].

Lemma 4.2. *Let $\mathcal{G}_0$ be a complete 3-partite graph and $\mathcal{H}_0$ be a 3-partite 3-uniform hypergraph with the same partition $U = U_1 \cup U_2 \cup U_3$, where $\left|U_i\right| = n$ for $1 \le i \le 3$. If $\boldsymbol{den}(\mathcal{H}_0) = d$, then the following properties are equivalent:*

(1) *$\mathcal{H}_0$ is $(\delta, d, 1)$-regular,*
(2) $\boldsymbol{com}(\mathcal{H}_0) = (1 \pm \delta')d^8 n^6/8$.

The equivalence of properties (1) and (2) is understood in the following sense. For two properties $\boldsymbol{P} = \boldsymbol{P}(\delta)$ and $\boldsymbol{P'} = \boldsymbol{P'}(\delta')$, "$\boldsymbol{P} \Rightarrow \boldsymbol{P'}$" means that for every $\delta' > 0$ there is a $\delta > 0$ so that any 3-uniform hypergraph $\mathcal{H}_0$ satisfying $\boldsymbol{P}(\delta)$ must also satisfy $\boldsymbol{P'}(\delta')$, provided $n > M_0(\delta')$.

By Lemma 4.2, to prove Theorem 1.13, it is sufficient to show the following theorem.

Theorem 4.3. *There exists a positive function $f(\delta)$ with the property $f(\delta) \to 0$ as $\delta \to 0$ such that the following statement holds.*

For all but at most $f(\delta)\prod_{i=1}^{s}\binom{M_i}{h}$ s-tuples $(L_1, L_2, \dots, L_s) \in \mathcal{C}_{h,s}(\mathcal{G})$, the induced subhypergraph $\mathcal{H}[\cup_{i=1}^{s} L_i]$ satisfies these two properties:

(P_1) ***den***$(\mathcal{H}[L_i \cup L_j \cup L_k]) = (1 \pm f(\delta))d_3$ *for every* $\{i, j, k\} \in [s]^3$, *and*
(P_2) ***com***$(\mathcal{H}[L_i \cup L_j \cup L_k]) = (1 \pm f(\delta))d_3^8 h^6/8$ *for every* $\{i, j, k\} \in [s]^3$.

In order to prove this theorem, for $t = 3, 4, \dots, s$, we will introduce statements $\mathbf{Den}_t(ppp)$ regarding the 'density' of subhypergraphs of $\mathcal{H}$ and statements $\mathbf{Com}_t(ppp)$ regarding the number of copies of $K^{(3)}_{2,2,2}$ in subhypergraphs of $\mathcal{H}$. These statements will be proved by induction on t. $\mathbf{Den}_s(ppp)$ and $\mathbf{Com}_s(ppp)$ will then imply conditions (P_1) and (P_2) of Theorem 4.3.

We start with some definitions. Set

$$
\begin{aligned}
M_p &= d_2^{h(p-1)} m, \\
M_p^+ &= \left(1 + \epsilon^{1/4}\right)^{h(p-1)} M_p, \\
M_p^- &= \left(1 - \epsilon^{1/4}\right)^{h(p-1)} M_p.
\end{aligned}
$$

Definition 4.4. *For $1 \le t \le s$, we call a complete t-tuple (cf. Definition 3.1) of h-subsets $(L_1, \dots, L_t)$ good if*

$$
M_{p+1}^- \le \left|\mathcal{G}(L_1 \cup \dots \cup L_p) \cap V_i\right| \le M_{p+1}^+ \tag{4.1}
$$

for every $1 \le p \le t$ and $p + 1 \le i \le s$.

Remark 4.5. *All but $\epsilon^{1/2^{t+1}} \prod_{p=1}^{t} \binom{M_p}{h}$ complete t-tuples of h-subsets $(L_1, \dots, L_t)$ are good (cf. Fact A.7(2)).*

Let t, $3 \le t \le s$, be given and let $(L_1, \dots, L_t)$ be a good t-tuple of h-subsets. For a triple $(L_{p_1}, L_{p_2}, L_{p_3})$, $1 \le p_1 < p_2 < p_3 \le t$, we define the following edge-extension property $\mathbf{EEP}_t(p_1, p_2, p_3)$ and C_4-extension property $\mathbf{C4EP}_t(p_1, p_2, p_3)$ regarding the density of subhypergraphs of $\mathcal{H}$ and the number of copies of $K^{(3)}_{2,2,2}$ in subhypergraphs of $\mathcal{H}$, respectively.

$\mathbf{EEP}_t(p_1, p_2, p_3)$: *All but at most $\delta^{1/4^t} h^2$ edges $e = \{x, y\}$ in $\mathcal{G}[L_{p_2} \cup L_{p_3}]$ satisfy*

$$
\left|\mathcal{H}(e) \cap L_{p_1}\right| = (1 \pm 4\delta) d_3 h.
$$

Observe that a triple $(L_{p_1}, L_{p_2}, L_{p_3})$, where $1 \le p_1 < p_2 < p_3 \le t$, having property $\mathbf{EEP}_t(p_1, p_2, p_3)$ implies that the induced subhypergraph $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup L_{p_3}]$ has density about d_3. The second property regards the number of copies of $K^{(3)}_{2,2,2}$ in the hypergraph $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup L_{p_3}]$.

C4EP$_t(p_1, p_2, p_3)$: *All but at most* $\delta^{1/4^{t+5}} h^4/4$ *four-cycles* C_4 *in* $\mathcal{G}[L_{p_2} \cup L_{p_3}]$ *satisfy*

$$\left|\mathcal{H}(C_4) \cap L_{p_1}\right| = \left(1 \pm \delta^{1/4^6}\right) d_3^4 h.$$

Here, we view C_4 as a set of four 2-subsets. Notice that a four-cycle C_4 in $\mathcal{G}[L_{p_2} \cup L_{p_3}]$ together with any two vertices in $\mathcal{H}(C_4) \cap L_{p_1}$ form a $K^{(3)}_{2,2,2}$ in $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup L_{p_3}]$. Thus the property **C4EP**$_t(p_1, p_2, p_3)$ implies that the induced subhypergraph $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup L_{p_3}]$ has the 'right' number of copies of $K^{(3)}_{2,2,2}$.

Now we are ready to state **Den**$_t(ppp)$ and **Com**$_t(ppp)$.

Den$_t(ppp)$: *All but at most* $\delta^{1/4^{t+1}} \prod_{p=1}^{t} \binom{M_p}{h}$ *good* t*-tuples of* h*-subsets* $(L_1, \ldots, L_t)$ *satisfy the following condition:*

(*) *all triples* $(L_{p_1}, L_{p_2}, L_{p_3})$, $1 \le p_1 < p_2 < p_3 \le t$, *have the edge-extension property* ***EEP***$_t(p_1, p_2, p_3)$.

Com$_t(ppp)$: *All but at most* $\delta^{1/4^{t+7}} \prod_{p=1}^{t} \binom{M_p}{h}$ *good* t*-tuples of* h*-subsets* $(L_1, \ldots, L_t)$ *satisfy the following condition:*

(*) *all triples* $(L_{p_1}, L_{p_2}, L_{p_3})$, $1 \le p_1 < p_2 < p_3 \le t$, *have the* C_4*-extension property* ***C4EP***$_t(p_1, p_2, p_3)$.

We note that **Den**$_s(ppp)$ and Fact A.7(2) imply that for almost all complete s-tuples of h-subsets $(L_1, \ldots, L_s)$, all triple systems $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup L_{p_3}]$, where $1 \le p_1 < p_2 < p_3 \le s$, have density $(1 \pm \delta')d_3$, where $\delta' \to 0$ as $\delta \to 0$. Therefore **Den**$_s(ppp)$ and Fact A.7(2) imply property (P_1) of Theorem 4.3. Similarly, property (P_2) of Theorem 4.3 follows from **Com**$_s(ppp)$ and Fact A.7(2).

Hence, our goal is to prove **Den**$_s(ppp)$ and **Com**$_s(ppp)$. The proofs of these two statements are very similar. In this paper, we present only the proof of **Com**$_s(ppp)$. It can be easily modified to prove **Den**$_s(ppp)$. Details of proving **Den**$_s(ppp)$ are given in [8] and we omit them here.

We will prove **Com**$_t(ppp)$ for $3 \le t \le s$ by induction on t. Our induction scheme is quite complex and we need several other auxiliary statements defined in the following section.

5. Induction scheme

While proving **Com**$_{t+1}(ppp)$, our assumption will reflect the situation when a 'typical' good t-tuple of h-subsets $(L_1, L_2, \ldots, L_t)$ is chosen. Recall that $\mathcal{G}(\bigcup_{p=1}^t L_p)$ is the neighborhood of the set $\bigcup_{p=1}^t L_p$ in the graph $\mathcal{G}$. Clearly, $\mathcal{G}(\bigcup_{p=1}^t L_p) \subset V_{t+1} \cup \cdots \cup V_s$. We will consider the graph $\mathcal{G}[\bigcup_{p=1}^t L_p \cup \mathcal{G}(\bigcup_{p=1}^t L_p)]$ and the hypergraph $\mathcal{H}[\bigcup_{p=1}^t L_p \cup \mathcal{G}(\bigcup_{p=1}^t L_p)]$. For every $t+1 \le f \le s$, set

$$W_f^{(t)} = \mathcal{G}(L_1 \cup \ldots \cup L_t) \cap V_f.$$

Note that

$$M^-_{t+1} \le \left|W^{(t)}_f\right| \le M^+_{t+1}.$$

We regard sets $L_1, L_2 \dots, L_t$ as the "past" and sets $W^{(t)}_{t+1}, W^{(t)}_{t+2} \dots, W^{(t)}_s$ as the "future". There are four possible types of triple systems in the hypergraph $\mathcal{H}[\bigcup_{p=1}^t L_p \cup \bigcup_{f=t+1}^s W^{(t)}_f]$:

"fff" type: $\mathcal{H}[W^{(t)}_{f_1} \cup W^{(t)}_{f_2} \cup W^{(t)}_{f_3}]$, where $t+1 \le f_1 < f_2 < f_3 \le s$, that is, hypergraphs induced on the union of three sets from the future.

"pff" type: $\mathcal{H}[L_p \cup W^{(t)}_{f_1} \cup W^{(t)}_{f_2}]$, where $1 \le p \le t$ and $t+1 \le f_1 < f_2 \le s$, that is, hypergraphs induced on the union of three sets one of which is from the past and two are from the future.

"ppf" type: $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup W^{(t)}_f]$, where $1 \le p_1 < p_2 \le t$ and $t+1 \le f \le s$, that is, hypergraphs induced on the union of three sets two of which are from the past and one is from the future.

"ppp" type: $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup L_{p_3}]$, where $1 \le p_1 < p_2 < p_3 \le t$, that is, hypergraphs induced on the union of three sets from the past.

For the 'fff' type triple systems, we are interested in their regularity. For the remaining three types of triple systems, we are interested in the number of copies of $K^{(3)}_{2,2,2}$ in them. To deal with this situation we are going to use assertion $\mathbf{Com}_t(ppp)$ together with assertions $\mathbf{Reg}_t(fff)$, $\mathbf{Com}_t(pff)$, $\mathbf{Com}_t(ppf)$ formulated below.

For $2 \le p \le t$, we set

$$r_1 = r$$
$$r_p = \begin{cases} \frac{2r_{p-1}d_2^{3h}}{\delta^{1/2\times 4^{p-1}}d_3} & \text{if } 2d_2^{3h} < \delta^{1/2\times 4^{p-1}}d_3, \\ r_{p-1} & \text{otherwise.} \end{cases}.$$

Now we formulate $\mathbf{Reg}_t(fff)$.

$\mathbf{Reg}_t(fff)$: *All but* $\epsilon^{1/24\times 4^{t-1}} \prod_{p=1}^t \binom{M_p}{h}$ *good t-tuples of h-subsets $(L_1, \dots, L_t)$ satisfy the following conditions.*

(1) The induced subgraph $\mathcal{G}[\mathcal{G}(L_1 \cup \dots \cup L_t)]$ is $(\epsilon^{1/2}, d_2)$-regular.

(2) The induced subhypergraph $\mathcal{H}[\mathcal{G}(L_1 \cup \dots \cup L_t)]$ is $(\delta^{1/4^t}, d_3, r_t)$-regular with respect to $\mathcal{G}[\mathcal{G}(L_1 \cup \dots \cup L_t)]$.

Before stating $\mathbf{Com}_t(pff)$ we need one related definition. For a given t, $1 \le t \le s-2$, and a triple $(L_p, W^{(t)}_{f_1}, W^{(t)}_{f_2})$, where $1 \le p \le t$ and $t+1 \le f_1 < f_2 \le s$, we define the following C_4-extension property.

$\mathbf{C4EP}_t(p, f_1, f_2)$: *All but at most* $\delta^{1/4^{t+5}} d_2^4 M^4_{t+1}/4$ *four-cycles C_4 in $\mathcal{G}[W^{(t)}_{f_1} \cup W^{(t)}_{f_2}]$ satisfy*

$$\left|\mathcal{H}(C_4) \cap L_p\right| = \left(1 \pm \delta^{1/4^6}\right) d_3^4 h. \tag{5.1}$$

Note that a triple $(L_p, W^{(t)}_{f_1}, W^{(t)}_{f_2})$ possessing property $\mathbf{C4EP}_t(p, f_1, f_2)$ implies that the induced subhypergraph $\mathcal{H}[L_p \cup W^{(t)}_{f_1} \cup W^{(t)}_{f_2}]$ has the 'right' number of copies of $K^{(3)}_{2,2,2}$. Now we are ready to state $\mathbf{Com}_t(pff)$.

$\mathbf{Com}_t(pff)$: *All but at most* $\delta^{1/4^{t+7}} \prod_{p=1}^{t} \binom{M_p}{h}$ *good* t*-tuples of* h*-subsets* $(L_1, \dots, L_t)$ *satisfy the following condition:*

(*) *all triples* $(L_p, W_{f_1}^{(t)}, W_{f_2}^{(t)})$*, where* $1 \le p \le t$ *and* $t+1 \le f_1 < f_2 \le s$*, posses* $\boldsymbol{C4EP}_t(p, f_1, f_2)$*.*

Similarly, we introduce assertion $\mathbf{Com}_t(ppf)$. First, for a given t, $2 \le t \le s-1$, and a triple $(L_{p_1}, L_{p_2}, W_f^{(t)})$, where $1 \le p_1 < p_2 \le t$ and $t+1 \le f \le s$, we define the following C_4-extension property.

$\mathbf{C4EP}_t(p_1, p_2, f)$: *All but at most* $\delta^{1/4^{t+5}} h^2 M_{t+1}^2/4$ *four-cycles* C_4 *in* $\mathcal{G}[L_{p_2} \cup W_f^{(t)}]$ *satisfy*

$$\left|\mathcal{H}(C_4) \cap L_{p_1}\right| = \left(1 \pm \delta^{1/4^6}\right) d_3^4 h. \tag{5.2}$$

Observe that a triple $(L_{p_1}, L_{p_2}, W_f^{(t)})$ having property $\mathbf{C4EP}_t(p_1, p_2, f)$ implies that the induced subhypergraph $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup W_f^{(t)}]$ has the 'right' number of copies of $K_{2,2,2}^{(3)}$. Then, we state $\mathbf{Com}_t(ppf)$ as follows.

$\mathbf{Com}_t(ppf)$: *All but at most* $\delta^{1/4^{t+7}} \prod_{p=1}^{t} \binom{M_p}{h}$ *good* t*-tuples of* h*-subsets* $(L_1, \dots, L_t)$ *satisfy the following condition:*

(*) *all triples* $(L_{p_1}, L_{p_2}, W_f^{(t)})$*, where* $1 \le p_1 < p_2 \le t$ *and* $t+1 \le f \le s$*, posses property* $\boldsymbol{C4EP}_t(p_1, p_2, f)$*.*

In sections to come, we are going to prove the following statements.

(i) Statement $\mathbf{Reg}_1(fff)$ is true.

(ii) Implication $\mathbf{Reg}_t(fff) \Rightarrow \mathbf{Reg}_{t+1}(fff)$ is true for every $t \in [s-4]$.

(iii) Statement $\mathbf{Com}_1(pff)$ is true.

(iv) Implication $\mathbf{Reg}_t(fff) \wedge \mathbf{Com}_t(pff) \Rightarrow \mathbf{Com}_{t+1}(pff)$ is true for every $t \in [s-3]$.

(v) Implication $\mathbf{Com}_1(pff) \Rightarrow \mathbf{Com}_2(ppf)$ is true.

(vi) Implication $\mathbf{Com}_t(pff) \wedge \mathbf{Com}_t(ppf) \Rightarrow \mathbf{Com}_{t+1}(ppf)$ is true for every $t \in [s-2] \setminus \{1\}$.

(vii) Implication $\mathbf{Com}_2(ppf) \Rightarrow \mathbf{Com}_3(ppp)$ is true.

(viii) Implication $\mathbf{Com}_t(ppf) \wedge \mathbf{Com}_t(ppp) \Rightarrow \mathbf{Com}_{t+1}(ppp)$ is true for every $t \in [s-1] \setminus \{1, 2\}$.

From *(i)–(viii)*, one may deduce by induction (see the diagram below) that $\mathbf{Com}_t(ppp)$ holds for every t, $3 \le t \le s$.

$$
\begin{array}{ccccc}
\underbrace{\mathbf{Reg}_1(fff),\ \mathbf{Com}_1(pff)}_{(i,\ iii)} & \xrightarrow{(v)} & \mathbf{Com}_2(ppf) & \xrightarrow{(vii)} & \mathbf{Com}_3(ppp) \\
\big\downarrow (ii,\ iv) & & (vi)\big\downarrow & & (viii)\big\downarrow \\
\mathbf{Reg}_2(fff),\ \mathbf{Com}_2(pff) & \xrightarrow{(vi)} & \mathbf{Com}_3(ppf) & \xrightarrow{(viii)} & \mathbf{Com}_4(ppp) \\
\big\downarrow (ii,\ iv) & & (vi)\big\downarrow & & (viii)\big\downarrow \\
\vdots & & \vdots & & \vdots \\
\big\downarrow (ii,\ iv) & & (vi)\big\downarrow & & (viii)\big\downarrow \\
\mathbf{Reg}_{s-3}(fff),\ \mathbf{Com}_{s-3}(pff) & \xrightarrow{(vi)} & \mathbf{Com}_{s-2}(ppf) & \xrightarrow{(viii)} & \mathbf{Com}_{s-1}(ppp) \\
\big\downarrow (iv) & & (vi)\big\downarrow & & (viii)\big\downarrow \\
\mathbf{Com}_{s-2}(pff) & \xrightarrow{(vi)} & \mathbf{Com}_{s-1}(ppf) & \xrightarrow{(viii)} & \mathbf{Com}_{s}(ppp)
\end{array}
$$

Statement *(i)* is verified in the next section and Section 7 contains the proof of *(ii)*. Section 8 shows *(iii)* and the proof of *(iv)* is given in Section 9. Implications *(v)* and *(vi)* are deduced in Section 10 and implications *(vii)* and *(viii)* in Section 11.

6. Proof of $\mathbf{Reg}_1(fff)$

The proof is based on Claim 6.1 and Claim 6.2 which regard conditions *(1)* and *(2)* in $\mathbf{Reg}_1(fff)$, respectively.

Claim 6.1. *$\mathcal{G}[\mathcal{G}(L_1)]$ is $(\epsilon^{1/2}, d_2)$-regular for all good h-subsets $L_1 \subset V_1$.*

Proof of Claim 6.1. Let $L_1 \subset V_1$ be a *good* h-subset. By (4.1) and $\left(1-\epsilon^{1/4}\right)^h d_2^h m \geq \epsilon^{1/4} m$ (recall that $\epsilon \ll d_2, 1/h$), Fact A.1 implies that $\mathcal{G}[W_{f_1}^{(1)} \cup W_{f_2}^{(1)}]$ is $(\epsilon^{1/2}, d_2)$-regular for every f_1, f_2, $2 \leq f_1 < f_2 \leq s$. □

The proof of $\mathbf{Reg}_1(fff)$ will be completed by proving the following claim.

Claim 6.2. *Fix any $\{f_1, f_2, f_3\}$, where $2 \leq f_1 < f_2 < f_3 \leq s$. Then, all but at most $2\epsilon^{1/12}\binom{m}{h}$ good h-subsets $L_1 \subset V_1$ are such that $\mathcal{H}[W_{f_1}^{(1)} \cup W_{f_2}^{(1)} \cup W_{f_3}^{(1)}]$ is $(\delta^{1/4}, d_3, r_1)$-regular with respect to $\mathcal{G}[W_{f_1}^{(1)} \cup W_{f_2}^{(1)} \cup W_{f_3}^{(1)}]$.*

Indeed, combining Claims 6.1 and 6.2, we obtain that all but $2\binom{s-1}{3}\epsilon^{1/12}\binom{m}{h} \leq \epsilon^{1/24}\binom{m}{h}$ good h-subsets $L_1 \subset V_1$ satisfy conditions *(1)* and *(2)* in $\mathbf{Reg}_1(fff)$.

Proof of Claim 6.2. We prove this claim by contradiction. Fix any f_1, f_2, f_3, where $2 \leq f_1 < f_2 < f_3 \leq s$. Suppose that there exist $\epsilon^{1/12}\binom{m}{h}$ good h-subsets $L^i \subset V_1$ satisfying condition:

(*) *There exist subgraphs* $\mathcal{Q}^i_1, \dots, \mathcal{Q}^i_{r_1}$ *of* $\mathcal{G}[\mathcal{G}(L^i) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})]$ *such that*

$$\Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}^i_t)\Big| \geq \delta^{1/4}\big|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(L^i) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})])\big|, \tag{6.1}$$

but,

$$\Big|\mathcal{H} \cap \bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}^i_t)\Big| > \big(1+\delta^{1/4}\big)d_3\Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}^i_t)\Big|. \tag{6.2}$$

We will derive a contradiction by applying the fact that $\mathcal{H}$ is (δ, d_3, r)-regular with respect to $\mathcal{G}$. We distinguish two cases.

Case 1. Suppose that $2d_2^{3h}/\delta^{1/2}d_3 \geq 1$. Then we consider a good h-subset $L \subset V_1$ for which **(*)** holds. In other words, there exist r_1 subgraphs $\mathcal{Q}_1, \dots, \mathcal{Q}_{r_1}$ of $\mathcal{G}[\mathcal{G}(L) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})]$ such that

$$\Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big| \geq \delta^{1/4}\big|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(L) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})])\big|, \tag{6.3}$$

but

$$\Big|\mathcal{H} \cap \bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big| > \big(1+\delta^{1/4}\big)d_3\Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big|. \tag{6.4}$$

In order to apply the (δ, d_3, r)-regularity of $\mathcal{H}$ (note that $r_1 = r$ in this case), we are going to prove that

$$\Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big| \geq \delta\big|\mathcal{K}_3(\mathcal{G}[V_{f_1} \cup V_{f_2} \cup V_{f_3}])\big|. \tag{6.5}$$

Due to (4.1), the $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}[\mathcal{G}(L) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})]$, Fact A.9, and $2d_2^{3h}/\delta^{1/2}d_3 \geq 1$, we have

$$\begin{aligned}\Big|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(L) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})])\Big| &\geq \big(1-\epsilon^{1/2}\big)^3\big(1-2\epsilon^{1/2}\big)d_2^3\big(d_2^h m\big(1-\epsilon^{1/4}\big)^h\big)^3 \\ &\geq \frac{1}{2}\delta^{1/2}d_3 \times \big(1-\epsilon^{1/4}\big)^{3h+4}d_2^3 m^3 \end{aligned}\tag{6.6}$$

Combining the above inequality with (6.3) and (1.2), we obtain

$$\Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big| \geq \delta \times \big((1+\epsilon)^3 + 4\epsilon/d_2^3\big)d_2^3 m^3.$$

Fact A.9 implies that

$$\big|\mathcal{K}_3(\mathcal{G}[V_{f_1} \cup V_{f_2} \cup V_{f_3}])\big| \leq \big((1+\epsilon)^3 + 4\epsilon/d_2^3\big)d_2^3 m^3,$$

and, therefore,

$$\Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big| \geq \delta\big|\mathcal{K}_3(\mathcal{G}[V_{f_1} \cup V_{f_2} \cup V_{f_3}])\big|.$$

Since $\mathcal{H}$ is (δ, d_3, r)-regular with respect to $\mathcal{G}$ and $r_1 = r$ in this case, we have

$$\Big|\mathcal{H} \cap \bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big| \le (1+\delta) d_3 \Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t)\Big|. \tag{6.7}$$

This is a contradiction to (6.4) since $\delta \ll 1$.

Case 2. Suppose that $2d_2^{3h}/\delta^{1/2}d_3 < 1$. Observe that in this case inequality (6.6) and consequently inequality (6.5) cannot be guaranteed. To overcome this problem, we will use the existence of subgraphs $\mathcal{Q}_1^i, \dots, \mathcal{Q}_{r_1}^i$ (cf. **(*)**) for a family of $r'' = r/r_1$ h-subsets L^i satisfying condition **(*)** and prove an inequality similar to (6.5) (cf. inequality (6.9) below). Then we apply the (δ, d_3, r)-regularity of $\mathcal{H}$.

We define an auxiliary graph $\mathcal{D}$ with $V(\mathcal{D}) = [V_1]^h$ and

$$E(\mathcal{D}) = \big\{\{L, L'\} \colon \big|\mathcal{G}(L \cup L') \cap V_f\big| \ne (1 \pm \epsilon^{1/4}) d_2^{2h} m \text{ for some } f \in \{2, \dots, s\}\big\}.$$

By Fact A.6, all but at most $\epsilon^{1/4}\binom{m}{h}^2$ pairs (L, L'), where L and L' are h-subsets of V_1, satisfy

$$\big|\mathcal{G}(L \cup L') \cap V_f\big| = \big(1 \pm \epsilon^{1/4}\big) d_2^{2h} m$$

for every $2 \le f \le s$. Consequently, $\big|E(\mathcal{D})\big| \le \epsilon^{1/4}\binom{m}{h}^2$.

We are going to apply Fact A.10 to the graph $\mathcal{D}$ with parameters $n = \binom{m}{h}$, $\sigma = \epsilon^{1/4}$, $c = \epsilon^{1/12}$, and $t = (1/d_2)^{3h}$. Set $W = \{L : L \text{ satisfies condition } \textbf{(*)}\}$ and observe that $|W| \ge cn$ by the assumption.

Using Fact A.10, we obtain the existence of $r'' = \mu(1/d_2)^{3h} < t$ subsets $L^i \in W$, where $\mu = \delta^{1/2} d_3/2$, such that for $2 \le f \le s$ and $1 \le i < j \le r''$,

$$\big|\mathcal{G}(L^i \cup L^j) \cap V_f\big| = \big(1 \pm \epsilon^{1/4}\big) d_2^{2h} m. \tag{6.8}$$

Note that $r'' > 1$ since $2d_2^{3h}/\delta^{1/2}d_3 < 1$.

In order to apply the (δ, d_3, r)-regularity of $\mathcal{H}$ (note that $r_1 \times r'' = r$), we are going to prove that

$$\Big|\bigcup_{i=1}^{r''} \bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t^i)\Big| \ge \delta \big|\mathcal{K}_3(\mathcal{G}[V_{f_1} \cup V_{f_2} \cup V_{f_3}])\big|. \tag{6.9}$$

We apply the Inclusion-Exclusion Principle and obtain

$$\Big|\bigcup_{i=1}^{r''} \bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t^i)\Big| \ge \sum_{i=1}^{r''} \Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t^i)\Big| - \sum_{1 \le i < j \le r''} \Big|\bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t^i) \cap \bigcup_{t=1}^{r_1} \mathcal{K}_3(\mathcal{Q}_t^j)\Big|. \tag{6.10}$$

Now we estimate both sums on the right-hand side of (6.10). Due to (4.1), the $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}[\mathcal{G}(L^i) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})]$, and Fact A.9, we have

$$\begin{aligned}\Big|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(L^i) \cap (V_{f_1} \cup V_{f_2} \cup V_{f_3})])\Big| &\ge \big(1 - \epsilon^{1/2}\big)^3 \big(1 - 2\epsilon^{1/2}\big) d_2^3 \big(\big(1 - \epsilon^{1/4}\big)^h d_2^h m\big)^3 \\ &\ge \big(1 - \epsilon^{1/4}\big)^{3h+4} d_2^{3h+3} m^3.\end{aligned}$$

We apply this to (6.1) and obtain

$$\sum_{i=1}^{r''}\Big|\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big| \geq r'' \times \delta^{1/4}\big(1-\epsilon^{1/4}\big)^{3h+4} d_2^{3h+3} m^3. \tag{6.11}$$

Furthermore, by (6.8), the $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}[\mathcal{G}(L^i \cup L^j)]$, and Fact A.9, we have

$$\Big|\bigcup_{t=1}^{r'}\mathcal{K}_3(\mathcal{Q}_t^i) \cap \bigcup_{t=1}^{r'}\mathcal{K}_3(\mathcal{Q}_t^j)\Big| \leq \big((1+\epsilon^{1/2})^3 + 4\epsilon^{1/2}/d_2^3\big) d_2^3\big((1+\epsilon^{1/4})d_2^{2h} m\big)^3,$$

and, consequently,

$$\sum_{1\leq i<j\leq r''}\Big|\bigcup_{t=1}^{r'}\mathcal{K}_3(\mathcal{Q}_t^i) \cap \bigcup_{t=1}^{r'}\mathcal{K}_3(\mathcal{Q}_t^j)\Big| \leq \binom{r''}{2} \times 2d_2^{6h+3}m^3 = (r'')^2 d_2^{6h+3} m^3. \tag{6.12}$$

Combining (6.11), (6.12,) and (6.10) yields

$$\Big|\bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big| \geq r'' \times \delta^{1/4}\big(1-\epsilon^{1/4}\big)^{3h+4} d_2^{3h+3} m^3 - (r'')^2 d_2^{6h+3} m^3.$$

Since $r'' = \mu(1/d_2)^{3h}$, $\mu = \delta^{1/2} d_3/2$, and (1.2), we get

$$\begin{aligned}\Big|\bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big| &\geq \mu(1/d_2)^{3h} \times \delta^{1/4}\big(1-\epsilon^{1/4}\big)^{3h+4} d_2^{3h+3} m^3 - \mu^2(1/d_2)^{6h} d_2^{6h+3} m^3 \\ &\geq \delta\big((1+\epsilon)^3 + 4\epsilon/d_2^3\big) d_2^3 m^3.\end{aligned}$$

The last inequality follows from $\epsilon \ll 1/h, d_2, \delta$ and $\delta \ll 1$. Fact A.9 implies that $\big|\mathcal{K}_3(\mathcal{G}[V_{f_1} \cup V_{f_2} \cup V_{f_3}])\big| \leq d_2^3 m^3((1+\epsilon)^3 + 4\epsilon/d_2^3)$, therefore,

$$\Big|\bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big| \geq \delta\big|\mathcal{K}_3(\mathcal{G}[V_{f_1} \cup V_{f_2} \cup V_{f_3}])\big|.$$

Since $\mathcal{H}$ is (δ, d_3, r)-regular with respect to $\mathcal{G}$, we have

$$\Big|\mathcal{H} \cap \bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big| \leq (1+\delta)d_3\Big|\bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big| \leq (1+\delta)d_3 \sum_{i=1}^{r''}\Big|\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big|. \tag{6.13}$$

It also follows from (6.11), (6.12), $\epsilon \ll 1/h$, and $\delta \ll 1$, that

$$\frac{\sum_{1\leq i<j\leq r''}\Big|\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i) \cap \bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^j)\Big|}{\sum_{i=1}^{r''}\Big|\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\Big|} \leq \frac{2}{3}\delta^{1/4} d_3. \tag{6.14}$$

Now we will obtain a lower bound on $\big|\mathcal{H} \cap \bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\big|$ and derive a contradiction to (6.13). Applying (6.2) and the Inclusion-Exclusion Principle, we get

$$\left|\mathcal{H}\cap\bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\right|\geq\left(1+\delta^{1/4}\right)d_3\sum_{i=1}^{r''}\left|\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\right| - \sum_{1\leq i<j\leq r''}\left|\mathcal{H}\cap\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\cap\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^j)\right|.$$

Since $\left|\mathcal{H}\cap\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\cap\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^j)\right|\leq\left|\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\cap\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^j)\right|$, by applying (6.14), we have

$$\left|\mathcal{H}\cap\bigcup_{i=1}^{r''}\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\right|\geq\left(1+\frac{\delta^{1/4}}{3}\right)d_3\sum_{i=1}^{r''}\left|\bigcup_{t=1}^{r_1}\mathcal{K}_3(\mathcal{Q}_t^i)\right|.$$

This contradicts (6.13) because $\delta\ll 1$.

Similarly, we can prove that at most $\epsilon^{1/12}\binom{m}{h}$ good h-subsets $L_1\subset V_1$ satisfy (6.1) and

$$\left|\mathcal{H}\cap\bigcup_{t=1}^{r'}\mathcal{K}_3(\mathcal{Q}_t)\right|<\left(1-\delta^{1/4}\right)d_3\left|\bigcup_{t=1}^{r'}\mathcal{K}_3(\mathcal{Q}_t)\right|.$$

Therefore, we proved that for any fixed $2\leq f_1<f_2<f_3\leq s$, all but $2\epsilon^{1/12}\binom{m}{h}$ good h-subsets $L_1\subset V_1$ are such that $\mathcal{H}[\mathcal{G}(L_1)\cap(V_{f_1}\cup V_{f_2}\cup V_{f_3})]$ is $(\delta^{1/4},d_3,r_1)$-regular with respect to $\mathcal{G}[\mathcal{G}(L_1)\cap(V_{f_1}\cup V_{f_2}\cup V_{f_3})]$. This completes the proof. □

7. Proof of $\mathbf{Reg}_t(fff)\Rightarrow\mathbf{Reg}_{t+1}(fff)$

Before proving implication (ii), we introduce one additional definition.

Definition 7.1. *For a good t-tuple of h-subsets $(L_1,\dots,L_t)$, we say that an h-subset $L_{t+1}\subset W_{t+1}^{(t)}$ is good for $(L_1,\dots,L_t)$ (sometimes we simply say L_{t+1} is good) if*

$$M_{t+2}^-\leq|W_f^{(t+1)}|\leq M_{t+2}^+$$

for every f, $t+2\leq f\leq s$.

Remark 7.2. *A $(t+1)$-tuple of h-subsets $(L_1,\dots,L_{t+1})$ is good if and only if $(L_1,\dots,L_t)$ is good and L_{t+1} is good for $(L_1,\dots,L_t)$.*

The proof of implication (ii) is a consequence of the following lemma.

Lemma 7.3. *If $(L_1,\dots,L_t)$ is a good t-tuple of h-subsets satisfying conditions (1) and (2) in $\boldsymbol{Reg}_t(fff)$, then all but at most $\epsilon^{1/24\times 2}\binom{M_{t+1}^+}{h}$ good h-subsets $L_{t+1}\subset W_{t+1}^{(t)}$ are such that $(L_1,\dots,L_{t+1})$ satisfies conditions (1) and (2) in $\boldsymbol{Reg}_{t+1}(fff)$.*

Sketch of the proof. Since a good l-tuple of h-subsets $(L_1,\dots,L_t)$ satisfies conditions (1) and (2) in $\mathbf{Reg}_t(fff)$, we know that the $(s-t)$-partite graph $\mathcal{G}[\mathcal{G}(L_1\cup L_2\cup\dots\cup L_t)]$ is $(\epsilon^{1/2},d_2)$-regular and the $(s-t)$-partite 3-uniform hypergraph $\mathcal{H}[\mathcal{G}(L_1\cup L_2\cup\dots\cup L_t)]$

is $(\delta^{1/4^t}, d_3, r_t)$-regular. This enables us to select L_{t+1} in a similar situation as for L_1. Replacing ϵ by $\epsilon^{1/2}$, δ by $\delta^{1/4^{t-1}}$, m by $M_{t+1}(1 \pm \epsilon^{1/4})^{ht}$, and r by r_t, we can prove Lemma 7.3 in the same way as we proved $\mathbf{Reg}_1(fff)$. □

Now we prove implication $\mathbf{Reg}_t(fff) \Rightarrow \mathbf{Reg}_{t+1}(fff)$ by applying Lemma 7.3.

Proof. Let $(L_1, \ldots, L_{t+1})$ be a good $(t+1)$-tuple of h-subsets which does not satisfy conditions (1) and (2) in $\mathbf{Reg}_{t+1}(fff)$. We distinguish two cases.

Case 1. $(L_1, \ldots, L_t)$ satisfies conditions (1) and (2) in $\mathbf{Reg}_t(fff)$, but L_{t+1} is such that $(L_1, \ldots, L_{t+1})$ violates either condition (1) or (2) in $\mathbf{Reg}_{t+1}(fff)$. By Lemma 7.3, there are at most $\epsilon^{1/24\times 2}\binom{M^+_{t+1}}{h}$ choices for such L_{t+1}.

From Fact A.7(3) we know that there are at most $\left(1+2\epsilon^{1/2^{t+1}}\right)\prod_{p=1}^{t}\binom{M_p}{h}$ good t-tuples $(L_1, \ldots, L_t)$. Consequently, the number of $(t+1)$-tuples $(L_1, \ldots, L_{t+1})$ as described above is at most

$$\left(1+2\epsilon^{1/2^{t+1}}\right)\prod_{p=1}^{t}\binom{M_p}{h} \times \epsilon^{1/24\times 2}\binom{M^+_{t+1}}{h} \le 2\epsilon^{1/24\times 2}\prod_{p=1}^{t+1}\binom{M^+_p}{h} \tag{7.1}$$

since $\epsilon \ll 1$.

Case 2. $(L_1, \ldots, L_t)$ violates either condition (1) or (2) in $\mathbf{Reg}_t(fff)$. By $\mathbf{Reg}_t(fff)$, there are at most $\epsilon^{1/24\times 4^{t-1}}\prod_{p=1}^{t}\binom{M_p}{h}$ such good t-tuples. For each such $(L_1, \ldots, L_t)$, there are at most $\binom{M^+_{t+1}}{h}$ choices for L_{t+1}. Therefore, the number of $(t+1)$-tuples $(L_1, \ldots, L_{t+1})$ in this case is at most

$$\epsilon^{1/24\times 4^{t-1}}\prod_{p=1}^{t}\binom{M_p}{h} \times \binom{M^+_{t+1}}{h} < \epsilon^{1/24\times 4^{t-1}}\prod_{p=1}^{t+1}\binom{M^+_p}{h}. \tag{7.2}$$

Combining (7.2) and (7.1), we obtain that all but at most

$$\left(\epsilon^{1/24\times 4^{t-1}} + 2\epsilon^{1/24\times 2}\right)\prod_{p=1}^{t+1}\binom{M^+_p}{h} \le \epsilon^{1/24\times 4^{t}}\prod_{p=1}^{t+1}\binom{M_p}{h}$$

good $(t+1)$-tuples of h-subsets $(L_1, \ldots, L_{t+1})$ satisfy both conditions ((1) and (2)) in $\mathbf{Reg}_{t+1}(fff)$. □

8. Proof of $\mathbf{Com}_1(pff)$

The proof of $\mathbf{Com}_1(pff)$ will be completed by proving the following claim.

Claim 8.1. *Let f_1, f_2, where $2 \le f_1 < f_2 \le s$, be fixed. Then all but at most $\delta^{1/4^6}\binom{m}{h}$ good h-subsets $L_1 \subset V_1$ satisfy the following:*

(5.1) holds for all but at most $\delta^{1/4^6} d_2^{4h+4} m^4/4$ four-cycles C_4 in $\mathcal{G}[W^{(1)}_{f_1} \cup W^{(1)}_{f_2}]$.

Indeed, Claim 8.1 implies that all but at most $\binom{s-1}{2}\delta^{1/4^6}\binom{m}{h} \le \delta^{1/4^7}\binom{m}{h}$ good h-subsets L_1 satisfy condition **(*)** in $\mathbf{Com}_1(pff)$: for all but at most $\delta^{1/4^6}d_2^{4h+4}m^4/4$ four-cycles C_4 in $\mathcal{G}[W_{f_1}^{(1)} \cup W_{f_2}^{(1)}]$,

$$\Big|\mathcal{H}(C_4) \cap L_1\Big| = \big(1 \pm \delta^{1/4^6}\big)d_3^4 h$$

holds for every f_1, f_2, $2 \le f_2 < f_2 \le s$.

To prove Claim 8.1, we define an auxiliary bipartite graph $\Gamma = (U_1 \cup U_2, E)$, where U_1 consists of all good h-subsets $L_1 \subset V_1$ and U_2 consists of all four-cycles C_4 in $\mathcal{G}[V_{f_1} \cup V_{f_2}]$. We join $L_1 \in U_1$ and $C_4 \in U_2$ by an edge if and only if $C_4 \in \mathcal{G}[\mathcal{G}(L_1)]$ and (5.1) holds. Note that $\big(1-\epsilon^{1/4}\big)\binom{m}{h} \le \big|U_1\big| \le \binom{m}{h}$ (cf. Fact A.5) and $\big|U_2\big| = \big(1 \pm \epsilon^{1/8}\big)d_2^4 m^4/4$ (cf. Fact A.8).

Then, Claim 8.1 translates into showing that

$$\deg_\Gamma(L_1) \ge \big|\big\{C_4 \colon C_4 \in \mathcal{G}[W_{f_1}^{(1)} \cup W_{f_2}^{(1)}]\big\}\big| - \delta^{1/4^6}d_2^{4h+4}m^4/4$$

for all but at most $\delta^{1/4^6}\binom{m}{h}$ sets $L_1 \in U_1$.

By (4.1), the $(\epsilon^{1/2}, d)$-regularity of $\mathcal{G}[W_{f_1}^{(1)} \cup W_{f_2}^{(1)}]$, Fact A.8, and $\epsilon \ll 1$, we have

$$\big|\big\{C_4 : C_4 \in \mathcal{G}[W_{f_1}^{(1)} \cup W_{f_2}^{(1)}]\big\}\big| \le \big(1 + \epsilon^{1/2^5}\big)d_2^{4h+4}m^4/4.$$

Consequently, the proof of Claim 8.1 follows from the following claim.

Claim 8.2. *In the graph* Γ*, all but at most* $\delta^{1/4^6}\binom{m}{h}$ *sets* L_1 *in* U_1 *satisfy*

$$\deg_\Gamma(L_1) \ge \big(1 + \epsilon^{1/2^5} - \delta^{1/4^6}\big)d_2^{4h+4}m^4/4. \tag{8.1}$$

Since the proof of this claim requires additional claims and lemmas, we put it as a separate subsection.

8.1. Proof of Claim 8.2

We will state and prove three auxiliary statements first, then we return to Claim 8.2.

Claim 8.3. *For every* f_1*,* f_2*,* $2 \le f_1 < f_2 \le s$*, all but at most* $\epsilon^{1/8}d_2^4m^4$ *four-cycles* $C_4 = \{\{x,y\},\{y,x'\},\{x',y'\},\{y',x\}\}$ *in* $\mathcal{G}[V_{f_1} \cup V_{f_2}]$ *satisfy*

$$\big(1-\epsilon^{1/4}\big)^4 d_2^4 m \le \big|\mathcal{G}(x,x',y,y') \cap V_1\big| \le \big(1+\epsilon^{1/4}\big)^4 d_2^4 m. \tag{8.2}$$

Proof. Fix arbitrary f_1, f_2 so that $2 \le f_1 < f_2 \le s$. It follows from Fact A.4 that all but at most $4\epsilon^{1/2}m^2$ pairs $\{x,x'\} \in [V_{f_1}]^2$ satisfy

$$\big(1-\epsilon^{1/2}\big)^2 d_2^2 m \le \big|\mathcal{G}(x,x') \cap V_1\big| \le \big(1+\epsilon^{1/2}\big)^2 d_2^2 m. \tag{8.3}$$

Consider a pair $\{x,x'\} \in [V_{f_1}]^2$ satisfying (8.3). Since $d_2^2(1-\epsilon^{1/2})^2 \gg \epsilon^{1/4}$, $\mathcal{G}[\mathcal{G}(x,x') \cap (V_1 \cup V_{f_2})]$ is $(\epsilon^{1/2}, d_2)$-regular by Fact A.1. Consequently, Fact A.4 implies that all but at most $4\epsilon^{1/4} \times (1+\epsilon^{1/2})^4 d_2^4 m^2 \le 6\epsilon^{1/4}d_2^4m^2$ pairs of vertices $y, y' \in \mathcal{G}(x,x') \cap V_{f_2}$ satisfy

$$\big|\mathcal{G}(x,x',y,y') \cap V_1\big| = \big(1 \pm \epsilon^{1/4}\big)^2 d_2^2 \big|\mathcal{G}(x,x') \cap V_1\big|. \tag{8.4}$$

Combining (8.3) and (8.4), we obtain that all but at most $4\epsilon^{1/2}m^2 \times \binom{m}{2} + \binom{m}{2} \times 6\epsilon^{1/4}d_2^4 m^2 \le \epsilon^{1/8}d_2^4 m^4$ four-cycles $C_4 = \{\{x,y\},\{y,x'\},\{x',y'\},\{y',x\}\}$ satisfy (8.2). □

Lemma 8.4. *For every f_1, f_2, $2 \le f_1 < f_2 \le s$, all but at most $\delta^{1/2^{10}} d_2^4 m^4/4$ four-cycles $C_4 = \{\{x,y\},\{y,x'\},\{x',y'\},\{y',x\}\}$ in $\mathcal{G}[V_{f_1} \cup V_{f_2}]$ satisfy both (8.2) and*

$$\big|\mathcal{H}(C_4) \cap V_1\big| = \big(1 \pm \delta^{1/2^{10}}\big) d_2^4 d_3^4 m. \tag{8.5}$$

Recall that $\epsilon \ll d_2, \delta \ll d_3$ (cf. (1.2)). The fact that $\epsilon \ll d_2$ allows to prove Claim 8.3 in a standard way. The proof of Lemma 8.4 is, however, more complicated. This is because d, the density of the graph $\mathcal{G}$, can be smaller than δ which measures the regularity of $\mathcal{H}$. The proof of this lemma is given in Section 8.2.

The following claim is a consequence of Lemma 8.4.

Claim 8.5. *In the graph Γ, all but at most $\delta^{1/2^{10}} d_2^4 m^4/4$ four-cycles C_4 in U_2 satisfy*

$$\deg_\Gamma(C_4) \ge \big(1 - 2e^{-\delta^{1/2^{10}} d_3^4 h}\big) \binom{\big(1-\epsilon^{1/4}\big)^4 d_2^4 m}{h} - \epsilon^{1/4}\binom{m}{h}. \tag{8.6}$$

Proof of Claim 8.5. Recall that in Γ, $U_1 = \binom{V_1}{h}$ and U_2 consists of all copies of C_4 in $\mathcal{G}[V_{f_1} \cup V_{f_2}]$. By Lemma 8.4, all but at most $\delta^{1/2^{10}} d_2^4 m^4/4$ four-cycles $C_4 \in U_2$ satisfy both (8.2) and (8.5). We will show that (8.6) holds for each such $C_4 = \big\{\{x,y\},\ \{y,x'\},\ \{x',y'\},\ \{y',x\}\big\}$.

Set $M_{C_4} = \mathcal{G}(x,x',y,y') \cap V_1$ and $N_{C_4} = \mathcal{H}(C_4) \cap V_1$. Note that $N_{C_4} \subset M_{C_4}$. In a view of the definition of Γ, to prove (8.6), we need to estimate the number of $L \in U_1$ for which $\big||N_{C_4} \cap L| - d_3^4 h\big| \le \delta^{1/4^6} d_3^4 h$ holds. To accomplish this, we use Chernoff's inequality for the hypergeometric distribution [5].

Let L be a random h-subset of M_{C_4} and $X = \big|N_{C_4} \cap L\big| = \big|\mathcal{H}(C_4) \cap L\big|$. Then X has the hypergeometric distribution with parameters $\big|M_{C_4}\big|$, h, and $\big|N_{C_4}\big|$.

Observe that $\big|M_{C_4}\big| = \big(1 \pm \epsilon^{1/4}\big)^4 d_2^4 m$ (cf. (8.2)), $\big|N_{C_4}\big| = \big(1 \pm \delta^{1/2^{10}}\big) d_2^4 d_3^4 m$ (cf. (8.5)), and $\mathbb{E}(X) = \big|N_{C_4}\big| h / \big|M_{C_4}\big| = \big(1 \pm 2\delta^{1/2^{10}}\big) d_3^4 h$. Applying Chernoff's inequality, we get

$$\mathbb{P}\Big(\big|X - d_3^4 h\big| > \delta^{1/4^6} d_3^4 h\Big) \le \mathbb{P}\Big(\big|X - \mathbb{E}(X)\big| > \delta^{1/4^6}\mathbb{E}(X)/4\Big) \le 2e^{-\delta^{1/2^{10}} d_3^4 h}. \tag{8.7}$$

By the definition of Γ, C_4 and L are adjacent in Γ, if and only if, $\big|X - d_3^4 h\big| \le \delta^{1/4^6} d_3^4 h$. Hence, (8.7) and Fact A.5 imply that

$$\begin{aligned}\deg_\Gamma(C_4) &\ge \big(1 - 2e^{-\delta^{1/2^{10}} d_3^4 h}\big)\binom{|M_{C_4}|}{h} - \epsilon^{1/4}\binom{m}{h} \\ &\ge \big(1 - 2e^{-\delta^{1/2^{10}} d_3^4 h}\big)\binom{\big(1-\epsilon^{1/4}\big)^4 d_2^4 m}{h} - \epsilon^{1/4}\binom{m}{h}.\end{aligned}$$

□

Now we are ready to prove Claim 8.2.

Proof of Claim 8.2. We first use Claim 8.5 to find a lower bound on the number of edges $e(\Gamma)$ of Γ. Then, assuming Claim 8.2 is false, we derive an upper bound on $e(\Gamma)$. Comparing these two bounds will yield a contradiction.

By Claim 8.5, we have

$$\begin{aligned} e(\Gamma) \geq & \big(1-\epsilon^{1/8}-\delta^{1/2^{10}}\big) d_2^4 m^4/4 \\ & \times \left(\big(1-2e^{-\delta^{1/2^{10}} d_3^4 h}\big) \left(\binom{\big(1-\epsilon^{1/4}\big)^4 d_2^4 m}{h} - \epsilon^{1/4} \binom{m}{h} \right) \right) \\ \geq & \big(1-2\delta^{1/2^{10}}\big) \binom{m}{h} d_2^{4h+4} m^4/4. \end{aligned} \tag{8.8}$$

The last inequality follows from $\epsilon \ll \delta$, d_2 and the fact that $2e^{-\delta^{1/2^{10}} d_3 h} \leq \delta^{1/2^{10}}/2$ when $h \gg 1/\delta$.

Now suppose that Claim 8.2 is not true, i.e., there exist more than $\delta^{1/4^6}\binom{m}{h}$ sets $L_1 \in U_1$ such that

$$\deg_\Gamma(L_1) < \big(1+\epsilon^{1/2^5}-\delta^{1/4^6}\big) d_2^{4h+4} m^4/4. \tag{8.9}$$

We are going to derive a contradiction to (8.8).

By (4.1), the $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}[\mathcal{G}(L_1)]$, and Fact A.8, we know that

$$\deg_\Gamma(L_1) \leq \big(1+\epsilon^{1/4}\big)^{4h} \big(1+\epsilon^{1/2^4}\big) d_2^{4h+4} m^4/4. \tag{8.10}$$

for every $L_1 \in U_1$. Combining (8.9) and (8.10) yields

$$\begin{aligned} e(\Gamma) \quad < \quad & \delta^{1/4^6} \binom{m}{h} \times \big(1+\epsilon^{1/2^5}-\delta^{1/4^6}\big) d_2^{4h+4} m^4/4 \\ & +\big(1-\delta^{1/4^6}\big) \binom{m}{h} \times \big(1+\epsilon^{1/4}\big)^{4h} \big(1+\epsilon^{1/2^4}\big) d_2^{4h+4} m^4/4 \\ \overset{(S2)}{\leq} \quad & \big(1-\delta^{1/2^{11}}/2\big) \binom{m}{h} d_2^{4h+4} m^4/4. \end{aligned}$$

This contradicts (8.8) since $\delta \ll 1$. □

8.2. Proof of Lemma 8.4

In order to verify Lemma 8.4, we need to show that all but at most $\delta^{1/2^{10}} d_2^4 m^4/4$ four-cycles satisfy both (8.2) and (8.5). In Claim 8.3, we proved that (8.2) holds for all but at most $\epsilon^{1/4} d_2^4 m^4$ four-cycles. Therefore, it suffices to show that (8.5) is true for all but at most $(1/2)\delta^{1/2^{10}} d_2^4 m^4/4$ four-cycles C_4 in $\mathcal{G}[V_{f_1} \cup V_{f_2}]$.

To this end, we construct a bipartite graph $\mathcal{B}_0 = (U_0 \cup W_0, E)$, where $U_0 = V_1$, W_0 consists of all four-cycles C_4 in $\mathcal{G}[V_{f_1} \cup V_{f_2}]$, and $x \in U_0$ and $C_4 \in W_0$ are adjacent in $\mathcal{B}_0$ if and only if $C_4 \subset \mathcal{H}(x)$.

Observe that $|W_0| = \big(1 \pm \epsilon^{1/8}\big) d_2^4 m^4/4$ (cf. Fact A.8). In order to prove Lemma 8.4, it is sufficient to show that all but at most $(1/2)\delta^{1/2^{10}} d_2^4 m^4/4$ four-cycles $C_4 \in W_0$ satisfy

$$\deg_{\mathcal{B}_0}(C_4) = \big(1 \pm \delta^{1/2^{10}}\big) d_2^4 d_3^4 m.$$

To prove this, we will apply the following three lemmas.

Lemma 8.6. *In the graph $\mathcal{B}_0$, all but at most $\delta^{1/4}m$ vertices $x \in U_0$ satisfy*

$$\deg_{\mathcal{B}_0}(x) = \left(1 \pm \delta^{1/2^5}\right) d_2^4 d_3^4 \left|W_0\right|. \tag{8.11}$$

Lemma 8.7. *In the graph $\mathcal{B}_0$, all but at most $\delta^{1/2^4}m^2$ pairs $\{x, x'\} \in [U_0]^2$ satisfy*

$$\deg_{\mathcal{B}_0}(x, x') = \left(1 \pm \delta^{1/2^7}\right) d_2^8 d_3^8 \left|W_0\right|. \tag{8.12}$$

The proofs of these two lemmas are given in Sections 8.3 and 8.4. In addition to these two lemmas,we will also use the following result, which is a modification of an earlier result of Duke-Lefmann-Rödl [3].

Lemma 8.8. *Let $\tilde{\delta}$, d be constants, $0 < \tilde{\delta}$, $d < 1$, and let $\mathcal{B} = (U \cup W, E)$ be a bipartite graph with $\left|U\right| \geq 1/(d^2\tilde{\delta})$. Denote by $\mathcal{D}$ the collection of all pairs $\{x, x'\} \in [U]^2$ for which either (i) or (ii) below fails:*

(i) $\deg_{\mathcal{B}}(x)$, $\deg_{\mathcal{B}}(x') > (1 - \tilde{\delta})d\left|W\right|$,

(ii) $\deg_{\mathcal{B}}(x, x') < (1 + \tilde{\delta})d^2\left|W\right|$.

If $|\mathcal{D}| < \tilde{\delta}|U|^2$ and

$$\sum_{\{x,x'\} \in \mathcal{D}} \deg_{\mathcal{B}}(x, x') \leq \tilde{\delta} d^2 |U|^2 |W|, \tag{8.13}$$

then $\mathcal{B}$ is (δ', d)-regular, where $\delta' = (11\tilde{\delta})^{1/5}$.

Although Lemma 8.8 resembles Proposition 2.5 in [3], this proposition cannot be applied to our situation because it is designed for the case when d is larger than $\tilde{\delta}$. In our situation, we consider graph $\mathcal{B}_0$ and set

$$\tilde{\delta} = \delta^{1/2^7} \text{ and } d = d_2^4 d_3^4.$$

Let $\mathcal{D}_0$ be the collection of all pairs $\{x, x'\} \in [U_0]^2$ for which either *(i)* or *(ii)* fails (replace $\mathcal{B}$ by $\mathcal{B}_0$, and W by W_0). By Lemma 8.6 and Lemma 8.7,

$$\left|\mathcal{D}_0\right| \leq \left(\delta^{1/4} + \delta^{1/2^4}\right)m^2 \leq \delta^{1/2^5}m^2 \leq \tilde{\delta}m^2. \tag{8.14}$$

Here, due to (1.2), we cannot rule out the situation when $\tilde{\delta} \geq d$. The purpose of Lemma 8.8 is to be applied to this situation.

The proof of this lemma is a modification of the earlier proof of Duke-Lefmann-Rödl result (or an earlier similar result given in [1]) and it is given in Appendix B.

Proof of Lemma 8.4. We are going to apply Lemmas 8.6, 8.7, and 8.8 to the bipartite graph $\mathcal{B}_0$ constructed at the beginning of this section. Let $\tilde{\delta}, d$ and $\mathcal{D}_0$ be defined as above. We know that $\left|\mathcal{D}_0\right| \leq \tilde{\delta}m^2$ (cf. (8.14). We will verify that

$$\sum_{\{x_i,x_j\} \in \mathcal{D}_0} \deg_{\mathcal{B}}(x, x') \leq \tilde{\delta} d^2 |U_0|^2 |W_0|$$

holds. Call a pair $\{x, x'\} \in [U_0]^2$ good if

$$\left|\mathcal{G}(x, x') \cap V_f\right| = \left(1 \pm \epsilon^{1/2}\right)^2 d_2^2 m \geq \epsilon^{1/4} m$$

for $f \in \{f_1, f_2\}$. Let $\mathcal{D}_0^{\rm good}$ be the set of all good pairs in $\mathcal{D}_0$ and $\mathcal{D}_0^{\rm bad} = \mathcal{D}_0 \setminus \mathcal{D}_0^{\rm good}$.

By Fact A.4, all but at most $4\epsilon^{1/2}m^2 \le \epsilon^{1/4}m^2$ pairs $\{x, x'\} \in [U_0]^2$ are good, that is,

$$\big|\mathcal{D}_0^{\rm bad}\big| \le \epsilon^{1/4}m^2. \tag{8.15}$$

For $\{x, x'\} \in \mathcal{D}_0^{\rm good}$, since $\big|\mathcal{G}(x, x')\cap V_f\big| \ge \epsilon^{1/4}m$ for $f \in \{f_1, f_2\}$, the graph $\mathcal{G}[\mathcal{G}(x, x')\cap (V_{f_1} \cup V_{f_2})]$ is $(\epsilon^{1/2}, d)$-regular (cf. Fact A.1). Consequently (cf. Fact A.8), the number of four-cycles in the graph $\mathcal{G}[\mathcal{G}(x, x') \cap (V_{f_1} \cup V_{f_2})]$ is less than

$$\big(1 + \epsilon^{1/32}\big)d_2^{12}m^4/4 \le \big(1 + \epsilon^{1/64}\big)d_2^8\big|W_0\big|.$$

Since $\mathcal{H}(x, x')[V_{f_1} \cup V_{f_2}]$ is a subgraph of $\mathcal{G}[\mathcal{G}(x, x') \cap (V_{f_1} \cup V_{f_2})]$ and $\deg_{\mathcal{B}_0}(x, x')$ is the number of four-cycles in the graph $\mathcal{H}(x, x')[V_{f_1} \cup V_{f_2}]$, we obtain that for $\{x, x'\} \in \mathcal{D}_0^{\rm good}$,

$$\deg_{\mathcal{B}_0}(x, x') \le \big(1 + \epsilon^{1/64}\big)d_2^8\big|W_0\big|. \tag{8.16}$$

Combining the fact that $\deg_{\mathcal{B}_0}(x, x') \le |W_0|$ for $\{x, x'\} \in \mathcal{D}_0^{\rm bad}$, (8.15), (8.14), and (8.16), we have

$$\begin{aligned}\sum_{\{x,x'\}\in D_0} \deg_{\mathcal{B}_0}(x, x') &= \sum_{\{x,x'\}\in D_0^{\rm bad}} \deg_{\mathcal{B}_0}(x, x') + \sum_{\{x,x'\}\in D_0^{\rm good}} \deg_{\mathcal{B}_0}(x, x')\\ &\le \epsilon^{1/4}m^2 \times \big|W_0\big| + \delta^{1/2^5}m^2 \times \big(1 + \epsilon^{1/64}\big)d_2^8\big|W_0\big|\\ &\overset{\textit{(S1), (S2)}}{\le} \tilde{\delta}d^2m^2\big|W_0\big|.\end{aligned}$$

Thus, we have verified that $\mathcal{B}_0$ satisfies the assumptions of Lemma 8.8. Applying this lemma to the graph $\mathcal{B}_0$, we obtain that $\mathcal{B}_0$ is (δ', d)-regular, where $\delta' = (11\tilde{\delta})^{1/5} = (11\delta^{1/2^7})^{1/5}$ and $d = d_2^4d_3^4$. By Fact A.2, all but at most

$$2\delta'|W_0| \le 2\big(11\delta^{1/2^7}\big)^{1/5} \times \big(1 + \epsilon^{1/8}\big)d_2^4m^4/4 \overset{\textit{(S1)}}{\le} (1/2)\delta^{1/2^{10}}d_2^4m^4/4$$

four-cycles C_4 in $\mathcal{G}[V_{f_1} \cup V_{f_2}]$ satisfy

$$\deg_{\mathcal{B}_0}(C_4) = (1 \pm \delta')dm = \big(1 \pm \delta^{1/2^{10}}\big)d_2^4d_3^4m.$$

This completes the proof of Lemma 8.4. □

8.3. Proof of Lemma 8.7

We start with some definitions and notation.

Definition 8.9. *Let $0 < \delta', d < 1$ be constants and let r' be a positive integer. A bipartite graph $\mathcal{B} = (U \cup W, E)$ is called (δ', d, r')-regular if whenever sets $U_1, U_2, \ldots, U_{r'} \subset U$ and $W_1, W_2, \ldots, W_{r'} \subset W$ are taken such that*

$$\left|\bigcup_{i=1}^{r'}(U_i \times W_i)\right| \ge \delta'|U \times W|,$$

then

$$\left|\mathcal{B} \cap \bigcup_{i=1}^{r'}(U_i \times W_i)\right| = (1 \pm \delta')d|U||W|.$$

Definition 8.10. *Let $\mathcal{D} = (U, E)$ be a t-partite graph with partition $U = \bigcup_{i=1}^{t} U_i$. $\mathcal{D}$ is called (δ', d, r')-regular if all pairs (U_i, U_j), $1 \le i < j \le s$ are (δ', d, r')-regular.*

We are going to use the following two lemmas, of which the first was proved by Dementieva, Haxell, Nagle, and Rödl (cf. Lemma 5.1 with the choice of constants given by (13)-(15) in [2]).

Lemma 8.11. *(see [2]) All but at most $\delta^{1/2^4} m^2$ pairs $\{x, x'\} \in [V_1]^2$ satisfy the following properties:*

(1) for any $2 \le f \le s$,

$$\big|\mathcal{G}(x, x') \cap V_f\big| = \big(1 \pm \epsilon^{1/2}\big)^2 d_2^2 m. \tag{8.17}$$

(2) $\mathcal{H}(x, x')$ is $(\delta^{1/2^4}, d_2 d_3^2, r')$-regular. Here, the vertex set of graph $\mathcal{H}(x, x')$ is $\mathcal{G}(x, x')$ and r' is an integer satisfying $\epsilon \ll 1/r' \ll d_2$.

Lemma 8.12. *Let $\{x, x'\}$ be a pair satisfying (8.17). If $\mathcal{H}(x, x')$ is $(\delta^{1/2^4}, d_2 d_3^2, r')$-regular, then $\mathcal{H}(x, x')[V_{f_1} \cup V_{f_2}]$ contains $\big(1 \pm \delta^{1/2^7}\big) d_2^8 d_3^8 \big|W_0\big|$ four-cycles for every $2 \le f_1 < f_2 \le s$.*

We postpone the proof of Lemma 8.12 until we finish the proof of Lemma 8.7.

Proof of Lemma 8.7. Due to the definition of the graph $\mathcal{B}_0$, $\deg_{\mathcal{B}_0}(x, x')$ equals the number of four-cycles in $\mathcal{H}(x, x')[V_{f_1} \cup V_{f_2}]$. By Lemma 8.11, $\mathcal{H}(x, x')$ is $(\delta^{1/2^4}, d_2 d_3^2, r')$-regular for all but at most $\delta^{1/2^4} m^2$ pairs $\{x, x'\} \in [V_1]^2$.

Furthermore, by Lemma 8.12, the $(\delta^{1/2^4}, d_2 d_3^2, r')$-regularity of $\mathcal{H}(x, x')$ ensures the existence of $\big(1 \pm \delta^{1/2^7}\big) d_2^8 d_3^8 |W_0|$ four-cycles in $\mathcal{H}(x, x')[V_{f_1} \cup V_{f_2}]$. Consequently,

$$\deg_{\mathcal{B}_0}(x, x') = \big(1 \pm \delta^{1/2^7}\big) d_2^8 d_3^8 |W_0|.$$

□

Now what is left, is to prove Lemma 8.12. In order to do so, we will need the following two facts.

Fact 8.13. *Let $\{x, x'\}$ be a pair satisfying (8.17). Then all but at most $8\epsilon^{1/4} d_2^4 m^2$ pairs $\{y, y'\} \in [\mathcal{G}(x, x') \cap V_{f_1}]^2$ satisfy*

$$\big|\mathcal{G}[\mathcal{G}(x, x') \cap (V_{f_1} \cup V_{f_2})](y, y')\big| = \big(1 \pm \epsilon^{1/4}\big)^4 d_2^4 m. \tag{8.18}$$

Proof. Let $\{x, x'\}$ be a pair satisfying (8.17). Again, by the $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}[\mathcal{G}(x, x') \cap (V_{f_1} \cup V_{f_2})]$ and Fact A.4, all but at most

$$4\epsilon^{1/4} \big|\mathcal{G}(x, x') \cap V_{f_1}\big|^2 \le 4\epsilon^{1/4} \big[\big(1 + \epsilon^{1/2}\big)^2 d_2^2 m\big]^2 \le 8\epsilon^{1/4} d_2^4 m^2$$

pairs $\{y, y'\} \in [\mathcal{G}(x, x') \cap V_{f_1}]^2$ satisfy

$$\big|\mathcal{G}[\mathcal{G}(x, x') \cap (V_{f_1} \cup V_{f_2})](y, y')\big| = \big(1 \pm \epsilon^{1/4}\big)^2 d_2^2 \big|\mathcal{G}(x, x') \cap V_{f_2}\big| = \big(1 \pm \epsilon^{1/4}\big)^4 d_2^4 m.$$

□

Fact 8.14. *Let $\{x,x'\}$ be a pair satisfying (8.17). If $\mathcal{H}(x,x')$ is $(\delta^{1/2^4}, d_2d_3^2, r')$-regular, then all but at most $\delta^{1/2^6}d_2^4m^2$ pairs $\{y,y'\} \in [\mathcal{G}(x,x')\cap V_{f_1}]^2$ satisfy*

$$\big|\mathcal{H}(x,x')[V_{f_1}\cup V_{f_2}](y,y')\big| = \big(1\pm\delta^{1/2^6}\big)d_2^4d_3^4m. \tag{8.19}$$

Proof. We call a vertex $y\in\mathcal{G}(x,x')\cap V_{f_1}$ *good* if it satisfies

$$\big|\mathcal{H}(x,x')[V_{f_1}\cup V_{f_2}](y)\big| = \big(1\pm\delta^{1/2^4}\big)d_2^3d_3^2m. \tag{8.20}$$

Since $\mathcal{H}(x,x')$ is $(\delta^{1/2^4}, d_2d_3^2, r')$-regular, and consequently $(\delta^{1/2^4}, d_2d_3^2)$-regular, by Fact A.2, at most $2\delta^{1/2^4}\big|\mathcal{G}(x,x')\cap V_{f_1}\big| \le 2\delta^{1/2^4}\big(1+\epsilon^{1/2}\big)^2d_2^2m \le 4\delta^{1/2^4}d_2^2m$ (cf. (8.17)) vertices y in $\mathcal{G}(x,x')\cap V_{f_1}$ are not good.

For a good vertex y in $\mathcal{G}(x,x')\cap V_{f_1}$, a vertex $y'\in\mathcal{G}(x,x')\cap V_{f_1}$ is called a *bad friend of y* if (8.19) fails.

We are going to show that there are less than $2\epsilon^{1/2^4}d_2^2m$ good vertices, each having at least $2\delta^{1/2^5}d_2^2m$ bad friends. Then at most

$$\big(4\delta^{1/2^4}d_2^2m + 2\epsilon^{1/2^4}d_2^2m\big)\times\big(1+\epsilon^{1/2}\big)^2d_2^2m + \big(1+\epsilon^{1/2}\big)^2d_2^2m\times 2\delta^{1/2^5}d_2^2m \le \delta^{1/2^6}d_2^4m^2$$

pairs $\{y,y'\}$ fail to satisfy (8.19) since $\epsilon \ll d_2,\delta$ and $\delta\ll 1$.

Suppose to the contrary that there are at least $2\epsilon^{1/2^4}d_2^2m$ good vertices, each having at least $2\delta^{1/2^5}d_2^2m$ bad friends.

Let C^- be the set of all good vertices y, each having $\delta^{1/2^5}d_2^2m$ bad friends y' with the property that the right hand side of (8.19) is small, i.e.

$$\big|\mathcal{H}(x,x')[V_{f_1}\cup V_{f_2}](y,y')\big| < \big(1-\delta^{1/2^6}\big)d_2^4d_3^4m. \tag{8.21}$$

Similarly, we define the set C^+ of all good vertices y, each having $\delta^{1/2^5}d_2^2m$ bad friends y' for which the right hand side of (8.19) is big. Without loss of generality, we may assume $\big|C^-\big|\ge\epsilon^{1/2^4}d_2^2m$.

This will yield a contradiction to our assumption that $\mathcal{H}(x,x')$ is $(\delta^{1/2^4}, d_2d_3^2, r')$-regular. We distinguish two cases.

Case 1. Suppose that $d_2\ge d_3^2\delta^{1/2^6}$. Take a good vertex $y\in C^-$ and let U be the set of $\delta^{1/2^5}d_2^2m$ bad friends of y satisfying (8.21). Then,

$$\big|U\big| = \delta^{1/2^5}d_2^2m. \tag{8.22}$$

Let $W=\mathcal{H}(x,x')[V_{f_1}\cup V_{f_2}](y)$. From (8.20), we know that

$$\big|W\big| = \big(1\pm\delta^{1/2^4}\big)d_2^3d_3^2m. \tag{8.23}$$

Combining (8.22), (8.23), $d_2\ge d_3^2\delta^{1/2^6}$, and $\epsilon\ll\delta\ll d_3$, we obtain

$$\big|U\big|\big|W\big| \ge \delta^{1/2^4}\big(1+\epsilon^{1/2}\big)^4d_2^4m^2 \ge \delta^{1/2^4}\big|\mathcal{G}(x,x')\cap V_{f_1}\big|\big|\mathcal{G}(x,x')\cap V_{f_2}\big|$$

because $\big|\mathcal{G}(x,x')\cap V_{f_1}\big|$, $\big|\mathcal{G}(x,x')\cap V_{f_2}\big| \le (1+\epsilon^{1/2})^2d_2^2m$ (cf. (8.17)). Since $\mathcal{H}(x,x')$ is

$(\delta^{1/2^4}, d_2 d_3^2, r')$-regular, we have

$$\big|\mathcal{H}(x,x')[U \cup W]\big| \geq \big(1-\delta^{1/2^4}\big) d_2 d_3^2 \big|U\big|\big|W\big| \geq \delta^{1/2^5}\big(1-2\delta^{1/2^4}\big) d_2^6 d_3^4 m^2. \tag{8.24}$$

On the other hand, since the degree of each vertex $y' \in U$ in $\mathcal{H}(x,x')[U \cup W]$ is bounded by (8.21), we get

$$\big|\mathcal{H}(x,x')[U \cup W]\big| < |U|\big(1-\delta^{1/2^6}\big) d_2^4 d_3^4 m = \delta^{1/2^5}\big(1-\delta^{1/2^6}\big) d_2^6 d_3^4 m^2.$$

This, however, contradicts (8.24) because of the fact that $2\delta^{1/2^4} \ll \delta^{1/2^6}$.

Case 2. Suppose that $d_2 < d_3^2 \delta^{1/2^6}$. In order to apply the $(\delta^{1/2^4}, d_2 d_3^2, r')$-regularity of $\mathcal{H}(x,x')$, we need to find sets $U_1, \ldots, U_{r''} \subset \mathcal{G}(x,x') \cap V_{f_1}$ and $W_1, \ldots, W_{r''} \subset \mathcal{G}(x,x') \cap V_{f_2}$ for some $r'' \leq r'$ such that

$$\left|\bigcup_{i=1}^{r''}(U_i \times W_i)\right| \geq \delta^{1/2^4}\big|\mathcal{G}(x,x') \cap V_{f_1}\big|\big|\mathcal{G}(x,x') \cap V_{f_2}\big|. \tag{8.25}$$

For $y_i \in C^-$, set $W_i = \mathcal{H}(x,x')[V_{f_1} \cup V_{f_2}](y_i)$. From (8.20), we know that

$$\big|W_i\big| = \big(1 \pm \delta^{1/2^4}\big) d_2^3 d_3^2 m. \tag{8.26}$$

Let U_i be the set of $\delta^{1/2^5} d_2^2 m$ bad friends of y_i satisfying (8.21). Then

$$\big|U_i\big| = \delta^{1/2^5} d_2^2 m. \tag{8.27}$$

Now we apply Fact A.10 to choose $y_1, y_2, \ldots, y_{r''} \in C$, and then use the Inclusion-Exclusion Principle to derive (8.25).

First, we define an auxiliary graph $\mathcal{D}$ with vertex set $V(\mathcal{D}) = \mathcal{G}(x,x') \cap V_{f_1}$ and edge set $E(\mathcal{D}) = \big\{\{y,y'\} : \big|\mathcal{G}[\mathcal{G}(x,x') \cap (V_{f_1} \cup V_{f_2})](y,y')\big| \neq (1 \pm \epsilon^{1/4})^4 d_2^4 m\big\}$. We note that $|V(\mathcal{D})| = (1 \pm \epsilon^{1/2})^2 d_2^2 m$. Since the graph $\mathcal{G}[\mathcal{G}(x,x') \cap (V_{f_1} \cup V_{f_2})]$ is $(\epsilon^{1/2}, d_2)$-regular, by Fact 8.13, we also have

$$|E(\mathcal{D})| \leq 16\epsilon^{1/4}|V(\mathcal{D})|^2.$$

Second, we set $\sigma = 16\epsilon^{1/4}$, $\mu = d_3^2 \delta^{1/2^6}$, $c = \epsilon^{1/2^4}/2$, and $t = 1/d_2$. We apply Fact A.10 to the graph $\mathcal{D}$ and find $r'' = \mu/d_2$ vertices $y_1, \ldots, y_{r''} \in W$ satisfying

$$\big|\mathcal{G}[\mathcal{G}(x,x') \cap (V_{f_1} \cup V_{f_2})](y_i, y_j)\big| = \big(1 \pm \epsilon^{1/4}\big)^4 d_2^4 m.$$

This immediately implies

$$\big|W_i \cap W_j\big| \leq \big(1 + \epsilon^{1/4}\big)^4 d_2^4 m \tag{8.28}$$

for $1 \leq i < j \leq r''$. Subsequently, by (8.27) and (8.28),

$$\big|(U_i \times W_i) \cap (U_j \times W_j)\big| \leq \delta^{1/2^5}\big(1 + \epsilon^{1/4}\big)^4 d_2^6 m^2. \tag{8.29}$$

Now we are going to estimate $\big|\bigcup_{i=1}^{r''} U_i \times W_i\big|$. By the Inclusion-Exclusion Principle, (8.26), (8.27), (8.29), and $r'' = d_3^2\delta^{1/2^6}/d_2$, we obtain that

$$\begin{aligned}\left|\bigcup_{i=1}^{r''} U_i \times W_i\right| &\geq \sum_{i=1}^{r''} \left|U_i \times W_i\right| - \sum_{1\leq i<j\leq r''} \left|(U_i \times W_i) \cap (U_j \times W_j)\right| \\ &\geq r''\delta^{1/2^5}\left(1-\delta^{1/2^4}\right)d_2^5 d_3^2 m^2 - \binom{r''}{2}\delta^{1/2^5}\left(1+\epsilon^{1/4}\right)^4 d_2^6 m^2 \\ &\geq \delta^{1/2^4}\left(1+\epsilon^{1/2}\right)^4 d_2^4 m^2 \\ &\overset{(8.17)}{\geq} \delta^{1/2^4}\left|\mathcal{G}(x,x')\cap V_{f_1}\right|\left|\mathcal{G}(x,x')\cap V_{f_2}\right|.\end{aligned} \tag{8.30}$$

Since $\mathcal{H}(x,x')$ is $(\delta^{1/2^4}, d_2 d_3^2, r')$-regular, we have

$$\left|\mathcal{H}(x,x') \cap \bigcup_{i=1}^{r''}(U_i \times W_i)\right| \geq \left(1-\delta^{1/2^4}\right)d_2 d_3^2 \left|\bigcup_{i=1}^{r''}(U_i \times W_i)\right|. \tag{8.31}$$

Now we are going to use our assumption on vertices in C^- to get a contradiction to (8.31). By the Inclusion-Exclusion principle, (8.26), (8.27), (8.29), and $\epsilon, \delta \ll 1$, we also note that

$$\begin{aligned}\frac{\left|\bigcup_{i=1}^{r''} U_i \times W_i\right|}{\sum_{i=1}^{r''}\left|U_i \times W_i\right|} &\geq 1 - \frac{\sum_{1\leq i<j\leq r''}\left|(U_i \times W_i)\cap(U_j \times W_j)\right|}{\sum_{i=1}^{r''}\left|U_i \times W_i\right|} \\ &\geq 1 - \frac{\binom{r''}{2}(1+\epsilon^{1/4})^4\delta^{1/2^5}d_2^6 m^2}{r''\delta^{1/2^5}(1-\delta^{1/2^4})d_2^5 d_3^2 m^2} \\ &\geq 1 - \frac{2}{3}\delta^{1/2^6}.\end{aligned} \tag{8.32}$$

For every vertex $y_i \in C$, recall that W_i is the set of all neighbors of y_i in $\mathcal{H}(x,x')[V_{f_1} \cup V_{f_2}]$ and U_i is the set of all bad friends of y_i satisfying (8.21). Thus, the degree of each vertex $y' \in U_i$ in $\mathcal{H}(x,x')[U_i \cup W_i]$ is bounded by (8.21), and, therefore,

$$\left|\mathcal{H}(x,x')[U_i \times W_i]\right| < \left|U_i\right|\left(1-\delta^{1/2^6}\right)d_2^4 d_3^4 m.$$

Combining the above inequality and (8.26) yields

$$\left|\mathcal{H}(x,x')[U_i \times W_i]\right| \leq \frac{\left(1-\delta^{1/2^6}\right)d_2 d_3^3}{(1-\delta^{1/2^4})}\left|U_i \times W_i\right|.$$

Consequently,

$$\left|\mathcal{H}(x,x') \cap \bigcup_{i=1}^{r''}(U_i \times W_i)\right| \leq \frac{\left(1-\delta^{1/2^6}\right)d_2 d_3^2}{\left(1-\delta^{1/2^4}\right)} \sum_{i=1}^{r''}\left|U_i \times W_i\right|.$$

Applying (8.32) with the above inequality, we have

$$\frac{\left|\mathcal{H}(x,x')\cap\bigcup_{i=1}^{r''}(U_i \times W_i)\right|}{\left|\bigcup_{i=1}^{r''}(U_i \times W_i)\right|} < \frac{(1-\delta^{1/2^6})d_2^2 d_3}{(1-\delta^{1/2^4})(1-\frac{2}{3}\delta^{1/2^6})} < \left(1-\delta^{1/2^4}\right)d_2^2 d_3$$

since $\delta \ll 1$. This, however, contradicts (8.31). □

Proof of Lemma 8.12. Let $\{x, x'\}$ be a pair satisfying (8.17), i.e.,

$$\big|\mathcal{G}(x,x') \cap V_f\big| = \big(1 \pm \epsilon^{1/2}\big)^2 d_2^2 m$$

holds for any f, $2 \le f \le s$, and suppose $\mathcal{H}(x,x')$ is $(\delta^{1/2^4}, d_2 d_3^2, r')$-regular. By Fact 8.14, we know that

(1) all but at most $\delta^{1/2^6} d_2^4 m^2$ pairs $\{y,y'\} \in [\mathcal{G}(x,x') \cap V_{f_1}]^2$ satisfy (8.19), i.e.,

$$\big|\mathcal{H}(x,x')[V_{f_1} \cup V_{f_2}](y,y')\big| = \big(1 \pm \delta^{1/2^6}\big) d_2^4 d_3^4 m.$$

Applying Fact 8.13, we obtain

(2) all but at most $8\epsilon^{1/4} d_2^4 m^2$ pairs $\{y,y'\} \in [\mathcal{G}(x,x') \cap V_{f_1}]^2$ satisfy (8.18), i.e.,

$$\big|\mathcal{G}[\mathcal{G}(x,x') \cap (V_{f_1} \cup V_{f_2})](y,y')\big| = \big(1 \pm \epsilon^{1/4}\big)^4 d_2^4 m.$$

Combining *(1)*, *(2)*, and (8.17), we obtain that for every such pair $\{x,x'\} \in [V_1]^2$, the number of copies of C_4 in $\mathcal{H}(x,x')[V_{f_1} \cup V_{f_2}]$ is bounded from above by

$$\left[\binom{(1+\epsilon^{1/2})^2 d_2^2 m}{2} - \delta^{1/2^6} d_2^4 m^2\right]\binom{(1+\delta^{1/2^6}) d_2^4 d_3^2 m}{2}$$
$$+ \big(\delta^{1/2^6} - 8\epsilon^{1/4}\big) d_2^4 m^2 \binom{(1+\epsilon^{1/4})^4 d_2^4 m}{2} + 8\epsilon^{1/4} d_2^4 m^2 \times m^2$$
$$\overset{(S1)}{\le} \big(1 + \delta^{1/2^7}\big) d_2^{12} d_3^4 m^4/4.$$

Similarly, by *(1)* and (8.17), we obtain the following lower bound on the number of copies of C_4 in $\mathcal{H}(x,x')[V_{f_1} \cup V_{f_2}]$.

$$\left[\binom{(1-\epsilon^{1/2})^2 d_2^2 m}{2} - \delta^{1/2^6} d_2^4 m^2\right]\binom{(1-\delta^{1/2^6}) d_2^4 d_3^2 m}{2} \overset{(S1)}{\ge} \big(1 - \delta^{1/2^7}\big) d_2^{12} d_3^4 m^4/4.$$

□

8.4. Proof of Lemma 8.6

The proof of Lemma 8.6 is very similar to the one of Lemma 8.7. We are going to use the following two results.

Lemma 8.15. *(see [2]) All but $\delta^{1/4} m$ vertices $x \in V_1$ satisfy the following properties:*

(1) For any $2 \le f \le s$,

$$\big|\mathcal{G}(x) \cap V_f\big| = \big(1 \pm \epsilon\big) d_2 m. \tag{8.33}$$

(2) $\mathcal{H}(x)$ is $(\delta^{1/4}, d_2 d_3, r)$-regular. Here, the vertex set of the graph $\mathcal{H}(x)$ is $\mathcal{G}(x)$.

Lemma 8.16. *Let $x \in V_1$ be a vertex satisfying (8.33). If $\mathcal{H}(x)$ is $(\delta^{1/4}, d_2 d_3, r)$-regular, then $\mathcal{H}(x)[V_{f_1} \cup V_{f_2}]$ contains $\big(1 \pm \delta^{1/2^5}\big) d_2^4 d_3^4 |W_0|$ four-cycles for every $2 \le f_1 < f_2 \le s$.*

The proof of Lemma 8.15 is given in [2] and Lemma 8.16 can be proved along the lines of the proof of Lemma 8.12. We omit details here.

Proof of Lemma 8.6. Due to the definition of the graph $\mathcal{B}_0$, $\deg_{\mathcal{B}_0}(x)$ equals the number of four-cycles in $\mathcal{H}(x)[V_{f_1} \cup V_{f_2}]$. By Lemma 8.15, $\mathcal{H}(x)$ is $(\delta^{1/2^4}, d_2 d_3, r)$-regular for all but at most $\delta^{1/4} m^2$ vertices $x \in V_1$. Furthermore, by Lemma 8.16, the $(\delta^{1/4}, d_2 d_3, r)$-regularity of $\mathcal{H}(x)$ ensures the existence of $\left(1 \pm \delta^{1/2^5}\right) d_2^4 d_3^4 |W_0|$ four-cycles in $\mathcal{H}(x)[V_{f_1} \cup V_{f_2}]$. Consequently,

$$\deg_{\mathcal{B}_0}(x) = \left(1 \pm \delta^{1/2^5}\right) d_2^4 d_3^4 \left|W_0\right|.$$

□

9. Proof of $\mathbf{Reg}_t(fff) \wedge \mathbf{Com}_t(pff) \Rightarrow \mathbf{Com}_{t+1}(pff)$

In order to prove this implication, we need to consider two types of triple systems
Type 1: $\mathcal{H}[L_{t+1} \cup W_{f_1}^{(t+1)} \cup W_{f_2}^{(t+1)}]$, where $t + 2 \le f_1 < f_2 \le s$, and
Type 2: $\mathcal{H}[L_p \cup W_{f_1}^{(t+1)} \cup W_{f_2}^{(t+1)}]$, where $1 \le p \le t$ and $t + 2 \le f_1 < f_2 \le s$.

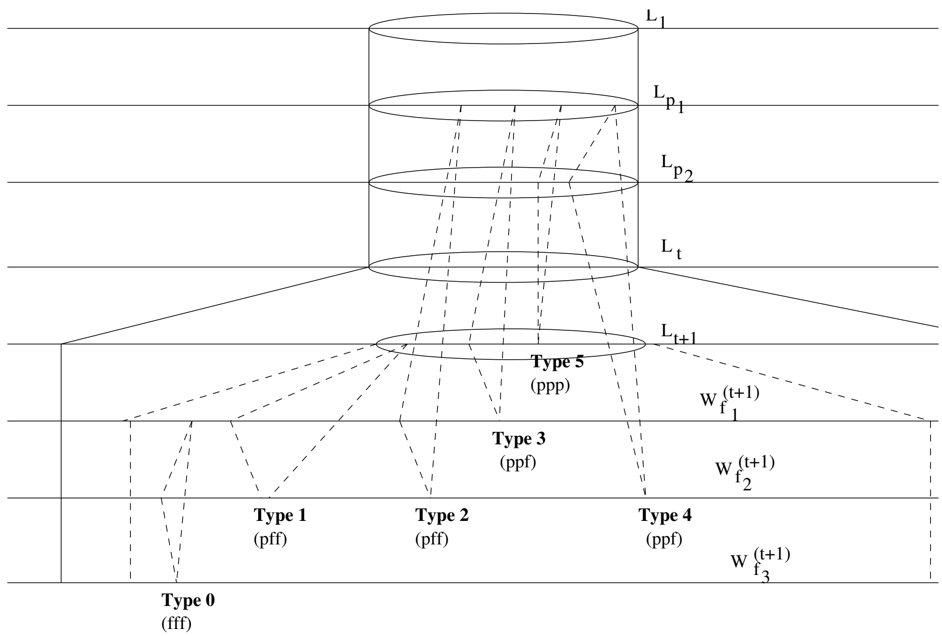

Figure 9.1 Different types of triple systems considered

We prove two auxiliary lemmas (one for each type of triple systems), which are then used to prove implication *(iv)*.

9.1. A lemma for Type 1 triple systems

Lemma 9.1. *Let $(L_1, \dots, L_t)$ be a good t-tuple of h-subsets satisfying conditions (1) and (2) in $\mathbf{Reg}_t(fff)$. Then all but at most $\delta^{1/4^{t+7}} \binom{M_{t+1}^+}{h}$ good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ are such that $(L_1, \dots, L_{t+1})$ satisfies the following condition:*

($\diamond$) *all triples* $(L_{t+1}, W_{f_1}^{(t+1)}, W_{f_2}^{(t+1)})$, *where* $t+2 \le f_1 < f_2 \le s$, *possess property* $\boldsymbol{C4EP}_{t+1}(t+1, f_1, f_2)$.

Sketch of the proof. Since a good t-tuple of h-subsets $(L_1, \ldots, L_t)$ satisfies conditions (1) and (2) in $\mathbf{Reg}_t(fff)$, the graph $\mathcal{G}[\mathcal{G}(L_1 \cup L_2 \cup \cdots \cup L_t)]$ is $(\epsilon^{1/2}, d_2)$-regular and the 3-uniform hypergraph $\mathcal{H}[\mathcal{G}(L_1 \cup L_2 \cup \cdots \cup L_t)]$ is $(\delta^{1/4^t}, d_3, r_t)$-regular. Hence, we are choosing L_{t+1} in a similar situation as for L_1.

Consequently, the proof of Lemma 9.1 is the same as the proof of $\mathbf{Com}_1(pff)$. The only modification is to replace ϵ by $\epsilon^{1/2}$, δ by $\delta^{1/4^t}$, m by $\left(1 \pm \epsilon^{1/4}\right)^{ht} M_{t+1}$, and r by r_t. $\square$

9.2. A lemma for Type 2 triple systems

Lemma 9.2. *Let* $(L_1, \ldots, L_t)$ *be a good* t*-tuple of* h*-subsets satisfying condition* **(*)** *in* $\boldsymbol{Com}_t(pff)$. *Then all but at most* $\delta^{1/4^{t+7}} \binom{M_{t+1}}{h}$ *good* h*-subsets* $L_{t+1} \subset W_{t+1}^{(t)}$ *are such that* $(L_1, \ldots, L_{t+1})$ *satisfies the following condition:*

($\diamond$) *all triples* $(L_p, W_{f_1}^{(t+1)}, W_{f_2}^{(t+1)})$, *where* $1 \le p \le t$ *and* $t+2 \le f_1 < f_2 \le s$ *have property* $\boldsymbol{C4EP}_{t+1}(p, f_1, f_2)$.

Proof. We will complete our proof by proving:

Claim 9.3. *For any fixed triple of integers* (p, f_1, f_2), *where* $1 \le p \le t$ *and* $t+2 \le f_1 < f_2 \le s$, *all but at most* $\delta^{1/4^{t+6}} \binom{M_{t+1}}{h}$ *good* h*-subsets* $L_{t+1} \subset W_{t+1}^{(t)}$ *are such that the triple* $(L_p, W_{f_1}^{(t+1)}, W_{f_2}^{(t+1)})$ *has property* $\boldsymbol{C4EP}_{t+1}(p, f_1, f_2)$.

Indeed, Claim 9.3 implies that all but at most $s^3 \times \delta^{1/4^{t+6}} \binom{M_{t+1}}{h} \le \delta^{1/4^{t+7}} \binom{M_{t+1}}{h}$ good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ satisfy condition ($\diamond$) given in Lemma 9.2. $\square$

Proof of Claim 9.3. Fix any triple of integers (p, f_1, f_2), where $1 \le p \le t$ and $t+2 \le f_1 < f_2 \le s$. We find it convenient to reformulate Claim 9.3 as an equivalent statement (Claim 9.6) and prove it instead. Before stating this claim, we need a definition related to the relevant property $\mathbf{C4EP}_{t+1}(p, f_1, f_2)$.

Definition 9.4. *We call a four-cycle* C_4 *in* $\mathcal{G}[W_{f_1}^{(t)} \cup W_{f_2}^{(t)}]$ *bad if*

$$\big| |\mathcal{H}(C_4) \cap L_p| - d_3^4 h \big| > \delta^{1/4^6} d_3^4 h.$$

Remark 9.5. *In other words, a four-cycle* C_4 *is bad if (*5.1*) is not satisfied (cf. the definition of* $\boldsymbol{C4EP}_t(p, f_1, f_2)$*). Consequently, a triple* $(L_p, W_{f_1}^{(t+1)}, W_{f_2}^{(t+1)})$ *does not have property* $\boldsymbol{C4EP}_{t+1}(p, f_1, f_2)$ *if and only if* $\mathcal{G}[W_{f_1}^{(t+1)} \cup W_{f_2}^{(t+1)}]$ *contains more than* $\delta^{1/4^{t+6}} d_2^4 M_{t+2}^4/4$ *bad four-cycles* C_4 .

To reformulate Claim 9.3, we construct an auxiliary bipartite graph $\Gamma = (U \cup W, E)$. The set U consists of all bad four-cycles in $\mathcal{G}[W_{f_1}^{(t)} \cup W_{f_2}^{(t)}]$, the set W consists of all good

h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$, and $C_4 \in U$ and $L_{t+1} \in W$ are adjacent in Γ if and only if $V(C_4) \subset \mathcal{G}(L_{t+1})$ (this is equivalent to $V(C_4) \subset W_{f_1}^{(t+1)} \cup W_{f_2}^{(t+1)}$).

In a view of Remark 9.5, we can reformulate Claim 9.3 as follows.

Claim 9.6. *The graph Γ contains at most $\delta^{1/4^{t+6}} \binom{M_{t+1}}{h}$ vertices in W with degree at least $\delta^{1/4^{t+6}} d_2^4 M_{t+2}^4/4$.*

Proof of Claim 9.6. We first estimate $e(\Gamma)$ and then apply a double counting argument to bound the number of vertices in W with 'big' degree.

We observe that Γ satisfies the following conditions:

(a) $|U| \leq \delta^{1/4^{t+5}} d_2^4 M_{t+1}^4/4$;

(b) for all but at most $8\epsilon^{1/4}(M_{t+1}^+)^4 \leq \epsilon^{1/16} d_2^4 M_{t+1}^4/4$ four-cycles $C_4 \in U$,

$$\deg_\Gamma(C_4) \leq \binom{(1+\epsilon^{1/4})^4 d_2^4 M_{t+1}^+}{h};$$

(c) for any $C_4 \in U$,

$$\deg_\Gamma(C_4) \leq \binom{M_{t+1}^+}{h}.$$

Indeed, $\mathbf{Com}_t(pff)$ implies *(a)*. The $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}[\mathcal{G}(L_1 \cup \cdots \cup L_t)]$ and Fact A.4 imply *(b)*. Since $|W_{t+1}^{(t)}| \leq M_{t+1}^+$, we have *(c)*.

By *(a)*, *(b)*, and *(c)*, we infer that

$$\begin{aligned} e(\Gamma) &\leq \delta^{1/4^{t+5}} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{(1+\epsilon^{1/4})^4 d_2^4 M_{t+1}^+}{h} + \epsilon^{1/16} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{M_{t+1}^+}{h} \\ &\overset{(S1)}{\leq} 2\delta^{1/4^{t+5}} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{d_2^4 M_{t+1}}{h}. \end{aligned}$$

Therefore by a double counting argument, the number of vertices in W with degree at least $\delta^{1/4^{t+6}} d_2^4 M_{t+2}^4/4 = \delta^{1/4^{t+6}} d_2^{4h+4} M_{t+1}^4/4$ is no more than

$$\frac{2\delta^{1/4^{t+5}} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{d_2^4 M_{t+1}}{h}}{\delta^{1/4^{t+6}} \frac{1}{4} d_2^{4h+4} M_{t+1}^4} \overset{(S1)}{\leq} \delta^{1/4^{t+6}} \binom{M_{t+1}}{h}.$$

This completes the proof of Claims 9.6 and 9.3. □

9.3. Proof of $\mathbf{Reg}_t(fff) \wedge \mathbf{Com}_t(pff) \Rightarrow \mathbf{Com}_{t+1}(pff)$

Let $(L_1, \ldots, L_{t+1})$ be a good $(t+1)$-tuple of h-subsets not satisfying condition **(*)** in $\mathbf{Com}_{t+1}(pff)$. We distinguish two cases.

Case 1. a good t-tuple $(L_1, \ldots, L_t)$ violates either condition (1) or (2) in $\mathbf{Reg}_t(fff)$ or **(*)** in $\mathbf{Com}_t(pff)$. Since we assume that $\mathbf{Reg}_t(fff)$ and $\mathbf{Com}_t(pff)$ are true, the number of $(t+1)$-tuples $(L_1, \ldots, L_{t+1})$ of this kind is at most

$$2\delta^{1/4^{t+7}} \prod_{p=1}^{t} \binom{M_p}{h} \times \binom{M_{t+1}^+}{h} < 2\delta^{1/4^{t+7}} \prod_{p=1}^{t+1} \binom{M_p^+}{h}. \tag{9.1}$$

Case 2. a good t-tuple $(L_1, \dots, L_t)$ satisfies conditions (1), (2) in $\mathbf{Reg}_t(fff)$ and **(*)** in $\mathbf{Com}_t(pff)$, but L_{t+1} is selected in such a way that $(t+1)$-tuple $(L_1, \dots, L_{t+1})$ violates condition **(*)** in $\mathbf{Com}_{t+1}(pff)$. In particular, this means that L_{t+1} does not satisfy condition **(⋄)** in either Lemma 9.1 or Lemma 9.2.

By Fact A.7(3), Lemma 9.1, and Lemma 9.2, the number of $(t+1)$-tuples $(L_1, \dots, L_{t+1})$ in this case is at most

$$\left(1 + 2\epsilon^{1/2^{t+1}}\right) \prod_{p=1}^{t} \binom{M_p}{h} \times 2\delta^{1/4^{t+7}} \binom{M^+_{t+1}}{h} \le 3\delta^{1/4^{t+7}} \prod_{p=1}^{t+1} \binom{M^+_p}{h} \tag{9.2}$$

since $\epsilon \ll 1$. Combining (9.1) and (9.2), we obtain that all but at most

$$\left(2\delta^{1/4^{t+7}} + 3\delta^{1/4^{t+7}}\right) \prod_{p=1}^{t+1} \binom{M^+_p}{h} \overset{(S1)}{\le} \delta^{1/4^{t+8}} \prod_{p=1}^{t+1} \binom{M_p}{h}$$

good $(t+1)$-tuples of h-subsets $(L_1, \dots, L_{t+1})$ satisfy condition **(*)** in $\mathbf{Com}_{t+1}(pff)$. □

10. Proof of $\mathbf{Com}_1(pff) \Rightarrow \mathbf{Com}_2(ppf)$ and $\mathbf{Com}_t(pff) \wedge \mathbf{Com}_t(ppf) \Rightarrow \mathbf{Com}_{t+1}(ppf)$

For the first implication, we need to consider Type 3 triple systems. For the second implication, we need to consider Type 3 and Type 4 triple systems:

Type 3: $\mathcal{H}[L_p \cup L_{t+1} \cup W_f^{(t+1)}]$, where $1 \le p \le t$ and $t+2 \le f \le s$.

Type 4: $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup W_f^{(t+1)}]$, where $1 \le p_1 < p_2 \le t$ and $t+2 \le f \le s$.

For each types of triple systems, we prove an auxiliary lemma, which is later used in proofs of both implications.

10.1. A lemma for Type 3 triple systems

Lemma 10.1. *Let $(L_1, \dots, L_t)$ be a good t-tuple of h-subsets satisfying condition **(*)** given in $\mathbf{Com}_t(pff)$. Then all but $\delta^{1/4^{t+7}} \binom{M_{t+1}}{h}$ good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ are such that $(L_1, \dots, L_{t+1})$ satisfies the following condition:*

(⋄) all triples $(L_p, L_{t+1}, W_f^{(t+1)})$, where $1 \le p \le t$ and $t+2 \le f \le s$, have property $\mathbf{C4EP}_{t+1}(p, t+1, f)$.

Proof. We complete our proof by proving the following:

Claim 10.2. *For any fixed triple of integers $(p, t+1, f)$, where $1 \le p \le t$ and $t+2 \le f \le s$, all but at most $\delta^{1/4^{t+6}} \binom{M_{t+1}}{h}$ good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ are such that the triple $(L_p, L_{t+1}, W_f^{(t+1)})$ has property $\mathbf{C4EP}_{t+1}(p, t+1, f)$.*

Indeed, Claim 10.2 implies that all but at most $s^2 \times \delta^{1/4^{t+6}} \binom{M_{t+1}}{h} \le \delta^{1/4^{t+7}} \binom{M_{t+1}}{h}$ (recall that $\delta \ll 1$) good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ satisfy condition **(⋄)** given in Lemma 10.1. □

Proof of Claim 10.2. We fix any triple of integers $(p, t+1, f)$, where $1 \le p \le t$ and $t+2 \le f \le s$. Now we reformulate Claim 10.2 as an equivalent statement (Claim 10.5) and then we prove this new statement. We start with a definition related to the relevant property $\mathbf{C4EP}_{t+1}(p, t+1, f)$.

Definition 10.3. *We call a four-cycle C_4 in $\mathcal{G}[W_{t+1}^{(t)} \cup W_f^{(t)}]$ bad if*

$$\big||\mathcal{H}(C_4) \cap L_p| - d_3^4 h\big| > \delta^{1/4^6} d_3^4 h.$$

Remark 10.4. *By Definition 10.3, a triple $(L_p, L_{t+1}, W_f^{(t+1)})$ does not have property $\boldsymbol{C4EP}_{t+1}(p, t+1, f)$ if and only if the graph $\mathcal{G}[L_{t+1} \cup W_f^{(t+1)}]$ contains more than $\delta^{1/4^{t+6}} \frac{1}{4} h^2 M_{t+2}^2$ bad four-cycles C_4.*

In order to reformulate Claim 10.2, we construct an auxiliary bipartite graph $\Gamma = (U \cup W, E)$, where U consists of all bad four-cycles C_4 in $\mathcal{G}[W_{t+1}^{(t)} \cup W_f^{(t)}]$, and W consists of all good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$. We join $C_4 \in U$ and $L_{t+1} \in W$ by an edge in Γ if and only if $V(C_4) \subset L_{t+1} \cup W_f^{(t+1)}$.

It follows from Remark 10.4 that we can reformulate Claim 10.2 in the following way.

Claim 10.5. *In the graph Γ, all but at most $\delta^{1/4^{t+6}} \binom{M_{t+1}}{h}$ vertices in W have degree at least $\delta^{1/4^{t+6}} h^2 M_{t+2}^2 / 4$.*

Proof of Claim 10.5. We first estimate $e(\Gamma)$ and then apply a double counting argument to bound the number of vertices in W with 'big' degree.

We observe that Γ has the following properties:

(a) $\big|U\big| \le \delta^{1/4^{t+5}} d_2^4 M_{t+1}^4 / 4$.

(b) For all but $4\epsilon^{1/4} (M_{t+1}^+)^4 \le \epsilon^{1/16} d_2^4 M_{t+1}^4 / 4$ four-cycles $C_4 \in U$,

$$\deg_\Gamma(C_4) \le \binom{(1+\epsilon^{1/4})^2 d_2^2 M_{t+1}^+}{h-2}.$$

(c) For any $C_4 \in U$,

$$\deg_\Gamma(C_4) \le \binom{M_{t+1}^+}{h-2}.$$

Statement $\mathbf{Com}_t(pff)$ implies *(a)*. The $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}[W_{t+1}^{(t)} \cup W_f^{(t)}]$ and Fact A.4 imply that for all but $4\epsilon^{1/4} (M_{t+1}^+)^2$ pairs $\{x, x'\} \in [W_f^{(t)}]^2$,

$$\big|\mathcal{G}(x, x') \cap W_{t+1}^{(t)}\big| = \big(1 \pm \epsilon^{1/4}\big)^2 d_2^2 M_{t+1}^+,$$

and this implies *(b)*. Since $|W_{t+1}^{(t)}| \le M_{t+1}^+$, we have *(c)*.

By properties *(a)*, *(b)*, and *(c)*, we infer that

$$\begin{aligned} e(\Gamma) &\leq \delta^{1/4^{t+5}} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{\left(1+\epsilon^{1/4}\right)^2 d_2^2 M_{t+1}^+}{h-2} + \epsilon^{1/16} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{M_{t+1}^+}{h-2} \\ &\leq 2\delta^{1/4^{t+5}} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{d_2^2 M_{t+1}}{h-2} \end{aligned}$$

since $\epsilon \ll 1/h, d_2, \delta$.

Therefore, by a double counting argument, the number of vertices in W with degree at least $\delta^{1/4^{t+6}} \frac{1}{4} h^2 M_{t+2}^2 = \delta^{1/4^{t+6}} \frac{1}{4} h^2 d_2^{2h} M_{t+1}^2$ is not more than

$$\frac{2\delta^{1/4^{t+5}} \frac{1}{4} d_2^4 M_{t+1}^4 \binom{d_2^2 M_{t+1}}{h-2}}{\delta^{1/4^{t+6}} \frac{1}{4} h^2 d_2^{2h} M_{t+1}^2} \leq \delta^{1/4^{t+6}} \binom{M_{t+1}}{h}.$$

This completes the proof of Claims 10.2 and 10.5. □

□

10.2. A lemma for Type 4 triple systems

Lemma 10.6. *Let $(L_1, \ldots, L_t)$ be a good t-tuple of h-subsets satisfying condition **(*)** in $\boldsymbol{Com}_t(ppf)$. Then all but at most $\delta^{1/4^{t+7}} \binom{M_{t+1}}{h}$ good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ are such that $(L_1, \ldots, L_{t+1})$ satisfies the following condition:*

($\diamond$) *all triples $(L_{p_1}, L_{p_2}, W_f^{(t+1)})$, where $1 \leq p_1 < p_2 \leq t$ and $t+2 \leq f \leq s$, posses property $\boldsymbol{C4EP}_{t+1}(p_1, p_2, f)$.*

Proof. We will complete our proof by proving:

Claim 10.7. *For any fixed triple of integers (p_1, p_2, f), where $1 \leq p_1 < p_2 \leq t$ and $t+2 \leq f \leq s$, all but at most $\delta^{1/4^{t+6}} \binom{M_{t+1}}{h}$ good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ are such that the triple $(L_{p_1}, L_{p_2}, W_f^{(t+1)})$ has property $\boldsymbol{C4EP}_{t+1}(p_1, p_2, f)$.*

Indeed, Claim 10.7 implies that all but at most $s^3 \times \delta^{1/4^{t+6}} \binom{M_{t+1}}{h} \leq \delta^{1/4^{t+7}} \binom{M_{t+1}}{h}$ (recall that $\delta \ll 1$) good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ satisfy condition ($\diamond$) given in Lemma 10.6. □

Proof of Claim 10.7. Fix any triple (p_1, p_2, f), where $1 \leq p_1 < p_2 \leq t$ and $t+2 \leq f \leq s$. Similarly to Claim 10.2, we prove an equivalent statement (Claim 10.10) to Claim 10.7. We first introduce a definition related to the property $\mathbf{C4EP}_{t+1}(p_1, p_2, f)$.

Definition 10.8. *We call a four-cycle C_4 in $\mathcal{G}[L_{p_2} \cup W_f^{(t)}]$ bad if*

$$\left| |\mathcal{H}(C_4) \cap L_{p_1}| - d_3^4 h \right| > \delta^{1/4^6} d_3^4 h.$$

Remark 10.9. *By Definition* 10.8*, a triple* $(L_{p_1}, L_{p_2}, W_f^{(t+1)})$ *does not have property* $\boldsymbol{C4EP}_{t+1}(p_1, p_2, f)$ *if and only if the graph* $\mathcal{G}[L_{p_2} \cup W_f^{(t+1)}]$ *contains more than* $\delta^{1/4^{t+6}} h^2 M_{t+2}^2/4$ *bad four-cycles* C_4*.*

As before, we construct an auxiliary bipartite graph $\Gamma = (U \cup W, E)$ such that U consists of all bad four-cycles C_4 in $\mathcal{G}[L_{p_2} \cup W_f^{(t)}]$, W consists of all good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$, and $C_4 \in U$ and $L_{t+1} \in W$ are adjacent in Γ if and only if $V(C_4) \subset \mathcal{G}(L_{t+1})$. This is equivalent to saying that $V(C_4) \subset L_{p_2} \cup W_f^{(t+1)}$.

In a view of Remark 10.9, we can reformulate Claim 10.7 as follows.

Claim 10.10. *In the graph* Γ*, there are at most* $\delta^{1/4^{t+6}} \binom{M_{t+1}}{h}$ *vertices in* W *with degree at least* $\delta^{1/4^{t+6}} h^2 M_{t+2}^2/4$*.*

Proof of Claim 10.10. We first estimate $e(\Gamma)$ and then apply a double counting argument to bound the number of vertices in W with 'big' degrees.

We note that Γ has the following properties:

(a) $|U| \leq \delta^{1/4^{t+5}} h^2 M_{t+1}^2/4$.

(b) For all but at most $4\epsilon^{1/4}(M_{t+1}^+)^2 h^2$ four-cycles $C_4 \in U$,
$$\deg_\Gamma(C_4) \leq \binom{\left(1+\epsilon^{1/4}\right)^2 d_2^2 M_{t+1}^+}{h}.$$

(c) For every $C_4 \in U$,
$$\deg_\Gamma(C_4) \leq \binom{M_{t+1}^+}{h}.$$

(a) follows from $\mathbf{Com}_t(ppf)$. The $(\epsilon^{1/2}, d_2)$-regularity of $\mathcal{G}(L_1 \cup \cdots \cup L_t)$ and Fact A.4 imply that for all but at most $4\epsilon^{1/4}(M_{t+1}^+)^2$ pairs $\{x, x'\} \in [W_f^{(t)}]^2$,
$$\left|\mathcal{G}(x, x') \cap W_{t+1}^{(t)}\right| \leq \left(1+\epsilon^{1/4}\right)^2 d_2^2 M_{t+1}^+,$$
and this implies *(b)*. Since $|W_{t+1}^{(t)}| \leq M_{t+1}^+$, we have *(c)*.

By *(a)*, *(b)*, and *(c)*, we claim that
$$\begin{aligned} e(\Gamma) &\leq \delta^{1/4^{t+5}} \frac{1}{4} h^2 M_{t+1}^2 \binom{\left(1+\epsilon^{1/4}\right)^2 d_2^2 M_{t+1}^+}{h} + 4\epsilon^{1/4}(M_{t+1}^+)^2 h^2 \binom{M_{t+1}^+}{h} \\ &\overset{(S1),\,(S2)}{\leq} 2\delta^{1/4^{t+5}} \frac{1}{4} h^2 M_{t+1}^2 \binom{d_2^2 M_{t+1}}{h}. \end{aligned}$$

Therefore, by a double counting argument, the number of vertices in W with degree at least $\delta^{1/4^{t+6}} h^2 M_{t+2}^2/4 = \delta^{1/4^{t+6}} h^2 d_2^{2h} M_{t+1}^2/4$ is no more than
$$\frac{2\delta^{1/4^{t+5}} \frac{1}{4} h^2 M_{t+1}^2 \binom{d_2^2 M_{t+1}}{h}}{\delta^{1/4^{t+6}} \frac{1}{4} h^2 d_2^{2h} M_{t+1}^2} \leq \delta^{1/4^{t+6}} \binom{M_{t+1}}{h}.$$

This completes the proof of Claims 10.10 and 10.7. □

□

10.3. The proof of $\mathbf{Com}_1(pff) \Rightarrow \mathbf{Com}_2(ppf)$

Since $\mathbf{Com}_1(ppf)$ is vacuously satisfied, $\mathbf{Com}_1(pff) \Rightarrow \mathbf{Com}_2(ppf)$ follows from the proof of $\mathbf{Com}_t(pff) \wedge \mathbf{Com}_t(ppf) \Rightarrow \mathbf{Com}_{t+1}(ppf)$.

10.4. The proof of $\mathbf{Com}_t(pff) \wedge \mathbf{Com}_t(ppf) \Rightarrow \mathbf{Com}_{t+1}(ppf)$

The proof of this implication follows from Lemma 10.1 and Lemma 10.6.

Proof. If a good $(t+1)$-tuple of h-subsets $(L_1, \dots, L_{t+1})$ does not satisfy condition **(*)** in $\mathbf{Com}_{t+1}(ppf)$, then there are two possible cases.

Case 1: a good t-tuple $(L_1, \dots, L_t)$ violates condition **(*)** in either $\mathbf{Com}_t(pff)$ or $\mathbf{Com}_t(ppf)$. Since we assume that $\mathbf{Com}_t(pff)$ and $\mathbf{Com}_t(ppf)$ are true, the number of $(t+1)$-tuples $(L_1, \dots, L_{t+1})$ of this kind is at most

$$2\delta^{1/4^{t+7}} \prod_{p=1}^{t} \binom{M_p}{h} \times \binom{M^+_{t+1}}{h} < 2\delta^{1/4^{t+7}} \prod_{p=1}^{t+1} \binom{M^+_p}{h}. \tag{10.1}$$

Case 2: a good t-tuple $(L_1, \dots, L_t)$ satisfies condition **(*)** in both $\mathbf{Com}_t(pff)$ and $\mathbf{Com}_t(ppf)$, but L_{t+1} is such that $(t+1)$-tuple $(L_1, \dots, L_{t+1})$ violates condition **(*)** in $\mathbf{Com}_{t+1}(ppf)$. This is indeed equivalent to that L_{t+1} does not satisfy condition **(⋄)** in either Lemma 10.1 or Lemma 10.6.

By Fact A.7(3), Lemma 10.1, and Lemma 10.6, the number of good $(t+1)$-tuples $(L_1, \dots, L_{t+1})$ in this case is at most

$$\left(1 + 2\epsilon^{1/2^{t+1}}\right) \prod_{p=1}^{t} \binom{M_p}{h} \times 2\delta^{1/4^{t+7}} \binom{M_{t+1}}{h} \le 3\delta^{1/4^{t+7}} \prod_{p=1}^{t+1} \binom{M_p}{h}. \tag{10.2}$$

Combining (10.1) and (10.2), we obtain that all but at most

$$\left(2\delta^{1/4^{t+7}} + 3\delta^{1/4^{t+7}}\right) \prod_{p=1}^{t+1} \binom{M^+_p}{h} \overset{(S1),\,(S2)}{\le} \delta^{1/4^{t+8}} \prod_{p=1}^{t+1} \binom{M_p}{h}$$

good $(t+1)$-tuples of h-subsets $(L_1, \dots, L_{t+1})$ satisfy condition **(*)** in $\mathbf{Com}_{t+1}(ppf)$. □

11. Proof of $\mathbf{Com}_2(ppf) \Rightarrow \mathbf{Com}_3(ppp)$ and $\mathbf{Com}_t(ppf) \wedge \mathbf{Com}_t(ppp) \Rightarrow \mathbf{Com}_{t+1}(ppp)$

In the proof of these two implications, we need to consider only one type of triple systems:

Type 5: $\mathcal{H}[L_{p_1} \cup L_{p_2} \cup L_{t+1}]$, where $1 \le p_1 < p_2 \le t$.

The core of the proof of both implications lies in the following lemma.

Lemma 11.1. *Let $(L_1, \dots, L_t)$ be a good t-tuple of h-subsets satisfying condition **(*)** in $\mathbf{Com}_t(ppf)$. Then all but at most $\delta^{1/4^{t+7}} \binom{M_{t+1}}{h}$ good h-subsets $L_{t+1} \subset W^{(t)}_{t+1}$ are such that $(L_1, \dots, L_{t+1})$ satisfies the following condition:*

(◇) $(L_{p_1}, L_{p_2}, L_{t+1})$ *posses the property* $\boldsymbol{C4EP}_{t+1}(p_1, p_2, t+1)$ *for every* $1 \le p_1 < p_2 \le t$.

Proof. The proof will be completed by proving:

Claim 11.2. *For any fixed triple of integers* $(p_1, p_2, t+1)$*, where* $1 \le p_1 < p_2 \le t$*, all but at most* $\delta^{1/4^{t+6}}\binom{M_{t+1}}{h}$ *good* h*-subsets* $L_{t+1} \subset W_{t+1}^{(t)}$ *are such that the triple* $(L_{p_1}, L_{p_2}, L_{t+1})$ *has the property* $\boldsymbol{C4EP}_{t+1}(p_1, p_2, t+1)$.

Indeed, Claim 11.2 implies that all but at most $s^2 \times \delta^{1/4^{t+6}}\binom{M_{t+1}}{h} \le \delta^{1/4^{t+7}}\binom{M_{t+1}}{h}$ good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$ satisfy condition (◇) in Lemma 11.1. □

Proof of Claim 11.2. Fix any triple of integers $(p_1, p_2, t+1)$, where $1 \le p_1 < p_2 \le t$. We reformulate Claim 11.2 as an equivalent statement (Claim 11.5) and then we prove this new claim. We start with a definition related to the property $\mathbf{C4EP}_{t+1}(p_1, p_2, t+1)$.

Definition 11.3. *We call a four-cycle* C_4 *in* $\mathcal{G}[L_{p_2} \cup W_{t+1}^{(t)}]$ *bad if*

$$\big||\mathcal{H}(C_4) \cap L_{p_1}| - d_3^4 h\big| > \delta^{1/4^6} d_3^4 h.$$

Remark 11.4. *By Definition* 11.3*, a triple* $(L_{p_1}, L_{p_2}, L_{t+1})$ *does not have the property* $\boldsymbol{C4EP}_{t+1}(p_1, p_2, t+1)$ *if and only if the graph* $\mathcal{G}[L_{p_2} \cup L_{t+1}]$ *contains more than* $\delta^{1/4^{t+6}} h^4/4$ *bad four-cycles* C_4.

As before, we construct an auxiliary bipartite graph $\Gamma = (U \cup W, E)$, where U consists of all bad four-cycles C_4 in $\mathcal{G}[L_{p_2} \cup W_{t+1}^{(t)}]$ and W consists of all good h-subsets $L_{t+1} \subset W_{t+1}^{(t)}$. We join $C_4 \in U$ and $L_{t+1} \in W$ by an edge in Γ if and only if $V(C_4) \subset L_{p_2} \cup L_{t+1}$.

In a view of Remark 11.4, Claim 11.2 can be reformulated as follows.

Claim 11.5. *In the graph* Γ*, the number of vertices in* W *with degree at least* $\delta^{1/4^{t+6}} h^4/4$ *is not more than* $\delta^{1/4^{t+6}}\binom{M_{t+1}}{h}$.

Proof of Claim 11.5. We first estimate the number of edges $e(\Gamma)$ of Γ and then use a double counting argument to bound the number of vertices in W with 'big' degree.

First we note that Γ has the following properties:

(a) $|U| \le \delta^{1/4^{t+5}} h^2 M_{t+1}^2/4$, and

(b) For every four-cycle $C_4 \in U$,

$$\deg_\Gamma(C_4) \le \binom{M_{t+1}^+}{h-2}.$$

Statement $\mathbf{Com}_t(ppf)$ implies *(a)*. Since $|W_{t+1}^{(t)}| \le M_{t+1}^+$, we have *(b)*.

By *(a)* and *(b)*, we claim that

$$e(\Gamma) \le \delta^{1/4^{t+5}} h^2 M_{t+1}^2 \binom{M_{t+1}^+}{h-2}/4 \overset{(S2)}{\le} 2\delta^{1/4^{t+5}} h^2 M_{t+1}^2 \binom{M_{t+1}}{h-2}/4.$$

Therefore, by a double counting argument, the number of vertices in W with degree at least $\delta^{1/4^{t+6}} h^4/4$ is no more than

$$\frac{2\delta^{1/4^{t+5}}\frac{1}{4}h^2 M_{t+1}^2\binom{M_{t+1}}{h-2}}{\delta^{1/4^{t+6}}\frac{1}{4}h^4} \le \delta^{1/4^{t+6}}\binom{M_{t+1}}{h}.$$

The last inequality follows from the fact that $1/m \ll \epsilon \ll 1/h$ and $\delta \ll 1$. This completes the proof of Claims 11.5 and 11.2. □

□

11.1. Sketch of the proof of $\mathbf{Com}_2(ppf) \Rightarrow \mathbf{Com}_3(ppp)$

In this case, statement $\mathbf{Com}_2(ppp)$ is vacuously satisfied. Hence, the proof of this implication follows from the proof of $\mathbf{Com}_t(ppf) \wedge \mathbf{Com}_t(ppp) \Rightarrow \mathbf{Com}_{t+1}(ppp)$, which is based on Lemma 11.1.

11.2. Proof of $\mathbf{Com}_t(ppf) \wedge \mathbf{Com}_t(ppp) \Rightarrow \mathbf{Com}_{t+1}(ppp)$

Now we prove the implication $\mathbf{Com}_t(ppf) \wedge \mathbf{Com}_t(ppp) \Rightarrow \mathbf{Com}_{t+1}(ppp)$ for $3 \le t \le s-1$ by applying Lemma 11.1 (we, indeed, prove it also for $t = 2$).

Proof. If a good $(t+1)$-tuples of h-subsets $(L_1, \dots, L_{t+1})$ does not satisfy condition **(*)** in $\mathbf{Com}_{t+1}(ppp)$, then one of the following two cases occurs.

Case 1: a good t-tuple $(L_1, \dots, L_t)$ violates condition **(*)** in either $\mathbf{Com}_t(ppf)$ or $\mathbf{Com}_t(ppp)$. By $\mathbf{Com}_t(ppf)$ and $\mathbf{Com}_t(ppp)$, the number of $(L_1, \dots, L_{t+1})$ of this kind is at most

$$2\delta^{1/4^{t+7}} \prod_{p=1}^{t}\binom{M_p}{h} \times \binom{M_{t+1}}{h} < 2\delta^{1/4^{t+7}} \prod_{p=1}^{t+1}\binom{M_p^+}{h}. \tag{11.1}$$

Case 2: a good t-tuple $(L_1, \dots, L_t)$ satisfies condition **(*)** in both $\mathbf{Com}_t(ppf)$ and $\mathbf{Com}_t(ppp)$, but L_{t+1} is such that $(L_1, \dots, L_{t+1})$ violates condition **(*)** in $\mathbf{Com}_{t+1}(ppp)$. This is equivalent to saying that L_{t+1} does not satisfy condition **(◇)** in Lemma 11.1.

By Fact A.7(3) and Lemma 11.1, the number of $(t+1)$-tuples $(L_1, \dots, L_{t+1})$ in this case is at most

$$(1 + 2\epsilon^{1/2^{t+1}}) \prod_{p=1}^{t}\binom{M_p}{h} \times \delta^{1/4^{t+7}}\binom{M_{t+1}}{h} \overset{(S1)}{\le} 2\delta^{1/4^{t+7}} \prod_{p=1}^{t+1}\binom{M_p}{h}. \tag{11.2}$$

Combining (11.1) and (11.2), we obtain that all but at most

$$\left(2\delta^{1/4^{t+7}} + 2\delta^{1/4^{t+7}}\right) \prod_{p=1}^{t+1}\binom{M_p^+}{h} \overset{(S1),\,(S2)}{\le} \delta^{1/4^{t+8}} \prod_{p=1}^{t+1}\binom{M_p}{h}$$

good $(t+1)$-tuples of h-subsets $(L_1, \dots, L_{t+1})$ satisfy condition **(*)** in $\mathbf{Com}_{t+1}(ppp)$. □

Appendix A. Some facts related to the regularity of graphs

In this appendix, we state a few facts which are related to the regularity of graphs. The proofs are given in [8].

For a graph $\mathcal{D} = (V, E)$ and a subset V' of V, recall that $\mathcal{D}[V']$ denote the subgraph of $\mathcal{D}$ induced on V'.

Fact A.1. *Let $\mathcal{D}$ be an (ϵ, d)-regular s-partite graph with partition $\bigcup_{i=1}^{s} U_i$, and let W_i be a subset of U_i with $|W_i| \geq \epsilon^{1/4}|U_i|$ for all $i \in [s]$. Then $\mathcal{D}[\cup_{i=1}^{s} W_i]$ is $(\epsilon^{1/2}, d)$-regular.*

Fact A.2. *Let $0 < \epsilon, d < 1$ and suppose that $\mathcal{D} = (U_1 \cup U_2, E)$ is an (ϵ, d)-regular bipartite graph. Then all but at most $2\epsilon|U_1|$ vertices $x \in U_1$ satisfy*

$$(1 - \epsilon)d\big|U_2\big| \leq \big|\mathcal{D}(x)\big| \leq (1 + \epsilon)d\big|U_2\big|.$$

This fact can be further extended in the following two ways.

Fact A.3. *Suppose $(1 - \epsilon^{1/2})^{4(s-1)} d^{4(s-1)} \geq \epsilon$. Let $\mathcal{D}$ be an (ϵ, d)-regular s-partite graph with partition $\bigcup_{i=1}^{s} U_i$. Then for any integer q with $1 \leq q \leq s - 1$, all but at most $2q\epsilon^{1/2}\big|U_2\big| \cdots \big|U_{q+1}\big|$ q-tuples of vertices $(a_2, \ldots, a_{q+1}) \in U_2 \times \cdots \times U_{q+1}$, satisfy*

$$\big(1 - \epsilon^{1/2}\big)^q d^q \big|U_1\big| \leq \big|\mathcal{D}(a_2, \ldots, a_{q+1}) \cap U_1\big| \leq \big(1 + \epsilon^{1/2}\big)^q d^q \big|U_1\big|. \tag{A.1}$$

Fact A.4. *Let q be a positive integer such that $\big(1 - \epsilon^{1/2}\big)^{4(q-1)} d^{4(q-1)} \geq \epsilon$. Let $\mathcal{D}$ be an (ϵ, d)-regular s-partite graph with partition $\bigcup_{i=1}^{s} U_i$. Then, all but at most $2q(s-1)\epsilon^{1/2}\big|U_1\big|^q$ q-subsets $\{x_1, \ldots, x_q\} \in [U_1]^q$ satisfy*

$$\big(1 - \epsilon^{1/2}\big)^q d^q \big|U_j\big| \leq \big|\mathcal{D}(x_1, \ldots, x_q) \cap U_j\big| \leq \big(1 + \epsilon^{1/2}\big)^q d^q \big|U_j\big|,$$

for all $j \in [s - 1]$.

Applying the above fact to the graph $\mathcal{G}$ from Setup, we have the following consequences.

Fact A.5. *All but at most $\epsilon^{1/4}\binom{m}{h}$ h-subsets $L_1 \subset V_1$ are good.*

Fact A.6. *All but at most $\epsilon^{1/4}\binom{m}{h}^2$ pairs (L, L') of h-subsets L, L' of V_1 satisfy*

$$\big|\mathcal{G}(L \cup L') \cap V_j\big| = \big(1 \pm \epsilon^{1/4}\big) d_2^{2h} m \tag{A.2}$$

for all $2 \leq j \leq s$.

Fact A.7. *Let s and n be positive integers. Then for every d, $0 < d < 1$, there exists $\epsilon_0 = \epsilon_0(d)$ such that for every $\epsilon \leq \epsilon_0$, every (ϵ, d)-regular s-partite graph with partition $\bigcup_{i=1}^{s} U_i$, where $\big|U_1\big| = \cdots = \big|U_s\big| = n$, satisfies the following property.*

For every $t \in [s]$ and $q \in [n]$, the following conditions hold.

(1) *The number of complete t-tuples of q-subsets $(B_1, \dots, B_t)$ is*

$$\left(1 \pm \epsilon^{1/2^{t+1}}\right) \prod_{p=1}^{t} \binom{d^{q(p-1)}n}{q}.$$

(2) *All but at most $\epsilon^{1/2^{t+1}} \prod_{p=1}^{t} \binom{d^{q(p-1)}n}{q}$ complete t-tuples of q-subsets $(B_1, \dots, B_t)$ are good.*

(3) *The number of good t-tuples of q-subsets $(B_1, \dots, B_t)$ is*

$$\left(1 \pm 2\epsilon^{1/2^{t+1}}\right) \prod_{p=1}^{t} \binom{d^{q(p-1)}n}{q}.$$

Fact A.8. *For $d > 0$, there exist $\epsilon_0 = \epsilon_0(d)$ and $n_0 = n_0(\epsilon_0)$, such that for every $\epsilon \le \epsilon_0$, any (ϵ, d)-regular bipartite graph $\mathcal{D} = (U_1 \cup U_2, E)$ with $|U_1|$, $|U_2| \ge n_0$ contains $\left(1 \pm \epsilon^{1/8}\right) d^4 |U_1|^2 |U_2|^2 / 4$ copies of C_4.*

The next fact counts the number of triangles K_3 in a 3-partite regular graph. It is an explicit version of Fact 1.2 for the case $s = 3$.

Fact A.9. *[4] Let d, ϵ be positive real numbers such that $\epsilon^{1/4} \le d^2(1-\epsilon)^2$. Then, the number of triangles K_3 in an arbitrary (ϵ, d)-regular 3-partite graph $\mathcal{D}$ with partition $U_1 \cup U_2 \cup U_3$ is given by*

$$(1-2\epsilon)(1-\epsilon)^3 d^3 |U_1||U_2||U_3| \le |\mathcal{K}_3(\mathcal{D})| \le \left((1+\epsilon)^3 d^3 + 4\epsilon\right)|U_1||U_2||U_3|. \tag{A.3}$$

The next fact guarantees that an independent set of certain size can be found in every big subset inside a graph with small density.

Fact A.10. *Let U be a set of size n and $\mathcal{D}$ be an arbitrary graph with vertex set U and $|\mathcal{D}| \le \sigma n^2$. Then for every subset $W \subset U$ with at least cn vertices and a positive integer t such that*

$$2\sigma t^2 < c^2, \tag{A.4}$$

there exists an independent set $\{x_1, \dots, x_t\} \subset W$ in the graph $\mathcal{D}$.

Appendix B. Proof of Lemma 8.8

Proof. Recall that $\delta' = (11\tilde{\delta})^{1/5}$ and set $\lambda = 1 - d$. Let $U = \{x_1, x_2, \dots, x_u\}$ and $W = \{y_1, y_2, \dots, y_w\}$ and define a $u \times w$ matrix M for the pair (U, W) with rows indexed by the elements of U and columns by the elements of W as follows.

For each $x_i \in U$ and $y_j \in W$ the entry $m(x_i, y_j)$ in the row of x_i and column of y_j is given by

$$m(x_i, y_j) = \begin{cases} \lambda & \text{if } (x_i, y_j) \in E, \\ -d & \text{if } (x_i, y_j) \notin E. \end{cases}$$

Let $U' \subseteq U$ and $W' \subseteq W$ be two subsets with $|U'||W'| \geq \delta'|U||W|$ (note that this implies that $|U'| \geq \delta'|U|$ and $|W'| \geq \delta'|W|$). Our goal is to show that

$$d(U', W') = (1 \pm \delta')d.$$

Let E' be the subset of E consisting of all edges of $\mathcal{B}$ joining a vertex from U' to a vertex from W'. By reordering, we may assume that $U' = \{x_1, x_2, \ldots, x_{u'}\}$ and $W' = \{y_1, y_2, \ldots, y_{w'}\}$. Let M' be the $u' \times w'$ submatrix of M associated with U' and W'. That is,

$$M' = (m(x_i, y_j))_{1 \leq i \leq u', 1 \leq j \leq w'}.$$

The sum of all of the entries of M' is equal to λ times the number of edges in E' minus d times the number of non-edges in E'.

$$\sum_{i=1}^{u'} \sum_{j=1}^{w'} m(x_i, y_j) = \lambda|E'| - d\big(u'w' - |E'|\big) = |E'| - du'w'. \tag{B.5}$$

For $x_i \in U'$, let $\vec{x_i}$ be the corresponding row vector of M and let $\vec{x'_i}$ be the corresponding row vector of M'. Then by the Cauchy-Schwartz inequality,

$$\left(\sum_{i=1}^{u'} \sum_{j=1}^{w'} m(x_i, y_j)\right)^2 \leq w' \sum_{j=1}^{w'} \left(\sum_{i=1}^{u'} m(x_i, y_j)\right)^2 = w' \left\| \sum_{i=1}^{u'} \vec{x'_i} \right\|^2, \tag{B.6}$$

where for vectors $\vec{x}$ and $\vec{y}$ the expression $\vec{x} \cdot \vec{y}$ means the usual scalar product and $||\vec{x}||^2 = \vec{x} \cdot \vec{x}$. Clearly

$$\left\| \sum_{i=1}^{u'} \vec{x'_i} \right\|^2 \leq \left\| \sum_{i=1}^{u'} \vec{x_i} \right\|^2. \tag{B.7}$$

Therefore, by (B.5), (B.6), and (B.7), we have

$$\big(|E'| - du'w'\big)^2 \leq w' \left\| \sum_{i=1}^{u'} \vec{x_i} \right\|^2. \tag{B.8}$$

In what follows, we will find an upper bound for

$$\left\| \sum_{i=1}^{u'} \vec{x_i} \right\|^2 = \sum_{i=1}^{u'} ||\vec{x_i}||^2 + 2 \sum_{1 \leq i < j \leq u'} \vec{x_i} \cdot \vec{x_j}.$$

For each $x_i \in U$ we have

$$\|\vec{x_i}\|^2 = \lambda^2 \deg_{\mathcal{B}}(x_i) + d^2(w - \deg_{\mathcal{B}}(x_i)) \leq \max\{d^2 w, \lambda^2 w\} \leq w,$$

and hence,

$$\sum_{i=1}^{u'} ||\vec{x_i}||^2 \leq u'w. \tag{B.9}$$

For $x_i \neq x_j \in U$, we obtain

$$\begin{aligned}\vec{x_i} \cdot \vec{x_j} =&\lambda^2 \deg_{\mathcal{B}}(x_i, x_j) - \lambda d(\deg_{\mathcal{B}}(x_i) - \deg_{\mathcal{B}}(x_i, x_j)) \\ &- \lambda d(\deg_{\mathcal{B}}(x_j) - \deg_{\mathcal{B}}(x_i, x_j)) \\ &+ d^2(w - \deg_{\mathcal{B}}(x_i) - \deg_{\mathcal{B}}(x_j) + \deg_{\mathcal{B}}(x_i, x_j)) \\ =&(\lambda^2 + 2\lambda d + d^2) \deg_{\mathcal{B}}(x_i, x_j) - (\lambda d + d^2)(\deg_{\mathcal{B}}(x_i) + \deg_{\mathcal{B}}(x_j)) + d^2 w.\end{aligned} \tag{B.10}$$

Since $\lambda + d = 1$, the right hand side of (B.10) simplifies to

$$\vec{x_i} \cdot \vec{x_j} = \deg_{\mathcal{B}}(x_i, x_j) - d(\deg_{\mathcal{B}}(x_i) + \deg_{\mathcal{B}}(x_j)) + d^2 w. \tag{B.11}$$

If $\{x_i, x_j\} \in \mathcal{D}$, then omitting the negative terms in the equation above yields

$$\vec{x_i} \cdot \vec{x_j} \leq \deg_{\mathcal{B}}(x_i, x_j) + d^2 w \tag{B.12}$$

If $\{x_i, x_j\} \notin \mathcal{D}$, then $\deg_{\mathcal{B}}(x_i) \geq (1 - \tilde{\delta})dw$, $\deg_{\mathcal{B}}(x_j) \geq (1 - \tilde{\delta})dw$, and $\deg_{\mathcal{B}}(x_i, x_j) < (1 + \tilde{\delta})d^2 w$. Consequently, for such a pair $\{x_i, x_j\}$ we get

$$\vec{x_i} \cdot \vec{x_j} \leq (1 + \tilde{\delta})d^2 w - 2d(1 - \tilde{\delta})dw + d^2 w < 3\tilde{\delta} d^2 w. \tag{B.13}$$

Since $|\mathcal{D}| < \tilde{\delta} u^2$ and $\sum_{\{x_i, x_j\} \in \mathcal{D}} \deg_{\mathcal{B}}(x_i, x_j) \leq \tilde{\delta} d^2 u^2 w$, we have

$$\begin{aligned}2 \sum_{1 \leq i < j \leq u'} \vec{x_i} \cdot \vec{x_j} =& 2 \sum_{\{x_i, x_j\} \in \mathcal{D}} \vec{x_i} \cdot \vec{x_j} + 2 \sum_{\{x_i, x_j\} \notin \mathcal{D}} \vec{x_i} \cdot \vec{x_j} \\ \overset{(B.12),(B.13)}{\leq}& 2 \sum_{\{x_i, x_j\} \in D} \big(\deg_{\mathcal{B}}(x_i, x_j) + d^2 w\big) + 2u^2 \times 3\tilde{\delta} d^2 w \\ \leq& 10\tilde{\delta} u^2 d^2 w.\end{aligned} \tag{B.14}$$

Combining (B.9) and (B.14) yields

$$\left\| \sum_{i=1}^{u'} \vec{x_i} \right\|^2 \leq u'w + 10\tilde{\delta} d^2 u^2 w.$$

Hence equation (B.8) becomes

$$\big(|E'| - du'w'\big)^2 < w'\big(u'w + 10\tilde{\delta} d^2 u^2 w\big).$$

Therefore,

$$\left| \frac{|E'|}{u'w'} - d \right| < \left(\frac{w}{u'w'} + \frac{10\tilde{\delta} d^2 u^2 w}{u'^2 w'} \right)^{1/2}.$$

Since $u' \geq \delta' u$ and $w' \geq \delta' w$, we have

$$\left| \frac{|E'|}{u'w'} - d \right| < \left(\frac{1}{\delta'^2 u} + \frac{10\tilde{\delta} d^2}{\delta'^3} \right)^{1/2}.$$

Recall that $u = |U| \geq 1/(d^2 \tilde{\delta})$ and $\delta' = (11\tilde{\delta})^{1/5}$, therefore

$$\left| \frac{|E'|}{u'w'} - d \right| < \left(\frac{11\tilde{\delta} d^2}{\delta'^3} \right)^{1/2} < \delta'.$$

Hence, we proved that

$$d(U', W') = (1 \pm \delta')d,$$

which completes the proof of Lemma 8.8. □

Acknowledgment

We would like to thank Andrzej Ruciński, Brendan Nagle, and the referee for their valuable comments.

References

[1] Alon, N., Duke, R., Leffmann, H., Rödl, V. and Yuster, R. (1994) The algorithmic aspects of the regularity lemma, *J. Algorithms* **16**, no. 1, 80–109.
[2] Dementieva, Y., Haxell, P., Nagle, B. and Rödl, V. (2002) On characterizing hypergraph regularity, *Random Structures Algorithms* **21**, no. 3-4, 293–335.
[3] Duke, R., Leffmann, H. and Rödl, V. (1995) A fast approximation algorithm for computing the frequencies of subgraphs in a given graph, *SIAM J. Comput.* **24**, no. 3, 598–620.
[4] Frankl, P. and Rödl, V. (2002) Extremal problems on set systems, *Random Structures Algorithms* **20**, no. 2, 131–164.
[5] Janson, S, Łuczak, T. and Ruciński, A. (2000) *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, xii + 333pp.
[6] Kohayakawa, Y., Rödl, V. and Skokan, J. (2002) Hypergraphs, Quasi-randomness, and Conditions for Regularity, *J. Combin. Theory Ser. A* **97**, no. 2, 307–352.
[7] Nagle, B. and Rödl, V. (2003) Regularity properties for triple systems, *Random Structures Algorithms*, to appear.
[8] Peng, Y. (2001) Counting small cliques in 3-uniform hypergraphs, PhD Thesis, Emory University (available at `http://www.mathcs.emory.edu/~rodl/grads.html`).
[9] Szemerédi, E. (1978) Regular partitions of graphs, *in* "Problèmes Combinatoires et Théorie des Graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)", 399–401, Colloq. Internat. CNRS n. 260, CNRS, Paris.