APPLICATIONS OF THE REGULARITY LEMMA FOR UNIFORM HYPERGRAPHS

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Abstract. In this note we discuss several combinatorial problems that can be addressed by the Regularity Method for hypergraphs. Based on recent results of Nagle, Schacht and the authors, we give here solutions to these problems.

In particular, we prove the following: *Let $\mathcal{F}$ be a k-uniform hypergraph on t vertices and suppose an n-vertex k-uniform hypergraph $\mathcal{H}$ contains only $o(n^t)$ copies of $\mathcal{F}$. Then one can delete $o(n^k)$ edges of $\mathcal{H}$ to make it $\mathcal{F}$-free.*

Similar results were recently obtained by W. T. Gowers.

1. Introduction

In 1976, Szemerédi proved the Regularity Lemma [31], a theorem which asserts that any graph can be partitioned into bounded number of random-like blocks (ε-regular pairs).

The Regularity Lemma proved to be a very powerful tool in graph theory with many applications (see [13, 12] for a survey). Many of these applications are based on the fact that random-like blocks ensured by the Regularity Lemma allow to find small subgraphs. A regularity lemma for 3-uniform hypergraphs that allows the same phenomenon (i.e. finding fixed size subhypergraphs) was considered in [9]. This lemma was extended to the case of k-uniform hypergraphs in [22].

This paper presents several applications of the lemma from [22] combined with the result of [16] and provides complete solutions to the following problems.

1.1. **Erdős-Stone type problem.**

For a set V and an integer $k \geq 1$, let $\binom{V}{k}$ be the set of all k-element subsets of V. We call a subset $\mathcal{G} \subseteq \binom{V}{k}$ a *k-uniform hypergraph* with the vertex set V. For a given k-uniform hypergraph $\mathcal{G}$, we denote by $V(\mathcal{G})$ and

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$E(\mathcal{G})$ its vertex and edge set, respectively. We identify hypergraphs with their edge sets and, consequently, use $|\mathcal{G}|$ for $|E(\mathcal{G})|$.

Let $\mathcal{G}$ and $\mathcal{H}$ be two k-uniform hypergraphs. We say that $\mathcal{H}$ is *$\mathcal{G}$-free* if $\mathcal{H}$ does not contain a subgraph isomorphic to $\mathcal{G}$. Erdős, Frankl, and Rödl [3] proved the following theorem.

Theorem 1.1. *For every $\varepsilon > 0$ and a fixed graph G with chromatic number χ, there exists $n_0(\varepsilon, G) \in \mathbb{N}$ so that every G-free graph H on $n > n_0(\varepsilon, G)$ vertices can be made K_χ-free by removing εn^2 edges.*

As an extension of Theorem 1.1, they proposed to study the following question: For integers $t \geq k \geq 2$, $s \geq 1$, let $K_t^{(k)}$ be the complete k-uniform hypergraph on t vertices and $K_t^{(k)}(s)$ be the complete t-partite k-uniform hypergraph with s vertices in each partite class. Note that $K_t^{(k)}(1) = K_t^{(k)}$.

For $k < t$, denote by $\varphi(k, t, s, n)$ the maximum number of edges needed to be deleted from a $K_t^{(k)}(s)$-free k-uniform hypergraph on n vertices to get a $K_t^{(k)}$-free k-uniform hypergraph. Erdős, Frankl, and Rödl [3] conjectured that for fixed $t > k \geq 2$ and $s \geq 1$ the function $\varphi(k, t, s, n) = o(n^k)$ as n tends to infinity. So far the above conjecture was confirmed to be true for $k = 3$, $t = 4$ in [9] and for $k = 3$, $t > 4$ and $k = 4$, $t = 5$ it follows from results in [15] and [23], respectively. Based on the recent results of Nagle, Rödl, Schacht and Skokan [16, 22], in this paper, we establish the conjecture for all suitable choices of t, k, and s.

Theorem 1.2. *For an arbitrary real number $\varepsilon > 0$ and integers $t > k \geq 2$, $s \geq 1$, there exists $n_0(\varepsilon, k, t, s)$ with the following property. Let $\mathcal{H}$ be any $K_t^{(k)}(s)$-free k-uniform hypergraph on $n > n_0(\varepsilon, k, t, s)$ vertices. Then it is possible to remove εn^k edges from $\mathcal{H}$ so that the resulting hypergraph is $K_t^{(k)}$-free. In other words,*

$$\varphi(k, t, s, n) = o(n^k)\,.$$

For graphs, i.e. when $k = 2$, this theorem implies that the Turán number $\mathrm{ex}\left(n, K_t^{(2)}(s)\right)$ (the maximum number of edges in a $K_t^{(2)}(s)$-free graph on n vertices) does not differ from the Turán number $\mathrm{ex}\left(n, K_t^{(2)}\right)$ by more than εn^2 for n sufficiently large. This combined with the well-known Turán Theorem [32] yields

$$\mathrm{ex}\left(n, K_t^{(2)}(s)\right) = \left(1 - \frac{1}{t-1} + o(1)\right)\binom{n}{2}\,. \tag{1.1}$$

Since (1.1) is the statement of the Erdős-Stone Theorem [7], Theorem 1.2 can be viewed as a generalization of the Erdős-Stone Theorem to hypergraphs.

In this paper we prove the following more general theorem, which answers a question of Füredi [10]. The case when $\mathcal{F} = K_{k+1}^{(k)}$ also appears in [11, 16].

Theorem 1.3. *For all $t \geq k \geq 2$, every k-uniform hypergraph $\mathcal{F}$ on t vertices, and $\varepsilon > 0$ there exist $\delta = \delta(\mathcal{F}, \varepsilon) > 0$ and $n_0 = n_0(\mathcal{F}, \varepsilon) \in \mathbb{N}$ such that the following statement holds.*

Suppose that an n-vertex k-uniform hypergraph $\mathcal{H}$, with $n > n_0$, contains only δn^t copies of $\mathcal{F}$. Then one can delete εn^k edges of $\mathcal{H}$ to make it $\mathcal{F}$-free.

As it turns out, it suffices to establish Theorem 1.3 for $\mathcal{F} = K_t^{(k)}$ in order to verify Theorem 1.2. We formally prove this observation in Section 2.

Proposition 1.4. *Theorem 1.3 implies Theorem 1.2.*

Theorem 1.2 and Theorem 1.3 have several applications. Some of them regard density theorems, among which are Szemerédi's theorem (see Section 1.2 and [9]) and related results due to Furstenberg and Katznelson [11, 27, 21]. It also has applications in discrete geometry [26] and to extremal hypergraph problems [17].

Below we will discuss some of these as well as some other applications in more detail.

1.2. **Szemerédi's Density Theorem.**

Let $r_k(n)$ be the maximum cardinality of a set $A \subseteq [n] := \{1, \ldots, n\}$ containing no arithmetic progression of length k. Answering an old question of Erdős and Turán [8], Szemerédi [30] established that $r_k(n) = o(n)$ for any fixed integer k.

There are several extremal hypergraph problems that are closely related to the value of $r_k(n)$. Such a problem (related to a well-known $(6,3)$-configuration) was perhaps first suggested by Brown, Erdős and Sós [1, 28] and considered by Ruzsa and Szemerédi in [24]. Some other problems of this type were discussed in [24, 5, 2]. The extremal problem related to the configuration $\mathcal{F}(k)$ (defined below) was investigated in [9] (see also [18]). The particular configuration $\mathcal{F}(k)$ was originally suggested by Frankl.

Let $A_i = \{a_i, b_i\}$ be pairwise disjoint 2-element sets for $i \in [k]$. Define $F_i = \{a_1, \ldots, a_k, b_i\} \setminus \{a_i\}$ and $\mathcal{F}(k) = \{F_1, \ldots, F_k\}$. Note that $|F_j \cap A_i| = 1$ for $1 \leq i, j \leq k$, that is, $\mathcal{F}(k)$ is a k-partite k-uniform hypergraph. Also, $F_i \cap F_j = \{a_1, \ldots, a_k\} \setminus \{a_i, a_j\}$; in particular, $|F_i \cap F_j| = k - 2 < k - 1$ holds for $1 \leq i < j \leq k$. We note that the triple system $\mathcal{F}(3)$ is the $(6,3)$-configuration considered in [24].

Let $\tilde{\text{ex}}(n, \mathcal{F}(k))$ denote $\max |\mathcal{H}|$, $\mathcal{H} \subset \binom{X}{k}$, $|X| = n$, such that

$$\begin{aligned} &\text{(i)}\ \ |H \cap H'| \leq k-2 \text{ holds for all distinct } H,\ H' \in \mathcal{H}, \text{ and} \\ &\text{(ii)}\ \ \mathcal{H} \text{ is } \mathcal{F}(k)\text{-free.} \end{aligned} \tag{1.2}$$

Note that for any $\mathcal{H}$ satisfying (i),

$$|\mathcal{H}| \leq \frac{\binom{n}{k-1}}{\binom{k}{k-1}} \leq \frac{n^{k-1}}{k}$$

must hold. In [9, Proposition 2.1-2.2] it was shown that

$$c_k n^{k-2} \times r_k(n) \leq \tilde{\text{ex}}(n, \mathcal{F}(k)) \leq \varphi(k-1, k, 2, n)\,, \tag{1.3}$$

where c_k is a constant only depending on k. Consequently, Theorem 1.2 implies $r_k(n) = o(n)$, i.e., the famous Density Theorem of Szemerédi.

1.3. **Székely's jack problem.**

The following problem was formulated by Székely [29] (see also [14, pages 226-7]).

For a point $\boldsymbol{c} = (c_1, c_2, \ldots, c_k) \in [n]^k$ we define a *jack* $J(\boldsymbol{c})$ with *center* $\boldsymbol{c}$ as the set of all points that differ from c in at most one coordinate. For i, $1 \leq i \leq k$, and fixed $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k \in [n]$, we also define a *line* as the set of n points of the form

$$\big\{(c_1, c_2, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_k), 1 \leq x \leq n\big\}.$$

Note that there are kn^{k-1} lines in $[n]^k$ and each jack contains exactly k lines.

Let $LS(n,k)$ be the maximum cardinality of a system $\mathcal{J}$ of jacks for which

(1) no two distinct jacks share a common line, and
(2) $\bigcap_{i=1}^{k} J_i = \emptyset$ for all distinct jacks $J_1, \ldots, J_k \in \mathcal{J}$.

Condition (1) immediately implies $LS(n,k) \leq kn^{k-1}/k = n^{k-1}$. Székely suggested that more is true and conjectured that $LS(n,k)/n^{k-1}$ tends to 0 as $n \to \infty$.

One can show that $LS(n,k)$ is closely related to $\tilde{\text{ex}}\big(kn, \mathcal{F}(k)\big)$. Indeed, in Section 3 we show the following.

Proposition 1.5. *For every integer $k > 1$*

$$\frac{k!}{k^k}\tilde{\text{ex}}\big(kn, \mathcal{F}(k)\big) \leq LS(n,k) \leq \tilde{\text{ex}}\big(kn, \mathcal{F}(k)\big).$$

Hence, in view of (1.3) and Theorem 1.2 we infer:

Theorem 1.6. $LS(n,k) = o(n^{k-1})$.

1.4. **Organization.**

The paper is organized as follows: in the next section we show Proposition 1.4, i.e., how Theorem 1.3 implies Theorem 1.2. Proposition 1.5 is verified in Section 3. In Section 4, we describe the notation and statement of our main tool - the Hypergraph Regularity Lemma. Other results needed in our proof are presented in Section 5. Then, in Section 6, we prove Theorem 1.3.

2. Proof of Proposition 1.4

In the proof of this proposition, we make use of the following lemma, which follows from the theorem of Erdős from [4] by a supersaturation argument (see also [6]).

Lemma 2.1. *For every $c > 0$ and positive integers $t \geq 2$ and $s \geq 1$ there exist $n_1 = n_1(c, t, s)$ and $c' > 0$ such that if $\mathcal{G}$ is a t-uniform hypergraph with $n > n_1$ vertices and at least cn^t edges, then $\mathcal{G}$ contains $c'n^{ts}$ copies of $K_t^{(t)}(s)$.*

Proof of Proposition 1.4. Let $\varepsilon > 0$ and $k, s, t \in \mathbb{N}$ be given. We must show that for any $K_t^{(k)}(s)$-free k-uniform hypergraph $\mathcal{H}$ on n vertices, n sufficiently large, it is possible to delete εn^k edges from $\mathcal{H}$ to obtain a $K_t^{(k)}$-free k-uniform hypergraph. Consequently, $\varphi(k, t, s, n) \leq \varepsilon n^k$ holds.

We start with defining the constants. With intention to apply Theorem 1.3 later, let $\delta > 0$ and $n_0 = n_0(t, k, \varepsilon)$ be the numbers guaranteed by Theorem 1.3. Furthermore, let $n_1 = n_1(\delta, t, s)$ be the number guaranteed by Lemma 2.1 applied with $c = \delta$.

Suppose $\mathcal{H}$ is an arbitrary $K_t^{(k)}(s)$-free k-uniform hypergraph on $n > \max\{n_0, n_1\}$ vertices. Let $\mathcal{G}$ be a t-uniform hypergraph with vertex set $V(\mathcal{G}) = V(\mathcal{H})$ and edge set formed by all cliques $K_t^{(k)}$ of $\mathcal{H}$. Then $\mathcal{G}$ is $K_t^{(t)}(s)$-free because $\mathcal{H}$ is $K_t^{(k)}(s)$-free. By Lemma 2.1, we obtain $|\mathcal{G}| \leq \delta n^t$ and, therefore, $\mathcal{H}$ contains at most δn^t copies of $K_t^{(k)}$ as subgraphs.

Applying Theorem 1.3 yields that $\mathcal{H}$ can be made $K_t^{(k)}$-free by omitting εn^k edges. □

3. Proof of Proposition 1.5

We start with the second inequality. Let $\mathcal{J}$ be the system of jacks satisfying (1) and (2) of maximum size. Our goal is to construct a k-partite k-uniform hypergraph $\mathcal{H}$ of size $|\mathcal{H}| = |\mathcal{J}|$ that also satisfies conditions (i) and (ii) in (1.2).

Let $V_1, \dots, V_k$ be k copies of $\{1, \dots, n\}$. Then we define $\mathcal{H}$ by setting

$$\begin{aligned} V(\mathcal{H}) &= V_1 \cup \ldots \cup V_k\,, \\ E(\mathcal{H}) &= \big\{\{a_1, \dots, a_k\} \colon J(a_1, \dots, a_k) \in \mathcal{J}, a_i \in V_i, i = 1, \dots, k\big\}\,. \end{aligned}$$

Clearly, $\mathcal{H}$ is a k-partite k-uniform hypergraph on kn vertices with $|\mathcal{H}| = |\mathcal{J}|$. We now prove that $\mathcal{H}$ also satisfies (1.2).

By (1), no two jacks share a line and, therefore, the centers of any two jacks in $\mathcal{J}$ differ in more than one coordinate. Consequently, every two edges of $\mathcal{H}$ differ in at least two vertices and (i) holds.

Suppose that (ii) is not true and $\mathcal{H}$ contains a copy of $\mathcal{F}(k) = \{F_1, \dots, F_k\}$ and $F_i = \{a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_k\}$, where $a_i, b_i \in V_i$, $i \in [k]$. By the definition of $\mathcal{H}$, every F_i corresponds to the jack $J_i = J(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_k) \in \mathcal{J}$. Since $(a_1, a_2, \dots, a_k)$ differs from $(a_1, \dots, a_{i-1}, b_i,$

$a_{i+1}, \dots, a_k)$ in exactly one coordinate, we have $(a_1, a_2, \dots, a_k) \in J_i$ for every $i \in [k]$. Then, however, $(a_1, a_2, \dots, a_k) \in \bigcap_{i=1}^{k} J_i$, which is a contradiction to (2). Consequently

$$LS(n,k) = |\mathcal{J}| = |\mathcal{H}| \le \tilde{\mathrm{ex}}(kn, \mathcal{F}(k)) .$$

On the other hand, let $\tilde{\mathcal{H}}$ be a k-uniform hypergraph on kn vertices satisfying conditions (i) and (ii) in (1.2) such that $|\tilde{\mathcal{H}}| = \tilde{\mathrm{ex}}\big(nk, \mathcal{F}(k)\big)$. It is a well-known fact[1] that $\tilde{\mathcal{H}}$ contains a k-partite subgraph with k-partition $V(\mathcal{H}) = V_1 \cup \ldots \cup V_k$ such that each partite set has size n and $|\mathcal{H}| \ge (k!/k^k)|\tilde{\mathcal{H}}| = (k!/k^k)\tilde{\mathrm{ex}}\big(nk, \mathcal{F}(k)\big)$. Let $\mathcal{J}$ be a system of jacks defined by

$$\mathcal{J} = \big\{ J(a_1, \dots, a_k) \colon \{a_1, \dots, a_k\} \in E(\mathcal{H}) \big\} .$$

Since $|\mathcal{J}| = |\mathcal{H}|$, if we prove that $\mathcal{J}$ satisfies (1) and (2), then

$$\frac{k!}{k^k} \tilde{\mathrm{ex}}\big(kn, \mathcal{F}(k)\big) \le |\mathcal{H}| = |\mathcal{J}| \le LS(n,k) ,$$

and we will be able to conclude that Proposition 1.5 holds.

Indeed, condition (i) of (1.2) implies that every two centers of jacks in $\mathcal{J}$ differ by at least two coordinates and, thus, no two jacks in $\mathcal{J}$ share a line. Hence $\mathcal{J}$ satisfies (1).

Suppose that (2) is not true and $(a_1, a_2, \dots, a_k) \in \bigcap_{i=1}^{k} J_i$ for some distinct jacks $J_1, \dots, J_k \in \mathcal{J}$. By reordering, we may assume that the center of J_i differs from $(a_1, a_2, \dots, a_k)$ at the i-th coordinate. Therefore, $J_i = J(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_k)$ for some $b_i \in [n]$. Consequently, $\{F_1, \dots, F_k\}$, where $F_i = \{a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_k\}$, is a copy of $\mathcal{F}(k)$ in $\mathcal{H} \subset \tilde{\mathcal{H}}$, which is a contradiction to (ii). □

4. Hypergraph Regularity Lemma

In this section, we present one of our two main tools for the proof of Theorem 1.3 – the Hypergraph Regularity Lemma from [22]. To this end, we need to introduce some notation. This notation, which simplifies our earlier description of regular partition in [22], is taken from a recent paper of Rödl and Schacht [19] (see also [20]).

4.1. **Cylinders and Complexes.**

This paper deals mainly with ℓ-partite k-uniform hypergraphs. We shall refer to such hypergraphs as (ℓ, k)-*cylinders*.

[1]This follows, for example, from the observation that there are $\binom{kn}{n}\binom{(k-1)n}{n}\dots\binom{2n}{n}$ partitions of $V(\mathcal{H})$ into k parts of size n and any given k-tuple of vertices is crossing (i.e., it intersects each of the k parts) in $\binom{k(n-1)}{n-1}\binom{(k-1)(n-1)}{n-1}\dots\binom{2(n-1)}{n-1}$ partitions.

Definition 4.1 (**cylinder**)**.** *Let $\ell \geq k \geq 2$ be two integers, V be a set, $|V| \geq \ell$, and $V = V_1 \cup \dots \cup V_\ell$ be a partition of V.*

A k-set $K \in \binom{V}{k}$ is crossing if $|V_i \cap K| \leq 1$ for every $i \in [\ell]$. We shall denote by $K_\ell^{(k)}(V_1, \dots, V_\ell)$ the complete (ℓ, k)-cylinder with vertex partition $V_1 \cup \dots \cup V_\ell$, i.e. the set of all crossing k-sets. Then, an (ℓ, k)-cylinder $\mathcal{G}$ is any subset of $K_\ell^{(k)}(V_1, \dots, V_\ell)$.

Definition 4.2. *For an (ℓ, k)-cylinder $\mathcal{G}$, where $k > 1$, we shall denote by $\mathcal{K}_j(\mathcal{G})$, $k \leq j \leq \ell$, the j-uniform hypergraph with the same vertex set as $\mathcal{G}$ and whose edges are precisely those j-element subsets of $V(\mathcal{G})$ that span cliques of order j in $\mathcal{G}$.*

Clearly, the quantity $|\mathcal{K}_j(\mathcal{G})|$ counts the total number of cliques of order j in an (ℓ, k)-cylinder $\mathcal{G}$, $1 < k \leq j \leq \ell$, and $\mathcal{K}_k(\mathcal{G}) = \mathcal{G}$.

For formal reasons, we find it convenient to extend the above definitions to the case when $k = 1$.

Definition 4.3. *We define an $(\ell, 1)$-cylinder $\mathcal{G}$ as a partition $V_1 \cup \dots \cup V_\ell$. For an $(\ell, 1)$-cylinder $\mathcal{G} = V_1 \cup \dots \cup V_\ell$ and $1 \leq j \leq \ell$, we set $\mathcal{K}_j(\mathcal{G}) = K_\ell^{(j)}(V_1, \dots, V_\ell)$.*

The concept of "cliques in 1-uniform hypergraphs" is certainly artificial. It fits well, however, to our general description of a complex (see Definition 4.6).

For an (ℓ, k)-cylinder $\mathcal{G}$ and a subset L of vertices in $\mathcal{G}$, where $k \leq |L| \leq \ell$, we say that L *belongs to* $\mathcal{G}$ if L induces a clique in $\mathcal{G}$.

We will often face a situation when we need to describe that one cylinder 'lies on' another cylinder. To this end, we define the term *underlying cylinder*.

Definition 4.4 (**underlying cylinder**)**.** *Let $\mathcal{F}$ be an $(\ell, k-1)$-cylinder and $\mathcal{G}$ be an (ℓ, k)-cylinder with the same vertex set. We say that $\mathcal{F}$ underlies $\mathcal{G}$ if $\mathcal{G} \subset \mathcal{K}_k(\mathcal{F})$.*

Note that if $k = 2$ and $\mathcal{F} = V_1 \cup \dots \cup V_\ell$, then $\mathcal{G}$ is an ℓ-partite graph with ℓ-partition $V_1 \cup \dots \cup V_\ell$.

Definition 4.5 (**density**)**.** *Let $\mathcal{G}$ be a k-uniform hypergraph and $\mathcal{F}$ be a $(k, k-1)$-cylinder. We define the density of $\mathcal{F}$ with respect to $\mathcal{G}$ by*

$$d_{\mathcal{G}}(\mathcal{F}) = \begin{cases} \frac{\left|\mathcal{G} \cap \mathcal{K}_k(\mathcal{F})\right|}{\left|\mathcal{K}_k(\mathcal{F})\right|} & \text{if } \left|\mathcal{K}_k(\mathcal{F})\right| > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Through this paper, we will work with a sequence of underlying cylinders. To accommodate this situation, we introduce the notion of *complex*.

Definition 4.6 (**complex**)**.** *Let ℓ and k, $\ell \geq k \geq 1$, be two integers. An (ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ is a system of cylinders $\left\{\mathcal{G}^{(j)}\right\}_{j=1}^{k}$ such that*

(a) *$\mathcal{G}^{(1)}$ is an $(\ell, 1)$-cylinder, i.e. $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$,*

(b) (ℓ, j)-cylinder $\mathcal{G}^{(j)}$ underlies $(\ell, j+1)$-cylinder $\mathcal{G}^{(j+1)}$ for every $j \in [k-1]$, i.e., $\mathcal{G}^{(j+1)} \subset \mathcal{K}_{j+1}(\mathcal{G}^{(j)})$.

4.2. Regularity of Cylinders and Complexes.

Now we define the notion of *regularity* of cylinders.

Definition 4.7. *Let $r \in \mathbb{N}$, $\mathcal{G}$ be a k-uniform hypergraph, and $\tilde{\mathcal{F}}$ be a system of $(k, k-1)$-cylinders $\mathcal{F}_1, \dots, \mathcal{F}_r$ with the same vertex set as $\mathcal{G}$. We define the density of $\tilde{\mathcal{F}}$ with respect to $\mathcal{G}$ by*

$$d_{\mathcal{G}}(\tilde{\mathcal{F}}) = \begin{cases} \dfrac{\left|\mathcal{G} \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j)\right|}{\left|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j)\right|} & \text{if } \left|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j)\right| > 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.2}$$

Now we define a *regular cylinder*.

Definition 4.8 (**(δ, d, r)-regular cylinder**). *Let $r \in \mathbb{N}$, δ, d be two positive real numbers such that $0 < \delta < d \le 1$, $\mathcal{F}$ be a $(k, k-1)$-cylinder, and $\mathcal{G}$ be a k-uniform hypergraph. We say that $\mathcal{G}$ is (δ, d, r)-regular with respect to $\mathcal{F}$ if the following condition is satisfied: whenever $\tilde{\mathcal{F}} = \{\mathcal{F}_1, \dots, \mathcal{F}_r\}$ is a system of subcylinders of $\mathcal{F}$ such that*

$$\left|\bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{F}_j)\right| \ge \delta \left|\mathcal{K}_k(\mathcal{F})\right|,$$

we have

$$d - \delta \le d_{\mathcal{G}}(\tilde{\mathcal{F}}) \le d + \delta\,.$$

We say that $\mathcal{G}$ is (δ, d, r)-irregular with respect to $\mathcal{F}$ if it is not (δ, d, r)-regular with respect to $\mathcal{F}$. If $r = 1$, we simply say that $\mathcal{G}$ is (δ, d)-regular with respect to $\mathcal{F}$.

Moreover, we sometimes say $\mathcal{G}$ is (δ, r)-regular (or simply δ-regular if $r = 1$) with respect to $\mathcal{F}$ if $\mathcal{G}$ is (δ, d, r)-regular with respect to $\mathcal{F}$ for some $d \in [0, 1]$.

We extend the above definition to the case of an $(\ell, k-1)$-cylinder $\mathcal{F}$.

Definition 4.9. *Let $k, \ell, r \in \mathbb{N}$, $\ell \ge k$, δ, d be two positive real numbers such that $0 < \delta < d \le 1$, $\mathcal{F}$ be an $(\ell, k-1)$-cylinder with an ℓ-partition $\bigcup_{i=1}^{\ell} V_i$, and $\mathcal{G}$ be a k-uniform hypergraph. We say that $\mathcal{G}$ is (δ, d, r)-regular with respect to $\mathcal{F}$ if the restriction $\mathcal{G}\big[\bigcup_{j \in I} V_j\big]$ is (δ, d, r)-regular with respect to $\mathcal{F}\big[\bigcup_{j \in I} V_j\big]$ for all $I \in \binom{[\ell]}{k}$.*

Now we are ready to introduce the concept of regularity for an (ℓ, k)-complex $\boldsymbol{\mathcal{G}}$.

Definition 4.10 (**($\boldsymbol{\delta}, \boldsymbol{d}, r$)-regular complex**). *Let $r \in \mathbb{N}$ and let $\boldsymbol{d} = (d_2, \dots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ be two vectors of positive real numbers such that $0 < \delta_j < d_j \leq 1$ for all $j = 2, \dots, k$. We say that an (ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular if*

(a) *$\mathcal{G}^{(2)}$ is (δ_2, d_2)-regular with respect to $\mathcal{G}^{(1)}$, and*
(b) *$\mathcal{G}^{(j+1)}$ is $(\delta_{j+1}, d_{j+1}, r)$-regular with respect to $\mathcal{G}^{(j)}$ for every $j \in [k-1] \setminus \{1\}$.*

4.3. **Partitions.**

Fix a non-empty set V, an arbitrary integer $k > 1$ and a vector $\boldsymbol{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$. We will define a family of partitions $\mathscr{P} = \{\mathscr{P}^{(i)}\}_{i=1}^{k-1}$ with properties described below.

Let $\mathscr{P}^{(1)}$ be a partition $V_1 \cup \dots \cup V_{a_1}$ of V. For every $j \in [a_1]$, denote by $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ the set of all crossing sets J of cardinality j, i.e.,

$$\mathrm{Cross}_j(\mathscr{P}^{(1)}) = K_{a_1}^{(j)}(V_1, \dots, V_{a_1}) \,.$$

For $j = 2, \dots, k-1$, $\mathscr{P}^{(j)}$ is going to be a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ in which each partition class $\mathcal{P}^{(j)}$ is a (j,j)-cylinder. We denote by $\mathcal{P}^{(j)}(J)$ the partition class of $\mathscr{P}^{(j)}$ that contains a given set $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$.

Each set $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$ defines a disjoint union

$$\hat{\mathcal{P}}^{(i)}(J) = \bigcup_{I \in \binom{J}{i}} \mathcal{P}^{(i)}(I)$$

of $\binom{j}{i}$ partition classes of $\mathscr{P}^{(i)}$, $i = 1, 2 \dots, j-1$. We use " ˆ " to stress the fact that $\hat{\mathcal{P}}^{(i)}(J)$ is not a single class of $\mathscr{P}^{(i)}$, but a union of $\binom{j}{i}$ of them. Also observe that $\hat{\mathcal{P}}^{(i)}(J)$ is a (j,i)-cylinder. When $i = j-1$, we call $\hat{\mathcal{P}}^{(j-1)}(J)$ a *j-polyad* and denote by $\hat{\mathscr{P}}^{(j)}$ the set of all j-polyads:

$$\hat{\mathscr{P}}^{(j)} = \left\{ \hat{\mathcal{P}}^{(j-1)}(J) \colon J \in \mathrm{Cross}_j(\mathscr{P}^{(1)}) \right\} .$$

This set induces a partition

$$\left\{ \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)} \right\}$$

of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$. Using this partition we will describe an interaction between $\mathscr{P}^{(j-1)}$ and $\mathscr{P}^{(j)}$ that we will require in our family of partitions $\mathscr{P} = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ for each $j = 2, \dots, k-1$.

We say that partitions $\mathscr{P}^{(j-1)}$ and $\mathscr{P}^{(j)}$ are *cohesive* if

$$\mathscr{P}^{(j)} \text{ refines } \left\{ \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)} \right\} .$$

In other words, each partition class of $\mathscr{P}^{(j)}$ is a subset of $\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big)$ for some j-polyad $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j)}$. The family of partitions $\mathscr{P} = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ is cohesive if $\mathscr{P}^{(j-1)}$ and $\mathscr{P}^{(j)}$ are cohesive for each $j = 2, \dots, k-1$.

Finally, we will require that the number of partition classes is bounded by a function independent of the number of vertices of V. This is accomplished in the following formal definition of the family of partitions.

Definition 4.11 (**family of partitions**). *Let k be a positive integer, V be a non-empty set, and $\boldsymbol{a} = (a_1, a_2, \dots, a_{k-1})$ be a vector of positive integers. Then we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ is a family of partitions on V if*

- $|\mathscr{P}^{(1)}| = a_1$,
- *$\mathscr{P}$ is cohesive,*
- $\big|\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)} \colon \mathcal{P}^{(j)} \subset \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big)\}\big| \le a_j$ *for every* $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$.

We also say that $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is L-bounded if $\max\{a_1, \dots, a_{k-1}\} \le L$.

Note that if $\mathscr{P}$ is cohesive, then

$$\hat{\boldsymbol{\mathcal{P}}}^{(j-1)}(J) = \big\{\hat{\mathcal{P}}^{(i)}(J)\big\}_{i=1}^{j-1} \tag{4.3}$$

is a $(j, j-1)$-complex for every set $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$ and $1 < j \le k$. Furthermore, observe that $\hat{\boldsymbol{\mathcal{P}}}^{(j-1)}(J)$ is determined by its "top layer", the j-polyad $\hat{\mathcal{P}}^{(j-1)}(J)$. We refer to $\hat{\boldsymbol{\mathcal{P}}}^{(j-1)}(J)$ as a *j-polyad complex* and denote by $\mathrm{Com}_{j-1}(\mathscr{P})$ the set of all j-polyad complexes.

4.4. **Regular partitions.**

Definition 4.12 (**$(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-equitable**). *Let $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$ and $\boldsymbol{d} = (d_2, \dots, d_{k-1})$ be two arbitrary but fixed vectors of real numbers between 0 and 1, μ be a number in interval $(0, 1]$, and r be a positive integer. We say that a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-equitable if*

(a) $|V_1| \le |V_2| \le \dots \le |V_{a_1}| \le |V_1| + 1$,

(b) *all but at most $\mu\binom{n}{k}$ many k-tuples $K \in \binom{V}{k}$ belong to $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(k, k-1)$-complexes $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)} \in \mathrm{Com}_{k-1}(\mathscr{P})$.*

The following definition describes a type of partition we are interested in.

Definition 4.13 (**regular family of partitions**). *We say $\mathscr{P}$ is (δ_k, r)-regular*[2] *with respect to $\mathcal{H}$, if for all but at most $\delta_k\binom{n}{k}$ many k-tuples $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ we have that $\mathcal{H} \cap \mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}(K)\big)$ is (δ_k, r)-regular with respect to the polyad $\hat{\mathcal{P}}^{(k-1)}(K)$.*

In [22], a regularity lemma for k-uniform hypergraphs was proved.

Theorem 4.14 (**Hypergraph Regularity Lemma**). *For every integer $k \in \mathbb{N}$, all numbers $\delta_k > 0$ and $\mu > 0$, and any non-negative functions $\delta_{k-1}(d_{k-1})$, $\delta_{k-2}(d_{k-2}, d_{k-1})$, $\dots$, $\delta_2(d_2, \dots, d_{k-1})$, $r = r(a_1, d_2, \dots, d_{k-1})$, there exist integers N_k and L_k such that the following holds.*

For every k-uniform hypergraph $\mathcal{H}$ with $|V(\mathcal{H})| \ge N_k$ there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on $V(\mathcal{H})$ and a vector $\boldsymbol{d} = (d_2, \dots, d_{k-1}) \in (0, 1]^{k-2}$ so that

[2] δ_2-regular for $k = 2$

(i) $\mathscr{P}$ *is a* $(\mu, \boldsymbol{\delta}(\boldsymbol{d}), \boldsymbol{d}, r(a_1, \boldsymbol{d}))$*-equitable family of partitions,*
(ii) $\mathscr{P}$ *is* $(\delta_k, r(a_1, \boldsymbol{d}))$*-regular with respect to* $\mathcal{H}$*, and*
(iii) $\mathscr{P}$ *is* L_k*-bounded.*

5. Counting Lemma

The aim of this section is to state the second main ingredient to our proof of Theorem 1.3. This is the Counting Lemma which was proved by Nagle, Rödl, and Schacht [16] (special cases of Theorem 5.2 were shown in [9, 15, 23]). In fact, here we use a variant of this lemma which can be derived from the main result of [16] by a standard argument (see [16, Corollary 12] and [25, Section 9] for a proof).

Before stating Theorem 5.2 we introduce some notation. First note that in the previous section vectors $\boldsymbol{d} = (d_2, \dots, d_{k-1})$ and $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$ were of length $k-2$. In the set up below we need to consider quantities d_k and δ_k as well. In order to be consistent we will use the notation $(\boldsymbol{d}, d_k)$ and $(\boldsymbol{\delta}, \delta_k)$ to denote the corresponding vectors of length $k-1$.

Definition 5.1. *Let* $\mathcal{F}$ *be a* k*-uniform hypergraph with vertex set* $[t]$*,* $\boldsymbol{d} = (d_2, \dots, d_{k-1})$*,* $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1}) \in (0,1]^{k-2}$*,* d_k*,* $\delta_k \in (0,1]$*, and* r*,* $m \in \mathbb{N}$*.*

We say a (t,k)*-complex* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(k)}\}$ *is an* $((\boldsymbol{\delta}, \delta_k), \geq(\boldsymbol{d}, d_k), r)$*-regular* $(m, \mathcal{F})$*-complex if the following holds:*

(a) $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_t$ *with* $|V_1| = \dots = |V_t| = m$*, and*
(b) *for every edge* $K \in \mathcal{F}$ *the* (k,k)*-complex*

$$\boldsymbol{\mathcal{H}}_K = \Big\{ \bigcup_{\alpha \in K} V_\alpha, \mathcal{H}^{(2)}\big[\bigcup_{\alpha \in K} V_\alpha\big], \dots, \mathcal{H}^{(k)}\big[\bigcup_{\alpha \in K} V_\alpha\big] \Big\}$$

is $((\boldsymbol{\delta}, \delta_k), (\boldsymbol{d}, d(K)), r)$*-regular for some* $d(K) \geq d_k$*.*

Theorem 5.2 (**Counting Lemma**). *For every* k*-uniform hypergraph* $\mathcal{F}$ *with vertex set* $V(F) = [t]$ *and* $\nu > 0$ *the following statement holds. There exist functions* $\delta'_k(d_k)$*,* $\delta'_{k-1}(d_{k-1}, d_k)$*, ...,* $\delta'_2(d_2, \dots, d_k)$*,* $r'(d_2, \dots, d_k)$ *and* $m_0(d_2, \dots, d_k)$ *so that for every choice of* $\boldsymbol{d} = (d_2, \dots, d_{k-1}) \in (0,1]^{k-2}$ *and* $d_k \in (0,1]$ *the following holds.*

If $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(k)}\}$ *is a* $((\boldsymbol{\delta}, \delta_k), \geq(\boldsymbol{d}, d_k), r)$*-regular* $(m, \mathcal{F})$*-complex with* $\boldsymbol{\delta} = \big(\delta'_2(d_2, \dots, d_k), \dots, \delta'_{k-1}(d_{k-1}, d_k)\big)$*,* $\delta_k = \delta'_k(d_k)$*,* $r = r'(\boldsymbol{d}, d_k)$*, and* $m \geq m_0(\boldsymbol{d}, d_k)$ *then* $\mathcal{H}^{(k)}$ *contains at least*

$$(1-\nu) \prod_{i=2}^{k-1} d_i^{|\Delta_i(\mathcal{F})|} \times d_k^{|\mathcal{F}|} \times m^t$$

copies of $\mathcal{F}$*, where* $\Delta_i(\mathcal{F}) = \{I \in \binom{[t]}{i} \colon$ *there exists* $K \in \mathcal{F}$ *so that* $I \subseteq K\}$*.*

Furthermore, we may assume that the function $m_0(d_2, \dots, d_k)$ *is non-increasing in every variable.*

Remark 5.3. Note that under the assumption that $d(K) = d_k$ for all $K \in \mathcal{F}$ Theorem 5.2 gives the asymptotic bound for the number of crossing and unlabeled copies of $\mathcal{F}$ in $\mathcal{H}^{(k)}$, i.e., copies with vertex set $v_1, \ldots, v_t$ with $v_\alpha \in V_\alpha$ for every $\alpha = 1, \ldots, t$.

6. Proof of Theorem 1.3

The proof of Theorem 1.3 is based on Theorem 4.14 and Theorem 5.2 and follows the lines of [3, 9]. First we will apply the Hypergraph Regularity Lemma (Theorem 4.14) to $\mathcal{H}$ with $\delta_k \ll \varepsilon$. Then we delete all k-tuples in irregular and sparse polyads. Our choice of δ_k will guarantee that at most εn^k edges are deleted. We conclude the proof by showing that if $\mathcal{H}' = \mathcal{H} \setminus \{\text{deleted edges}\}$ is not $\mathcal{F}$-free, then by the Counting Lemma (Theorem 5.2) it must contain more than δn^t copies of $\mathcal{F}$, which contradicts the assumption of Theorem 1.3.

Proof of Theorem 1.3. Suppose that $\varepsilon > 0$ and a k-uniform hypergraph $\mathcal{F}$ with vertex set $[t]$ are given. Set $\nu = 1/4$ and let $\delta'_k(d_k)$, $\delta'_{k-1}(d_{k-1}, d_k)$, $\ldots, \delta'_2(d_2, \ldots, d_k)$, $r'(d_2, \ldots, d_k)$, and $m_0(d_2, \ldots, d_k)$ be the functions guaranteed by Theorem 5.2. We also set $d_k = \varepsilon/100$. With intention to apply Theorem 4.14 we choose

$$\begin{aligned} \delta_k &= \min\{\varepsilon/100, \delta'_k(d_k)\}, \\ \mu &= \varepsilon/100, \end{aligned} \tag{6.1}$$

$$\delta_i(d_i, \ldots, d_{k-1}) = \min\left\{\delta'_i(d_i, \ldots, d_{k-1}, \tfrac{\varepsilon}{100}), \tfrac{d_i}{2}\right\},\ i = 2, \ldots, k-1, \tag{6.2}$$

$$r(a_1, d_2, \ldots, d_{k-1}) = r'(d_2, \ldots, d_{k-1}, \varepsilon/100), \tag{6.3}$$

and obtain integers N_k and L_k. Set

$$\delta = \frac{1}{2} \times \left(L_k^{-2^k}\right)^{2^t} \times \left(\frac{\varepsilon}{100}\right)^{\binom{t}{k}} \times \left(\frac{1}{L_k}\right)^t, \tag{6.4}$$

and

$$n_0 = \max\left\{N_k, m_0(L_k^{-2^k}, \ldots, L_k^{-2^k}, \varepsilon/100) \times L_k, \frac{2t}{\delta}\right\}. \tag{6.5}$$

Suppose that $\mathcal{H}$ is a k-uniform hypergraph with $n > n_0$ vertices and with at most δn^t copies of $\mathcal{F}$. Applying Theorem 4.14 to $\mathcal{H}$ yields a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ and a vector $\boldsymbol{d} = (d_2, \ldots, d_{k-1}) \in (0, 1]^{k-2}$ such that

(i) $\mathscr{P}$ is $(\mu, \boldsymbol{\delta}(\boldsymbol{d}), \boldsymbol{d}, r(a_1, \boldsymbol{d}))$-equitable family of partitions,
(ii) $\mathscr{P}$ is $(\delta_k, r(a_1, \boldsymbol{d}))$-regular with respect to $\mathcal{H}$,
(iii) $a_1, \ldots, a_{k-1} \le L_k$.

Observe that now, when the family of partitions $\mathscr{P}$ is found, the parameters $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$ and r become constants. More precisely, since Theorem 4.14 gave rise to a family of partitions $\mathscr{P}$ and a vector $\boldsymbol{d} = (d_2, \ldots, d_{k-1})$, in view of (6.2) and (6.3) these densities fix the values of $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_{k-1})$ and r.

We now delete all edges K from $\mathcal{H}$ for which one of the following holds:

(a) $K \notin \mathrm{Cross}_k(\mathscr{P}^{(1)})$,

(b) $K \in \text{Cross}_k(\mathscr{P}^{(1)})$ but $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)}(K)$ (cf. (4.3)) is not a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(k, k-1)$-complex,
(c) $K \in \text{Cross}_k(\mathscr{P}^{(1)})$ but $\mathcal{H} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K))$ is not (δ_k, r)-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(K)$, or
(d) $K \in \text{Cross}_k(\mathscr{P}^{(1)})$ and $d_{\mathcal{H}}(\hat{\mathcal{P}}^{(k-1)}(K)) \leq \varepsilon/100$.

It follows from (i) that there are at most $\mu\binom{n}{k}$ edges in $\mathcal{H}$ satisfying (a) or (b). Moreover, (ii) implies that the number of edges considered in (c) is bounded by $\delta_k\binom{n}{k}$. Consequently, we removed at most $(\mu+\delta_k+\varepsilon/100)\binom{n}{k} \leq \varepsilon n^k$ edges from $\mathcal{H}$ to obtain $\mathcal{H}'$. We claim this yields a subhypergraph $\mathcal{H}'$ without a copy of $\mathcal{F}$.

To the contrary, suppose there is a copy $\mathcal{F}_0$ of $\mathcal{F}$ in $\mathcal{H}'$. Let $V(\mathcal{F}_0) = \{v_1, v_2, \dots, v_t\} \subseteq V(\mathcal{H}')$ and suppose $v_\alpha \in V_{h_\alpha}$ for $\alpha = 1, \dots, t$. Unfortunately, for different vertices $v_\alpha \neq v_{\alpha'}$ the set V_{h_α} may still be equal to $V_{h_{\alpha'}}$ and Theorem 5.2 is not equipped to directly address this problem.

To overcome this difficulty, we construct an auxiliary $(m, \mathcal{F})$-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$, which satisfies the assumptions of Theorem 5.2 and such that the number of crossing, unlabeled copies of $\mathcal{F}$ in $\mathcal{G}^{(k)}$ (see Remark 5.3) gives a lower bound to the number of copies of $\mathcal{F}$ in $\mathcal{H}'$. More precisely, for each $\alpha = 1, \dots, t$ let W_α be a copy of the set V_{h_α} such that for all $\alpha \neq \alpha'$ we have $W_\alpha \neq W_{\alpha'}$. Let $\varphi_\alpha \colon W_\alpha \to V_{h_\alpha}$ be a bijection and for every edge $K \in \mathcal{F}_0$ let $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)}(K) \cup \{\mathcal{H}' \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)\}$ be the unique (k,k)-complex which contains K and is determined by the partition $\mathscr{P}$ and $\mathcal{H}'$. We denote this (k,k)-complex by $\boldsymbol{\mathcal{H}}'(K, \mathscr{P})$.

Consider a copy $\boldsymbol{\mathcal{G}}_K$ of $\boldsymbol{\mathcal{H}}'(K, \mathscr{P})$ on the vertex set $\bigcup_{\alpha \in K} W_\alpha$ with

$$\varphi_K = \bigcup_{\alpha \in K} \varphi_\alpha \colon \bigcup_{\alpha \in K} W_\alpha \to \bigcup_{\alpha \in K} V_{h_\alpha}$$

being an isomorphism between $\boldsymbol{\mathcal{G}}_K$ and $\boldsymbol{\mathcal{H}}'(K, \mathscr{P})$, i.e., an edge preserving bijection for every layer of both complexes. We then set

$$\boldsymbol{\mathcal{G}} = \bigcup_{K \in \mathcal{F}_0} \boldsymbol{\mathcal{G}}_K\,.$$

It follows from the definition of $\boldsymbol{\mathcal{G}}$, that $\boldsymbol{\mathcal{G}}$ is a $((\boldsymbol{\delta}, \delta_k), \geq(\boldsymbol{d}, \varepsilon/100), r)$-regular $(m, \mathcal{F})$-complex. Moreover, all but at most tm^{t-1} crossing, unlabeled copies of $\mathcal{F}$ in $\boldsymbol{\mathcal{G}}$, correspond to copies of $\mathcal{F}$ in $\bigcup_{K \in \mathcal{F}} \boldsymbol{\mathcal{H}}'(K, \mathscr{P})$ and hence to copies of $\mathcal{F}$ in $\mathcal{H}'$. (Possible exceptions are those copies which contain two distinct vertices $w \in W_\alpha$ and $w' \in W_{\alpha'}$ for which $V_{h_\alpha} = V_{h_{\alpha'}}$ and

$\varphi_\alpha(w) = \varphi_{\alpha'}(w').$) In view of Theorem 5.2, therefore, there are at least

$$(1-\nu)\prod_{i=2}^{k-1} d_i^{|\Delta_i(\mathcal{F})|} \times \left(\frac{\varepsilon}{100}\right)^{|\mathcal{F}|} \times m^t - tm^{t-1}$$

$$\stackrel{(6.5)}{\geq} \frac{3}{4}\prod_{i=2}^{k-1} d_i^{\binom{t}{i}} \times \left(\frac{\varepsilon}{100}\right)^{\binom{t}{k}} \times \left(\frac{n}{a_1}\right)^t - \frac{\delta}{2}n^t \tag{6.6}$$

copies of $\mathcal{F}$ in $\mathcal{H}'$.

In order to complete the argument, we will prove that

$$\frac{1}{2}\prod_{i=2}^{k-1} d_i^{\binom{t}{i}} \times \left(\frac{\varepsilon}{100}\right)^{\binom{t}{k}} \times \left(\frac{1}{a_1}\right)^t > \delta\,, \tag{6.7}$$

which in view of (6.6) contradicts the assumption that $\mathcal{H} \supseteq \mathcal{H}'$ contains less than δn^t copies of $\mathcal{F}$. Since $a_1 \leq L_k$ (cf. (iii) above), the next claim implies (6.7) and, hence, concludes the proof of Theorem 1.3. □

Claim 6.1. *$d_j > L_k^{-2^k}$ for $j = 2, \ldots, k-1$.*

Proof. Let $2 \leq j \leq k-1$ and suppose $d_j \leq L_k^{-2^k}$. Recall that $a_i \leq L_k$ (cf. (iii)) for all $i = 1, \ldots, k-1$. Consequently, the number of $(j, j-1)$-polyads of the partition is at most

$$\binom{a_1}{j}\prod_{i=2}^{j-1} a_i^{\binom{j}{i}} \leq L^{2^j-2} \leq L^{2^{k-1}}.$$

We now bound the number of j-tuples in (δ_j, d_j, r)-regular j-polyads of $\mathscr{P}$. For that we observe $m^j = (n/a_1)^j \leq (n/j)^j \leq \binom{n}{j}$ and, consequently, the number of j-tuples in (δ_j, d_j, r)-regular polyads is at most

$$\begin{aligned}(d_j+\delta_j)\times m^j \times L^{2^{k-1}} &\leq \frac{3d_j}{2}\times\binom{n}{j}\times L^{2^{k-1}} \\ &\leq \frac{3L_k^{-2^k}}{2}\times\binom{n}{j}\times L_k^{2^{k-1}} \\ &< \frac{3}{4}\binom{n}{j}.\end{aligned} \tag{6.8}$$

The last line follows from the obvious inequality $L_k \geq 2$.

On the other hand, each j-tuple J, which is either not crossing or does not belong to a $((\delta_2, \ldots, \delta_{j-1}), (d_2, \ldots, d_{j-1}), r)$-regular $(j, j-1)$-complex (call those J *bad*) extends to $\binom{n-j}{k-j}$ k-tuples in V containing J. Each such k-tuple necessarily is either not crossing with respect to $\mathscr{P}^{(1)}$ or belongs to $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-irregular $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)} \in \mathrm{Com}_{k-1}(\mathscr{P})$. Since $\mathscr{P}$ is $(\mu, \boldsymbol{\delta}, \boldsymbol{d}, r)$-equitable (see (i) above) there are at most $\mu\binom{n}{k}$ such k-tuples (containing bad j-tuples).

Consequently the number of bad j-tuples must be less than

$$\frac{\mu\binom{n}{k}\binom{k}{j}}{\binom{n-j}{k-j}} = \mu\binom{n}{j}.$$

Since $\mu \leq 1/4$, this contradicts (6.8) and hence the claim follows. □

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