REGULARITY LEMMA FOR k-UNIFORM HYPERGRAPHS

VOJTĚCH RÖDL AND JOZEF SKOKAN

Abstract. Szemerédi's Regularity Lemma proved to be a very powerful tool in extremal graph theory with a large number of applications. Chung [*Regularity lemmas for hypergraphs and quasi-randomness*, Random Structures and Algorithms 2 (1991), 241–252], Frankl and Rödl [*The uniformity lemma for hypergraphs*, Graphs and Combinatorics 8 (1992), 309–312, *Extremal problems on set systems*, Random Structures and Algorithms 20 (2002), 131–164] considered several extensions of Szemerédi's Regularity Lemma to hypergraphs.

In particular, [*Extremal problems on set systems*, Random Structures and Algorithms 20 (2002), 131–164] contains a regularity lemma for 3-uniform hypergraphs that was applied to a number of problems. In this paper, we present a generalization of this regularity lemma to k-uniform hypergraphs. Similar results were independently and alternatively obtained by W. T. Gowers.

1. Introduction

While proving his famous Density Theorem [14], E. Szemerédi found an auxiliary lemma which later proved to be a powerful tool in extremal graph theory. This lemma [15] states that all sufficiently large graphs can be approximated, in some sense, by random graphs. Since "random-like" graphs are often easier to handle than arbitrary graphs, the Regularity Lemma is especially useful in situations when the problem in question is easier to prove for random graphs.

This paper is an attempt to expand Szemerédi's Regularity Lemma to $(k+1)$-uniform hypergraphs for $k \geq 2$. Unlike for graphs, there are several natural ways to define "regularity" (quasi-randomness) for k-uniform hypergraphs. Consequently, various forms of a regularity lemma for hypergraphs have been already considered in [1, 11, 3, 5, 2, 4].

One of the main reasons for the wide applicability of Szemerédi's Regularity Lemma is the fact that it enables one to find all small graphs as subgraphs of a regular graph (see [8, 7] for a survey). In [4], this issue is addressed for 3-uniform hypergraphs (i.e. case $k = 2$). The Regularity Lemma

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proved by Frankl and Rödl produces a quasi-random setup in which one can find small subhypergraphs (see also [9]). The aim of this paper is to discuss a generalization of this lemma to $(k+1)$-uniform hypergraphs for $k > 2$ that similarly as [4] allows one to find small subhypergraphs in its regular partition.

We first recall the Regularity Lemma of Szemerédi.

Definition 1.1. *Let $G = (V, E)$ be a graph and δ be a positive real number, $0 < \delta \le 1$. We say that a pair (A, B) of two disjoint subsets of V is δ-regular if*

$$|d(A', B') - d(A, B)| < \delta$$

for any two subsets $A' \subset A$, $B' \subset B$, $|A'| \ge \delta|A|$, $|B'| \ge \delta|B|$. Here, $d(A, B) = |E(A, B)|/(|A||B|)$ stands for the density of the pair (A, B).

This definition states that a regular pair has uniformly distributed edges. The Regularity Lemma of Szemerédi [15] guarantees a partition of the vertex set $V(G)$ of a graph G into t sets $V_1 \cup \ldots \cup V_t$ in such a way that most of the pairs (V_i, V_j) satisfy Definition 1.1. The precise statement is following.

Theorem 1.2 (Regularity Lemma [15])**.** *For every $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exist two integers $N_0 = N_0(\varepsilon, t_0)$ and $T_0 = T_0(\varepsilon, t_0)$ with the following property: for every graph $\mathcal{H}$ with $n \ge N_0$ vertices there is a partition $\mathcal{P}$ of the vertex set into t classes*

$$\mathcal{P}\colon V = V_1 \cup \ldots \cup V_t$$

such that

(i) $t_0 \le t \le T_0$,
(ii) $||V_i| - |V_j|| \le 1$ *for every* $1 \le i < j \le t$, *and*
(iii) *all but at most* $\varepsilon\binom{t}{2}$ *pairs* (V_i, V_j), $1 \le i < j \le t$, *are* ε*-regular.*

For technical reasons, in this paper, we consider a slightly weaker version of this lemma.

Theorem 1.3. *For every $\varepsilon > 0$ there exist two integers $N_0 = N_0(\varepsilon)$ and $T_0 = T_0(\varepsilon)$ with the property that for every graph $\mathcal{H}$ with $n \ge N_0$ vertices there is a partition $\mathcal{P}$ of the vertex set V into t classes*

$$\mathcal{P}\colon V = V_1 \cup \ldots \cup V_t$$

such that

(i) $t \le T_0$, *and*
(ii) *all but at most* $\varepsilon\binom{n}{2}$ *pairs of vertices* $\{v, w\} \subset V$ *belong to* ε*-regular pairs* (V_i, V_j), $1 \le i < j \le t$, *i.e.,* $v \in V_i$, $w \in V_j$. *Consequently,* $\sum_{i=1}^{t}\binom{|V_i|}{2} \le \varepsilon\binom{n}{2}$ *holds.*

Observe that Theorem 1.3 follows from the Regularity Lemma applied with ε replaced by $\varepsilon/8$ and $t_0 = 8/\varepsilon$.

The aim of this paper is to establish a Regularity Lemma for $(k+1)$-uniform hypergraphs (we will refer to it as statement $\mathbf{Regularity}(\boldsymbol{k})$) which extends Theorem 1.3.

For $k+1=2$ (the case of Szemerédi's Regularity Lemma), the underlying structure $\mathscr{P}_1$ of a graph $\mathcal{H}=(V,E)$ is an auxiliary partition $V_1 \cup \ldots \cup V_t$ of the set of vertices V.

For $k+1>2$ (the case discussed in this paper), the underlying structure of a $(k+1)$-uniform hypergraph $\mathcal{H}$ will be an auxiliary partition $\mathscr{P}_k$ of $[V]^k$, where $[V]^k$ is the set of all k-tuples from V. It turns out that in order to take a full advantage of the "regular behavior" of $\mathcal{H}$ with respect to $\mathscr{P}_k$, one needs more information about partition $\mathscr{P}_k$ itself. To "gain control" over the partition classes of $\mathscr{P}_k$, we view them as k-uniform hypergraphs and regularize them (applying $\mathbf{Regularity}(\boldsymbol{k-1})$ as an induction argument), getting partitions $\mathscr{P}_{k-1}$, $\mathscr{P}_{k-2}, \ldots, \mathscr{P}_1$ of $[V]^{k-1}$, $[V]^{k-2}, \ldots,$ V respectively. Unfortunately, this leads to a fairly technical concept of a partition.

The advantage of this concept is, however, that similarly to [15], [4] and [9], it allows to find and count small subhypergraphs in a "regular situation" using so called Counting Lemma. In the graph case, the proof of the Counting Lemma is rather simple (cf. Fact A in [4] or the Key Lemma in [8]). On the other hand, in the k-uniform hypergraph case, this is a very technical statement which has been proved for $k=3$ in [9] and for $k=4$ in [12, 13]. The general case (i.e. k is arbitrary) has been recently verified in [10]. We have been also informed [6] that W. T. Gowers proved the Regularity Lemma and the corresponding Counting Lemma for k-uniform hypergraphs independently, using a different approach.

2. Organization

As mentioned before, the most technical part of this paper is the description of the environment in which we work. The proof of the Regularity Lemma itself is then straightforward, based on ideas from [15, 4]. The structure of the paper is as follows.

In Section 3, we introduce cylinders and complexes, which are the basic building blocks of auxiliary partitions considered here.

In Section 4, we describe the structure of this auxiliary partition, whereas in Section 5, we introduce a concept of polyad that extends the concept of a pair (V_i, V_j) in partitions considered by Szemerédi.

In Section 7, we introduce an equitable $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-partition which is a concept ensuring that all but at most $\mu\binom{n}{k+1}$ $(k+1)$-tuples of vertices from V are "under control", that is, they belong to regular polyads (similarly as all but at most $\varepsilon\binom{n}{2}$ pairs are in regular pairs, cf. *(ii)* in Theorem 1.3). We also define a (δ_{k+1}, r)-regular partition corresponding to a regular partition of Szemerédi and present our main results – Theorems 7.14 and 7.17.

The proof of the main result is in Sections 8 – 11. In Section 8, we describe our induction scheme and in Section 9, we show its easier part. Section 10 contains auxiliary results for the proof of implication *(I3)*, which is the key part of our induction scheme for the Main Theorem. In Section 11, we give the proof of this implication.

3. Concepts

We start with a basic notation. We denote by $[\ell]$ the set $\{1,\dots,\ell\}$. For a set V and an integer $k \geq 2$, let $[V]^k$ be the system of all k-element subsets of V. A subset $\mathcal{G} \subset [V]^k$ is called a *k-uniform hypergraph*. We sometimes use the notation $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G})) = (V, E)$. For every subset $V' \subset V$, we denote by $\mathcal{G}[V']$ the subhypergraph induced on V', in other words, $\mathcal{G}[V'] = \mathcal{G} \cap [V']^k$. If there is no danger of confusion, we shall identify the hypergraphs with their edge sets.

3.1. **Cylinders and Complexes.**

This paper deals mainly with ℓ-partite k-uniform hypergraphs. We shall refer to such hypergraphs as (ℓ, k)-*cylinders*.

Definition 3.1 (**cylinder**). *Let $\ell \geq k \geq 2$ be two integers, V be a set, $|V| \geq \ell$, and $V = V_1 \cup \dots \cup V_\ell$ be a partition of V.*

A k-set $K \in [V]^k$ is crossing if $|V_i \cap K| \leq 1$ for every $i \in [\ell]$. We shall denote by $K_\ell^{(k)}(V_1,\dots,V_\ell)$ the complete (ℓ,k)-cylinder with vertex partition $V_1 \cup \dots \cup V_\ell$, i.e. the set of all crossing k-sets. Then, an (ℓ,k)-cylinder $\mathcal{G}$ is any subset of $K_\ell^{(k)}(V_1,\dots,V_\ell)$.

Definition 3.2. *For an (ℓ,k)-cylinder $\mathcal{G}$, where $k > 1$, we shall denote by $\mathcal{K}_j(\mathcal{G})$, $k \leq j \leq \ell$, the j-uniform hypergraph with the same vertex set as $\mathcal{G}$ and whose edges are precisely those j-element subsets of $V(\mathcal{G})$ that span cliques of order j in $\mathcal{G}$.*

Clearly, the quantity $|\mathcal{K}_j(\mathcal{G})|$ counts the total number of cliques of order j in an (ℓ,k)-cylinder $\mathcal{G}$, $1 < k \leq j \leq \ell$, and $\mathcal{K}_k(\mathcal{G}) = \mathcal{G}$.

For formal reasons, we find it convenient to extend the above definitions to the case when $k = 1$.

Definition 3.3. *We define an $(\ell,1)$-cylinder $\mathcal{G}$ as a partition $V_1 \cup \dots \cup V_\ell$. For an $(\ell,1)$-cylinder $\mathcal{G} = V_1 \cup \dots \cup V_\ell$ and $1 \leq j \leq \ell$, we set $\mathcal{K}_j(\mathcal{G}) = K_\ell^{(j)}(V_1,\dots,V_\ell)$.*

The concept of “cliques in 1-uniform hypergraphs” is certainly artificial. It fits well, however, to our general description of a complex (see Definition 3.6).

For an (ℓ,k)-cylinder $\mathcal{G}$ and a subset L of vertices in $\mathcal{G}$, where $k \leq |L| \leq \ell$, we say that L *belongs to* $\mathcal{G}$ if L induces a clique in $\mathcal{G}$.

We will often face a situation when one cylinder ‘lies on’ another cylinder. To this end, we define the term *underlying cylinder*.

Definition 3.4 (**underlying cylinder**). *Let $\mathcal{F}$ be an $(\ell, k-1)$-cylinder and $\mathcal{G}$ be a (ℓ, k)-cylinder. We say that $\mathcal{F}$ underlies $\mathcal{G}$ if $\mathcal{G} \subset \mathcal{K}_k(\mathcal{F})$.*

Note that if $k = 2$ and $\mathcal{F} = V_1 \cup \cdots \cup V_\ell$, then $\mathcal{G}$ is an ℓ-partite graph with ℓ-partition $V_1 \cup \cdots \cup V_\ell$.

Definition 3.5 (**density**). *Let $\mathcal{G}$ be a k-uniform hypergraph and $\mathcal{F}$ be a $(k, k-1)$-cylinder. We define the* density *of $\mathcal{F}$ with respect to $\mathcal{G}$ by*

$$d_{\mathcal{G}}(\mathcal{F}) = \begin{cases} \frac{|\mathcal{G} \cap \mathcal{K}_k(\mathcal{F})|}{|\mathcal{K}_k(\mathcal{F})|} & \text{if } |\mathcal{K}_k(\mathcal{F})| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Through this paper, we will work with a sequence of underlying cylinders. To accommodate this situation, we introduce the notion of *complex*.

Definition 3.6 (**complex**). *Let ℓ and k, $\ell \geq k \geq 1$, be two integers. An (ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ is a system of cylinders $\left\{\mathcal{G}^{(j)}\right\}_{j=1}^{k}$ such that*

- (a) *$\mathcal{G}^{(1)}$ is an $(\ell, 1)$-cylinder, i.e. $\mathcal{G}^{(1)} = V_1 \cup \cdots \cup V_\ell$,*
- (b) *(ℓ, j)-cylinder $\mathcal{G}^{(j)}$ underlies $(\ell, j+1)$-cylinder $\mathcal{G}^{(j+1)}$ for every $j \in [k-1]$, i.e. $\mathcal{G}^{(j+1)} \subset \mathcal{K}_{j+1}(\mathcal{G}^{(j)})$.*

3.2. Regularity of Cylinders and Complexes.

Now we define the notion of *regularity* of cylinders:

Definition 3.7 (**regular cylinder**). *Let δ, d be real numbers, $0 \leq \delta < d \leq 1$, $\mathcal{F}$ be a $(k, k-1)$-cylinder, and $\mathcal{G}$ be a k-uniform hypergraph with the same vertex set. We say that $\mathcal{G}$ is (δ, d)-regular with respect to $\mathcal{F}$ if the following condition is satisfied: whenever $\mathcal{F}^{'} \subset \mathcal{F}$ is a $(k, k-1)$-cylinder such that*

$$\left|\mathcal{K}_k(\mathcal{F}^{'})\right| \geq \delta \left|\mathcal{K}_k(\mathcal{F})\right|$$

then

$$d - \delta \leq d_{\mathcal{G}}(\mathcal{F}') \leq d + \delta.$$

We also say that $\mathcal{G}$ is $(\delta, \geq d)$-regular if $\mathcal{G}$ is (δ, d')-regular for some $d' \geq d$.

For $k = 2$ this definition means that a bipartite graph $G = (V_1 \cup V_2, E)$ is (δ, d)-regular if for any two subsets $V_1' \subset V_1$ and $V_2' \subset V_2$ such that $|V_1'||V_2'| \geq \delta |V_1||V_2|$, we have

$$|d(V_1', V_2') - d| < \delta,$$

where $d(V_1', V_2') = |G[V_1' \cup V_2']|/|V_1'||V_2'|$ is the density of the pair (V_1', V_2'). This differs from Definition 1.1. However, it is easy to observe that

- (δ, d)-regularity implies $2\delta^{1/2}$-regularity in a sense of Definition 1.1, and
- δ-regularity implies (δ, d)-regularity, where $d = d(V_1, V_2)$, in a sense of Definition 3.7.

We further extend this definition to the case when $\mathcal{F}$ is an $(\ell, k-1)$-cylinder.

Definition 3.8. *Let $\ell \geq k$ be positive integers, $\mathcal{F}$ be an $(\ell, k-1)$-cylinder with an ℓ-partition $\bigcup_{i=1}^{\ell} V_i$ and $\mathcal{G}$ be a k-uniform hypergraph with the same vertex set. We say that $\mathcal{G}$ is (δ, d)-regular ($(\delta, \geq d)$-regular respectively) with respect to $\mathcal{F}$ if the restriction $\mathcal{G}\left[\bigcup_{j \in I} V_j\right]$ is (δ, d)-regular ($(\delta, \geq d)$-regular respectively) with respect to $\mathcal{F}\left[\bigcup_{j \in I} V_j\right]$ for all $I \in [\ell]^k$.*

For $k > 2$, the situation becomes more complicated and due to the quantification of constants in a hypergraph regularity lemma (Remark 4.6, [4]), it is not obvious that Definition 3.7 has an effect comparable to the case $k = 2$.

To overcome this difference, Frankl and Rödl introduced in [4] the concept of (δ, r)-*regularity*. Here we present this concept in more general form. We start with the definition of the density of a system of cylinders.

Definition 3.9. *Let $r \in \mathbb{N}$, $\mathcal{G}$ be a k-uniform hypergraph, and $\tilde{\mathcal{F}}$ be a system of $(k, k-1)$-cylinders $\mathcal{F}_1, \ldots, \mathcal{F}_r$ with the same vertex set as $\mathcal{G}$. We define the* density *of $\tilde{\mathcal{F}}$ with respect to $\mathcal{G}$ by*

$$d_{\mathcal{G}}(\tilde{\mathcal{F}}) = \begin{cases} \frac{|\mathcal{G} \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j)|}{|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j)|} & \text{if } |\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j)| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Now we define a regular cylinder.

Definition 3.10 (**(δ, d, r)-regular cylinder**). *Let $r \in \mathbb{N}$, $\mathcal{F}$ be a $(k, k-1)$-cylinder, and $\mathcal{G}$ be a k-uniform hypergraph. We say that $\mathcal{G}$ is (δ, d, r)-regular with respect to $\mathcal{F}$ if the following condition is satisfied: whenever $\tilde{\mathcal{F}} = \{\mathcal{F}_1, \ldots, \mathcal{F}_r\}$ is a system of subcylinders of $\mathcal{F}$ such that*

$$\left| \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j) \right| \geq \delta \left| \mathcal{K}_k(\mathcal{F}) \right|,$$

then

$$d - \delta \leq d_{\mathcal{G}}(\tilde{\mathcal{F}}) \leq d + \delta.$$

We also say that

- *$\mathcal{G}$ is (δ, d, r)-irregular with respect to $\mathcal{F}$ if it is not (δ, d, r)-regular with respect to $\mathcal{F}$;*
- *$\mathcal{G}$ is $(\delta, \geq d, r)$-regular with respect to $\mathcal{F}$ if $\mathcal{G}$ is (δ, d', r)-regular with respect to $\mathcal{F}$ for some $d' \geq d$;*
- *$\mathcal{G}$ is (δ, r)-regular with respect to $\mathcal{F}$ if $\mathcal{G}$ is (δ, d', r)-regular with respect to $\mathcal{F}$ for some $d' \geq 0$.*

We extend the above definition to the case of an $(\ell, k-1)$-cylinder $\mathcal{F}$.

Definition 3.11. *Let $k, \ell, r \in \mathbb{N}$, $\ell \geq k$, $\mathcal{F}$ be an $(\ell, k-1)$-cylinder with an ℓ-partition $\bigcup_{i=1}^{\ell} V_i$, and $\mathcal{G}$ be a k-uniform hypergraph. We say that $\mathcal{G}$ is (δ, d, r)-regular ($(\delta, \geq d, r)$-regular respectively) with respect to $\mathcal{F}$ if* *the restriction* $\mathcal{G}\left[\bigcup_{j\in I} V_j\right]$ *is* (δ, d, r)*-regular* ($(\delta, \geq d, r)$*-regular respectively*) *with respect to* $\mathcal{F}\left[\bigcup_{j\in I} V_j\right]$ *for all* $I \in [\ell]^k$.

Notice that if a k-uniform hypergraph $\mathcal{G}$ is $(\delta, \geq d, r)$-regular with respect to $\mathcal{F}$, then each restriction $\mathcal{G}\left[\bigcup_{j\in I} V_j\right]$ can be (δ, d', r)-regular with a different $d' \geq d$. Similarly to Definition 3.10, we say

- $\mathcal{G}$ is (δ, d, r)-irregular with respect to $\mathcal{F}$ if it is not (δ, d, r)-regular with respect to $\mathcal{F}$;
- $\mathcal{G}$ is (δ, r)-regular with respect to $\mathcal{F}$ if $\mathcal{G}$ is (δ, d', r)-regular with respect to $\mathcal{F}$ for some $d' \geq 0$.

Now we are ready to introduce the concept of regularity for an (ℓ, k)-complex $\boldsymbol{\mathcal{G}}$.

Definition 3.12 (**$(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex**)**.** *Let* $\boldsymbol{d} = (d_2, \ldots, d_k)$ *and* $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$ *be two vectors of positive real numbers such that* $0 < \delta_j < d_j \leq 1$ *for all* $j = 2, \ldots, k$ *and* $r \in \mathbb{N}$. *We say that an* (ℓ, k)*-complex* $\boldsymbol{\mathcal{G}}$ *is* $(\boldsymbol{\delta}, \boldsymbol{d}, r)$*-regular if*

(a) $\mathcal{G}^{(2)}$ *is* (δ_2, d_2)*-regular with respect to* $\mathcal{G}^{(1)}$, *and*
(b) $\mathcal{G}^{(j+1)}$ *is* $(\delta_{j+1}, d_{j+1}, r)$*-regular with respect to* $\mathcal{G}^{(j)}$ *for every* $j \in [k-1]\backslash\{1\}$.

We say that an (ℓ, k)*-complex* $\boldsymbol{\mathcal{G}}$ *is* $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r)$*-regular if there exits a vector* $\boldsymbol{d'} = (d'_2, \ldots, d'_k)$, $d'_j \geq d_j$, $j = 2, 3, \ldots, k$, *so that* $\boldsymbol{\mathcal{G}}$ *is* $(\boldsymbol{\delta}, \boldsymbol{d'}, r)$*-regular.*

Remark 3.13. We owe the reader an explanation of the above definition for $k = 1$ and 2.

When $k = 1$, vector $\boldsymbol{d}$ is empty, conditions (a) and (b) do not apply, and, thus, every $(\ell, 1)$-complex is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular.

When $k = 2$, only condition (a) applies. Therefore, an $(\ell, 2)$-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2\}$ is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular if $\mathcal{G}^{(2)}$ is (δ_2, d_2)-regular with respect to $\mathcal{G}^{(1)}$.

Note that parameter r is relevant only in the case when $k > 2$.

As mentioned in the Introduction, regular complexes are basic building elements of an auxiliary partition used in the formulation and proof of our regularity lemma. The next sections describe these auxiliary partitions.

4. Partitions

For every $j \in [k]$, let $a_j \in \mathbb{N}$ and $\psi_j \colon [V]^j \to [a_j]$ be a mapping. Clearly, mapping ψ_1 defines a partition $V = V_1 \cup \ldots \cup V_{a_1}$, where $V_i = \psi_1^{-1}(i)$ for all $i \in [a_1]$.

For $j \in [a_1]$, let $\mathrm{Cross}_j(\psi_1)$ be the set of all crossing sets $J \in [V]^j$, i.e. sets for which $|J \cap V_i| \leq 1$ for all $i \in [a_1]$. Note that $\mathrm{Cross}_j(\psi_1) = K_{a_1}^{(j)}(V_1, \ldots, V_{a_1})$.

Let $([a_1])^j_< = \{(\lambda_1,\dots,\lambda_j)\colon\ 1 \le \lambda_1 < \dots < \lambda_j \le a_1\}$ be the set of vectors naturally corresponding to the totally ordered j-element subsets of $[a_1]$. More generally, for a totally ordered set Π of cardinality at least j, let $(\Pi)^j_<$ be the family of totally ordered j-element subsets of Π.

For every $j \in [k]$, we consider the projection π_j of $\mathrm{Cross}_j(\psi_1)$ to $([a_1])^j_<$, mapping a set $J \in \mathrm{Cross}_j(\psi_1)$ to the set $\pi_j(J) = (\lambda_1,\dots,\lambda_j) \in ([a_1])^j_<$ so that $|J \cap V_{\lambda_h}| = 1$ for every $h \in [j]$.

Moreover, for every $1 \le h \le \min\{j,k\}$, let

$$\Psi_h(J) = (x_{\pi_h(H)} = \psi_h(H))_{H\in[J]^h}$$

be a vector with $\binom{j}{h}$ entries indexed by elements from $(\pi_j(J))^h_<$. For our purposes it will be convenient to assume that the entries of $\Psi_h(J)$ are ordered lexicographically with respect to their indices. Notice that

$$\Psi_1(J) \in ([a_1])^j_< \text{ and } \Psi_h(J) \in \underbrace{[a_h] \times \dots \times [a_h]}_{\binom{j}{h}-\text{times}} = [a_h]^{\binom{j}{h}} \text{ for } h > 1.$$

We define

$$\Psi^{(j)}(J) = (\Psi_1(J), \Psi_2(J), \dots, \Psi_j(J)).$$

Then $\Psi^{(j)}(J)$ is a vector with $2^j - 1$ entries. Also observe that if we set $\boldsymbol{a} = (a_1, a_2, \dots, a_k)$ and

$$A(j,\boldsymbol{a}) = ([a_1])^j_< \times \prod_{h=2}^{j} [a_h]^{\binom{j}{h}}, \tag{4.1}$$

then $\Psi^{(j)}(J) \in A(j,\boldsymbol{a})$ for every crossing set $J \in \mathrm{Cross}_j(\psi_1)$. In other words, to each crossing set J we assign a vector $(x_{\pi_h(H)})_{H\subset J}$ with each entry $x_{\pi_h(H)}$ corresponding to a non-empty subset H of J such that $x_{\pi_h(H)} = \psi_h(H) \in [a_h]$, where $h = |H|$.

For two crossing sets $J_1, J_2 \in \mathrm{Cross}_j(\psi_1)$, let us write

$$J_1 \sim J_2 \text{ if } \Psi^{(j)}(J_1) = \Psi^{(j)}(J_2). \tag{4.2}$$

The equivalence relation (4.2) defines a partition of $\mathrm{Cross}_j(\psi_1)$ into at most

$$|A(j,\boldsymbol{a})| = \binom{a_1}{j} \times \prod_{h=2}^{j} a_h^{\binom{j}{h}}$$

parts. Now we describe these parts explicitly using (2^j-1)-dimensional vectors from $A(j,\boldsymbol{a})$.

For each $j \in [k]$, let $\mathscr{P}^{(j)}$ be the partition of $\mathrm{Cross}_j(\psi_1)$ given by the equivalence relation (4.2). This way, each partition class in $\mathscr{P}^{(j)}$ has its unique address $\boldsymbol{x}^{(j)} \in A(j,\boldsymbol{a})$. While $\boldsymbol{x}^{(j)}$ is a (2^j-1)-dimensional vector, we will frequently view it as a j-dimensional vector $(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_j)$, where $\boldsymbol{x}_1 = (x_1,\dots,x_j) \in ([a_1])^j_<$ is a totally ordered set and $\boldsymbol{x}_h = (x_\Xi)_{\Xi\in(\boldsymbol{x}_1)^h_<} \in [a_h]^{\binom{j}{h}}$, $1 < h \le j$, is a $\binom{j}{h}$-dimensional vector with entries from $[a_h]$. For each address $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$ we denote its corresponding partition class from $\mathscr{P}^{(j)}$ by

$$\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) = \big\{P \in \mathrm{Cross}_j(\psi_1) \colon \Psi^{(j)}(P) = \boldsymbol{x}^{(j)}\big\}.$$

This way we will ensure some structure between the classes from $\mathscr{P}^{(j)}$ and $\mathscr{P}^{(j-1)}$.

More precisely, for each partition class $\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) \in \mathscr{P}^{(j)}$ there exist j partition classes $\mathcal{P}_1^{(j-1)}, \dots, \mathcal{P}_j^{(j-1)} \in \mathscr{P}^{(j-1)}$ such that for $\mathcal{P}^{(j-1)}(\boldsymbol{x}^{(j)}) = \bigcup_{h\in[j]} \mathcal{P}_h^{(j-1)}$ we have

$$\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) \subseteq \mathcal{K}_j(\mathcal{P}^{(j-1)}(\boldsymbol{x}^{(j)})).$$

In other words, $\mathcal{P}^{(j-1)}(\boldsymbol{x}^{(j)})$ forms an underlying $(j, j-1)$-cylinder of $\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)})$ consisting of $\binom{j}{j-1}$ classes from $\mathscr{P}^{(j-1)}$. Given $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$ (and the corresponding $\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) \in \mathscr{P}^{(j)}$), we give a formal definition of $\mathcal{P}^{(j-1)}(\boldsymbol{x}^{(j)})$ below. In fact, for every $h < j$ we introduce a notation for a (j, h)-cylinder $\mathcal{P}^{(h)}(\boldsymbol{x}^{(j)})$ which consists of $\binom{j}{h}$ partition classes of $\mathscr{P}^{(h)}$ and satisfies $\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) \subseteq \mathcal{K}_j(\mathcal{P}^{(h)}(\boldsymbol{x}^{(j)}))$.

To this end, we need the following notation. Let $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_j) \in A(j, \boldsymbol{a})$, where $\boldsymbol{x}_1 \in ([a_1])^j_<$ is a totally ordered set and $\boldsymbol{x}_u = (x_\Upsilon)_{\Upsilon\in(\boldsymbol{x}_1)^u_<} \in [a_u]^{\binom{j}{u}}$, $1 < u \le j$. For a given h-element subset Ξ of $\boldsymbol{x}_1 = (x_1, \dots, x_j)$ we are interested in a vector $\boldsymbol{x}^{(j)}(\Xi)$ which is "the restriction of $\boldsymbol{x}^{(j)}$ to Ξ". More precisely, we define $\boldsymbol{x}^{(j)}(\Xi)$ as the vector consisting of precisely those entries of $\boldsymbol{x}^{(j)}$ that are indexed by subsets of Ξ. Finally, $\boldsymbol{x}^{(j)}(\Xi) = (\boldsymbol{x}_1^\Xi, \boldsymbol{x}_2^\Xi, \dots, \boldsymbol{x}_h^\Xi)$, where for $1 \le u \le h$,

$$\boldsymbol{x}_u^\Xi = (x_\Upsilon)_{\Upsilon\in(\Xi)^u_<}$$

is the $\binom{h}{u}$-dimensional vector consisting of those entries of $\boldsymbol{x}_u$ that are labeled with ordered u-element subsets of Ξ.

Remark. For example, if $\boldsymbol{x}^{(4)} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)$, where

$$\boldsymbol{x}_1 = (2,3,5,7),\ \boldsymbol{x}_2 = (x_{(2,3)}, x_{(2,5)}, x_{(2,7)}, x_{(3,5)}, x_{(3,7)}, x_{(5,7)}),$$
$$\boldsymbol{x}_3 = (x_{(2,3,5)}, x_{(2,3,7)}, x_{(2,5,7)}, x_{(3,5,7)}),\ \boldsymbol{x}_4 = (x_{(2,3,5,7)}),$$

and $\Xi = (2,5,7)$, then

$$\boldsymbol{x}_1^\Xi = (2,5,7),\ \boldsymbol{x}_2^\Xi = (x_{(2,5)}, x_{(2,7)}, x_{(5,7)}),\ \boldsymbol{x}_3^\Xi = (x_{(2,5,7)}).$$

Definition 4.1. *For each $h \in [j]$ and $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_j) \in A(j, \boldsymbol{a})$, we set*

$$\mathcal{P}^{(h)}(\boldsymbol{x}^{(j)}) = \bigcup_{\Xi\in(\boldsymbol{x}_1)^h_<} \big\{P \in \mathrm{Cross}_h(\psi_1) \colon \Psi^{(h)}(P) = (\boldsymbol{x}_1^\Xi, \dots, \boldsymbol{x}_h^\Xi)\big\}. \tag{4.3}$$

Then, the following claim holds.

Claim 4.2. *For every $j \in [k]$ and every $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_j) \in A(j, \boldsymbol{a})$, the following is true.*

(a) *For all* $h \in [j]$, $\mathcal{P}^{(h)}(\boldsymbol{x}^{(j)})$ *is a* (j,h)*-cylinder;*
(b) $\boldsymbol{\mathcal{P}}(\boldsymbol{x}^{(j)}) = \left\{\mathcal{P}^{(h)}(\boldsymbol{x}^{(j)})\right\}_{h=1}^{j}$ *is a* (j,j)*-complex.*

Now we define formally the notion of a partition.

Definition 4.3 (**Partition**)**.** *Let* k *be a positive integer,* V *be a non-empty set,* $\boldsymbol{a} = \boldsymbol{a}_{\mathscr{P}} = (a_1, a_2, \ldots, a_k)$ *be a vector of positive integers, and* $\psi_j \colon [V]^j \to [a_j]$ *be a mapping,* $j \in [k]$*. Set* $\boldsymbol{\psi} = \{\psi_j \colon j \in [k]\}$*. Then, we define a partition* $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ *of* $\mathrm{Cross}_k(\psi_1)$ *by*[1]

$$\mathscr{P} = \left\{\mathcal{P}^{(k)}(\boldsymbol{x}) \colon \boldsymbol{x} \in A(k, \boldsymbol{a})\right\}. \tag{4.4}$$

We also define the rank of $\mathscr{P}$ *by*

$$\mathrm{rank}(\mathscr{P}) = |A(k, \boldsymbol{a})|. \tag{4.5}$$

Remark 4.4. Without loss of generality, we may assume that mappings $\psi_j \colon [V]^j \to [a_j]$ are onto for all $j \in [k]$. Then we have

$$\binom{a_1}{k} \times \prod_{h=2}^{k} a_h^{\binom{k}{h}} = \mathrm{rank}(\mathscr{P}) \geq a_h$$

for all $h \in [k]$.

Remark 4.5. It follows from Definition 4.3 that for every $j \in [k]$,

$$\mathscr{P}^{(j)} = \mathscr{P}(j, \boldsymbol{a}, \boldsymbol{\psi}) = \left\{\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) \colon \boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})\right\} \tag{4.6}$$

is a partition of $\mathrm{Cross}_j(\psi_1)$. Therefore, with every partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ of $\mathrm{Cross}_k(\psi_1)$ we have associated a system of partitions $\left\{\mathscr{P}^{(j)}\right\}_{j=1}^{k}$ defined by (4.6). This system represents the "underlying structure" of $\mathcal{P}$ in the following sense:

Every $\mathcal{P} \in \mathscr{P}$ can be written as $\mathcal{P}^{(k)}(\boldsymbol{x})$ for some $\boldsymbol{x} \in A(k, \boldsymbol{a})$ (see (4.4)). Since $\mathscr{P} = \mathscr{P}^{(k)}$, every $\mathcal{P} \in \mathscr{P}$ uniquely defines (k,k)-complex $\boldsymbol{\mathcal{P}}(\boldsymbol{x}) = \left\{\mathcal{P}^{(h)}(\boldsymbol{x})\right\}_{h=1}^{k}$ (see Claim 4.2) such that

- $\mathcal{P} = \mathcal{P}^{(k)}(\boldsymbol{x}) \in \boldsymbol{\mathcal{P}}(\boldsymbol{x})$,
- $\mathcal{P}^{(h)}(\boldsymbol{x})$ consists of $\binom{k}{h}$ elements of $\mathscr{P}^{(h)}$ for every $h \in [k]$, and
- $\mathcal{P}^{(h+1)}(\boldsymbol{x}) \subseteq \mathcal{K}_{h+1}(\mathcal{P}^{(h)}(\boldsymbol{x}))$ for every $h \in [k-1]$.

Remark 4.6. For $k = 1$, $\mathscr{P}$ is simply the partition $V = V_1 \cup \ldots \cup V_{a_1}$, where $V_i = \psi_1^{-1}(i)$. Such partition is considered in Theorem 1.3.

Remark 4.7. For $k = 2$, $\mathscr{P}$ is composed of bipartite graphs ($(2,2)$-cylinders) $\mathcal{P}^{(2)}(\boldsymbol{x})$ with bipartition $\mathcal{P}^{(1)}(\boldsymbol{x})$ ($(2,1)$-cylinders). If we write $\boldsymbol{x} \in A(2, \boldsymbol{a})$ as $\boldsymbol{x} = (i, j, \alpha)$, where $1 \leq i < j \leq a_1$ and $\alpha \in [a_2]$, then $(2,2)$-cylinders $\mathcal{P}^{(2)}(\boldsymbol{x})$ correspond to bipartite graphs P_{α}^{ij} with bipartition $V_i \cup V_j$ that were considered in [4].

[1]If there is no danger of confusion, we will omit the superscript ${}^{(k)}$ in $\boldsymbol{x}^{(k)} \in A(k, \boldsymbol{a})$ to simplify the text.

Later, we will also need to describe when one partition *refines* another one.

Definition 4.8. *Let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{\psi}, \boldsymbol{a})$ and $\mathscr{S} = \mathscr{S}(k, \boldsymbol{\varphi}, \boldsymbol{b})$ be two partitions. We say that $\mathscr{S}$ refines $\mathscr{P}$, and write $\mathscr{S} \prec \mathscr{P}$, if for every $\mathcal{P}^{(k)} \in \mathscr{P}$ there are $\mathcal{S}_i^{(k)} \in \mathscr{S}$, $i \in I(\mathcal{S}^{(k)})$, so that*

$$\mathcal{P}^{(k)} = \bigcup \big\{ \mathcal{S}_i^{(k)} \colon i \in I(\mathcal{S}^{(k)}) \big\}.$$

We remark that the above definition implies that $\mathrm{Cross}_k(\psi_1) \subseteq \mathrm{Cross}_k(\varphi_1)$.

Let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{\psi}, \boldsymbol{a})$ be a partition of $\mathrm{Cross}_k(\psi_1)$ and suppose that for every $\boldsymbol{x} \in A(k, \boldsymbol{a})$, we decompose $\mathcal{P}^{(k)}(\boldsymbol{x}) \in \mathscr{P}$ into mutually edge-disjoint (k,k)-cylinders $\mathcal{S}^{(k)}(\xi, \boldsymbol{x})$, where $1 \le \xi \le s$. In other words, $\mathcal{P}^{(k)}(\boldsymbol{x}) = \bigcup_{\xi=1}^{s} \mathcal{S}^{(k)}(\xi, \boldsymbol{x})$ for all $\boldsymbol{x} \in A(k, \boldsymbol{a})$. Then we claim the following.

Claim 4.9. *The system*

$$\mathscr{S} = \Big\{ \mathcal{S}^{(k)}(\xi, \boldsymbol{x}) \colon \boldsymbol{x} \in A(k, \boldsymbol{a}), \xi \in [s] \Big\}$$

is a partition of $\mathrm{Cross}_k(\psi_1)$ *that refines* $\mathscr{P}$.

5. Polyads

A regular pair played a central role in the definition of a regular partition for graphs (see Theorem 1.2). In [4], where the regularity lemma for triples was considered, this role was played by a 'triad' (which corresponds to a $(3,2)$-cylinder). In order to define a regular partition $\mathscr{P}$ for a k-uniform hypergraph, we extend these two concepts by introducing *polyads*. Polyads are $(k+1,k)$-cylinders consisting of selected $k+1$ members of $\mathscr{P}$.

We describe first the environment in which we work.

Setup 5.1. Let k be a positive integer, V be a non-empty set, $\boldsymbol{a} = \boldsymbol{a}_{\mathscr{P}} = (a_1, a_2, \ldots, a_k)$ be a vector of positive integers, $\boldsymbol{\psi} = \{\psi_j \colon j \in [k]\}$ be a set of mappings $\psi_j \colon [V]^j \to [a_j]$, $j \in [k]$. Let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ be the partition of $\mathrm{Cross}_k(\psi_1)$ (see Definition 4.3).

Recall that for every crossing set $K \in \mathrm{Cross}_{k+1}(\psi_1)$ and $h \in [k]$, we defined $\Psi_h(K)$ as the $\binom{k+1}{h}$-dimensional vector

$$\Psi_h(K) = (x_{\pi_h(H)} = \psi_h(H))_{H \in (K)^{h}_{<}},$$

where $\pi_h(H) = (\lambda_1, \ldots, \lambda_h) \in ([a_1])^h_<$ is such that $|H \cap V_{\lambda_u}| = 1$ for every $u \in [h]$. We set

$$\hat{\boldsymbol{\Psi}}^{(k)}(K) = (\Psi_1(K), \Psi_2(K), \ldots, \Psi_k(K))$$

and observe that $\hat{\boldsymbol{\Psi}}^{(k)}(K)$ is a vector having $\sum_{h=1}^{k} \binom{k+1}{h} = 2^{k+1} - 2$ entries. We define set $\hat{A}(k, \boldsymbol{a})$ of $(2^{k+1}-2)$-dimensional vectors by

$$\hat{A}(k, \boldsymbol{a}) = \hat{A}_{\mathscr{P}}(k, \boldsymbol{a}) = ([a_1])^{k+1}_< \times \prod_{h=2}^{k} [a_h]^{\binom{k+1}{h}}. \tag{5.1}$$

Then $\hat{\boldsymbol{\Psi}}^{(k)}(K) \in \hat{A}(k, \boldsymbol{a})$ for each crossing set $K \in \mathrm{Cross}_{k+1}(\psi_1)$.

Let $\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a})$. Then we write vector $\hat{\boldsymbol{x}}$ as $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \ldots, \hat{\boldsymbol{x}}_k)$, where $\hat{\boldsymbol{x}}_1 \in ([a_1])^{k+1}_<$ is an ordered set and $\hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon)_{\Upsilon \in (\hat{\boldsymbol{x}}_1)^u_<} \in [a_u]^{\binom{k+1}{u}}$, is a $\binom{k+1}{u}$-dimensional vector with entries from $[a_u]$ for every $u > 1$.

Given an ordered set $\Xi \subseteq \hat{\boldsymbol{x}}_1$ with $1 \leq |\Xi| = h \leq k$, we set $\hat{\boldsymbol{x}}^\Xi_u = (\hat{x}_\Upsilon)_{\Upsilon \in (\Xi)^u_<}$ for each $u \in [h]$. We also define

$$\hat{\mathcal{P}}^{(h)}(\hat{\boldsymbol{x}}) = \bigcup_{\Xi \in (\hat{\boldsymbol{x}}_1)^h_<} \left\{P \in \mathrm{Cross}_h(\psi_1) \colon \Psi^{(h)}(P) = (\hat{\boldsymbol{x}}^\Xi_1, \ldots, \hat{\boldsymbol{x}}^\Xi_h)\right\} \tag{5.2}$$

for each $h \in [k]$, and set $\hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}}) = \left\{\hat{\mathcal{P}}^{(h)}(\hat{\boldsymbol{x}})\right\}_{h=1}^k$. Similarly to Claim 4.2, we can prove the following.

Claim 5.2. *For every vector $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \ldots, \hat{\boldsymbol{x}}_k) \in \hat{A}(k, \boldsymbol{a})$, the following statements are true.*

(a) *For all $h \in [k]$, $\hat{\mathcal{P}}^{(h)}(\hat{\boldsymbol{x}})$ is a $(k+1, h)$-cylinder;*

(b) *$\hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}}) = \left\{\hat{\mathcal{P}}^{(h)}(\hat{\boldsymbol{x}})\right\}_{h=1}^k$ is a $(k+1, k)$-complex.*

In this paper, $(k+1, k)$-cylinders $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ will play a special role and we will call them polyads.

Definition 5.3 (Polyad). *Let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ be the partition of* $\mathrm{Cross}_k(\psi_1)$ *as described in the Setup 5.1. Then, for each vector $\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a})$, we refer to $(k+1, k)$-cylinder $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ as a polyad.*

We also define the set $\hat{\mathscr{P}}$ of all polyads of $\mathscr{P}$ by

$$\hat{\mathscr{P}} = \left\{\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}) \colon \hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a})\right\}. \tag{5.3}$$

For every polyad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ there exists a unique vector $\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a})$ such that $\hat{\mathcal{P}} = \hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$. Hence, each polyad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ uniquely defines $(k+1, k)$-complex $\hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}}) = \left\{\hat{\mathcal{P}}^{(i)}(\hat{\boldsymbol{x}})\right\}_{i=1}^k$ such that $\hat{\mathcal{P}} \in \hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}})$.

Remark 5.4. Similarly to Remark 4.4, if $\psi_j \colon [V]^j \to [a_j]$, $j \in [k]$, are mappings defining $\mathscr{P}$, then we have

$$\binom{a_1}{k+1} \times \prod_{h=2}^{k} a_h^{\binom{k+1}{h}} \geq |\hat{\mathscr{P}}|.$$

Remark 5.5. For $k = 1$, we have $\boldsymbol{a} = (a_1)$ and $\hat{A}(1, \boldsymbol{a})$ consists of 1-dimensional vectors $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1)$, where $\hat{\boldsymbol{x}}_1 = (i, j)$, $1 \leq i < j \leq a_1$. For a fixed $\hat{\boldsymbol{x}} = (i, j)$, we have $\hat{\boldsymbol{x}}^{\{i\}}_1 = (i)$ and $\hat{\boldsymbol{x}}^{\{j\}}_1 = (j)$. Consequently, a polyad

$$\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}) = \bigcup_{\Xi \in \{\{i\}, \{j\}\}} \left\{P \in \mathrm{Cross}_1(\psi_1) \colon \Psi^{(1)}(P) = (\hat{\boldsymbol{x}}^\Xi_1)\right\}$$

is the bipartition $V_i \cup V_j$ (see Remark 4.6).

Remark 5.6. For $k = 2$, a polyad $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}}) \in \hat{\mathscr{P}}$ is a $(3,2)$-cylinder and $\hat{\boldsymbol{x}} = ((i,j,\ell),(\alpha,\beta,\gamma)) \in \hat{A}(2,\boldsymbol{a})$ is a six-dimensional vector such that $1 \leq i < j < \ell \leq a_1$, $\alpha,\beta,\gamma \in [a_2]$. In view of Remark 4.7, $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}})$ is the 3-partite graph $P^{ij}_\alpha \cup P^{i\ell}_\beta \cup P^{j\ell}_\gamma$ and $\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}})$ is its 3-partition $V_i \cup V_j \cup V_\ell$. Note that the triple $(P^{ij}_\alpha, P^{i\ell}_\beta, P^{j\ell}_\gamma)$ corresponding to $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}})$ was called a triad in [4].

Every polyad $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}) \in \hat{\mathscr{P}}$ is a $(k+1,k)$-cylinder that is the union of $k+1$ elements ((k,k)-cylinders) of $\mathscr{P}$. We describe these elements using vector $\hat{\boldsymbol{x}}$.

Let $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \dots, \hat{\boldsymbol{x}}_k) \in \hat{A}(k,\boldsymbol{a})$ be given. Then, for every $1 \leq u \leq k$, vector $\hat{\boldsymbol{x}}_u$ can be written as $\hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon)_{\Upsilon \in (\hat{\boldsymbol{x}}_1)^u_<}$, i.e. its entries are labeled by u-element subsets of $\hat{\boldsymbol{x}}_1$ in lexicographic order. For every $x \in \hat{\boldsymbol{x}}_1$, we set

$$\partial_x \hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon \colon x \notin \Upsilon)_{\Upsilon \in (\hat{\boldsymbol{x}}_1)^u_<}. \tag{5.4}$$

In other words, vector $\partial_x \hat{\boldsymbol{x}}_u$ contains precisely those entries of $\hat{\boldsymbol{x}}$ which are labeled by an u-element subset of $\hat{\boldsymbol{x}}_1$ not containing x. Clearly, $\partial_x \hat{\boldsymbol{x}}_u$ has $\binom{k}{u}$ entries from $[a_u]$. Furthermore, we set

$$\partial_x \hat{\boldsymbol{x}} = (\partial_x \hat{\boldsymbol{x}}_1, \partial_x \hat{\boldsymbol{x}}_2, \dots, \partial_x \hat{\boldsymbol{x}}_k)$$

and observe that $\partial_x \hat{\boldsymbol{x}}$ is a (2^k-1)-dimensional vector belonging to $A(k,\boldsymbol{a})$. Then, the following fact is true.

Fact 5.7. *For every vector $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_k) \in \hat{A}(k,\boldsymbol{a})$,*

$$\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}) = \bigcup_{x \in \hat{\boldsymbol{x}}_1} \mathcal{P}^{(k)}(\partial_x \hat{\boldsymbol{x}}). \tag{5.5}$$

On the other hand, given a (k,k)-cylinder $\mathcal{P}^{(k)}(\boldsymbol{x}) \in \mathscr{P}$, we will also need to describe all polyads that contain this (k,k)-cylinder.

Let $\boldsymbol{x} \in A(k,\boldsymbol{a})$ and $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_k) \in \hat{A}(k,\boldsymbol{a})$ be given. We say that $\boldsymbol{x} \prec \hat{\boldsymbol{x}}$ if there exists $x \in \hat{\boldsymbol{x}}_1$ such that $\boldsymbol{x} = \partial_x \hat{\boldsymbol{x}}$. In this case, we say that $\hat{\boldsymbol{x}}$ is an *extension* of $\boldsymbol{x}$ and denote by $\mathrm{Ext}(\boldsymbol{x})$ the set of all extensions of $\boldsymbol{x}$, i.e.

$$\mathrm{Ext}(\boldsymbol{x}) = \left\{ \hat{\boldsymbol{x}} \in \hat{A}(k,\boldsymbol{a}) \colon \boldsymbol{x} \prec \hat{\boldsymbol{x}} \right\}.$$

Then, in view of Fact 5.7, we have that $\mathcal{P}^{(k)}(\boldsymbol{x}) \subset \hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ whenever $\boldsymbol{x} \prec \hat{\boldsymbol{x}}$.

Remark. For $k = 1$, a 1-dimensional vector $\boldsymbol{x} = (i)$ describes a subscript i of a set V_i in Szemerédi's partition and a 2-dimensional vector $\hat{\boldsymbol{x}} = (i,j)$ describes a pair of subscripts i, j of a pair V_i, V_j in Szemerédi's partition (see Theorem 1.3). Hence, if $\boldsymbol{x} = (i)$, then $\boldsymbol{x} \prec \hat{\boldsymbol{x}}$ if $\hat{\boldsymbol{x}}$ contains i and $\mathrm{Ext}(\boldsymbol{x})$ is the set of all 2-dimensional vectors containing i.

Remark. For $k = 2$, a 3-dimensional ($3 = 2^2 - 1$) vector $\boldsymbol{x}$ describes subscripts i, j, and α corresponding to a bipartite graph P^{ij}_α in a Frankl-Rödl partition. A 6-dimensional ($6 = 2^{2+1} - 2$) vector $\hat{\boldsymbol{x}} = (i,j,\ell,\alpha,\beta,\gamma)$ describes a triad in a Frankl-Rödl partition (see Definition 7.6). Hence, $\boldsymbol{x} \prec \hat{\boldsymbol{x}}$

if a triad determined by $\hat{\boldsymbol{x}}$ includes bipartite graph P^{ij}_{α} and $\mathrm{Ext}(\boldsymbol{x})$ is the set of all such vectors $\hat{\boldsymbol{x}}$.

We will prove the following fact.

Fact 5.8. *For every $\boldsymbol{x} \in A(k, \boldsymbol{a})$, $|\mathrm{Ext}(\boldsymbol{x})| \leq |A(k, \boldsymbol{a})|^k$.*

Proof. Let $\boldsymbol{x} \in A(k, \boldsymbol{a})$ be given. If $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_k) \in \mathrm{Ext}(\boldsymbol{x})$, i.e. $\boldsymbol{x} = \partial_x \hat{\boldsymbol{x}}$ for some $x \in \hat{\boldsymbol{x}}_1$, then notice that $2^k - 1$ of $2^{k+1} - 2$ components of vector $\hat{\boldsymbol{x}}$ are determined by $\boldsymbol{x}$. Therefore, the size of set $\mathrm{Ext}(\boldsymbol{x})$ is bounded by

$$(a_1 - k) \times \frac{\prod_{j=2}^{k} a_j^{\binom{k+1}{j}}}{\prod_{j=2}^{k} a_j^{\binom{k}{j}}} \leq a_1 \times \prod_{j=2}^{k} a_j^{\binom{k}{j-1}}.$$

Since $\binom{k}{j-1} \leq k \times \binom{k}{j}$ and $|A(k, \boldsymbol{a})| = \binom{a_1}{k} \times \prod_{j=2}^{k} a_j^{\binom{k}{j}}$, it is easy to observe that the above product is bounded by $|A(k, \boldsymbol{a})|^k$. $\square$

6. Glossary of terms

This section provides a brief summary of terms defined in the previous three sections. The reader may find it useful in the remainder of this paper.

Cylinders and Complexes.

- An (ℓ, k)-cylinder is an ℓ-partite k-uniform hypergraph.
- A complex $\boldsymbol{\mathcal{G}} = \left\{\mathcal{G}^{(j)}\right\}_{j=1}^{k}$ is a set of k-cylinders satisfying conditions (a) and (b) of Definition 3.6.

Partition.

Let a_1, $a_2, \ldots, a_k$ be fixed positive integers and $\boldsymbol{a} = (a_1, \ldots, a_k)$. Below, J is a set with j elements.

- $\psi_j \colon [V]^j \to [a_j]$, $j \in [k]$ are k mappings.
- $V_i = \psi_1^{-1}(i)$ for every $i \in [a_1]$.
- $\mathrm{Cross}_j(\psi_1) = \{J \in [V]^j \colon |J \cap V_i| \leq 1, i \in [a_1]\}$ is the set of all j-element crossing subsets J of V.
- $\pi_j(J) = (\lambda_1, \ldots, \lambda_j) \in ([a_1])^j_<$ is so that $|J \cap V_{\lambda_h}| = 1$ for every $h \in [j]$.
- $\Psi_h(J) = (x_{\pi_h(H)} = \psi_h(H))_{H \in [J]^h}$ is a vector with $\binom{j}{h}$ entries from $[a_h]$, where $h \in [j]$.
- $\Psi^{(j)}(J) = (\Psi_1(J), \Psi_2(J), \ldots, \Psi_j(J))$ is a vector with $\sum_{h=1}^{j} \binom{j}{h} = 2^j - 1$ entries.
- $A(j, \boldsymbol{a}) = ([a_1])^j_< \times \prod_{h=2}^{j} [a_h]^{\binom{j}{h}}$ is a set of $2^j - 1$ dimensional vectors.
- $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_j)$ is a vector from $A(j, \boldsymbol{a})$, where $\hat{\boldsymbol{x}}_1 \in ([a_1])^j_<$ and $\hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon)_{\Upsilon \in (\hat{\boldsymbol{x}}_1)^u_<} \in [a_u]^{\binom{j}{u}}$ for every $u > 1$.
- for an ordered set $\Xi \subseteq \hat{\boldsymbol{x}}_1$ with $1 \leq |\Xi| = h \leq j$, we set $\hat{\boldsymbol{x}}_u^{\Xi} = (\hat{x}_\Upsilon)_{\Upsilon \in (\Xi)^u_<}$ for each $u \in [h]$.

- $\mathcal{P}^{(h)}(\boldsymbol{x}^{(j)}) = \bigcup_{\Xi\in(\boldsymbol{x}_1)^h_<}\{P \in \mathrm{Cross}_h(\psi_1) \colon \Psi^{(h)}(P) = (\boldsymbol{x}_1^\Xi, \dots, \boldsymbol{x}_h^\Xi)\}$ is a (j,h)-cylinder for every $h \in [j]$.
- $\boldsymbol{\mathcal{P}}(\boldsymbol{x}^{(j)}) = \left\{\mathcal{P}^{(h)}(\boldsymbol{x}^{(j)})\right\}_{h=1}^{j}$ is a (j,j)-complex.
- $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi}) = \left\{\mathcal{P}^{(k)}(\boldsymbol{x}) \colon \boldsymbol{x} \in A(k, \boldsymbol{a})\right\}$ is a partition of the set $\mathrm{Cross}_k(\psi_1)$.

Polyads.
Let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ be any partition of $\mathrm{Cross}_k(\psi_1)$ and $K \in \mathrm{Cross}_{k+1}(\psi_1)$ is a $k+1$ element crossing set.

- $\Psi_h(K) = (\psi_h(H))_{H\in(K)^h_<}$ is a vector with $\binom{k+1}{h}$ entries from $[a_h]$ for every $h \in [k]$.
- $\hat{\boldsymbol{\Psi}}^{(k)}(K) = (\Psi_1(K), \dots, \Psi_k(K))$ is a vector with $\sum_{h=1}^{k}\binom{k+1}{h} = 2^{k+1} - 2$ entries.
- $\hat{A}(k, \boldsymbol{a}) = \hat{A}_{\mathscr{P}}(k, \boldsymbol{a}) = ([a_1])^{k+1}_< \times \prod_{h=2}^{k}[a_h]^{\binom{k+1}{h}}$ is a set of $2^{k+1} - 2$ dimensional vectors.
- $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \dots, \hat{\boldsymbol{x}}_k)$ is a vector from $\hat{A}(k, \boldsymbol{a})$, where $\hat{\boldsymbol{x}}_1 \in ([a_1])^{k+1}_<$ and $\hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon)_{\Upsilon\in(\hat{\boldsymbol{x}}_1)^u_<} \in [a_u]^{\binom{k+1}{u}}$ for every $u > 1$.
- for an ordered set $\Xi \subseteq \hat{\boldsymbol{x}}_1$ with $1 \le |\Xi| = h \le k$, we set $\hat{\boldsymbol{x}}_u^\Xi = (\hat{x}_\Upsilon)_{\Upsilon\in(\Xi)^u_<}$ for each $u \in [h]$.
- $\hat{\mathcal{P}}^{(h)}(\hat{\boldsymbol{x}}) = \bigcup_{\Xi\in(\hat{\boldsymbol{x}}_1)^h_<}\left\{P \in \mathrm{Cross}_h(\psi_1) \colon \Psi^{(h)}(P) = (\hat{\boldsymbol{x}}_1^\Xi, \dots, \hat{\boldsymbol{x}}_h^\Xi)\right\}$ is a $(k+1,h)$-cylinder for every $h \in [k]$.
- $\hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}}) = \left\{\hat{\mathcal{P}}^{(h)}(\hat{\boldsymbol{x}})\right\}_{h=1}^{k}$ is a $(k+1,k)$-complex.
- $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ is called a polyad.

Extensions.
Let $\boldsymbol{x} \in A(\boldsymbol{a}, k)$, $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \dots, \hat{\boldsymbol{x}}_k) \in \hat{A}(\boldsymbol{a}, k)$, $\hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon)_{\Upsilon\in(\hat{\boldsymbol{x}}_1)^u_<}$ for $u \in [k]$.

- $\partial_x\hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon \colon x \notin \Upsilon))_{\Upsilon\in(\hat{\boldsymbol{x}}_1)^u_<}$ is a $\binom{k}{u}$-dimensional vector from $[a_u]^{\binom{k}{u}}$ for every $u \in [k]$.
- $\partial_x\hat{\boldsymbol{x}} = \left(\partial_x\hat{\boldsymbol{x}}_1, \partial_x\hat{\boldsymbol{x}}_2, \dots, \partial_x\hat{\boldsymbol{x}}_k\right)$ is a $(2^k - 1)$-dimensional vector from $A(k, \boldsymbol{a})$.
- $\boldsymbol{x} \prec \hat{\boldsymbol{x}}$ if and only if $\boldsymbol{x} = \partial_x\hat{\boldsymbol{x}}$ for some $x \in \hat{\boldsymbol{x}}_1$.
- $\mathrm{Ext}(\boldsymbol{x}) = \left\{\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a}) \colon \boldsymbol{x} \prec \hat{\boldsymbol{x}}\right\}$.
- $\mathcal{P}^{(k)}(\boldsymbol{x}) \subset \hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ whenever $\boldsymbol{x} \prec \hat{\boldsymbol{x}}$.

7. Regular partition

Let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ be any partition of $\mathrm{Cross}_k(\psi_1)$ on n vertices as described in Setup 5.1. Then we define the *(relative) volume* of a polyad $\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}$ by

$$\mathrm{Vol}(\hat{\mathcal{P}}^{(k)}) = \frac{|\mathcal{K}_{k+1}(\hat{\mathcal{P}}^{(k)})|}{\binom{n}{k+1}}. \tag{7.1}$$

Remark 7.1. For $k = 1$ and for any vector $\hat{\boldsymbol{x}} = (i, j) \in \hat{A}(1, \boldsymbol{a})$, we have $1 \le i < j \le a_1$, and polyad $\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}})$ is simply bipartition $V_i \cup V_j$ (cf. Remark 5.5). Thus, $\mathcal{K}_2(\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}))$ is the complete bipartite graph $K(V_i, V_j)$ (cf. Definition 3.3) and $\mathrm{Vol}(\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}})) = |V_i||V_j|/\binom{n}{2}$.

Remark 7.2. For $k = 2$ and for any six-dimensional vector $\hat{\boldsymbol{x}} = (i, j, \ell, \alpha, \beta, \gamma) \in A(2, \boldsymbol{a})$, polyad $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}}) = P^{ij}_{\alpha} \cup P^{i\ell}_{\beta} \cup P^{j\ell}_{\gamma}$ is a $(3, 2)$-cylinder (see Remark 4.7). Hence, $|\mathcal{K}_3(\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}}))|$ counts the number of triangles in $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}})$ and $\mathrm{Vol}(\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}}))$ corresponds to a relative number of triangles $t((P^{ij}_{\alpha}, P^{i\ell}_{\beta}, P^{j\ell}_{\gamma}))$ in a triad $(P^{ij}_{\alpha}, P^{i\ell}_{\beta}, P^{j\ell}_{\gamma})$ defined in [4].

In this paper, we will work only with partitions with certain properties. These properties are summarized in the following definition.

Definition 7.3 (equitable $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-partition). *Let $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$ and $\boldsymbol{d} = (d_2, \ldots, d_k)$ be two arbitrary but fixed vectors of real numbers between 0 and 1, μ be a number in interval $(0, 1]$ and r be a positive integer. We say that a partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ is an equitable $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-partition if all but at most $\mu\binom{n}{k+1}$ many $(k+1)$-tuples $K \in [V]^{k+1}$ belong to $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complexes $\hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}}) = \left\{\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}})\right\}_{j=1}^{k}$, where $\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a})$. More precisely,*

$$\sum_{\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a})} \left\{ \mathrm{Vol}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})) \colon \hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}}) \text{ is } (\boldsymbol{\delta}, \boldsymbol{d}, r)\text{-regular} \right\} > 1 - \mu. \tag{7.2}$$

Remark 7.4. For $k = 1$, polyad $\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}})$ is a $(2, 1)$-cylinder (see Remark 5.5) and $\hat{\mathcal{P}}(\hat{\boldsymbol{x}}) = \{\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}})\}$ is a $(2, 1)$-complex that is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular for every vector $\hat{\boldsymbol{x}} \in \hat{A}(1, \boldsymbol{a})$ (see Remark 3.13). Thus, Definition 7.3 states that all but $\mu\binom{n}{2}$ pairs of vertices are crossing.

Remark 7.5. For $k = 2$ and a vector $\hat{\boldsymbol{x}} \in \hat{A}(2, \boldsymbol{a})$, $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(3, 2)$-complex $\hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}})$ consists of (δ_2, d_2)-regular tripartite graph $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}})$ and its tripartition $\mathcal{P}^{(1)}(\boldsymbol{x})$ (see Remarks 3.13 and 5.6). Due to (7.1), inequality (7.2) means that all but at most $\mu\binom{n}{3}$ triples of vertices from V are crossing and belong to (δ_2, d_2)-regular $(3, 2)$-cylinders from $\hat{\mathscr{P}}$.

In terms of the connection between Definition 7.3 and [4], we first recall the definition of an *equitable* $(\ell, t, \varepsilon_1, \varepsilon_2)$-*partition* (Definition 3.2 in [4]).

Definition 7.6. *Let V be a set. An equitable $(\ell, t, \varepsilon_1, \varepsilon_2)$-partition $\mathcal{P}$ of $[V]^2$ is an (auxiliary) partition $V = \bigcup_{i=0}^{t} V_i$ with $|V_0| \le t$ and $|V_1| = \ldots = |V_t| = m$, together with a family of graphs P^{ij}_{α}, where $1 \le i < j \le t$ and $0 \le \alpha \le \ell$, such that*

1. *$\bigcup_{\alpha=0}^{\ell} P^{ij}_{\alpha} = K(V_i, V_j)$ for all i, j, $1 \le i < j \le t$, and*
2. *for all but $\varepsilon_1\binom{t}{2}$ pairs i, j, $1 \le i < j \le t$, $|P^{ij}_0| \le \varepsilon_1 m^2$ and all bipartite graphs P^{ij}_{α}, $\alpha \in [\ell]$, are $(\varepsilon_2, 1/\ell)$-regular (see Definition 3.10).*

Remark 7.7. One can show that an equitable $(\ell, t, \varepsilon_1, \varepsilon_2)$-partition $\mathcal{P}$ is also an equitable $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-partition, provided that $\mu = 27\varepsilon_1$, $\boldsymbol{\delta} = (\delta_2) = (\varepsilon_2)$, $\boldsymbol{d} = (d_2) = (1/\ell)$, $t \geq 1/\varepsilon_1$, $m \geq 1/\varepsilon_1$, and $\boldsymbol{a} = (t+1, \ell+1)$. This means that we must prove that all but at most $\mu\binom{n}{3}$ triples are crossing and triangles in (δ_2, d_2)-regular $(3,2)$-cylinders. Indeed, there are at most

- $t \times n^2$ triples containing a vertex from V_0,
- $t \times \binom{m}{2} \times n + t \times \binom{m}{3}$ triples which are not crossing,
- $\varepsilon_1\binom{t}{2} \times m^2 \times n$ triples in $(3,2)$-cylinders containing P_0^{ij} with $|P_0^{ij}| > \varepsilon_1 m^2$ or in $(3,2)$-cylinders containing $(\varepsilon_2, 1/\ell)$-irregular P_α^{ij} (i.e. i, j is an exceptional pair from (2)),
- $\binom{t}{2} \times n \times \varepsilon_1 m^2$ triples in $(3,2)$-cylinders containing P_0^{ij} with $|P_0^{ij}| \leq \varepsilon_1 m^2$.

Thus, the number of triples of vertices which are not in $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular polyads is bounded by

$$tn^2 + t\binom{m}{2}n + t\binom{m}{3} + 2\varepsilon_1\binom{t}{2}m^2 n \leq \left(\frac{t}{n} + \frac{1}{2t} + \frac{1}{6t^2} + \varepsilon_1\right)n^3$$
$$\leq 9(\varepsilon_1 + \varepsilon_1/2 + \varepsilon_1/2 + \varepsilon_1)\binom{n}{3} = \mu\binom{n}{3}.$$

Hence, (7.2) holds.

Now we can define the notion of a regular partition – a partition we are looking for.

Definition 7.8 (**regular partition**). *Let $\mathcal{H}$ be a $(k+1)$-uniform hypergraph with vertex set V, $|V| = n$, and let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ be any equitable $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-partition of $\mathrm{Cross}_k(\psi_1)$.*

A polyad $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ is called (δ_{k+1}, r)-regular (w.r.t $\mathcal{H}$) if

(a) *complex $\hat{\boldsymbol{\mathcal{P}}}(\hat{\boldsymbol{x}}) = \left\{\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}})\right\}_{j=1}^{k}$ is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular, and*

(b) *$\mathcal{H}$ is (δ_{k+1}, r)-regular[2] with respect to $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$.*

We say $\mathscr{P}$ is (δ_{k+1}, r)-regular[1] (w.r.t $\mathcal{H}$) if all but at most $\delta_{k+1}\binom{n}{k+1}$ many $(k+1)$-tuples $K \in [V]^{k+1}$ are in (δ_{k+1}, r)-regular polyads $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$. In other words,

$$\sum_{\hat{\boldsymbol{x}} \in \hat{A}(k,\boldsymbol{a})} \left\{\mathrm{Vol}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})) \colon \hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}) \text{ is } (\delta_{k+1}, r)\text{-regular}\right\} > 1 - \delta_{k+1}. \tag{7.3}$$

Remark 7.9. For $k = 1$, each polyad $\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}})$, where $\hat{\boldsymbol{x}} = (i,j) \in \hat{A}(1, \boldsymbol{a})$, is just bipartition $V_i \cup V_j$. Moreover, by Remark 3.13, condition (a) is trivially satisfied and condition (b) means that pair (V_i, V_j) is δ_2-irregular (see discussion behind Definition 3.7).

[2] δ_2-regular for $k = 1$

Hence, Definition 7.8 states that partition $V = V_1 \cup \ldots \cup V_{a_1}$ is δ_2-regular if all but at most $\delta_2\binom{n}{2}$ pairs of vertices are crossing and in δ_2-regular pairs. This exactly fits the description of the partition from Theorem 1.3.

Remark 7.10. For $k = 2$, the concept of an equitable $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-partition plays the same role as that of an equitable $(\ell, t, \varepsilon_1, \varepsilon_2)$-partition in [4]. Similarly, a polyad $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}})$ corresponds to a triad defined in [4] (see Remarks 7.5 and 7.7). Then, (δ_3, r)-irregular polyad $\hat{\mathcal{P}}^{(2)}(\hat{\boldsymbol{x}})$ corresponds to a (δ_3, r)-irregular triad as defined in Definition 3.3 of [4]. Hence, Definition 7.8 corresponds to Definition 3.4 in [4].

In the previous two definitions, r was a fixed integer and $\boldsymbol{d}$ and $\boldsymbol{\delta}$ were two fixed vectors. For our regularity lemma to work, we need to extend these definitions to the case when vector $\boldsymbol{\delta}$ is a prescribed function of $\boldsymbol{d}$ and r is a prescribed function of a_1 (the number of vertex classes) and $\boldsymbol{d}$. We remark that the dependency of r on a_1 and $\boldsymbol{d}$ is not needed for the proof of our regularity lemma but it is essential for applications of this lemma.

Definition 7.11 (**functionally equitable partition**). *Let μ be a number in interval $(0, 1]$, $\delta_k(d_k)$, $\delta_{k-1}(d_{k-1}, d_k)$, $\ldots$, $\delta_2(d_2, \ldots, d_k)$, and $r = r(t, d_2, \ldots, d_k)$ be non-negative functions. Set $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$.*

A partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ of $\mathrm{Cross}_k(\psi_1)$ is a functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition if there exists a vector $\boldsymbol{d} = (d_2, \ldots, d_k)$ such that $\mathscr{P}$ is an equitable $(\mu, \boldsymbol{\delta}(\boldsymbol{d}), \boldsymbol{d}, r(a_1, \boldsymbol{d}))$-partition (see Definition 7.3).

Definition 7.12 (**regular functionally equitable partition**). *Let a $(k+1)$-uniform hypergraph $\mathcal{H}$ and a number δ_{k+1}, where $0 < \delta_{k+1} \leq 1$, be given. We say that a functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition $\mathscr{P}$ is (δ_{k+1}, r)-regular*[3] *(w.r.t. $\mathcal{H}$) if $\mathscr{P}$ is $(\delta_{k+1}, r(a_1, \boldsymbol{d}))$-regular*[2] *(w.r.t. $\mathcal{H}$), where $\boldsymbol{d}$ is the vector from Definition 7.11.*

Remark 7.13. Note that for $k = 1$ there are no functions given in the above definitions, and, therefore, a δ_2-regular functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition $\mathscr{P}$ corresponds to a δ_2-regular partition (see Remark 7.9).

The objective of this paper is to prove the following theorem.

Theorem 7.14 (**Main theorem**). *For every integer $k \in \mathbb{N}$, all numbers $\delta_{k+1} > 0$ and $\mu > 0$, and any non-negative functions $\delta_k(d_k)$, $\delta_{k-1}(d_{k-1}, d_k)$, $\ldots$, $\delta_2(d_2, \ldots, d_k)$, and $r = r(t, d_2, \ldots, d_k)$, there exist integers n_{k+1} and L_{k+1} such that the following holds.*

For every $(k+1)$-uniform hypergraph $\mathcal{H}$ with at least n_{k+1} vertices there exists a partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ of $\mathrm{Cross}_k(\psi_1)$ so that

(i) *$\mathscr{P}$ is a functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition,*
(ii) *$\mathscr{P}$ is (δ_{k+1}, r)-regular (w.r.t. $\mathcal{H}$), and*
(iii) $\mathrm{rank}(\mathscr{P}) = |A(k, \boldsymbol{a})| \leq L_{k+1}$.

[3] δ_2-regular for $k = 1$

Remark 7.15. For $k = 1$, Theorem 7.14 is equivalent to Theorem 1.3. Indeed, in view of Remark 7.13, (i) and (ii) mean that the partition $\mathscr{P}$ is δ_2-regular. Furthermore, $|A(1, \boldsymbol{a})| = a_1$, thus, condition (iii) means that the number of partition classes is bounded by L_2, independently of the given graph $\mathcal{H}$. This is precisely the statement of Theorem 1.3.

Remark 7.16. The proof of Theorem 7.14 is by induction and implicitly uses the Regularity Lemma of Szemerédi as the base case for the induction. Since the proof doesn't change the sizes of vertex classes once we apply the induction assumption, we may assume that every two vertex classes of every partition considered in this paper differ in sizes by at most 1. In other words, if $\mathscr{P}$ is a partition of $\mathrm{Cross}_k(\psi_1)$, then

$$|\psi_1^{-1}(1)| \le |\psi_1^{-1}(2)| \le \ldots \le |\psi_1^{-1}(a_1)| \le |\psi_1^{-1}(1)| + 1.$$

Note that similarly to Szemerédi's Lemma (case $k = 1$), one can show a version of Theorem 7.14 with the hypergraph $\mathcal{H}$ replaced by an s-tuple of hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2, \ldots$, $\mathcal{H}_s$.

Theorem 7.17 (**Statement Regularity$(\boldsymbol{k})$**). *Let $s, k \ge 1$ be fixed integers. Then, for all numbers $\delta_{k+1} > 0$ and $\mu > 0$, and any non-negative functions $\delta_k(d_k)$, $\delta_{k-1}(d_{k-1}, d_k)$, ..., $\delta_2(d_2, \ldots, d_k)$, and $r = r(t, d_2, \ldots, d_k)$, there exist integers n'_{k+1} and L'_{k+1} such that the following holds.*

For every $(k+1)$-uniform hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_s$ with common vertex set of size at least n'_{k+1} there exists a partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ of $\mathrm{Cross}_k(\psi_1)$ so that

(i) *$\mathscr{P}$ is a functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition,*
(ii) *$\mathscr{P}$ is (δ_{k+1}, r)-regular with respect to every $\mathcal{H}_i$, $i \in [s]$, and*
(iii) $\mathrm{rank}(\mathscr{P}) = |A(k, \boldsymbol{a})| \le L'_{k+1}$.

Remark 7.18. We are going to use **Regularity$(\boldsymbol{k})$** as an assumption to prove **Regularity$(\boldsymbol{k+1})$**. However, for simplicity and since there is no principle difference between the proof of Theorem 7.17 and Theorem 7.14, we will show **Regularity$(\boldsymbol{k+1})$** only for $s = 1$.

Remark 7.19. For $k = 1$, the above theorem appears (as Lemma 3.7) in [4] in the following form:

For any $\varepsilon_0 > 0$ and positive integers t and s, there exist integers $N(\varepsilon_0, t, s)$ and $T(\varepsilon_0, t, s)$ such that the following holds. If $|V| > N(\varepsilon_0, t, s)$, then for any partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$, with $|V_0| < t$ and $|V_1| = |V_2| = \ldots = |V_t|$, and any system of graphs $H_1, H_2, \ldots, H_s$, each on the vertex set V, there exists a partition $V = W_0 \cup W_1 \cup \ldots \cup W_{t'}$ such that

(1) $|W_0| < t' < T(\varepsilon_0, t, s)$,
(2) $|W_1| = |W_2| = \ldots = |W_{t'}|$,
(3) the partition $V = W_0 \cup W_1 \cup \ldots \cup W_{t'}$ "refines" the given partition $V = V_0 \cup V_1 \cup \ldots \cup V_{t'}$, that is, for all $1 \le i \le t'$, there exists $1 \le j \le t$ such that $W_i \subset V_j$,

(4) $W_0 \cup W_1 \cup \ldots \cup W_{t'}$ is ε_0-regular with respect to H_i for all $i = 1, 2, \ldots, s$.

Note that if $|V|$ is divisible by t and t', then W_0 and V_0 are empty and the partition $V = W_1 \cup \ldots \cup W_{t'}$ refines the given partition $V = V_1 \cup \ldots \cup V_t$.

Regularity$(\boldsymbol{k})$ is an assumption to prove **Regularity$(\boldsymbol{k+1})$** and, therefore, Lemma 3.7 in [4] is the base case for the induction. Since the proof doesn't change the sizes of the vertex classes once we apply the induction assumption and we apply **Regularity$(\boldsymbol{k})$** (and implicitly Lemma 3.7) only finitely many times, we may assume the following throughout the proof:

- the size of the vertex set V is divisible by the number of classes of each vertex partition considered (we can always add a constant number of vertices);
- when applying **Regularity$(\boldsymbol{k})$**, the resulting partition $\mathscr{R}$ refines any given initial vertex partition $V = V_1 \cup \ldots \cup V_t$. In other words, if $\mathscr{R}$ is a partition of $\mathrm{Cross}_k(\psi_1)$, then for every $1 \le i \le a_1$ there exists $1 \le j \le t$ such that
$$\psi_1^{-1}(i) \subset V_j.$$
Consequently, every crossing set in the partition $V = V_1 \cup \ldots \cup V_t$ remains crossing in $\mathscr{R}$. In particular, this delicate observation will be used in (11.10).

Since we introduced a number of various symbols in this section, we highlight the following:

- $\delta_2, \delta_3, \ldots, \delta_k, r$ are parameters that control the regularity properties of the underlying structure (partition);
- μ is a parameter describing what fraction of $(k+1)$-tuples are not "under control", that is, they are not crossing or do not belong to dense, regular polyads;
- δ_{k+1} controls the regularity of $\mathcal{H}$ with respect to underlying polyads;
- while μ and δ_{k+1} are fixed positive reals, $\delta_2, \delta_3, \ldots, \delta_k$ are functions of densities $d_2, d_3, \ldots, d_k$;
- r is a function of the number of partition classes a_1 and densities $d_2, d_3, \ldots, d_k$.

8. Proof of the main theorem.

Our proof of Theorem 7.14 resembles the proofs from [15, 4]. First, we define the notion of the index of a partition.

Definition 8.1 (**Index**). *Let $\mathcal{H}$ be a $(k+1)$-uniform hypergraph with vertex set V and let $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ be a partition of* $\mathrm{Cross}_k(\psi_1)$. *We define the index of partition $\mathscr{P}$ by*
$$\operatorname{ind} \mathscr{P} = \sum_{\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a})} \mathrm{Vol}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})) \times d^2_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})).$$

Then we observe that the index of every partition is bounded.

Fact 8.2. *For every $(k+1)$-uniform hypergraph $\mathcal{H}$ and every partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ of $\mathrm{Cross}_k(\psi_1)$, we have*

$$0 \leq \operatorname{ind} \mathscr{P} \leq 1.$$

Let δ_{k+1}, μ, $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$, $\delta_i = \delta_i(d_i, \ldots, d_k)$, where $i = 2, \ldots, k$, and $r = r(t, d_2, \ldots, d_k)$ be as in Theorem 7.14. In the actual proof of Theorem 7.14, we make a use of the following statement that enables to increase the index of the partition.

Lemma 8.3 (Index Pumping Lemma, **Statement Pump($\boldsymbol{k}$)**)**.** *For every functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$, which is not (δ_{k+1}, r)-regular, there exists a partition $\mathscr{T}$ and a function f (defined in context) such that*

$$\mathscr{T} \text{ is a functionally equitable } (\mu, \boldsymbol{\delta}, r)\text{-partition},$$

$$\operatorname{rank}(\mathscr{T}) \leq f(\operatorname{rank}(\mathscr{P}), \delta_{k+1}, \boldsymbol{\delta}, r),$$

and

$$\operatorname{ind} \mathscr{T} \geq \operatorname{ind} \mathscr{P} + \delta_{k+1}^4/2.$$

The proof of Theorem 7.14 will follow from the following facts

(I1) **Regularity(2)** holds;
(I2) **Pump($\boldsymbol{k}$)⇒Regularity($\boldsymbol{k}$)** holds for every $k > 2$;
(I3) **Regularity($\boldsymbol{k-1}$)⇒Pump($\boldsymbol{k}$)** holds for every $k > 2$.

Indeed, we have

$$\underbrace{\textbf{Regularity(2)}}_{(I1)} \overset{(I3)}{\Rightarrow} \textbf{Pump(3)} \overset{(I2)}{\Rightarrow} \textbf{Regularity(3)} \overset{(I3)}{\Rightarrow} \textbf{Pump(4)} \overset{(I2)}{\Rightarrow} \ldots$$

$$\ldots \overset{(I2)}{\Rightarrow} \textbf{Regularity}(\boldsymbol{k-1}) \overset{(I3)}{\Rightarrow} \textbf{Pump}(\boldsymbol{k}) \overset{(I2)}{\Rightarrow} \textbf{Regularity}(\boldsymbol{k}) \overset{(I3)}{\Rightarrow} \ldots.$$

What remains to prove are facts *(I1)–(I3)*. We start with the first two since they are easier to handle.

9. Proof of facts *(I1)*, *(I2)*.

Proof of (I1). First we write the statement of **Regularity(2)**:

Lemma 9.1. *Let $s \geq 1$ be a fixed integer. Then, for all numbers $\delta_3 > 0$ and $\mu > 0$, and any non-negative functions $\delta_2(d_2)$ and $r = r(t, d_2)$, there exist integers n_3 and L_3 such that the following holds.*

For all 3-uniform hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_s$ on the same vertex set with at least n_3 vertices there exists a partition $\mathscr{P} = \mathscr{P}(2, \boldsymbol{a}, \boldsymbol{\psi})$ of $\mathrm{Cross}_2(\psi_1)$ so that

(i) *$\mathscr{P}$ is a functionally equitable $(\mu, (\delta_2), r)$-partition,*
(ii) *$\mathscr{P}$ is (δ_3, r)-regular with respect to every $\mathcal{H}_i$, $i \in [s]$, and*
(iii) *$\operatorname{rank}(\mathscr{P}) = |A(2, \boldsymbol{a})| \leq L_3$.*

Lemma 9.1 is a consequence of Theorem 9.2 (see Theorem 3.11 in [4]).

Theorem 9.2. *For all integers s, t_0, and ℓ_0, for all δ and ε_1, $0 < \varepsilon_1 \leq 2\delta^4/s$, and for all integer-valued functions $r(t,\ell)$ and all functions $\varepsilon_2(\ell)$, there exist T_0, L_0, and N_0 such that if $\mathcal{H}_1, \ldots, \mathcal{H}_s$ are 3-uniform hypergraphs on the same vertex set V with $|V| > N_0$, then, for some t and ℓ satisfying $t_0 \leq t < T_0$ and $\ell_0 \leq \ell < L_0$, there exists an equitable $(\ell, t, \varepsilon_1, \varepsilon_2(\ell))$-partition (see Definition 7.6) which is $(\delta, r(t,\ell))$-regular with respect to each $\mathcal{H}_i$, $i \in [s]$.*

In order to get Lemma 9.1, we apply Theorem 9.2 and obtain an equitable $(\ell, t, \varepsilon_1, \varepsilon_2(\ell))$-partition $\mathscr{P}$ that is $(\delta, r(t,\ell))$-regular with respect to each $\mathcal{H}_i$, $i \in [s]$. The input parameters for Theorem 9.2 are chosen so that $\mathscr{P}$ is an equitable $(\mu, \boldsymbol{\delta}(\boldsymbol{d}), \boldsymbol{d}, r(t, \boldsymbol{d}))$-partition by Remark 7.7 and $(\delta_3, r(t, \boldsymbol{d}))$-regular with respect to each $\mathcal{H}_i$, $i \in [s]$, by Remark 7.10. We omit details here. □

Proof of I2. To prove $\mathbf{Pump}(\boldsymbol{k}) \Rightarrow \mathbf{Regularity}(\boldsymbol{k})$, we follow the idea of Szemerédi [15]. We define an initial partition $\mathscr{P}_0$ that is a functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition. If partition $\mathscr{P}_0$ is not (δ_{k+1}, r)-regular, then we apply $\mathbf{Pump}(\boldsymbol{k})$ and obtain a functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition $\mathscr{P}_1$ whose index exceeds $\operatorname{ind} \mathscr{P}_0$ by a positive constant. We repeat the whole procedure until we get a (δ_{k+1}, r)-regular functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition. This must happen in finite many steps because the index of every partition is bounded by 1 and we increase the index by a positive constant at each step.

Set $a_1 = \lceil 4(k+1)^2/\mu \rceil$ and let $\mathscr{P}_0 = \mathscr{P}_0(k, \boldsymbol{a}_0, \boldsymbol{\psi}_0)$ be a partition of $\mathrm{Cross}_k(\psi_{0,1})$, where $\boldsymbol{a}_0 = (a_1, 1, \ldots, 1)$, $\boldsymbol{\psi}_0 = (\psi_{0,1}, \ldots, \psi_{0,k})$, where $\psi_{0,1}$ is an arbitrary mapping $V \to [a_1]$ so that

$$|\psi_{0,1}^{-1}(1)| \leq |\psi_{0,1}^{-1}(2)| \leq \ldots \leq |\psi_{0,1}^{-1}(a_1)| \leq |\psi_{0,1}^{-1}(1)| + 1$$

and $\psi_{0,j} \colon [V]^j \to \{1\}$ for $j \in [k] \setminus \{1\}$.

Set $V_i = \psi_{0,1}^{-1}(i)$ for $i = 1, \ldots, a_1$. It follows from Definitions 4.3 and 4.1 that any $\mathcal{P} \in \mathscr{P}_0$ is of the form $K_k^{(k)}(V_{i_1}, \ldots, V_{i_k})$, where $1 \leq i_1 < \ldots < i_k \leq a_1$. Similarly, by Definition 5.3 and (5.2), every polyad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}_0$ is of the form $K_{k+1}^{(k)}(V_{i_1}, \ldots, V_{i_{k+1}})$, where $1 \leq i_1 < \ldots < i_{k+1} \leq a_1$.

Let $\hat{\mathcal{P}} = K_{k+1}^{(k)}(V_{i_1}, \ldots, V_{i_{k+1}}) \in \hat{\mathscr{P}}_0$ be any polyad. Then, by (5.2), the unique $(k+1,k)$-complex $\hat{\boldsymbol{\mathcal{P}}} = \left\{\hat{\mathcal{P}}^{(i)}\right\}_{i=1}^{k}$ such that $\hat{\mathcal{P}} \in \hat{\boldsymbol{\mathcal{P}}}$ is defined by

$$\hat{\mathcal{P}}^{(i)} = \begin{cases} V_{i_1} \cup \ldots \cup V_{i_{k+1}} & \text{for } i = 1, \\ K_{k+1}^{(i)}(V_{i_1}, \ldots, V_{i_{k+1}}) & \text{for } i > 1. \end{cases}$$

It follows from Definition 3.10 that the complete $(i+1)$-uniform $(k+1)$-partite hypergraph $K_{k+1}^{(i+1)}(V_{i_1}, \ldots, V_{i_{k+1}})$ is $(\delta', 1, r')$-regular with respect to $K^{(i)}_{k+1}(V_{i_1},\dots,V_{i_{k+1}})$ for any $\delta'>0$ and any $r'\in\mathbb{N}$. Therefore, $\hat{\boldsymbol{\mathcal{P}}}$ is $(\boldsymbol{\delta}(\boldsymbol{d}),\boldsymbol{d},r(a_1,\boldsymbol{d}))$-regular $(k+1,k)$-complex, where $\boldsymbol{d}=(\underbrace{1,\dots,1}_{k-1})$.

Consequently, to conclude that $\mathcal{P}_0$ is a functionally equitable $(\mu,\boldsymbol{\delta},r)$-partition, we need to show that the total volume of all polyads in $\hat{\mathscr{P}}_0$ is at least $1-\mu$ (see Definitions 7.11 and 7.3). This translates into proving that all but $\mu\binom{n}{k+1}$ many $(k+1)$-tuples are crossing.

This is, however, easy, since the number of $(k+1)$-tuples that are not crossing is at most $a_1\times\binom{1+n/a_1}{2}\times\binom{n}{k-1}\le\mu\binom{n}{k+1}$ since $a_1=\lceil 4(k+1)^2/\mu\rceil$.

If partition $\mathscr{P}_0$ is (δ_{k+1},r)-regular, then we are done. Otherwise, we apply $\mathbf{Pump}(\boldsymbol{k})$ and obtain a functionally equitable $(\mu,\boldsymbol{\delta},r)$-partition $\mathscr{P}_1$ with $\mathrm{rank}(\mathscr{P}_1)\le f(\mathrm{rank}(\mathscr{P}_0),\delta_{k+1},\boldsymbol{\delta},r)$ and $\mathrm{ind}\,\mathscr{P}_1\ge\mathrm{ind}\,\mathscr{P}_0+\delta_{k+1}^4/2$.

If $\mathscr{P}_1$ is not (δ_{k+1},r)-regular, we repeat the process and obtain partitions $\mathscr{P}_2,\mathscr{P}_3,\dots$ satisfying

$$\mathrm{rank}(\mathscr{P}_i)\le f^i(\mathrm{rank}(\mathscr{P}_0),\delta_{k+1},\boldsymbol{\delta},r),$$

where $f^i(\mathscr{P}_0)$ means i-times iterated function f, and

$$\mathrm{ind}\,\mathscr{P}_i\ge\mathrm{ind}\,\mathscr{P}_0+i\times\delta_{k+1}^4/2.$$

Since $0\le\mathrm{ind}\,\mathscr{P}\le 1$ for any partition $\mathscr{P}$, this process will stop after at most $2/\delta_{k+1}^4$ steps. The last partition $\mathscr{P}_{\mathrm{last}}$ must be a functionally equitable $(\mu,\boldsymbol{\delta},r)$-partition that is (δ_{k+1},r)-regular and $\mathrm{rank}(\mathscr{P}_{\mathrm{last}})\le L_{k+1}=f^{2\delta_{k+1}^{-4}}(\mathrm{rank}(\mathscr{P}_0),\delta_{k+1},\boldsymbol{\delta},r)$. $\square$

In order to prove *(I3)*, we first summarize all needed auxiliary results in the next section and then we provide the actual proof of implication *(I3)*.

10. Auxiliary results for the proof of implication *(I3)*.

In our proof we will need the following results. The first tool is statement $\mathbf{Regularity}(\boldsymbol{k-1})$ in which $\mathcal{H}_1$, $\mathcal{H}_2,\dots$, $\mathcal{H}_s$ are replaced with k-uniform hypergraphs $\mathcal{G}^{(k)}_1$, $\mathcal{G}^{(k)}_2,\dots,\mathcal{G}^{(k)}_s$ and which we assume by induction assumption. We use the notation $\mu',\delta'_k,\boldsymbol{\delta}'$, etc. to be consistent with the context in which we apply Lemma 10.1 and to distinguish the fact that it is an induction assumption.

Lemma 10.1. *Let s, $k\ge 1$ be fixed integers. Then, for all numbers $\delta'_k>0$ and $\mu'>0$, for any vector $\boldsymbol{\delta}'=(\delta'_2,\dots,\delta'_{k-1})$ of non-negative functions $\delta'_{k-1}(d_{k-1})$, $\delta'_{k-2}(d_{k-2},d_{k-1})$, $\dots$, $\delta'_2(d_2,\dots,d_{k-1})$, and for any positive integer function $r'=r'(t,d_2,\dots,d_{k-1})$, there exist integers n'_k and L'_k such that the following holds:*

For every k-uniform hypergraphs $\mathcal{G}^{(k)}_1$, $\mathcal{G}^{(k)}_2,\dots,\mathcal{G}^{(k)}_s$ with common vertex set of size at least n'_k there exists a partition $\mathscr{R}=\mathscr{R}(k-1,\boldsymbol{a}^{\mathscr{R}},\boldsymbol{\psi}^{\mathscr{R}})$ of $\mathrm{Cross}_{k-1}(\psi_1^{\mathscr{R}})$ *so that*

(i) *$\mathscr{R}$ is a functionally equitable $(\mu',\boldsymbol{\delta}',r')$-partition,*

(ii) $\mathscr{R}$ *is* (δ'_k, r')*-regular with respect to every* $\mathcal{G}_i^{(k)}$, $i \in [s]$, *and*
(iii) $\operatorname{rank}(\mathscr{R}) = |A(k-1, \boldsymbol{a}^{\mathscr{R}})| \le L'_k$.

The next lemma enables to decompose each sufficiently dense regular (k,k)-cylinder into a bounded number of regular (k,k)-cylinders with *smaller* relative densities. In order to preserve the flow of the proof, we postpone the proof of this lemma as well as the next lemma (Lemma 10.3) to the Appendix A.

Lemma 10.2 (Slicing Lemma)**.** *Suppose* α, δ *are two positive real numbers such that* $0 < 2\delta < \alpha \le 1$. *Let* $\mathcal{G}$ *be a* $(k, k-1)$*-cylinder satisfying* $|\mathcal{K}_k(\mathcal{G})| \ge m^k/\ln m$ *and* $\mathcal{H}$ *be a* (k,k)*-cylinder which is* (δ, α, r)*-regular with respect to* $\mathcal{G}$. *Then, for every* $0 < p < 1$, *where* $3\delta < p\alpha$ *and* $kr \ln m/m \le \delta^3/(3(\ln 4)\alpha p)$, *and* $u = \lfloor 1/p \rfloor$ *the following holds:*

There exists a decomposition of $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_u$ *such that* $\mathcal{H}_i$ *is* $(3\delta, p\alpha, r)$*-regular with respect to* $\mathcal{G}$ *for every* $i \in [u]$.

Remark. For $\mathcal{H}_0$ we have the following: $|\mathcal{H}_0| = |\mathcal{H}| - \sum_{j=1}^{u} |\mathcal{H}_j| \le (\alpha + \delta)\big|\mathcal{K}_k(\mathcal{G})\big| - u \times (p\alpha - 3\delta)\big|\mathcal{K}_k(\mathcal{G})\big|$. Since $pu = p\lfloor 1/p \rfloor \ge 1 - p$, we obtain $|\mathcal{H}_0| \le (p\alpha + 4u\delta)\big|\mathcal{K}_k(\mathcal{G})\big|$.

The proof of the Slicing Lemma is very similar to the proof of Lemma 3.8 in [4] (which is actually this lemma for $k = 2$ in a slightly different setting). The details can be found in the Appendix A.

By Definition 4.8, for two partitions $\mathscr{S}$ and $\mathscr{T}$, $\mathscr{T}$ refines $\mathscr{S}$ if for every $\mathcal{S}^{(k)} \in \mathscr{S}$ there exists $\mathcal{T}_i^{(k)} \in \mathscr{T}$, $i \in I(\mathcal{S}^{(k)})$, such that $\mathcal{S}^{(k)} = \bigcup_{i \in I(\mathcal{S}^{(k)})} \mathcal{T}^{(k)}$. Then we have the following lemma.

Lemma 10.3. *If* $\mathscr{T}$ *refines* $\mathscr{S}$, *then* $\operatorname{ind} \mathscr{T} \ge \operatorname{ind} \mathscr{S}$.

We will also need the following fact which is a consequence of the Cauchy-Schwarz inequality (for its proof see [4]).

Fact 10.4. *Let* σ_i, d_i, $i \in I$, *be positive real numbers satisfying* $\sum_{i \in I} \sigma_i = 1$. *Set* $d = \sum_{i \in I} \sigma_i d_i$. *Let* $J \subset I$ *be a proper subset of* I *such that* $\sum_{j \in J} \sigma_j = \sigma$ *and*

$$\sum_{j \in J} \sigma_j d_j = \sigma(d + \nu).$$

Then

$$\sum_{i \in I} \sigma_i d_i^2 \ge d^2 + \frac{\nu^2 \sigma}{1 - \sigma}$$

and, therefore, if $\sigma \ge \delta$ *and* $|\nu| \ge \delta$ *for some* $\delta > 0$, *then*

$$\sum_{i \in I} \sigma_i d_i^2 \ge d^2 + \delta^3.$$

Now we are ready for the proof of implication *(I3)*.

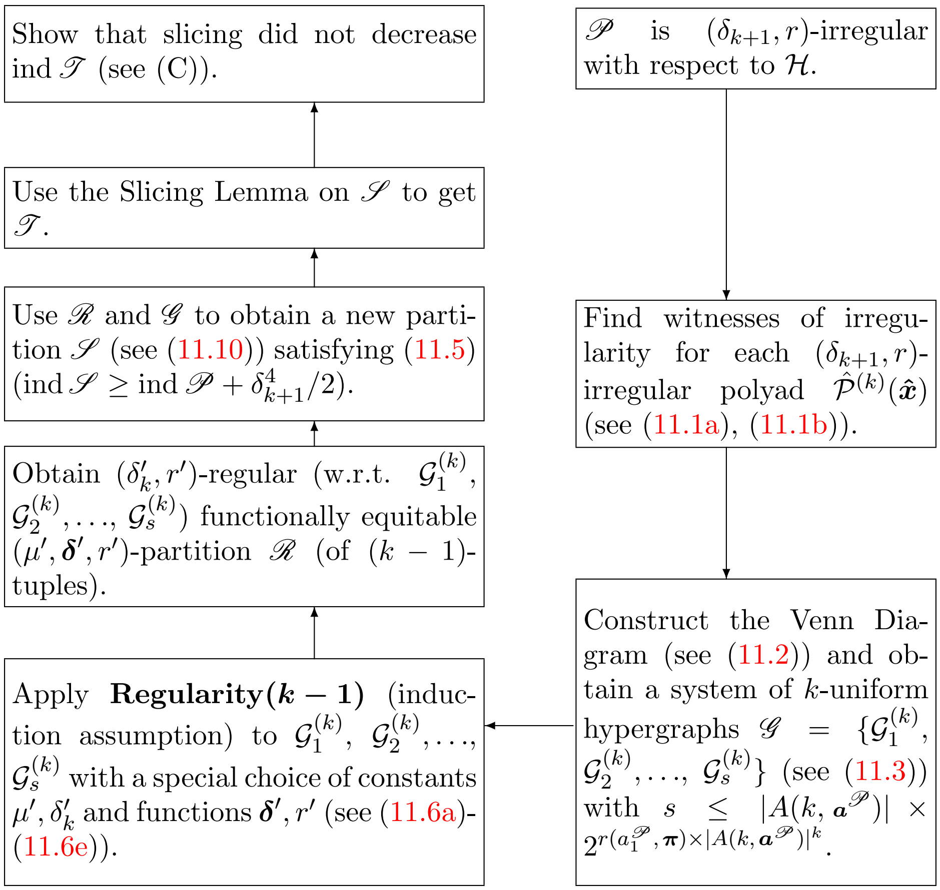

FIGURE 1. Scheme of the proof of **Regularity$(k-1)$⇒Pump(k)**.

11. Proof of implication *(I3)*

Proof of Lemma 8.3. We will follow the scheme outlined at Fig.1.

Let $\mathcal{H}$ be a $(k+1)$-uniform hypergraph, $\boldsymbol{a}^{\mathscr{P}} = (a_1^{\mathscr{P}}, \dots, a_k^{\mathscr{P}})$ be a vector of positive integers and let δ_{k+1}, μ, $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$, where $\mu \leq \delta_{k+1}/2$, $\delta_i = \delta_i(d_i, \dots, d_k)$, where $i = 2, \dots, k$, and $r = r(t, d_2, \dots, d_k)$ be as in Theorem 7.14 [4]. Furthermore, let

$$\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}^{\mathscr{P}}, \boldsymbol{\psi}^{\mathscr{P}}) = \{\mathcal{P}^{(k)}(\boldsymbol{x}) \colon \boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) \in A(k, \boldsymbol{a}^{\mathscr{P}})\}$$

[4]We may assume $\mu \leq \delta_{k+1}/2$ (needed for (11.21)) because any $(\mu, \boldsymbol{\delta}, r)$-partition is also a $(\mu^*, \boldsymbol{\delta}, r)$-partition for every $\mu^* > \mu$ (see Definitions 7.3 and 7.11).

be any functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition of $\mathrm{Cross}_k(\psi_1^{\mathscr{P}})$ which is not (δ_{k+1}, r)-regular.

This means (see Definition 7.11) that there exists a vector $\boldsymbol{\pi} = (\pi_2, \dots, \pi_k)$ of positive real numbers such that $\mathscr{P}$ is an equitable $(\mu, \boldsymbol{\delta}(\boldsymbol{\pi}), \boldsymbol{\pi}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-partition and (cf. (7.3))

$$\sum_{\hat{\boldsymbol{x}} \in \hat{A}(k, \boldsymbol{a}^{\mathscr{P}})} \big\{ \mathrm{Vol}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}))\colon \hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}) \text{ is } (\delta_{k+1}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))\text{-regular} \big\} \leq 1 - \delta_{k+1}.$$

For every $(\delta_{k+1}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-irregular polyad $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ satisfying (a) and violating (b) of Definition 7.8, there exist $r(a_1^{\mathscr{P}}, \boldsymbol{\pi})$ witnesses of irregularity, that is, an $r(a_1^{\mathscr{P}}, \boldsymbol{\pi})$-tuple of $(k+1, k)$-cylinders $\tilde{Q}(\hat{\boldsymbol{x}}) = \big\{ \mathcal{Q}_1^{(k)}(\hat{\boldsymbol{x}}), \dots, \mathcal{Q}_{r(a_1^{\mathscr{P}}, \boldsymbol{\pi})}^{(k)}(\hat{\boldsymbol{x}}) \big\}$ such that

$$\Big| \bigcup_{i=1}^{r(a_1^{\mathscr{P}}, \boldsymbol{\pi})} \mathcal{K}_{k+1}(\mathcal{Q}_i^{(k)}(\hat{\boldsymbol{x}})) \Big| \geq \delta_{k+1} \big| \mathcal{K}_{k+1}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})) \big| \tag{11.1a}$$

and

$$\big| d_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})) - d_{\mathcal{H}}(\tilde{Q}(\hat{\boldsymbol{x}})) \big| > \delta_{k+1}. \tag{11.1b}$$

Since each $\mathcal{Q}_i^{(k)}(\hat{\boldsymbol{x}})$ is a $(k+1, k)$-cylinder, it can be written as the union of $k+1$ (k, k)-cylinders $\mathcal{Q}_i^{(k)}(\hat{\boldsymbol{x}}) = \bigcup_{x \in \hat{\boldsymbol{x}}_1} \mathcal{Q}_i^{(k)}(\partial_x \hat{\boldsymbol{x}})$, where $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_k)$ and

$$\mathcal{Q}_i^{(k)}(\partial_x \hat{\boldsymbol{x}}) = \mathcal{Q}_i^{(k)}(\hat{\boldsymbol{x}}) \cap \mathcal{P}^{(k)}(\partial_x \hat{\boldsymbol{x}}).$$

Let $\boldsymbol{x}_0$ be an arbitrary but fixed vector from $A(k, \boldsymbol{a}^{\mathscr{P}})$. Observe that for given $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_k) \in \hat{A}(k, \boldsymbol{a}^{\mathscr{P}})$ there exists at most one $x = x(\hat{\boldsymbol{x}}) \in \hat{\boldsymbol{x}}_1$ such that $\mathcal{Q}_i^{(k)}(\partial_x \hat{\boldsymbol{x}}) \subset \mathcal{P}^{(k)}(\boldsymbol{x}_0)$ for every $i \in [r(a_1^{\mathscr{P}}, \boldsymbol{\pi})]$. Moreover, such x exists if, and only if, $\hat{\boldsymbol{x}}$ extends $\boldsymbol{x}_0$, i.e. $\boldsymbol{x}_0 \prec \hat{\boldsymbol{x}}$.

Consider system $\mathcal{X}(\boldsymbol{x}_0)$ of such hypergraphs, i.e.

$$\mathcal{X}(\boldsymbol{x}_0) = \left\{ \mathcal{Q}_i^{(k)}(\partial_x \hat{\boldsymbol{x}})\colon \begin{array}{l} \mathcal{Q}_i^{(k)}(\partial_x \hat{\boldsymbol{x}}) \subset \mathcal{P}^{(k)}(\boldsymbol{x}_0), \hat{\boldsymbol{x}} \in \mathrm{Ext}(\boldsymbol{x}_0), \\ x = x(\hat{\boldsymbol{x}}), i \in [r(a_1^{\mathscr{P}}, \boldsymbol{\pi})] \end{array} \right\}.$$

From Fact 5.8, we have that $\big|\mathcal{X}(\boldsymbol{x}_0)\big| \leq r(a_1^{\mathscr{P}}, \boldsymbol{\pi}) \times |A(k, \boldsymbol{a}^{\mathscr{P}})|^k$.

For every $\boldsymbol{x}_0 \in A(k, \boldsymbol{a}^{\mathscr{P}})$, let $\tilde{\mathcal{G}}(\boldsymbol{x}_0)$ be the system of edge disjoint (k, k)-cylinders given by regions of the Venn diagram of elements of $\mathcal{X}(\boldsymbol{x}_0)$ in $\mathcal{P}^{(k)}(\boldsymbol{x}_0)$. In other words, if $\mathcal{X}(\boldsymbol{x}_0) = \{\mathcal{Q}_1, \dots, \mathcal{Q}_c\}$, where $\mathcal{Q}_i \subset \mathcal{P}^{(k)}(\boldsymbol{x}_0)$ and $c = \big|\mathcal{X}(\boldsymbol{x}_0)\big| \leq r(a_1^{\mathscr{P}}, \boldsymbol{\pi}) \times |A(k, \boldsymbol{a}^{\mathscr{P}})|^k$, then

$$\tilde{\mathcal{G}}(\boldsymbol{x}_0) = \Big\{ \bigcap_{i=1}^{c} \mathcal{Q}_i^{\varepsilon_i}\colon (\varepsilon_1, \dots, \varepsilon_c) \in \{0,1\}^c \Big\}, \tag{11.2}$$

where

$$\mathcal{Q}_i^{\varepsilon_i} = \begin{cases} \mathcal{Q}_i & \text{for } \varepsilon_i = 1, \\ \mathcal{P}^{(k)}(\boldsymbol{x}_0) \setminus \mathcal{Q}_i & \text{for } \varepsilon_i = 0. \end{cases}$$

Note that

(1) $\mathcal{P}^{(k)}(\boldsymbol{x}_0) = \bigcup_{\mathcal{G}\in\tilde{\mathcal{G}}(\boldsymbol{x}_0)} \mathcal{G}$ and this union is disjoint, and
(2) the size of $\tilde{\mathcal{G}}(\boldsymbol{x}_0)$ is bounded by $2^{r(\boldsymbol{a}_1^{\mathscr{P}},\boldsymbol{\pi})\times|A(k,\boldsymbol{a}^{\mathscr{P}})|^k}$.

We remark that for those $\boldsymbol{x}_0$ for which $\mathcal{X}(\boldsymbol{x}_0)$ is empty (i.e. there are no irregular polyads $\hat{\mathcal{P}}(\hat{\boldsymbol{x}})$ with $\boldsymbol{x}_0 \prec \hat{\boldsymbol{x}}$), we have $\tilde{\mathcal{G}}(\boldsymbol{x}_0) = \{\mathcal{P}^{(k)}(\boldsymbol{x}_0)\}$. In view of Claim 4.9, system

$$\mathscr{G} = \bigcup_{\boldsymbol{x}\in A(k,\boldsymbol{a}^{\mathscr{P}})} \tilde{\mathcal{G}}(\boldsymbol{x}) \tag{11.3}$$

is a partition of $\mathrm{Cross}_k(\psi_1^{\mathscr{P}})$ that refines $\mathscr{P}$. Alternatively, we will write

$$\mathscr{G} = \{\mathcal{G}_1^{(k)}, \mathcal{G}_2^{(k)}, \dots, \mathcal{G}_s^{(k)}\},$$

where

$$s \le |A(k,\boldsymbol{a}^{\mathscr{P}})| \times 2^{r(a_1^{\mathscr{P}},\boldsymbol{\pi})\times|A(k,\boldsymbol{a}^{\mathscr{P}})|^k}. \tag{11.4}$$

Now, we are going to modify $\mathscr{G}$ to obtain a partition $\mathscr{S}$ with the following properties:

(a) $\mathscr{S}$ is an *almost equitable* $(\mu/2, \boldsymbol{\delta}, r)$-partition, that is there is a vector $\boldsymbol{\sigma} = (\sigma_2, \dots, \sigma_k)$ such that all but at most $(\mu/2)\binom{n}{k+1}$ many $(k+1)$-tuples $K \in [V]^{k+1}$ belong to *almost* $(\boldsymbol{\delta}(\boldsymbol{\sigma}), \boldsymbol{\sigma}, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-*regular* complexes $\hat{\boldsymbol{\mathcal{S}}} = \{\hat{\mathcal{S}}^{(j)}\}_{j=1}^k \in \hat{\mathscr{S}}$. Here, $\hat{\boldsymbol{\mathcal{S}}} = \{\hat{\mathcal{S}}^{(i)}\}_{i=1}^k$ is almost $(\boldsymbol{\delta}(\boldsymbol{\sigma}), \boldsymbol{\sigma}, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular if, for $\boldsymbol{\sigma}^i = (\sigma_i, \dots, \sigma_k)$, $i = 2, 3, \dots, k$, (11.5)
- $\mathcal{S}^{(2)}$ is $(\delta_2(\boldsymbol{\sigma}^2), \sigma_2)$-regular with respect to $\mathcal{S}^{(1)}$,
- $\mathcal{S}^{(i)}$ is $(\delta_i(\boldsymbol{\sigma}^i), \sigma_i, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(i-1)}$ for $i = 3, \dots, k-1$, and
- $\mathcal{S}^{(k)}$ is $(\delta_k(\boldsymbol{\sigma}^k), \rho, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(k-1)}$ and $\rho \ge \sigma_k$.

(b) $\mathrm{rank}(\mathscr{S}) \le f(\mathrm{rank}(\mathscr{P}), \delta_{k+1}, \boldsymbol{\delta}, r)$, and
(c) $\mathrm{ind}\,\mathscr{S} \ge \mathrm{ind}\,\mathscr{P} + \delta_{k+1}^4/2$.

This will be done in three steps.

Step 1. Define input parameters δ_2', δ_3', …, δ_k', r', and μ' appearing in Lemma 10.1. Then apply this lemma on k-uniform hypergraphs $\mathcal{G}_1^{(k)}$, $\mathcal{G}_2^{(k)}$, …, $\mathcal{G}_s^{(k)}$ to obtain a partition $\mathscr{R}$ of $\mathrm{Cross}_{k-1}(\psi^{\mathscr{R}})$.

Step 2. Combine hypergraphs $\mathcal{G}_1^{(k)}$, $\mathcal{G}_2^{(k)}$, …, $\mathcal{G}_s^{(k)}$ (which form a partition of $\mathrm{Cross}_k(\psi_1^{\mathscr{P}})$) and partition $\mathscr{R}$ to construct our partition $\mathscr{S}$.

Step 3. Show that $\mathscr{S}$ satisfies conditions 11.5*(a)*-*(c)*.

We continue with **Step 1.** Recall that δ_{k+1}, μ, $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$, $\delta_i = \delta_i(d_i, \dots, d_k)$, where $i = 2, \dots, k$, and $r = r(t, d_2, \dots, d_k)$ as in Theorem 7.14 are given. Moreover, $\boldsymbol{\pi} = (\pi_2, \dots, \pi_k)$ is a vector of positive real numbers such that $\mathscr{P}$ is an equitable $(\mu, \boldsymbol{\delta}(\boldsymbol{\pi}), \boldsymbol{\pi}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-partition.

Set

$$\rho_k = \mu/\big(8s(k+1)^2\big), \tag{11.6a}$$

and define

$$\delta'_k = \min\big\{\delta_k(\rho_k)/3, \mu/\big(8s(k+1)^2\big), \mu^2/\big(512s^2(k+1)^4\big)\big\}, \tag{11.6b}$$

$$\mu' = \min\big\{48{\delta'_k}^4/s, \mu/\big(8(k+1)^2\big)\big\}, \tag{11.6c}$$

be reals between 0 and 1. For $i = 2, 3, \ldots, k-1$, we also define the following functions (in variables $d_2, \ldots, d_{k-1}$):

$$\delta'_i(d_i, \ldots, d_{k-1}) = \delta_i(d_i, \ldots, d_{k-1}, \rho_k), \tag{11.6d}$$

and

$$r'(t, d_2, \ldots, d_{k-1}) = r(t, d_2, \ldots, d_{k-1}, \rho_k). \tag{11.6e}$$

Moreover, set $\boldsymbol{\delta}' = (\delta'_2, \ldots, \delta'_{k-1})$.

Applying Lemma 10.1 (i.e. the induction assumption **Regularity**$(\boldsymbol{k-1})$) with these choices of parameters to $\mathcal{G}_1^{(k)}, \mathcal{G}_2^{(k)}, \ldots, \mathcal{G}_s^{(k)}$, we obtain a partition $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}^{\mathscr{R}}, \boldsymbol{\psi}^{\mathscr{R}})$ such that for some vector $\boldsymbol{\rho} = (\rho_2, \ldots, \rho_{k-1}) \in (0,1]^{k-1}$,

$$\begin{aligned}
&\text{(i) } \mathscr{R} \text{ is an equitable } (\mu', \boldsymbol{\delta}'(\boldsymbol{\rho}), \boldsymbol{\rho}, r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))\text{-partition,}\\
&\text{(ii) } \mathscr{R} \text{ is } (\delta'_k, r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))\text{-regular with respect to every } \mathcal{G}_i^{(k)},\ i \in [s], \text{ and}\\
&\text{(iii) } \operatorname{rank}(\mathscr{R}) = |A(k-1, \boldsymbol{a}^{\mathscr{R}})| \le L'_k.
\end{aligned} \tag{11.7}$$

For **Step 2**, we will now extend partition $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}^{\mathscr{R}}, \boldsymbol{\psi}^{\mathscr{R}})$ to a partition $\mathscr{S}$ of k-tuples . For each $\xi \in [s]$, and every (2^k-2)-dimensional vector $\hat{\boldsymbol{y}} \in \hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}})$, we define (k,k)-cylinders $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})$, by

$$\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) = \begin{cases} \mathcal{G}_\xi^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})) & \text{if } \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})) \subset \mathrm{Cross}_k(\psi_1^{\mathscr{P}}),\\ \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})) & \text{otherwise.} \end{cases} \tag{11.8a}$$

Note that if $\mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})) \not\subset \mathrm{Cross}_k(\psi_1^{\mathscr{P}})$, then $\mathcal{S}^{(k)}(1, \hat{\boldsymbol{y}}) = \ldots = \mathcal{S}^{(k)}(s, \hat{\boldsymbol{y}})$. This may seem artificial, but we find it convenient to define it this way.

For each $i \in [k-1]$, we also define (k,i)-cylinders $\mathcal{S}^{(i)}(\xi, \hat{\boldsymbol{y}})$, by

$$\mathcal{S}^{(i)}(\xi, \hat{\boldsymbol{y}}) = \hat{\mathcal{R}}^{(i)}(\hat{\boldsymbol{y}}). \tag{11.8b}$$

Since we have $\mathrm{Cross}_k(\psi_1^{\mathscr{P}}) \subseteq \mathrm{Cross}_k(\psi_1^{\mathscr{R}})$ by Remark 7.19 and $\mathscr{G} = \{\mathcal{G}_1^{(k)}, \mathcal{G}_2^{(k)}, \ldots, \mathcal{G}_s^{(k)}\}$ is a partition of $\mathrm{Cross}_k(\psi_1^{\mathscr{P}})$, we obtain

$$\mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})) = \bigcup_{\xi=1}^{s} \mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) \tag{11.9}$$

for every $\hat{\boldsymbol{y}} \in \hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}})$.

Combining (11.9) with the fact that $\{\mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}}))\colon \hat{\boldsymbol{y}} \in \hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}})\}$ is a partition of $\mathrm{Cross}_k(\psi_1^{\mathscr{R}})$, and Claim 4.9 yields

$$\mathscr{S} = \{\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})\colon \hat{\boldsymbol{y}} \in \hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}}), \xi \in [s]\} \tag{11.10}$$

is a partition of $\mathrm{Cross}_k(\psi_1^{\mathscr{R}})$. Furthermore, $\mathscr{S}$ refines $\mathscr{P}$, that is, each $\mathcal{P} \in \mathscr{P}$ is a disjoint union of members of $\mathscr{S}$:

$$\mathcal{P} = \bigcup_{\mathcal{S} \subset \mathcal{P},\ \mathcal{S} \in \mathscr{S}} \mathcal{S}. \tag{11.11}$$

Now we establish **Step 3** by showing that $\mathscr{S}$ satisfies (11.5)*(a)*-*(c)*.

Proof of (11.5)*(a).* We will show that the following vector $\boldsymbol{\sigma}$ satisfies requirements of (11.5)*(a)*: $\boldsymbol{\sigma} = (\boldsymbol{\rho}, \rho_k) = (\rho_2, \dots, \rho_{k-1}, \rho_k)$, where ρ_k is given by (11.6a) and $(\rho_2, \dots, \rho_{k-1}) = \boldsymbol{\rho}$ comes from (11.7).

Set $a_1^{\mathscr{S}} = a_1^{\mathscr{R}}$, $\boldsymbol{\sigma}^i = (\rho_i, \dots, \rho_k)$, $\boldsymbol{\rho}^i = (\rho_i, \dots, \rho_{k-1})$ for $i = 2, \dots, k-1$, and $\boldsymbol{\sigma}^k = (\rho_k)$. Observe that $r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}) = r(a_1^{\mathscr{S}}, \boldsymbol{\sigma})$, $\delta_k' \le \delta_k(\rho_k) = \delta_k(\boldsymbol{\sigma}^k)$, and $\delta_i'(\boldsymbol{\rho}^i) = \delta_i(\boldsymbol{\sigma}^i)$ for $i = 2, \dots, k-1$.

We call a $(k+1)$-tuple of vertices K *bad* if one of the following cases occurs:

(1) K is not crossing.

(2) There exists a $(k+1, k)$-complex $\hat{\boldsymbol{\mathcal{S}}} = \{\hat{\mathcal{S}}^{(1)}, \hat{\mathcal{S}}^{(2)}, \dots, \hat{\mathcal{S}}^{(k)}\}$, where $\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}$, such that either

- *(2a)* K belongs[5] to a $(\delta_i(\boldsymbol{\sigma}^i), \rho_i, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-irregular $(k+1, i)$-cylinder $\hat{\mathcal{S}}^{(i)}$ for some $i \in \{2, \dots, k-1\}$, or
- *(2b)* K belongs to a $(\delta_k(\boldsymbol{\sigma}^k), r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-irregular $(k+1, k)$-cylinder $\hat{\mathcal{S}}^{(k)}$, or
- *(2c)* K belongs to a $(\delta_k(\boldsymbol{\sigma}^k), \rho, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular $(k+1, k)$-cylinder $\hat{\mathcal{S}}^{(k)}$ with $\rho < \rho_k$.

In view of (11.5)*(a)*, we need to show that at most $(\mu/2)\binom{n}{k+1}$ many $(k+1)$-tuples are bad. Now we estimate the number of $(k+1)$-tuples in *(1)* and *(2a)*-*(2c)*.

First, we estimate the number of $(k+1)$-tuples in *(1)* and *(2a)*. If a $(k+1)$-tuple K is not crossing, then it contains a k-tuple $K' \in [K]^k$ that is not crossing. Also, if K belongs to a $(\delta_i(\boldsymbol{\sigma}^i), \rho_i, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-irregular $(k+1, i)$-cylinder $\hat{\mathcal{S}}^{(i)}$ for some $i \in \{2, \dots, k-1\}$, then, in view of (11.8a), it must contain a k-tuple K' which belongs to some $(\delta_i'(\boldsymbol{\rho}^i), \rho_i, r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-irregular (k, i)-cylinder $\hat{\mathcal{R}}^{(i)}(\hat{\boldsymbol{y}}) \in \hat{\mathscr{R}}$.

Since $\mathscr{R}$ is an equitable $(\mu', \boldsymbol{\delta}'(\boldsymbol{\rho}), \boldsymbol{\rho}, r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-partition (cf. 11.7*(i)*) the number of k-tuples K' satisfying either of the above two properties is at most $\mu'\binom{n}{k}$. We have $\mu' \le \mu/\big(8(k+1)^2\big)$ (cf. (11.6c)). Therefore, the number of

[5] K belongs to $\hat{\mathcal{S}}^{(i)}$ if it induces a clique in $\hat{\mathcal{S}}^{(i)}$.

$(k+1)$-tuples K satisfying *(1)* or *(2a)* is at most

$$\mu/\big(8(k+1)^2\big)\binom{n}{k}\times n \le \mu/\big(4(k+1)\big)\binom{n}{k+1}. \tag{11.12}$$

Second, we estimate the number of $(k+1)$-tuples K to which *(2b)* applies. Assume that K belongs to $(k+1,k)$-cylinder $\hat{\mathcal{S}}^{(k)}$ which is $(\delta_k(\boldsymbol{\sigma}^k), r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-irregular.

This means (see Definition 3.11) that one of the $k+1$ (k,k)-subcylinders of $\hat{\mathcal{S}}^{(k)}$ (say $\mathcal{S}_{\mathrm{irreg}}$) is $(\delta_k(\boldsymbol{\sigma}^k), r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-irregular.

By (11.8a) and (11.10), there exist $\xi \in [s]$ and $\hat{\boldsymbol{y}} \in \hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}})$ such that

$$\mathcal{S}_{\mathrm{irreg}} = \mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) = \mathcal{G}_\xi^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})).$$

Hence, $\mathcal{G}_\xi^{(k)}$ is $(\delta_k', r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-irregular with respect to $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})$. This means, however, that $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}})$ violates condition (b) of Definition 7.8. Moreover, K contains a k-tuple $K' \in \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}}))$.

Since $\mathscr{R}$ is $(\delta_k', r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-regular with respect to all $\mathcal{G}_i^{(k)}$ (see 11.7*(ii)* and Definition 7.8), the number of such k-tuples K' is at most $s \times \delta_k' \times \binom{n}{k}$. Thus, the number of $(k+1)$-tuples K in this category is bounded by $s \times \delta_k' \times \binom{n}{k} \times n$.

Now we estimate the number of $(k+1)$-tuples satisfying *(2c)*. If K belongs to a $(k+1,k)$-cylinder $\hat{\mathcal{S}}^{(k)}$ that is $(\delta_k(\boldsymbol{\sigma}^k), \rho, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with $\rho < \rho_k$, then the number of such $(k+1)$-tuples is at most $\rho_k \times \binom{n}{k} \times n$.

Using (11.6a) and (11.6b), we obtain that the number of $(k+1)$-tuples satisfying *(2b) or (2c)* is at most

$$s \times \delta_k' \times \binom{n}{k} \times n + \rho_k \times \binom{n}{k} \times n \le \mu/\big(4(k+1)\big)\binom{n}{k+1}. \tag{11.13}$$

Combining (11.12) and (11.13) yields that at most

$$\mu/\big(4(k+1)\big)\binom{n}{k+1} + \mu/\big(4(k+1)\big)\binom{n}{k+1} \le (\mu/2)\binom{n}{k+1}$$

many $(k+1)$-tuples K satisfy either one of *(1)*, *(2a)*-*(2c)*. □

Proof of (11.5)*(b).* It follows from (11.10) that $\mathrm{rank}(\mathscr{S}) \le |\hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}})| \times s$. We know that $s \le |A(k, \boldsymbol{a}^{\mathscr{P}})| \times 2^{r(a_1^{\mathscr{P}}, \boldsymbol{\pi}) \times |A(k, \boldsymbol{a}^{\mathscr{P}})|^k}$ (see (11.4)). Moreover, by the induction assumption (see 11.7*(iii)*), we have $|A(k-1, \boldsymbol{a}^{\mathscr{R}})| \le L_k'$, where L_k' depends only on $\mathcal{G}_1^{(k)}, \dots, \mathcal{G}_s^{(k)}$ (i.e. on $\mathscr{P}$), $\boldsymbol{\delta}$, and r. Consequently,

$$|\hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}})| \le |A(k-1, \boldsymbol{a}^{\mathscr{R}})|^k \le {L_k'}^k$$

and $\mathrm{rank}(\mathscr{S}) \le f(\mathrm{rank}(\mathscr{P}), \delta_{k+1}, \boldsymbol{\delta}, r)$. □

Proof of (11.5)*(c).* Let $\hat{\mathscr{P}}_{\mathrm{reg}}$ be the set of all $(\delta_{k+1}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-regular polyads in $\hat{\mathscr{P}}$, $\hat{\mathscr{P}}_{\mathrm{a}}$ be the set of all polyads $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ violating (a) in Definition 7.8, and let $\hat{\mathscr{P}}_{\mathrm{b}}$ be the set of all polyads $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}) \in \hat{\mathscr{P}}$ which satisfy (a) and violate (b) in Definition 7.8. Observe that $\hat{\mathscr{P}} = \hat{\mathscr{P}}_{\mathrm{reg}} \cup \hat{\mathscr{P}}_{\mathrm{a}} \cup \hat{\mathscr{P}}_{\mathrm{b}}$ and this union is disjoint.

Then,

$$\operatorname{ind} \mathscr{P} = \xi_1 + \xi_2,$$

where

$$\xi_1 = \sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{reg}} \cup \hat{\mathscr{P}}_{\mathrm{a}}} \operatorname{Vol}(\hat{\mathcal{P}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)})$$

and

$$\xi_2 = \sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{b}}} \operatorname{Vol}(\hat{\mathcal{P}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}).$$

For any polyad $\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}$ with $|\mathcal{K}_{k+1}(\hat{\mathcal{P}}^{(k)})| > 0$, and for any polyad $\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}$ such that $\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}$, we set

$$\sigma_{\hat{\mathcal{S}}^{(k)}} = \frac{|\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})|}{|\mathcal{K}_{k+1}(\hat{\mathcal{P}}^{(k)})|} = \frac{\operatorname{Vol}(\hat{\mathcal{S}}^{(k)})}{\operatorname{Vol}(\hat{\mathcal{P}}^{(k)})}.$$

Since partition $\mathscr{S}$ refines $\mathscr{P}$ (see (11.11), for each polyad $\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}$, we have

$$\mathcal{K}_{k+1}(\hat{\mathcal{P}}^{(k)}) = \bigcup_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})$$

and this union is disjoint. Consequently, we have

$$1 = \sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \sigma_{\hat{\mathcal{S}}^{(k)}} \tag{11.14a}$$

and

$$d_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}) = \sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \sigma_{\hat{\mathcal{S}}^{(k)}} d_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}). \tag{11.14b}$$

Combining (11.14a), (11.14b), and the Cauchy-Schwarz inequality yields

$$d^2_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}) \le \sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \sigma_{\hat{\mathcal{S}}^{(k)}} d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) \tag{11.15}$$

for every polyad $\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}$. We use this to estimate ξ_1. Indeed, we use (11.15) and $\sigma_{\hat{\mathcal{S}}^{(k)}} = \operatorname{Vol}(\hat{\mathcal{S}}^{(k)}) / \operatorname{Vol}(\hat{\mathcal{P}}^{(k)})$ to conclude that

$$\begin{aligned} \xi_1 &\overset{(11.15)}{\le} \sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{reg}} \cup \hat{\mathscr{P}}_{\mathrm{a}}} \operatorname{Vol}(\hat{\mathcal{P}}^{(k)}) \sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \sigma_{\hat{\mathcal{S}}^{(k)}} d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) \\ &= \sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{reg}} \cup \hat{\mathscr{P}}_{\mathrm{a}}} \sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \operatorname{Vol}(\hat{\mathcal{S}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}). \end{aligned} \tag{11.16}$$

Let $\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$ be any polyad in $\hat{\mathscr{P}}_{\mathrm{b}}$ and let I be the set of all polyads $\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})$. Recall that $\tilde{Q}(\hat{\boldsymbol{x}}) = \{\mathcal{Q}_1^{(k)}(\hat{\boldsymbol{x}}), \dots, \mathcal{Q}_{r(a_1^{\mathscr{P}}, \boldsymbol{\pi})}^{(k)}(\hat{\boldsymbol{x}})\}$ is an $r(a_1^{\mathscr{P}}, \boldsymbol{\pi})$-tuple of witnesses of irregularity, that is, $(k+1, k)$-cylinders satisfying (11.1a) and (11.1b). Denote by J the set of all polyads $\hat{\mathcal{S}}^{(k)}$ so that $\hat{\mathcal{S}}^{(k)} \subset \mathcal{Q}_i^{(k)}(\hat{\boldsymbol{x}})$ for some $i \in [r(a_1^{\mathscr{P}}, \boldsymbol{\pi})]$, and set

$$\begin{aligned} \sigma &= \sum_{\hat{\mathcal{S}}^{(k)} \in J} \sigma_{\hat{\mathcal{S}}^{(k)}}, \\ \nu &= d_{\mathcal{H}}(\tilde{Q}(\hat{\boldsymbol{x}})) - d_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})). \end{aligned}$$

Then,

$$\sigma = \sum_{\hat{\mathcal{S}}^{(k)} \in J} \sigma_{\hat{\mathcal{S}}^{(k)}} = \frac{|\bigcup_{i=1}^{r(a_1^{\mathscr{P}}, \boldsymbol{\pi})} \mathcal{K}_{k+1}(\mathcal{Q}_i^{(k)}(\hat{\boldsymbol{x}}))|}{|\mathcal{K}_{k+1}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}}))|} \overset{(11.1a)}{\geq} \delta_{k+1} \tag{11.17}$$

and

$$\begin{aligned} \sum_{\hat{\mathcal{S}}^{(k)} \in J} \sigma_{\hat{\mathcal{S}}^{(k)}} d_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) &= \sigma \sum_{i=1}^{r(a_1^{\mathscr{P}}, \boldsymbol{\pi})} \sum \left\{ \frac{\sigma_{\hat{\mathcal{S}}^{(k)}}}{\sigma} d_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) \colon \hat{\mathcal{S}}^{(k)} \subset \mathcal{Q}_i^{(k)}(\hat{\boldsymbol{x}}) \right\} \\ &= \sigma d_{\mathcal{H}}(\tilde{Q}(\hat{\boldsymbol{x}})) = \sigma \left(d_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})) + \nu \right). \end{aligned} \tag{11.18}$$

Moreover,

$$|\nu| = \left| d_{\mathcal{H}}(\tilde{Q}(\hat{\boldsymbol{x}})) - d_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}(\hat{\boldsymbol{x}})) \right| \overset{(11.1b)}{>} \delta_{k+1}. \tag{11.19}$$

Thus, by Fact 10.4 applied with parameters I, J, σ, ν defined above and $\delta = \delta_{k+1}$, we obtain

$$\sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \sigma_{\hat{\mathcal{S}}^{(k)}} d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) \geq d^2_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}) + \delta^3_{k+1}. \tag{11.20}$$

for every polyad $\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{b}}$. We remark that (11.14a), (11.14b), (11.17), (11.18), and (11.19) verify the assumptions of Fact 10.4.

Now we use this to estimate ξ_2. Indeed, observe first that

$$\sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{b}}} \mathrm{Vol}(\hat{\mathcal{P}}^{(k)}) > \delta_{k+1} - \mu \geq \delta_{k+1}/2 \tag{11.21}$$

because at most $(1 - \delta_{k+1})\binom{n}{k+1}$ many $(k+1)$-tuples are in $(\delta_{k+1}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-regular polyads and at most $\mu\binom{n}{k+1}$ many $(k+1)$-tuples are either not crossing or not in $(\boldsymbol{\delta}(\boldsymbol{\pi}), \boldsymbol{\pi}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-regular $(k+1, k)$-complexes $\hat{\boldsymbol{\mathcal{P}}}$. Combining this with (11.20) yields

$$\xi_2 = \sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{b}}} \mathrm{Vol}(\hat{\mathcal{P}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{P}}^{(k)}) \overset{(11.20)}{\leq}$$

$$\sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{b}}} \mathrm{Vol}(\hat{\mathcal{P}}^{(k)}) \left(\sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \sigma_{\hat{\mathcal{S}}^{(k)}} d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) - \delta^3_{k+1} \right)$$

$$\leq \sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}_{\mathrm{b}}} \sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \mathrm{Vol}(\hat{\mathcal{S}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) - \delta^4_{k+1}/2. \tag{11.22}$$

We put equations (11.16) and (11.22) together and obtain

$$\mathrm{ind}\,\mathscr{S} = \sum_{\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}} \mathrm{Vol}(\hat{\mathcal{S}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) \geq \sum_{\hat{\mathcal{P}}^{(k)} \in \hat{\mathscr{P}}} \sum_{\hat{\mathcal{S}}^{(k)} \subset \hat{\mathcal{P}}^{(k)}} \mathrm{Vol}(\hat{\mathcal{S}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)})$$
$$\geq \xi_1 + \xi_2 + \delta^4_{k+1}/2 = \mathrm{ind}\,\mathscr{P} + \delta^4_{k+1}/2.$$

□

Observe that if we could show that $\mathscr{S}$ is an equitable $(\mu/2, \boldsymbol{\delta}, r)$-partition instead of an almost equitable $(\mu/2, \boldsymbol{\delta}, r)$-partition, then $\mathscr{S}$ would be a partition we are looking for. Note that the only difference would be to prove in (11.5)(a)

- $\mathcal{S}^{(k)}$ is $(\delta_k(\boldsymbol{\sigma}^k), \sigma_k, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(k-1)}$;

instead of

- $\mathcal{S}^{(k)}$ is $(\delta_k(\boldsymbol{\sigma}^k), \rho, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(k-1)}$ and $\rho \geq \sigma_k$

However, we are not able to prove this and, therefore, in the remaining part of the proof we will modify partition $\mathscr{S}$ into a functionally equitable $(\mu, \boldsymbol{\delta}, r)$-partition $\mathscr{T}$.

To this end, we will use the Slicing lemma (with appropriately chosen p) which enables to decompose each sufficiently dense (δ, α, r)-regular cylinder into $\lfloor 1/p \rfloor$ $(3\delta, \alpha p, r)$-regular cylinders. We apply this lemma to every $\mathcal{S}^{(k)} \in \mathscr{S}$ with relative density $\rho \geq \sigma_k$ and divide it into (k,k)-cylinders with the same density.

We need to verify that the rank of a new partition $\mathscr{T}$ will not increase by much and its index will not decrease.

Now we provide details of this construction. We call $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) \in \mathscr{S}$ *good* if it satisfies the following:

(S1) $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})$ is $(\delta'_k, \rho, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(k-1)}(\xi, \hat{\boldsymbol{y}})$ and $\rho = \rho(\xi, \hat{\boldsymbol{y}}) \geq \rho_k$,

(S2) $\mathcal{S}^{(i)}(\xi, \hat{\boldsymbol{y}}) = \hat{\mathcal{R}}^{(i)}(\hat{\boldsymbol{y}})$ is $(\delta_i(\boldsymbol{\sigma}^i), \rho_i, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(i-1)}(\xi, \hat{\boldsymbol{y}}) = \hat{\mathcal{R}}^{(i)}(\hat{\boldsymbol{y}})$ for $i = 2, \dots, k-1$, and

(S3) $\left|\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})\right| \geq (n/a_1)^k / \ln(n/a_1)$.

Otherwise, we call $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})$ *bad.* Denote by $\mathscr{S}_{\mathrm{good}}$ the set of all good (k,k)-cylinders $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) \in \mathscr{S}$ and let $\mathscr{S}_{\mathrm{bad}} = \mathscr{S} \setminus \mathscr{S}_{\mathrm{good}}$. Observe the following:

- $\mu' \binom{n}{k}$ bounds the size of the union of those (k,k)-cylinders $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})$, for which (S2) does not hold (recall $\mathscr{R}$ is an equitable $(\mu', \boldsymbol{\delta}'(\boldsymbol{\rho}), \boldsymbol{\rho}, r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-partition and $a_1^{\mathscr{S}} = a_1^{\mathscr{R}}$);
- $s \times \delta_k' \binom{n}{k} + s \times \rho_k \binom{n}{k}$ estimates the size of the union of those $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})$ for which (S1) does not hold: $s \times \delta_k' \binom{n}{k}$ is for the size of all $(\delta_k', \rho, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-irregular $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) = \mathcal{G}_\xi^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{y}}))$ ($\mathscr{R}$ is $(\delta_k', r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-regular with respect to all $\mathcal{G}_i^{(k)}$) and $s \times \rho_k \binom{n}{k}$ for ones with $\rho = \rho(\xi, \hat{\boldsymbol{y}}) < \rho_k$;
- $\operatorname{rank}(\mathscr{S}) \times (n/a_1)^k / \ln(n/a_1)$ estimates the size of the union of those $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})$ that are violating (S3).

Subsequently, for sufficiently large n, we obtain

$$\sum \left\{ |\mathcal{S}^{(k)}| \colon \mathcal{S}^{(k)} \in \mathscr{S}_{\mathrm{bad}} \right\}$$
$$\leq \mu' \binom{n}{k} + s \times \delta_k' \binom{n}{k} + s \times \rho_k \binom{n}{k} + \operatorname{rank}(\mathscr{S}) \times \frac{(n/a_1)^k}{\ln(n/a_1)}$$
$$\overset{(11.6a)-(11.6c)}{\leq} \frac{\mu}{2(k+1)^2} \binom{n}{k}. \tag{11.23}$$

Now we are going to define a new partition $\mathscr{T}$ as follows:

(α) First observe that every $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) \in \mathscr{S}_{\mathrm{good}}$ is $(\delta_k', \rho, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(k-1)}(\xi, \hat{\boldsymbol{y}})$, where $\rho = \rho(\xi, \hat{\boldsymbol{y}}) \geq \rho_k$, and

$$\left| \mathcal{K}_k(\mathcal{S}^{(k-1)}(\xi, \hat{\boldsymbol{y}})) \right| \geq \left| \mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) \right| \geq (n/a_1)^k / \ln(n/a_1).$$

Then we use the Slicing lemma with $\mathcal{H}$ replaced by $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}})$, $\mathcal{G}$ by $\mathcal{S}^{(k-1)}(\xi, \hat{\boldsymbol{y}})$, and r with $r(a_1^{\mathscr{S}}, \boldsymbol{\sigma})$, and with parameters

$$m = \frac{n}{a_1^{\mathscr{S}}}, \tag{11.24a}$$
$$\delta = \delta_k', \tag{11.24b}$$
$$\alpha = \rho(\xi, \hat{\boldsymbol{y}}), \tag{11.24c}$$
$$p = p(\xi, \hat{\boldsymbol{y}}) = \frac{\mu}{8s(k+1)^2 \rho(\xi, \hat{\boldsymbol{y}})} = \frac{\rho_k}{\rho(\xi, \hat{\boldsymbol{y}})} \leq 1, \tag{11.24d}$$
$$u = u(\xi, \hat{\boldsymbol{y}}) = \lfloor 1/p(\xi, \hat{\boldsymbol{y}}) \rfloor \leq \frac{8s(k+1)^2}{\mu}. \tag{11.24e}$$

This yields (k,k)-cylinders $\mathcal{T}^{(k)}(i, \xi, \hat{\boldsymbol{y}})$, $i = 0, 1, \dots u(\xi, \hat{\boldsymbol{y}})$, satisfying

(T1) $\mathcal{S}^{(k)}(\xi, \hat{\boldsymbol{y}}) = \bigcup_{i=0}^{u(\xi, \hat{\boldsymbol{y}})} \mathcal{T}^{(k)}(i, \xi, \hat{\boldsymbol{y}})$ and this union is disjoint,

(T2) $\mathcal{T}^{(k)}(i, \xi, \hat{\boldsymbol{y}})$ is $(3\delta_k', \rho_k, r(a_1^{\mathscr{S}}, \boldsymbol{\sigma}))$-regular w.r.t. $\mathcal{S}^{(k-1)}(\xi, \hat{\boldsymbol{y}})$ for every $i = 1, \dots, u(\xi, \hat{\boldsymbol{y}})$, and

(T3)

$$|\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})| \le (p\alpha+4u\delta)\big|\mathcal{K}_k(\mathcal{S}^{(k-1)}(\xi,\hat{\boldsymbol{y}}))\big| \le \frac{\mu}{4s(k+1)^2}|\mathcal{K}_k(\mathcal{S}^{(k-1)}(\xi,\hat{\boldsymbol{y}}))|.$$

(β) For every $\mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}}) \in \mathscr{S}_{\mathrm{bad}}$, we set

$$\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}}) = \mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}}). \tag{11.25}$$

Then we define $\mathscr{T}$ by

$$\mathscr{T} = \big\{\mathcal{T}^{(k)}(i,\xi,\hat{\boldsymbol{y}})\colon \mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}}) \in \mathscr{S}_{\mathrm{good}}, i\in[u(\xi,\hat{\boldsymbol{y}})]\big\} \\ \cup \big\{\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})\colon \hat{\boldsymbol{y}} \in \hat{A}(k-1,\boldsymbol{a}^{\mathscr{R}}), \xi\in[s]\big\}. \tag{11.26}$$

We need to show that $\mathscr{T}$ is a partition we are looking for. We accomplish this by proving

(A): $\mathscr{T}$ is a functionally equitable $(\mu,\boldsymbol{\delta},r)$-partition,
(B): $\operatorname{rank}(\mathscr{T}) \le \tilde{f}(\operatorname{rank}(\mathscr{P}),\delta_{k+1},\boldsymbol{\delta},r)$, and
(C): $\operatorname{ind}\mathscr{T} \ge \operatorname{ind}\mathscr{P} + \delta_{k+1}^4/2$.

Since parts *(B)* and *(C)* are easier to prove than *(A)*, we start with them.
Part *(B)*: It follows from (11.26) and (11.24e) that

$$\begin{aligned}\operatorname{rank}(\mathscr{T}) &\le \frac{8s(k+1)^2}{\mu}\times\operatorname{rank}(\mathscr{S}) + s\times\operatorname{rank}(\mathscr{S})\\ &\overset{(11.5)(b)}{\le} \left(\frac{8s(k+1)^2}{\mu}+s\right)\times f(\operatorname{rank}(\mathscr{P}),\delta_{k+1},\boldsymbol{\delta},r)\\ &= \tilde{f}(\operatorname{rank}(\mathscr{P}),\delta_{k+1},\boldsymbol{\delta},r).\end{aligned}$$

Part *(C)*: (T1) and (11.25) shows that $\mathscr{T}$ refines $\mathscr{S}$ (see Definition 4.8). Hence, applying Lemma 10.3 yields

$$\operatorname{ind}\mathscr{T} \ge \operatorname{ind}\mathscr{S} \overset{(11.5)(c)}{\ge} \operatorname{ind}\mathscr{P} + \frac{\delta_{k+1}^4}{2}.$$

Now we prove part *(A)*: For $a_1^{\mathscr{T}} = a_1^{\mathscr{S}}$ and $\boldsymbol{\sigma} = (\rho_2,\dots,\rho_k) = (\boldsymbol{\rho},\rho_k)$, we prove that

$$\mathscr{T} \text{ is an equitable } (\mu,\boldsymbol{\delta}(\boldsymbol{\sigma}),\boldsymbol{\sigma},r(a_1^{\mathscr{T}},\boldsymbol{\sigma}))\text{-partition.} \tag{11.27}$$

Then, by Definition 7.11, partition $\mathscr{T}$ is a functionally equitable $(\mu,\boldsymbol{\delta},r)$-partition. By Definition 7.3, all what remains to show is

> all but at most $\mu\binom{n}{k+1}$ $(k+1)$-tuples $K\in[V]^{k+1}$ belong to $(\boldsymbol{\delta}(\boldsymbol{\sigma}),\boldsymbol{\sigma},r(a_1^{\mathscr{T}},\boldsymbol{\sigma}))$-regular complexes.

For the following we recall that $\boldsymbol{\sigma}^i = (\rho_i,\dots,\rho_k)$ for $2\le i\le k$. Now, let $\mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}}) \in \mathscr{S}_{\mathrm{good}}$. We show then that

$$\{\mathcal{S}^{(1)}(\xi,\hat{\boldsymbol{y}}),\dots,\mathcal{S}^{(k-1)}(\xi,\hat{\boldsymbol{y}}),\mathcal{T}^{(k)}(i,\xi,\hat{\boldsymbol{y}})\}$$

is a $(\boldsymbol{\delta}(\boldsymbol{\sigma}),\boldsymbol{\sigma},r(a_1^{\mathscr{T}},\boldsymbol{\sigma}))$-regular complex for all $i\in[u(\xi,\hat{\boldsymbol{y}})]$. Indeed,

• $\mathcal{S}^{(2)}(\xi,\hat{\boldsymbol{y}})$ is $(\delta_2(\boldsymbol{\sigma}^2),\rho_2)$-regular with respect to $\mathcal{S}^{(1)}(\xi,\hat{\boldsymbol{y}})$ because of (S2),

• For $i=3,\dots,k-1$, $\mathcal{S}^{(i)}(\xi,\hat{\boldsymbol{y}})$ is $(\delta_i(\boldsymbol{\sigma}^i),\rho_i,r(a_1^{\mathscr{T}},\boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(i-1)}(\xi,\hat{\boldsymbol{y}})$ because of (S2),

• $\mathcal{T}^{(k)}(i,\xi,\hat{\boldsymbol{y}})$ is $(\delta_k(\rho_k),\rho_k,r(a_1^{\mathscr{S}},\boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(k-1)}(\xi,\hat{\boldsymbol{y}})$ because

– $\mathcal{T}^{(k)}(i,\xi,\hat{\boldsymbol{y}})$ is $(3\delta'_k,\rho_k,r(a_1^{\mathscr{S}},\boldsymbol{\sigma}))$-regular with respect to $\mathcal{S}^{(k-1)}(\xi,\hat{\boldsymbol{y}})$ (cf. (T2)), and

– $\delta'_k\le\delta_k(\rho_k)/3$ (cf. (11.6b)).

Denote by $\mathscr{T}_{\text{good}}$ the set of all $\mathcal{T}^{(k)}(i,\xi,\hat{\boldsymbol{y}})\in\mathscr{T}$, $i\in[u(\xi,\hat{\boldsymbol{y}})]$, such that $\mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}})\in\mathscr{S}_{\text{good}}$. Furthermore, set $\mathscr{T}_{\text{bad}}=\mathscr{T}\setminus\mathscr{T}_{\text{good}}$ and let $\hat{\mathscr{T}}_{\text{good}}$ be the set of polyads $\hat{\mathcal{T}}^{(k)}\in\hat{\mathscr{T}}$ which consists only of elements from $\mathscr{T}_{\text{good}}$.

In other words, every $\hat{\mathcal{T}}^{(k)}\in\hat{\mathscr{T}}_{\text{good}}$ belongs to a $(\boldsymbol{\delta}(\boldsymbol{\sigma}),\boldsymbol{\sigma},r(a_1^{\mathscr{T}},\boldsymbol{\sigma}))$-regular $(k+1,k)$-complex. Hence, we must prove

$$\sum\Big\{\operatorname{Vol}(\hat{\mathcal{T}}^{(k)})\colon\hat{\mathcal{T}}^{(k)}\in\hat{\mathscr{T}}_{\text{good}}\Big\}>1-\mu.$$

If $\hat{\mathcal{T}}^{(k)}\notin\hat{\mathscr{T}}_{\text{good}}$, then it must contain a (k,k)-cylinder $\mathcal{T}^{(k)}\in\mathscr{T}_{\text{bad}}$. This means however, that $\mathcal{T}^{(k)}=\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})$ for some $\hat{\boldsymbol{y}}\in\hat{A}(k-1,\boldsymbol{a}^{\mathscr{R}})$ and $\xi\in[s]$. By the definitions of $\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})$ (see $(\alpha),(\beta)$), we have

$$\begin{aligned}\sum\big\{|\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})|\colon\xi\in[s],\hat{\boldsymbol{y}}\in\hat{A}(k-1,\boldsymbol{a}^{\mathscr{R}})\big\}&\\ \le\sum\big\{|\mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}})|\colon\mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}})\in\mathscr{S}_{\text{bad}}\big\}&\\ +\sum\big\{|\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})|\colon\mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}})\in\mathscr{S}_{\text{good}}\big\}&.\end{aligned}$$

Then we use (11.23) and (T3) to conclude

$$\begin{aligned}\sum\big\{|\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})|\colon\xi\in[s],\hat{\boldsymbol{y}}\in\hat{A}(k-1,\boldsymbol{a}^{\mathscr{R}})\big\}\le\frac{\mu}{2(k+1)^2}\binom{n}{k}&\\ +\frac{\mu}{4s(k+1)^2}\sum\big\{|\mathcal{K}_k(\mathcal{S}^{(k-1)}(\xi,\hat{\boldsymbol{y}}))|\colon\xi\in[s],\hat{\boldsymbol{y}}\in\hat{A}(k-1,\boldsymbol{a}^{\mathscr{R}})\big\}&\\ \le\frac{\mu}{(k+1)^2}\binom{n}{k}.&\end{aligned}\tag{11.28}$$

The last inequality follows from the fact that $\{\mathcal{K}_k(\mathcal{S}^{(k-1)}(\xi,\hat{\boldsymbol{y}}))\colon\hat{\boldsymbol{y}}\in\hat{A}(k-1,\boldsymbol{a}^{\mathscr{R}})\}$ forms a partition of $\operatorname{Cross}_k(\psi_1^{\mathscr{R}})$ for every fixed $\xi\in[s]$. Therefore, by (11.28), we have

$$\begin{aligned}\sum\Big\{\operatorname{Vol}(\hat{\mathcal{T}}^{(k)})\colon\hat{\mathcal{T}}^{(k)}\notin\hat{\mathscr{T}}_{\text{good}}\Big\}&\\ \le\frac{n}{\binom{n}{k+1}}\times\sum\big\{|\mathcal{T}^{(k)}(0,\xi,\hat{\boldsymbol{y}})|\colon\xi\in[s],\hat{\boldsymbol{y}}\in\hat{A}(k-1,\boldsymbol{a}^{\mathscr{R}})\big\}&\\ \overset{(11.23)}{\le}\frac{n}{\binom{n}{k+1}}\times\frac{\mu}{(k+1)^2}\binom{n}{k}\le\mu.&\end{aligned}$$

$\square$

12. Concluding remarks

Definitions 7.11 and 7.12 describe the most important properties required from a partition $\mathscr{P}$ produced by the Regularity Lemma for k-uniform hypergraphs (Theorem 7.14).

For some applications of Szemerédi's Regularity Lemma, it turned out to be useful to have a version of this lemma that produces an ε-regular partition of vertices satisfying some additional conditions. As an example we mention Lemma 3.7 from [4] (see also Remark 7.19) in which a partition produced by Szemerédi's Regularity Lemma also refines a given initial partition of vertices.

Here we present a version of Theorem 7.14 in which we impose an additional "divisibility" condition on densities $d_2, \ldots, d_k$ in Definition 7.11 and we require $\mathscr{P}$ to "refine" an initial complex $\boldsymbol{\mathcal{G}}$. This modified regularity lemma is one of the key ingredients in the proof of the Counting Lemma in [10].

First, we need some additional notation. Suppose $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ is a partition of $\mathrm{Cross}_k(\psi_1)$ and $\{\mathscr{P}^{(j)}\}_{j=1}^{k}$ is a system of partitions associated with $\mathscr{P}$ (see Remark 4.5). For an (ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$, we say that $\mathscr{P}$ *respects* $\boldsymbol{\mathcal{G}}$ if for every $j \in [k]$ and every $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$ either $\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) \subseteq \mathcal{G}^{(j)}$ or $\mathcal{P}^{(j)}(\boldsymbol{x}^{(j)}) \cap \mathcal{G}^{(j)} = \emptyset$.

Then, our modified regularity lemma reads as follows:

Corollary 12.1. *Let $\ell \geq k \geq 2$ be arbitrary but fixed integers. Then for all positive numbers $\lambda_2, \ldots, \lambda_k, \delta_{k+1}$ and μ, and any non-negative functions $\delta_k(d_k)$, $\delta_{k-1}(d_{k-1}, d_k)$, $\ldots$, $\delta_2(d_2, \ldots, d_k)$, and $r = r(t, d_2, \ldots, d_k)$, there exist integers n_{k+1} and L_{k+1} such that the following holds.*

For every $(k+1)$-uniform hypergraph $\mathcal{H}$ and (ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ with common vertex set of size at least n_{k+1}, there exists a partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}, \boldsymbol{\psi})$ of $\mathrm{Cross}_k(\psi_1)$ and a vector $\boldsymbol{\pi} = (\pi_2, \ldots \pi_k)$, so that

$$
\begin{aligned}
&\text{(i) } \mathscr{P} \text{ is an equitable } (\mu, \boldsymbol{\delta}(\boldsymbol{\pi}), r(a_1, \boldsymbol{\pi}))\text{-partition,}\\
&\text{(ii) } \mathscr{P} \text{ is } (\delta_{k+1}, r(a_1, \boldsymbol{\pi}))\text{-regular with respect to } \mathcal{H},\\
&\text{(iii) } \mathrm{rank}(\mathscr{P}) = |A(k, \boldsymbol{a})| \leq L_{k+1},\\
&\text{(iv) } \mathscr{P} \text{ respects } \boldsymbol{\mathcal{G}}, \text{ and}\\
&\text{(v) } \lambda_j/\pi_j \text{ is an integer for } j = 2, \ldots, k.
\end{aligned}
\tag{12.29}
$$

Remark. The difference between Corollary 12.1 and Theorem 7.14 is that

(1) in Corollary 12.1, we have additional input parameters
   (a) numbers λ_j, $j = 2, \ldots, k$, and
   (b) an (ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$;

(2) in Corollary 12.1, we impose additional conditions (iv) and (v) on the output partition $\mathscr{P}$.

The proof of Corollary 12.1 closely follows the proof of Theorem 7.14. Here we point out only the differences between these proofs:

(D1) As the induction assumption we do not use Lemma 10.1 but the above corollary stated for $\mathcal{H}$ replaced by a family of hypergraphs $\mathcal{G}_1, \dots, \mathcal{G}_s$.

Lemma 12.2. *For all integers $s \geq 1$, $\ell \geq k \geq 2$ and all positive numbers $\lambda_2, \dots, \lambda_{k-1}$, δ'_k and μ', and any non-negative functions $\delta'_{k-1}(d_{k-1})$, $\delta'_{k-2}(d_{k-2}, d_{k-1})$, ..., $\delta'_2(d_2, \dots, d_{k-1})$, and $r' = r'(t, d_2, \dots, d_{k-1})$, there exist integers n'_k and L'_k such that the following holds.*

For all k-uniform hypergraphs $\mathcal{G}_1, \dots, \mathcal{G}_s$ and an $(\ell, k-1)$-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k-1}$ with common vertex set of size at least n_k, there exists a partition $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}^{\mathscr{R}}, \boldsymbol{\psi}^{\mathscr{R}})$ of $\mathrm{Cross}_{k-1}(\psi_1^{\mathscr{R}})$ and a vector $\boldsymbol{\rho} = (\rho_2, \dots \rho_{k-1})$, so that

- (i) *$\mathscr{R}$ is an equitable $(\mu', \boldsymbol{\delta}'(\boldsymbol{\rho}), r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-partition,*
- (ii) *$\mathscr{R}$ is $(\delta'_k, r'(a_1^{\mathscr{R}}, \boldsymbol{\rho}))$-regular (w.r.t. $\mathcal{H}$),*
- (iii) *$\mathrm{rank}(\mathscr{R}) = |A(k-1, \boldsymbol{a}^{\mathscr{R}})| \leq L'_k$,*
- (iv) *$\mathscr{R}$ respects $\boldsymbol{\mathcal{G}}$, and*
- (v) *λ_j/ρ_j is an integer for $j = 2, \dots, k-1$.*

(D2) In the proof of the Pumping Lemma, we start with an equitable $(\mu, \boldsymbol{\delta}(\boldsymbol{\pi}), \boldsymbol{\pi}, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-partition $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}^{\mathscr{P}}, \boldsymbol{\psi}^{\mathscr{P}})$ which is $(\delta_k, r(a_1^{\mathscr{P}}, \boldsymbol{\pi}))$-irregular and satisfies (12.29)(iii)-(v). Using the witnesses of irregularity (11.1a), (11.1b), we construct a system of k-uniform hypergraphs $\mathcal{G}_1, \dots, \mathcal{G}_s$ that form a partition of $\mathrm{Cross}_k(\psi_1^{\mathscr{P}})$ that refines $\mathscr{P}$.

This implies that every $\mathcal{G}_i$, $i \in [s]$, respects the given $\mathcal{G}^{(k)} \in \boldsymbol{\mathcal{G}}$, that is, either $\mathcal{G}_i \subset \mathcal{G}^{(k)}$ or $\mathcal{G}_i \cap \mathcal{G}^{(k)} = \emptyset$.

(D3) For **Step 1.**, we define ρ_k by

$$\mu/16s(k+1)^2 \leq \rho_k \leq \mu/8s(k+1)^2 \text{ and } \lambda_k/\rho_k \in \mathbb{N} \tag{12.30}$$

instead of (11.6a). Note that (12.30) is possible by setting $\rho_k = \lambda_k/L$, where L is an integer such that $\lambda_k/L \leq \mu/8s(k+1)^2 < \lambda_k/(L-1)$.

Then, instead of Lemma 10.1, we apply Lemma 12.2 with input parameters given by (12.30), (11.6b)-(11.6e), and with additional parameters $\lambda_2, \dots, \lambda_{k-1}$ and the $(\ell, k-1)$-complex $\{\mathcal{G}^{(j)}\}_{j=1}^{k-1}$.

This yields a partition $\mathscr{R}$ of $\mathrm{Cross}_{k-1}(\psi_1^{\mathscr{R}})$ satisfying (11.7). Moreover, $\mathscr{R}$ respects $\{\mathcal{G}^{(j)}\}_{j=1}^{k-1}$ and λ_j/ρ_j is an integer for $j = 2, \dots, k-1$.

(D4) The partition $\mathscr{S}$ defined by (11.8a) and (11.10) satisfies (11.5) again. Moreover, vector $\boldsymbol{\sigma} = (\rho_2, \dots, \rho_k)$ (see the proof of (11.5)*(a)*) satisfies $\lambda_j/\rho_j \in \mathbb{N}$ for $j = 2, \dots, k$. For $j = 2, \dots, k-1$ this comes from applying Lemma 12.2 and for $j = k$ from the definition of ρ_k (see (12.30)).

Finally, since $\mathscr{R}$ respects $\{\mathcal{G}^{(j)}\}_{j=1}^{k-1}$ (see *(D3)*) and every $\mathcal{G}_i$, $i \in [s]$, respects $\mathcal{G}^{(k)} \in \boldsymbol{\mathcal{G}}$ (see *(D2)*), (11.8a) implies that $\mathscr{S}$ respects $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$.

(D5) While defining the partition $\mathscr{T}$ (see (T1)-(T3), (11.25), (11.26)), we have (k,k)-cylinders

$$\mathcal{T}^{(k)}(i,\xi,\hat{\boldsymbol{y}}) \subseteq \mathcal{S}^{(k)}(\xi,\hat{\boldsymbol{y}}),$$

for $i = 0, 1, \dots u(\xi, \hat{\boldsymbol{y}})$ (or $i = 0$ only), $\xi \in [s]$, and $\hat{\boldsymbol{y}} \in \hat{A}(k-1, \boldsymbol{a}^{\mathscr{R}})$. Since $\mathscr{S}$ respects $\boldsymbol{\mathcal{G}}$ (see *(D4)*), the partition $\mathscr{T}$ defined by (11.26) also respects $\boldsymbol{\mathcal{G}}$. Note that calculations in parts *(A)–(C)* remain the same.

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Appendix A. Some proofs from Section 10

We first give a proof of the Slicing Lemma.

Proof of Lemma 10.2. For every edge $e \in \mathcal{H}$, we define a random variable X_e with values in $\{0, 1, \ldots, u\}$ by

$$\begin{aligned} \mathbb{P}(X_e = i) &= p \text{ for } i \in [u], \\ \mathbb{P}(X_e = 0) &= 1 - pu. \end{aligned}$$

Then, we define $\mathcal{H}_i$ by putting $e \in \mathcal{H}$ into $\mathcal{H}_i$ if and only if $X_e = i$. Clearly, $|\mathcal{H}_i|$ is a random variable with binomial distribution $\mathrm{Bi}(|\mathcal{H}|, p)$.

Let $\mathcal{G}_1, \ldots, \mathcal{G}_r$ be subcylinders of $\mathcal{G}$ such that

$$\Big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\Big| \geq 3\delta\big|\mathcal{K}_k(\mathcal{G})\big|. \tag{A.1}$$

Then, due to (δ, α, r)-regularity of $\mathcal{H}$, we have

$$\Big|\mathcal{H} \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\Big| = (\alpha \pm \delta)\Big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\Big|.$$

Subsequently, for every $i \in [u]$, the expected number of edges of $\mathcal{H}_i$ in $\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)$ is

$$E_i = \mathbb{E}\left(\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|\right) = (\alpha \pm \delta)p\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|. \tag{A.2}$$

Set $\gamma = \delta/p\alpha$, and observe that

$$p\alpha - 3\delta \leq (1-\gamma)p(\alpha - \delta), \tag{A.3a}$$
$$p\alpha + 3\delta \geq (1+\gamma)p(\alpha + \delta). \tag{A.3b}$$

Suppose that for some $i \in [u]$ we have

$$\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| - p\alpha\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| \geq 3\delta\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|.$$

Then, using (A.2) and (A.3b), we obtain

$$\begin{aligned} \big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| &\geq (p\alpha + 3\delta)\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| \\ &\overset{(A.3b)}{\geq} (1+\gamma)p(\alpha+\delta)\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| \overset{(A.2)}{\geq} (1+\gamma)E_i. \end{aligned}$$

Consequently,

$$\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| - E_i \geq \gamma E_i.$$

Similarly, assuming

$$\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| - p\alpha\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| \leq -3\delta\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|,$$

we obtain $\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| - E_i < -\gamma E_i$. As a result, we have

$$\mathbb{P}\left(\Big|\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| - p\alpha\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|\Big| \geq 3\delta\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|\right)$$
$$\leq \mathbb{P}\left(\Big|\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| - E_i\Big| \geq \gamma E_i\right). \quad \text{(A.4)}$$

Using the Chernoff inequality, we estimate the right-hand side of (A.4) by $2\exp(-\gamma^2 E_i/3)$. Moreover, from (A.1), (A.2), and $|\mathcal{K}_k(\mathcal{G})| \geq m^k/\ln m$, we conclude

$$E_i > (\alpha - \delta)p\Big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\Big| > \alpha\delta p\frac{m^k}{\ln m}.$$

Thus,

$$\mathbb{P}\left(\Big|\big|\mathcal{H}_i \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big| - p\alpha\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|\Big| \geq 3\delta\big|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\big|\right)$$
$$\leq 2\exp(-(\delta^3 m^k/3\alpha p\ln m)) \leq 2\exp(-(\ln 4)krm^{k-1}) = 2 \times 4^{-krm^{k-1}}.$$

There are at most $2^{krm^{k-1}}$ ways of selecting $(k,k-1)$-cylinders $\mathcal{G}_1, \dots, \mathcal{G}_r \subseteq \mathcal{G}$. Hence, the probability that at least one of (k,k)-cylinders $\mathcal{H}_i$ is $(3\delta, p\alpha, r)$-irregular is bounded by $u \times 2^{krm^{k-1}} \times 2 \times 4^{-krm^{k-1}} < 1$ for $r \geq 1$. Thus, we infer that there exists a choice of (k,k)-cylinders $\mathcal{H}_1, \dots, \mathcal{H}_u$ so that every $\mathcal{H}_i$ is $(3\delta, p\alpha, r)$-regular. □

Proof of Lemma 10.3. For each polyad $\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}$ with $|\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})| > 0$, and for every polyad $\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}$ such that $\hat{\mathcal{T}}^{(k)} \subset \hat{\mathcal{S}}^{(k)}$, we set

$$\sigma_{\hat{\mathcal{T}}^{(k)}} = \frac{|\mathcal{K}_{k+1}(\hat{\mathcal{T}}^{(k)})|}{|\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})|} = \frac{\mathrm{Vol}(\hat{\mathcal{T}}^{(k)})}{\mathrm{Vol}(\hat{\mathcal{S}}^{(k)})}.$$

Since partition $\mathscr{T}$ refines $\mathscr{S}$, similarly to the proof of (11.5)*(c)* (see also (11.14a) – (11.15)), we obtain

$$d^2_{\mathcal{H}}(\hat{\mathcal{S}}^{(k)}) \leq \sum_{\hat{\mathcal{T}}^{(k)} \subset \hat{\mathcal{S}}^{(k)}} \sigma_{\hat{\mathcal{T}}^{(k)}} d^2_{\mathcal{H}}(\hat{\mathcal{T}}^{(k)}) \quad \text{(A.5)}$$

for every polyad $\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}$. We use this to estimate $\operatorname{ind}\mathscr{S}$. Since $\sigma_{\hat{\mathcal{T}}^{(k)}} = \operatorname{Vol}(\hat{\mathcal{T}}^{(k)})/\operatorname{Vol}(\hat{\mathcal{S}}^{(k)})$, we have

$$\begin{aligned}\operatorname{ind}\mathscr{S} \overset{(A.5)}{\leq} \sum_{\hat{\mathcal{S}}^{(k)}\in\hat{\mathscr{S}}} \operatorname{Vol}(\hat{\mathcal{S}}^{(k)}) \sum_{\hat{\mathcal{T}}^{(k)}\subset\hat{\mathcal{S}}^{(k)}} \sigma_{\hat{\mathcal{S}}^{(k)}} d^2_{\mathcal{H}}(\hat{\mathcal{T}}^{(k)}) \\ = \sum_{\hat{\mathcal{S}}^{(k)}\in\hat{\mathscr{S}}} \sum_{\hat{\mathcal{T}}^{(k)}\subset\hat{\mathcal{S}}^{(k)}} \operatorname{Vol}(\hat{\mathcal{T}}^{(k)}) d^2_{\mathcal{H}}(\hat{\mathcal{T}}^{(k)}) \leq \operatorname{ind}\mathscr{T}.\end{aligned}$$

□

Department of Mathematics and Computer Science, Emory University, Atlanta, GA, 30322, USA

E-mail address: rodl@mathcs.emory.edu

Department of Mathematics, MC-382, University of Illinois at Urbana–Champaign, 1409 W. Green Street, Urbana, IL, 61801, USA and Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508–900 São Paulo, Brazil

E-mail address: jozef@member.ams.org