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Chapter 1

Introduction

While proving his famous Density Theorem [Sze75], E. Szemerédi invented an auxiliary lemma which later proved to be a powerful tool in extremal graph theory. This lemma and its improved version named the Regularity lemma [Sze78], states that all sufficiently large graphs can be approximated by "random-like" graphs. This feature is especially useful in situations when the problem in question is easier to prove for random graphs.

In particular, one such situation arises when the counting copies of a given small graph in another graph. Although this problem is very hard in general, there is a simple counting argument which counts these copies in the approximation produced by the Regularity Lemma. Since this counting argument applied together with the Regularity Lemma have had numerous applications (see [KS96] for survey), a natural question arises whether it can be generalized to hypergraphs.

There were generalizations of the Regularity Lemma [FR92, Chu91]. Yet, they have failed to produce "random-like" approximations in which one could count copies of given small hypergraphs and, therefore, the odds for many applications have remainded low.

A breakthrough came when P. Frankl and V. Rödl developed a regularity lemma for 3-uniform hypergraphs [FR00], which yields a copy of the complete 3-uniform hypergraph on 4 vertices $K_4^{(3)}$ in the approximations. This was was later generalized to the counting of arbitrary small 3-uniform hypergraphs by B. Nagle and V. Rödl [NR99], and they were also able to find a number of applications [KNR99, NR00, RR98].

It also turned out that developing a counting argument is a bigger problem than the generalization of the Regularity Lemma (although these two issues cannot be separated completely). The purpose of this work is to develop a counting argument for 4-uniform "random-like" hypergraphs and show that the level of technical complication is significantly higher than in the case of 3-uniform hypergraphs (c.f. [FR00] and Theorem 1.13 below).

However, there is good hope that our approach can be extended to the general case of k-uniform hypergraphs.

1.1 Notation and basic definitions

We start with some definitions. For a set V and an integer $k \geq 2$, let $[V]^k$ denote the system of all k-element subsets of V. An ordered pair $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G})) = (V, E)$, where $E = E(\mathcal{G})$ is a subset of $[V]^k$, is called a *k-uniform hypergraph.* If $k = 2$, we have a *graph.* We call the cardinality of $V(\mathcal{G})$ the *order* of $\mathcal{G}$ whereas the cardinality of the set $E(\mathcal{G})$ is called the *size* of $\mathcal{G}$.

Let $V = V_1 \cup \cdots \cup V_s$ be a partition, we say that a set $e \subset V$ is *crossing* if $|e \cap V_j| \leq 1$ for all $j = 1, 2, \ldots, s$. Furthermore, a hypergraph $\mathcal{G} = (V_1 \cup \cdots \cup V_s, E)$ is said to be s-partite if its all edges are crossing. We shall also denote by $K_s^{(k)}(V_1, \ldots, V_s)$ the complete k-uniform s-partite hypergraph with partition $V_1 \cup \cdots \cup V_s$.

Since this paper will deal mainly with partite hypergraphs, it is convenient to introduce the term *cylinder*.

Definition 1.1. *Let $s \geq k \geq 1$ be two integers. We define an (s,k)-cylinder $\mathcal{G}$ as follows.*

For $k = 1$, $\mathcal{G}$ is a partition $V(\mathcal{G}) = V_1 \cup \ldots \cup V_s$. For $k > 1$, $\mathcal{G}$ is any s-partite k-uniform hypergraph.

If there is no danger of confusion, we shall identify the hypergraphs (cylinders) with their edge sets.

Definition 1.2. *Let $k = 1$ and let $\mathcal{G}$, $\mathcal{G}'$ be two (s,k)-cylinders, $V(\mathcal{G}) = V_1 \cup \ldots \cup V_s$ and $V(\mathcal{G}') = V_1' \cup \ldots \cup V_s'$. We say that $\mathcal{G}'$ is a subcylinder of $\mathcal{G}$ if $V_i' \subset V_i$ for all $i = 1, 2, \ldots, s$. While for $k > 1$ and two (s,k)-cylinders $\mathcal{G}$, $\mathcal{G}'$, we say that $\mathcal{G}'$ is a subcylinder of $\mathcal{G}$ if $E(\mathcal{G}') \subset E(\mathcal{G})$. Moreover, we say that $\mathcal{G}'$ is an* induced *subcylinder of $\mathcal{G}$ if $E(\mathcal{G}') = E(\mathcal{G}) \cap [V(\mathcal{G}')]^k$.*

If $s = k+1$, we will often write an (s,k)-cylinder $\mathcal{G}$ as $\mathcal{G} = \bigcup_{i=1}^{s} \mathcal{G}_{\hat{i}}$, where $\mathcal{G}_{\hat{i}}$ is the subcylinder of $\mathcal{H}$ induced on $\bigcup_{j \neq i} V_j$.

A subcylinder $\mathcal{G}' = (V', E')$ of $\mathcal{G}$ is a *clique* in $\mathcal{G}$ if $E' = [V']^k$.

Definition 1.3. *For an $(s,1)$-cylinder $\mathcal{G} = V_1 \cup \cdots \cup V_s$ and $1 \leq j \leq s$, we define $\mathcal{K}_j(\mathcal{G}) = K_s^{(j)}(V_1, \ldots, V_s)$. For an (s,k)-cylinder $\mathcal{G}$, where $k > 1$, we shall denote by $\mathcal{K}_j(\mathcal{G})$, $k \leq j \leq s$, the j-uniform hypergraph whose edges are precisely those j-element subsets of $V(\mathcal{G})$ that span cliques of order j in $\mathcal{G}$.*

Clearly, for $k > 1$, the quantity $|\mathcal{K}_j(\mathcal{G})|$ counts the total number of cliques of order j in $\mathcal{G}$. We will often face a situation when we need to describe that one cylinder 'lies on' another cylinder. To this end, we define the term *underlying cylinder*.

Definition 1.4. *Let $\mathcal{G}$ be an $(s,k-1)$-cylinder and $\mathcal{H}$ be an (s,k)-cylinder with the same s-partition. We say that $\mathcal{G}$ underlies $\mathcal{H}$ if $\mathcal{H} \subset \mathcal{K}_k(\mathcal{G})$.*

Through this paper, we will work with a series of underlying cylinders. To accommodate this situation, we introduce the notion of a *complex*.

Definition 1.5. *Let s and k, $s \geq k \geq 2$, be two integers. An (s,k)-complex $\boldsymbol{\mathcal{H}}$ is a system of cylinders $\{\mathcal{H}^{(i)}\}_{i=1}^{k}$ such that*

(a) $\mathcal{H}^{(1)}$ *is an $(s,1)$-cylinder $V_1 \cup \cdots \cup V_s$,*

(b) *for every $i \in [k-1]$, $\mathcal{H}^{(i)}$ underlies $\mathcal{H}^{(i+1)}$, i.e. $\mathcal{H}^{(i+1)} \subset \mathcal{K}_{i+1}(\mathcal{H}^{(i)})$.*

1.2 Regularity for graphs

Before we state the Regularity Lemma, we must introduce the concept of *regular pairs*.

Definition 1.6 (Szemerédi, 1978). *Let $G = (V, E)$ be a graph and δ be a positive real number, $0 < \delta \leq 1$. We say that a pair (A, B) of two disjoint subsets of V is δ-regular if*

$$|d(A', B') - d(A, B)| < \delta$$

for any two subsets $A' \subset A$, $B' \subset B$, with $|A'| \geq \delta|A|$, $|B'| \geq \delta|B|$. Here, $d(A, B) = |E(A, B)|/(|A||B|)$ stands for the density *of the pair (A, B).*

This definition states that a regular pair has uniformly distributed edges. The Regularity Lemma of Szemerédi [Sze78] enables us to partition the vertex set $V(G)$ of a graph G into t sets $V_1 \cup \ldots \cup V_t$ in such a way that most of the pairs (V_i, V_j) satisfy Definition 1.6. The precise statement follows.

Theorem 1.7 (Regularity Lemma). *For every $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exist two integers $N_0 = N_0(\varepsilon, t_0)$ and $T_0 = T_0(\varepsilon, t_0)$ with the following property: for every graph G with $n \geq N_0$ vertices there is a partition of the vertex set into $t + 1$ classes*

$$V = V_0 \cup V_1 \cup \ldots \cup V_t$$

such that

(i) $t_0 \le t \le T_0$,

(ii) $|V_0| \le \varepsilon n$,

(iii) $|V_1| = \ldots = |V_t|$, *and*

(iv) *all but at most* $\varepsilon\binom{t}{2}$ *pairs* (V_i, V_j), $1 \le i < j \le t$, *are* ε*-regular.*

Moreover, this lemma is sufficiently strong to ensure the existence of various small subgraphs in G. The easiest case, when we count copies of K_3, is summarized in the next fact.

Fact 1.8. *If all* (V_i, V_j), (V_i, V_k), *and* (V_j, V_k) *are* ε*-regular pairs with density* d *and* $2\varepsilon < d$, *then*

$$(1-2\varepsilon)(d-\varepsilon)^3|V_i||V_j||V_k| \le |\mathcal{K}_3(G \cap K(V_i, V_j, V_k))| \le \left[2\varepsilon + (d+\varepsilon)^3\right]|V_i||V_j||V_k|.$$

This fact and its extensions (c.f. Fact 3.5) are a key to many applications of the Regularity Lemma (c.f. [CRST83, KSS97, KS96]) .

1.3 Regularity for hypergraphs

Now we define the notion of *regularity* for cylinders:

Definition 1.9. *Let* $\mathcal{G}$ *be a* $(k, k-1)$*-cylinder underlying a* (k, k)*-cylinder* $\mathcal{H}$*. We say that* $\mathcal{H}$ *is* (δ, d)*-regular with respect to* $\mathcal{G}$ *if the following condition is satisfied: whenever* $\mathcal{G}' \subset \mathcal{G}$ *is a* $(k, k-1)$*-cylinder such that*

$$\left|\mathcal{K}_k(\mathcal{G}')\right| \ge \delta\,|\mathcal{K}_k(\mathcal{G})|$$

then

$$(d-\delta)\left|\mathcal{K}_k(\mathcal{G}')\right| \leq \left|\mathcal{H}\cap\mathcal{K}_k(\mathcal{G}')\right| \leq (d+\delta)\left|\mathcal{K}_k(\mathcal{G}')\right|.$$

Note that for $k=2$, Definition 1.9 varies from Szemerédi's definition of a δ-regular pair (V_1, V_2) (cf. [Sze78]). It is easy to observe that

- (δ, d)-regularity implies $\delta^{1/2}$-regularity in the sense of Definition 1.6, and
- δ-regularity in the above sense gives also (δ, d)-regularity.

For $k>2$, the situation becomes more complicated and due to the quantification of constants in the hypergraph regularity lemma (Remark 4.6, [FR00]), Definition 1.9 is not strong enough to have the effect of Definition 1.6 in the case $k=2$. To overcome this problem, P. Frankl and V. Rödl introduced in [FR00] the concept of (δ, r)-*regularity.* Here we present this concept in a more general form.

Definition 1.10. *Let $r\in\mathbb{N}$ and $\mathcal{G}$ be a $(k,k-1)$-cylinder underlying a (k,k)-cylinder $\mathcal{H}$. We say that $\mathcal{H}$ is (δ,d,r)-regular with respect to $\mathcal{G}$ if the following condition is satisfied: whenever $\mathcal{G}_1,\ldots,\mathcal{G}_r\subset\mathcal{G}$ are $(k,k-1)$-cylinders such that*

$$\left|\bigcup_{j=1}^{r}\mathcal{K}_k(\mathcal{G}_j)\right| \geq \delta\left|\mathcal{K}_k(\mathcal{G})\right|,$$

then

$$(d-\delta)\left|\bigcup_{j=1}^{r}\mathcal{K}_k(\mathcal{G}_j)\right| \leq \left|\mathcal{H}\cap\bigcup_{j=1}^{r}\mathcal{K}_k(\mathcal{G}_j)\right| \leq (d+\delta)\left|\bigcup_{j=1}^{r}\mathcal{K}_k(\mathcal{G}_j)\right|. \tag{1.1}$$

We extend the above definition to the case of an (s,k)-cylinder $\mathcal{H}$.

Definition 1.11. *Let $r\in\mathbb{N}$ and $\mathcal{G}$ be an $(s,k-1)$-cylinder underlying an (s,k)-cylinder $\mathcal{H}$. We say that $\mathcal{H}$ is (δ,d,r)-regular with respect to $\mathcal{G}$ if $\mathcal{H}\left[\bigcup_{j\in I}V_j\right]$ is (δ,d,r)-regular with respect to $\mathcal{G}\left[\bigcup_{j\in I}V_j\right]$ for all $I\in[s]^k$.*

Now we are ready to introduce the concept of regularity for an (s,k)-complex $\boldsymbol{\mathcal{H}}$.

Definition 1.12. *Let $r \in \mathbb{N}$ and $\boldsymbol{d} = (d_2, \dots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ be two vectors of positive real numbers such that $0 < \delta_i < d_i \leq 1$ for all $i = 2, \dots k$. We say that an (s, k)-complex $\boldsymbol{\mathcal{H}}$ is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular if*

(a) $\mathcal{H}^{(2)}$ is (δ_2, d_2)-regular with respect to $\mathcal{H}^{(1)}$, and

(b) $\mathcal{H}^{(i+1)}$ is $(\delta_{i+1}, d_{i+1}, r)$-regular with respect to $\mathcal{H}^{(i)}$ for every $i \in [k-1] \setminus \{1\}$.

P. Frankl and V. Rödl proved the regularity lemma which allows splitting of the edge set of an arbitrary 3-uniform hypergraph into $(3, 3)$-complexes in a way similar to the manner in which the Szemerédi Lemma paritions the edge set of the graphs into biparitite graphs most of which are ε-regular. (Here ε-regularity has been replaced by (δ, d, r)-regularity of corresponding 3-complexes.) They also proved the following theorem extending Fact 1.8 from graphs to 3-uniform hypergraphs.

Theorem 1.13 (P. Frankl, V. Rödl [FR00]). *For any $\nu > 0$ and any $d_3 \in (0, 1]$ there is a real number δ_3 such that for any positive real number $d_2 \in (0, 1]$ there exists δ_2 and $r \in \mathbb{N}$ such that if $\boldsymbol{\mathcal{H}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(4, 3)$-complex, where $\boldsymbol{d} = (d_2, d_3)$ and $\boldsymbol{\delta} = (\delta_2, \delta_3)$, then $\mathcal{H}^{(3)}$ contains $(1 \pm \nu) d_3^{\binom{4}{3}} d_2^{\binom{4}{2}} n^4$ copies of $K_4^{(3)}$.*

Here $1 \pm \nu$ stands for a number in the interval $(1 - \nu, 1 + \nu)$. This theorem plays the role of Fact 1.8. Indeed, it enables us to find copies of the complete 3-uniform hypergraph on 4 vertices $K_4^{(3)}$ in 3-cylinders underlied by a regular sparse 2-cylinder. However, this theorem would be useless without an appropriate version of a regularity lemma for 3-uniform hypergraphs. Such a lemma was also introduced in [FR00]. Moreover, this result was extended by B. Nagle and V. Rödl in [NR99] who developed an argument for counting copies of complete 3-uniform hypergraphs on k vertices $K_k^{(3)}$.

1.4 The main result

The aim of this thesis is to extend Theorem 1.13 to 4-uniform hypergraphs.

Theorem 1.14 (Main Theorem). *For any $\nu > 0$ the following statement holds.*

For every $d_4 \in (0,1]$, there is a real number δ_4 such that for any $d_3 \in (0,1]$, there exists a real number δ_3 such that for any $d_2 \in (0,1]$, there is δ_2 and $r \in \mathbb{N}$ with the property that whenever $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \mathcal{H}^{(3)}, \mathcal{H}^{(4)}\}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(5,4)$-complex, where $\boldsymbol{d} = (d_2, d_3, d_4)$ and $\boldsymbol{\delta} = (\delta_2, \delta_3, \delta_4)$, then $\mathcal{H}^{(4)}$ contains

$$(1 \pm \nu) d_4^{\binom{5}{4}} d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5$$

copies of $K_5^{(4)}$.

An appropriate version of the regularity lemma for 4-uniform hypergraphs has been developed in [RS00]. Extending the regularity lemma of Frankl and Rödl, this lemma allows us to partition the edge set of an arbitrary 4-uniform hypergraph into $(4,4)$-complexes, most of which are regular in the sense of Theorem 1.14.

Perhaps surprisingly, it appears that it is not an extension of the the Regularity Lemma but rather generalizations of Fact 1.8 which cause the difficulties in the hypergraphs case. Unlike the proof of Fact 1.8, which is straightforward, the proof of Theorem 1.13 is already quite complex. Theorem 1.14, as an extension of Theorem 1.13 to 4-uniform hypergraphs, is yet much more difficult prove.

We believe that Theorem 1.14 is the last step towards a proof of the following general counting statement:

Conjecture 1.15. *For any $\nu > 0$ and any $k \in \mathbb{N}$, the following is true: $\forall d_k \in (0,1]$ $\exists \delta_k$ $\forall d_{k-1} \in (0,1]$ $\exists \delta_{k-1}$ $\ldots$ $\forall d_2 \in (0,1]$ $\exists r \in \mathbb{N}$ $\exists \delta_2$ such that if $\boldsymbol{\mathcal{H}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(k+1,k)$-complex, where $\boldsymbol{d} = (d_2, \ldots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$, then $\mathcal{H}^{(k)}$ contains*

$$(1 \pm \nu) \prod_{s=2}^{k} d_s^{\binom{k+1}{s}} \times n^{k+1}$$

copies of $K_{k+1}^{(k)}$.

This together with a general regularity lemma for k-uniform hypergraph introduced in [RS00] (which is proved under the assumption of validity of this conjecture) is the last step towards a fully applicable regularity lemma for k-uniform hypergraphs.

1.5 Structure of the thesis

The structure of this work is the following: in the next chapter, we first introduce Propositions 2.2-2.6 and later use these propositions to prove the Main Theorem. In Chapter 3 we describe various properties of 2-cylinders and prove Propositions 2.2 and 2.3. Then, in Chapter 4, we prove the so-called l-graphs Lemma which counts copies of K_3 in the series of nested regular 2-cylinders. This lemma plays an important role in the investigation of properties of 3- and 4-cylinders. In Chapter 5, we discuss and prove various properties of regular 3-cylinders. The proofs of Propositions 2.4 and 2.5 are also presented in this chapter. The last chapter, Chapter 6, provides some theory of regular 4-uniform hypergraphs and proves Proposition 2.6.

Chapter 2

Proof of the Main Theorem

The goal of this section is to prove the Main Theorem. We first state all necessary concepts and propositions and later use them in the actual proof. One of the central concepts in the proof of Theorem 1.14 is the notion of the *link* of a vertex.

Definition 2.1. *Let $\mathcal{G}$ be a k-uniform hypergraph and $x \in V(\mathcal{G})$. We will call the set*

$$\mathcal{G}(x) = \{e \setminus \{x\} \colon e \in \mathcal{G},\ x \in e\}$$

the link of the vertex x in $\mathcal{G}$. Note that $\mathcal{G}(x)$ is a $(k-1)$-uniform hypergraph. Moreover, if $\mathcal{G}$ is an (s,k)-cylinder, then $\mathcal{G}(x)$ is an $(s-1,k-1)$-cylinder. For a subset $W \subset V(\mathcal{G})$, we define $\mathcal{G}(W)$ by

$$\mathcal{G}(W) = \bigcap_{x \in W} \mathcal{G}(x). \tag{2.1}$$

For simplicity, if $W = \{x_1, \ldots, x_k\}$, we write $\mathcal{G}(x_1, \ldots, x_k)$.

Through the remainder of the paper we fix a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(5,4)$-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \mathcal{H}^{(3)}, \mathcal{H}^{(4)}\}$ and $\nu > 0$, and we will assume that $\mathcal{H}^{(1)} = V_1 \cup \ldots \cup V_5$ and $|V_1| = \ldots = |V_5| = n$. The purpose of this condition is to simplify the proof and all statements remain valid for partite sets with different sizes.

Let us recall the quantification of the constants in Theorem 1.14:

$$\forall d_4 \exists \delta_4 \forall d_3 \exists \delta_3 \forall d_2 \exists \delta_2 \exists r.$$

Due to this quantification we may assume

$$\begin{aligned} \nu \gg \delta_4 > 0 \\ 1 > d_4 \gg \delta_4 > 0 \\ 1 > d_3 \gg \delta_3 > 0 \\ 1 > d_2 \gg \delta_2 > 0 \\ \delta_4 \gg \delta_3 \gg \delta_2, \\ r > d_3^{-100} d_2^{-100}. \end{aligned} \tag{2.2}$$

The main role of the fourth condition is to simplify the error terms. Without it we would have to carry on long strings of error terms depending on δ_2, δ_3, and δ_4.

Our proof will be based on the following propositions.

Proposition 2.2. *For all but at most $8\delta_2^{1/2} n$ vertices $x \in V_1$*

$$|\mathcal{K}_4(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)])| \leq 2 d_2^{\binom{5}{2}} n^4. \tag{2.3}$$

Proposition 2.3. *For all but but at most $16\delta_2^{1/2} n$ pairs of vertices x, $x' \in V_5$*

$$|\mathcal{K}_4(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x, x')])| \leq 2 d_2^{14} n^4. \tag{2.4}$$

Proposition 2.4. *For all but but at most $46\delta_3^{1/2} n$ vertices $x \in V_1$*

$$\frac{1}{2} d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \leq |\mathcal{K}_4(\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x)))| \leq 2 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4. \tag{2.5}$$

Proposition 2.5. *For all but but at most $60\delta_3^{1/16} n^2$ pairs of vertices x, $x' \in V_1$*

$$|\mathcal{K}_4(\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x, x')))| \leq 2 d_3^{16} d_2^{14} n^4. \tag{2.6}$$

Proposition 2.6. *For all but but at most $10\delta_4^{1/4}n$ vertices $x \in V_1$*

$$|\mathcal{K}_4(\mathcal{H}^{(4)}(x))| = (1 \pm \nu/2)d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4. \tag{2.7}$$

Since the proofs of these propositions are rather complex, we defer them until later. Propositions 2.2 and 2.3 are proved in Chapter 3. In order to prove Propositions 2.4, 2.5, and 2.6, we need the so called k-graphs Lemma (see Chapter 4) and a number of additional claims. Therefore, the proofs of Propositions 2.4 and 2.5 are in Chapter 5, and the proof of Proposition 2.6 is given in Chapter 6.

We will also need the following lemma:

Lemma 2.7 (Picking Lemma). *Let V be a set of size m, k be a nonnegative integer, and $\mathcal{P}_1, \ldots \mathcal{P}_k$ be arbitrary graphs on V. Furthermore, suppose that $|\mathcal{P}_1| \leq \sigma_1 m^2$, $|\mathcal{P}_2| \leq \sigma_2 m^2, \ldots |\mathcal{P}_k| \leq \sigma_k m^2$. Then for every subset $W \subset V$ with at least cm elements and a positive integer t such that*

$$\frac{2\sigma_1 t^2}{c^2} < \frac{1}{k}, \tag{2.8}$$

there exists a choice of t vertices $x_1, x_2, \ldots, x_t \in W$ such that

i) $\{x_u, x_v\} \notin \mathcal{P}_1$ for all $1 \leq u < v \leq t$,

ii) for all $i \in [k]\backslash\{1\}$, $\{x_u, x_v\} \notin \mathcal{P}_i$ for all but at most $\frac{2k\sigma_i}{c^2}t^2$ pairs $1 \leq u < v \leq t$.

Proof. Choose randomly a t-element subset $R \subset W$. We will show that

$$\mathbb{P}(|\mathcal{P}_1 \cap [R]^2| \geq 1) < \frac{1}{k}, \tag{2.9}$$

and

$$\mathbb{P}(|\mathcal{P}_i \cap [R]^2| \geq \frac{2k\sigma_i}{c^2}t^2) < \frac{1}{k} \tag{2.10}$$

for all $i \in [k]\backslash\{1\}$, which implies the existence of an t-element subset satisfying conditions i) and ii).

Both (2.9) and (2.10) follow from Markov's inequality. Indeed, the probability that a randomly chosen pair $\{x, y\}$ is an edge of $\mathcal{P}_i$ can be bounded as follows

$$\mathbb{P}(xy \in E(\mathcal{P}_i)) \leq \frac{\sigma_i m^2}{\binom{|W|}{2}} \leq \frac{\sigma_i m^2}{\frac{|W|^2}{4}} \leq \frac{4\sigma_i}{c^2}.$$

Using (2.8), the expected number of edges in a random selection of t vertices from W is

$$\mathbb{E}(|\mathcal{P}_1 \cap [R]^2|) \leq \binom{t}{2}\frac{4\sigma_1}{c^2} \leq \frac{t^2}{2}\frac{4\sigma_1}{c^2} < \frac{1}{k}.$$

Similarly,

$$\mathbb{E}(|\mathcal{P}_i \cap [R]^2|) \leq \binom{t}{2}\frac{4\sigma_i}{c^2} < \frac{t^2}{2}\frac{4\sigma_i}{c^2} = \frac{2t^2\sigma_i}{c^2}.$$

At this point we apply Markov's inequality and we obtain (2.9) and (2.10):

$$\mathbb{P}(|\mathcal{P}_1 \cap [R]^2| \geq 1) \leq \mathbb{E}(|\mathcal{P}_1 \cap [R]^2|) < \frac{1}{k}$$

and

$$\mathbb{P}\left(|\mathcal{P}_i \cap [R]^2| \geq \frac{2k\sigma_i}{c^2}t^2\right) < \frac{\frac{2t^2\sigma_i}{c^2}}{k\frac{2t^2\sigma_i}{c^2}} = \frac{1}{k}.$$

□

Now we are ready to prove the Main Theorem.

Proof of Theorem 1.14. Let W be the set of all vertices $x \in V_1$ satisfying inequality (2.7). Thus, for every vertex $x \in W$ we have:

$$|\mathcal{K}_4(\mathcal{H}^{(4)}(x))| \quad = \quad (1 \pm \nu/2)d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4. \tag{2.11}$$

By Proposition 2.6 we know that

$$|W| \geq \left(1 - 100\delta_4^{1/4}\right) n.$$

Since the proof is rather complex and long, we outline its idea first. For every vertex $x \in W$ there are $(1 \pm \nu/2)d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ copies of $K_4^{(3)}$ in $\mathcal{H}^{(4)}(x)$. Notice that every such $K_4^{(3)}$ together with x form a copy of $\left(K_5^{(4)} \setminus \text{edge}\right)$ in $\mathcal{H}^{(4)}$. Therefore, we would like to apply the (δ_4, d_4, r)-regularity of $\mathcal{H}^{(4)}$ on these copies to obtain the uncounted for edge.

The number of copies of $K_4^{(3)}$ in $\mathcal{H}^{(4)}(x)$ is, however, insufficient to apply the (δ_4, d_4, r)-regularity of $\mathcal{H}^{(4)}$. Indeed, from Theorem 1.13 we have: $|\mathcal{K}_4(\mathcal{H}^{(3)})| \geq (1/2)d_2^6 d_3^4 n^4$. To apply the (δ_4, d_4, r)-regularity of $\mathcal{H}^{(4)}$ we need to satisfy

$$|\mathcal{K}_4(\mathcal{H}^{(4)}(x))| \geq \delta_4 |\mathcal{K}_4(\mathcal{H}^{(3)})|.$$

Since $|\mathcal{K}_4(\mathcal{H}^{(4)}(x))| \leq (1+\nu/2) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$, we obtain

$$(1+\nu/2) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \geq \delta_4 \times (1/2) d_2^6 d_3^4 n^4$$

or $(1+\nu/2) d_4^4 d_3^6 d_2^4 \geq \delta_4$. This is impossible to satisfy due to the order of constants and quantification of this theorem.

Thus, we must use the full power of r-regularity. We select $r = 2\delta_4^{1/2}/(d_2^4 d_3^6)$ vertices $x_1, \ldots, x_r$ from W in such a way that the size of $\bigcup_{j=1}^r \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))$ is sufficiently large to apply the regularity of $\mathcal{H}^{(4)}$, i.e.

$$\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq |\mathcal{K}_4(\mathcal{H}_{\hat{1}}^{(3)})|. \tag{2.12}$$

In order to choose this r-tuple of vertices with a large union, we will use the Picking Lemma and the fact that

$$\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq \sum_{j=1}^{r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| - \sum_{1 \leq i < j \leq r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i, x_j))\right|.$$

The Picking Lemma and Proposition 2.6 will guarantee the choice of the r-tuple $x_1, \ldots, x_r$ for which $\sum_{j=1}^r \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|$ is "large", whereas the same lemma and Propositions 2.3 and 2.5 will make the second term $\sum_{1 \leq i < j \leq r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i, x_j))\right|$ "small".

Since (2.12) holds, we wil be able to apply the (δ_4, d_4, r)-regularity of $\mathcal{H}^{(4)}$ to obtain

$$\left|\mathcal{H}^{(4)} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| = (d_4 \pm \delta_4)\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|.$$

Observe that the left-hand side of the above equation counts the number of copies of $K_5^{(4)}$ that use one of $x_1, \ldots, x_r$ as a vertex. Also note that this number is $O(n^4)$ which is far less than what Theorem 1.14 promises.

To get a full amount of copies of $k_5^{(4)}$ as claimed by the Theorem, we will iterate this process as long as we are able to use the Picking Lemma. At the end we take care of any remaining vertices, i.e. we estimate the number of $K_5^{(4)}$ that use vertices left in W and vertices not satisfying (2.11). Our main tools will be Propositions 2.2 and 2.4.

After describing the idea, we start with a detailed proof. We define two graphs $\mathcal{P}_1$ and $\mathcal{P}_2$, both with vertex set V_1 and edge sets defined by:

$$\begin{aligned} E(\mathcal{P}_1) &= \left\{xx' \colon |\mathcal{K}_4(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,x')])| > 2d_2^{14}n^4\right\}, \\ E(\mathcal{P}_2) &= \left\{xx' \colon |\mathcal{K}_4(\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x,x')))| > 2d_3^{16}d_2^{14}n^4\right\}. \end{aligned}$$

It follows from Propositions 2.3 and 2.5 that the sizes of $\mathcal{P}_1$ and $\mathcal{P}_2$ are bounded, more precisely, $|\mathcal{P}_1| \leq 16\delta_2^{1/2}n^2$ and $|\mathcal{P}_2| \leq 60\delta_3^{1/16}n^2$.

We apply the Picking Lemma on W with parameters $\sigma_1 = 16\delta_2^{1/2}$, $\sigma_2 = 60\delta_3^{1/16}$, $t = r = 2\delta_4^{1/2}/(d_2^4 d_3^6)$, $c = \delta_4^{1/2}$, and obtain r vertices $x_1, \ldots, x_r \in W$ such that all pairs $\{x_i, x_j\}$ satisfy

$$|\mathcal{K}_4(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x_i,x_j)])| \leq 2d_2^{14}n^4, \tag{2.13}$$

and all but $\left(2 \times 2 \times 60\delta_3^{1/16}/\delta_4\right) r^2 \leq \delta_3^{1/32} r^2$ pairs $\{x_i, x_j\}$ satisfy

$$|\mathcal{K}_4(\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x_i,x_j)))| \leq 2d_3^{16}d_2^{14}n^4. \tag{2.14}$$

This is possible as long as $|W| \geq c \times m = \delta_4^{1/2} m$ and condition (2.8) is satisfied, in other words, if

$$\frac{2 \times 16\delta_2^{1/2} \times t^2}{\left(\delta_4^{1/2}\right)^2} < \frac{1}{2} \tag{2.15}$$

holds. This is true because

$$\frac{2 \times 16\delta_2^{1/2} \times t^2}{\left(\delta_4^{1/2}\right)^2} = \frac{128\delta_2^{1/2}}{d_2^8 d_3^{12}} \leq 128 \times \frac{\delta_2^{1/4}}{d_2^8} \times \frac{\delta_3^{1/4}}{d_3^{12}} \leq 128 \times \delta_2^{1/8} \times \delta_3^{1/8} < \frac{1}{2}.$$

Here we used assumption (2.2): $\delta_2 \ll d_2 < 1$, $\delta_3 \ll d_3 < 1$, and $\delta_2 \ll \delta_3$.

Now we estimate the size of $\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))$. We first apply Observation 4.5:

$$\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq \sum_{j=1}^{r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| - \sum_{1 \leq i < j \leq r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i)) \cap \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|.$$

The next step is to estimate both terms on the right-hand side. The first term is easier to handle, we use inequality (2.11):

$$\sum_{j=1}^{r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq r \times (1 - \nu/2) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4. \tag{2.16}$$

To get an estimate for the second term, we must observe several facts:

- $\mathcal{K}_4(\mathcal{H}^{(4)}(x_i) \cap \mathcal{K}_4(\mathcal{H}^{(4)}(x_j)) = \mathcal{K}_4(\mathcal{H}^{(4)}(x_i, x_j))$ for every $1 \leq i < j \leq r$.
- It follows from the fact that $\mathcal{H}^{(4)} \subset \mathcal{K}_4(\mathcal{H}^{(3)})$ that every copy of $K_4^{(3)}$ in $\mathcal{H}^{(4)}(x_i, x_j)$ is also a copy of $K_4^{(3)}$ in $\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x_i, x_j))$.
- Every copy of $K_4^{(3)}$ in $\mathcal{H}^{(4)}(x_i, x_j)$ is also a copy of K_4 in $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x_i, x_j)]$). This again follows from the fact that $\mathcal{H}^{(4)} \subset \mathcal{K}_4(\mathcal{H}^{(3)})$ and $\mathcal{H}^{(3)} \subset \mathcal{K}_3(\mathcal{H}^{(2)})$.

Since we know that all but at most $\delta_3^{1/32} r^2$ pairs $\{x_i, x_j\}$ satisfy (2.14), for these pairs we use the estimate

$$\left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i, x_j))\right| \leq |\mathcal{K}_4(\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x_i, x_j)))| \leq 2 d_3^{16} d_2^{14} n^4. \tag{2.17}$$

The remaining $\delta_3^{1/32}r^2$ pairs $\{x_i, x_j\}$ satisfy (2.13), thus we estimate $|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i, x_j))|$ as

$$|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i, x_j))| \le |\mathcal{K}_4(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x_i, x_j)])| \le 2d_2^{14}n^4. \tag{2.18}$$

Now we combine (2.17) and (2.18) to obtain

$$\sum_{1\le i<j\le r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i)) \cap \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \le \binom{r}{2} \times 2d_3^{16}d_2^{14}n^4 + \delta_3^{1/32}r^2 \times 2d_2^{14}n^4.$$

We use the assumption $\delta_3 \ll d_3$ and conclude that $\delta_3^{1/32}r^2 \times 2d_2^{14}n^4 \le d_3^{16}r^2d_2^{14}n^4$. Then,

$$\sum_{1\le i<j\le r} \left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i)) \cap \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \le 2r^2d_3^{16}d_2^{14}n^4. \tag{2.19}$$

Using (2.16), (2.19), and the definition of r (recall $r = 2\delta_4^{1/2}/(d_2^4d_3^6)$), we obtain that

$$\begin{aligned}\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| &\ge r \times (1-\nu/2)d_4^4d_3^{\binom{5}{3}}d_2^{\binom{5}{2}}n^4 - 2r^2d_3^{16}d_2^{14}n^4 \\ &\ge \delta_4^{1/2}d_4^4d_3^4d_2^6n^4 - 8\delta_4d_3^4d_2^6n^4 \overset{(2.2)}{\ge} 2\delta_4d_3^4d_2^6n^4.\end{aligned} \tag{2.20}$$

Note that 3-cylinder $\mathcal{H}_{\hat{1}}^{(3)}$ is (δ_3, d_3, r)-regular with respect to $\mathcal{H}_{\hat{1}}^{(2)}$ and $\mathcal{H}_{\hat{1}}^{(2)}$ is (δ_2, d_2)-regular. Furthermore, the quantification of this theorem allows us to choose δ_3 and δ_2 so that the assumptions of Theorem 1.13 are satisfied. Thus, we infer that $\left|\mathcal{K}_4(\mathcal{H}_{\hat{1}}^{(3)})\right| \le 2d_3^4d_2^6n^4$. Therefore,

$$\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \ge \delta_4 \left|\mathcal{K}_4(\mathcal{H}_{\hat{1}}^{(3)})\right|,$$

so, by the (δ_4, d_4, r)-regularity of $\mathcal{H}^{(4)}$ with respect to $\mathcal{H}^{(3)}$, we obtain

$$d_4 - \delta_4 \le \frac{\left|\mathcal{H}^{(4)} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|}{\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|} \le d_4 + \delta_4. \tag{2.21}$$

From the above inequality and (2.11), one can easily conclude:

$$\left|\mathcal{H}^{(4)} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \leq (d_4+\delta_4)\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|$$

$$\leq (d_4+\delta_4)\sum_{j=1}^{r}\left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \leq (d_4+\delta_4) r(1+\nu/2) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4.$$

In order to get a lower bound on $\left|\mathcal{H}^{(4)} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|$, we first use (2.21):

$$\left|\mathcal{H}^{(4)} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq (d_4-\delta_4)\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|. \tag{2.22}$$

Second, we want to apply Observation 4.5 with $a = \delta_4^{1/4}$ and obtain:

$$\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq \left(1-\delta_4^{1/4}\right)\sum_{j=1}^{r}\left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|. \tag{2.23}$$

In order to do this, we must show that

$$\delta_4^{1/4} \times \sum_{j=1}^{r}\left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| - \sum_{1\leq i<j\leq r}\left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i)) \cap \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq 0.$$

This is easy to verify using (2.16), (2.19), and $d_4 \gg \delta_4$. Indeed,

$$\delta_4^{1/4} \times \sum_{j=1}^{r}\left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \overset{(2.16)}{\geq} \delta_4^{1/4} \times \delta_4^{1/2} d_4^4 d_3^4 d_2^6 n^4 \geq 8\delta_4 d_3^4 d_2^6 n^4,$$

and

$$\sum_{1\leq i<j\leq r}\left|\mathcal{K}_4(\mathcal{H}^{(4)}(x_i)) \cap \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \overset{(2.19)}{\leq} 2r^2 d_3^{16} d_2^{14} n^4 \leq 8\delta_4 d_3^4 d_2^6 n^4.$$

Then, we combine inequality (2.22) and (2.23) and get:

$$\left|\mathcal{H}^{(4)} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \geq d_4\left(1-\delta_4^{1/4}\right)^2\left|\bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right|$$

We apply (2.16) and get:

$$\left|\mathcal{H}^{(4)} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(\mathcal{H}^{(4)}(x_j))\right| \overset{(2.16)}{\geq} \left(1-\delta_4^{1/4}\right)^2 \times r \times (1-\nu/2) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4. \tag{2.24}$$

We remove vertices $x_1, \ldots, x_r$ from W and iterate the whole process again. Due to (2.15), we can repeat this process as long as

$$|W| > \delta_4^{1/2} n. \tag{2.25}$$

This way we produce a sequence of at least $\left(1 - 100\delta_4^{1/4} - \delta_4^{1/2}\right) n/r$ but not more than n/r r-tuples $X^{(1)} = \{x_1, \ldots, x_r\} = \left\{x_1^{(1)}, \ldots, x_r^{(1)}\right\}$, $X^{(2)} = \left\{x_1^{(2)}, \ldots, x_r^{(2)}\right\}$, etc.

Analoguously to (2.24), each iteration produces at least

$$\left(1 - \delta_4^{1/4}\right)^2 \times r \times (1 - \nu/2) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \geq (1 - 3\nu/4) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$$

copies of $K_5^{(4)}$ $\left(\text{each of which uses exactly one vertex from } X^{(i)} = \left\{x_1^{(i)}, \ldots, x_r^{(i)}\right\}\right)$. Note that $\nu \gg \delta_4$ and, therefore, the following lower bound on the number of $K_5^{(4)}$'s in $\mathcal{H}^{(4)}$ holds:

$$\begin{aligned} |\mathcal{K}_5(\mathcal{H}^{(4)})| &\geq (1 - 3\nu/4) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \times \left(1 - 100\delta_4^{1/4} - \delta_4^{1/2}\right) \frac{n}{r} \\ &\overset{(2.2)}{\geq} (1 - \nu) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5. \end{aligned}$$

The upper bound causes some extra difficulties - we must count not only

(i) the contribution of r-tuples of vertices taken from W, but also

(ii) contribution of vertices left in W, and

(iii) vertices not satisfying (2.11).

We will handle each of these categories of vertices separately:

(i) An upper bound on number of copies of $K_5^{(4)}$ produced by taking r-tuples from W can be obtained similar to the lower bound above: every r-tuple is in at most $(d_4 + \delta_4) \times r(1 + \nu/2) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ copies of $K_5^{(4)}$. There are at most n/r such r-tuples, together producing at most

$$(d_4 + \delta_4)(1 + \nu/2) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5 \leq (1 + 3\nu/4) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5$$

copies of $K_5^{(4)}$.

(ii) The number of vertices left in W is at most $\delta_4^{1/2}n$ (c.f.(2.25)). Each such vertex satisfies (2.7) and, consequently, is involved in not more than $2d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ copies of $K_5^{(4)}$. Therefore, this group of vertices contributes at most $\delta_4^{1/2}n \times 2d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 < \delta_4^{1/4} d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5$ copies of $K_5^{(4)}$. We used again the assumption $\delta_4 \ll d_4$.

(iii) Now we must estimate the contribution of vertices not satisfying (2.11). Recall that we have at most $100\delta_4^{1/4}n$ such vertices. We distinguish three categories of these vertices:

- Consider vertices satisfying (2.5). Proposition 2.4 implies that all but $46\delta_3^{1/16}n$ vertices belong to this category. We estimate contribution of every such vertex x by $|\mathcal{K}_4(\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x)))| \leq 2d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$. Therefore, they contribute by at most $100\delta_4^{1/4}n \times 2d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \leq \delta_4^{1/8} d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ copies of $K_5^{(4)}$.

- Consider vertices not satisfying (2.5) but satisfying (2.3). Proposition 2.4 implies that all but $8\delta_2^{1/2}n$ remaining vertices belongs here. Then, each such vertex x is in at most $|\mathcal{K}_4(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)])| \leq 2d_2^{\binom{5}{2}} n^4$ copies of $K_5^{(4)}$. The total contribution of these vertices is then bounded by $80\delta_3^{1/16}n \times 2d_2^{\binom{5}{2}} n^4 \leq \delta_3^{1/32} d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5 \leq \delta_4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5 \leq \delta_4^{1/2} d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5$. Here we used assumptions (2.2).

- The remaining at most $8\delta_2^{1/2}n$ vertices satisfy neither (2.5) nor (2.3). In this case, we use a rough estimate that every vertex is in at most n^4 copies of $K_5^{(4)}$ and, thus, the contribution of these vertices is at most $8\delta_2^{1/2}n \times n^4 \leq \delta_2^{1/4} d_2^{\binom{5}{2}} n^5 \leq \delta_3 d_2^{\binom{5}{2}} n^5 \leq \delta_3^{1/2} d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5 \leq \delta_4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5 \leq \delta_4^{1/2} d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5$.

At this point we are ready to derive the upper bound. We add the contributions of all vertices above and obtain

$$|\mathcal{K}_5(\mathcal{H}^{(4)})| \leq \left(1 + \nu/2 + \delta_4^{1/4} + \delta_4^{1/8} + 2\delta_4^{1/2}\right) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5 \leq (1+\nu) d_4^5 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^5.$$

□

Chapter 3

More Definitions and Facts about Cylinders

The main goal of this Chapter is to extend the notation from the Introduction and to provide some basic facts about cylinders. We will also prove Propositions 2.2 and 2.3.

Definition 3.1. *Let $\mathcal{G}$ be an $(s,2)$-cylinder with s-partition $V = V_1 \cup \ldots \cup V_s$. We define the neighborhood of a vertex $x \in V$ by $N(x) = N_{\mathcal{G}}(x) = \mathcal{G}(x)$ and the degree of x by $\deg(x) = \deg_{\mathcal{G}}(x) = |\mathcal{G}(x)|$. If W is a subset of vertices of V, we define $N(W) = N_{\mathcal{G}}(W) = \mathcal{G}(W)$ and $\deg(W) = \deg_{\mathcal{G}}(W) = |\mathcal{G}(W)|$.*

If $x \notin V_j$, $j \in [s]$, then we set $N_j(x) = N(x) \cap V_j$ and $\deg_j(x) = |N_j(x)|$. Similarly, if $W \cap V_j = \emptyset$, we set $N_j(W) = N(W) \cap V_j$ and $\deg_j(W) = |N_j(W)|$.

Note that almost all of vertices in a regular $(2,2)$-cylinder have nearly the same degree. More precisely, the following fact is true:

Fact 3.2. *Let $\mathcal{G} = (V_1 \cup V_2, E)$ be a (δ, d)-regular bipartite graph with $|V_1| = |V_2| = m$*

and $0 < \delta \le d$. *Then all but at most* $2\delta m$ *vertices* $x \in V_1$ *satisfy*

$$(d-\delta)m \le \deg(x) \le (d+\delta)m.$$

We extend this fact to an arbitrary subset W of vertices.

Fact 3.3. *Let* k *be a positive integer such that* $(d-\delta)^{2k} \ge \delta$, *and let* $\mathcal{G}$ *be a* (δ, d)-*regular* $(s,2)$-*cylinder with* s-*partition* $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$. *Then, all but at most* $2k(s-1)\delta^{1/2}m^k$ k-*tuples of vertices* $\{x_1, x_2, \ldots x_k\} \subseteq V_1$ *satisfy the following condition:*

For every $i \in [k]$ *and every* $j \in [s]\backslash\{1\}$, *if* W *is any subset of* $\{x_1, x_2, \ldots x_k\}$, $|W| = i$, *then*

$$(d-\delta)^i\, m \le \deg_j(W) \le (d+\delta)^i\, m. \tag{3.1}$$

Proof. Note that we can restrict ourselves to the case $s = 2$ because then we apply this result simultaneously to $s-1$ $(2,2)$-cylinders $\mathcal{G}[V_1 \cup V_j]$, $j \in [s]\backslash\{1\}$.

We proceed by induction on k. For $k = 1$, the statement follows from Fact 3.2. Furthermore, assume that the claim is true for $k \ge 1$ and we would like to verify it for $k+1$. There are two possible ways a $(k+1)$-tuple Y can violate condition (3.1).

First, there exists an $i \le k$ such that there is an i-tuple $W \subset Y$, $|W| = i$, which violates (3.1). But then, W is a part of a k-tuple violating (3.1). By the induction assumption, there are at most $2k\delta^{1/2}m^k$ such k-tuples. Therefore, one can find at most $m \times 2k\delta^{1/2}m^k = 2k\delta^{1/2}m^{k+1}$ "bad" $(k+1)$-tuples of this kind.

Second, a $(k+1)$-tuple Y satisfies (3.1) for all i, $1 \le i \le k$, however, either

$$\deg(Y) < (d-\delta)^{k+1}m, \tag{3.2}$$

or

$$\deg(Y) > (d+\delta)^{k+1}m. \tag{3.3}$$

We can estimate the number of such $(k+1)$-tuples as follows. Fix a k-tuple W satisfying (3.1). Let Z_W be the set of all vertices $x \in V_1$ such that the $(k+1)$-tuple $Y = W \cup \{x\}$ satisfies (3.2). If $|Z_W| \geq \delta^{1/2} m$, then

$$|\mathcal{K}_2(Z_W \cup N(W))| = |Z_W||N(W)| \geq \delta^{1/2} m \times (d-\delta)^k m \geq \delta m^2.$$

Since $\mathcal{G}$ is (δ, d)-regular, we get $e(Z_W, N(W)) \geq (d-\delta)|Z_W||N(W)|$. On the other hand, using (3.2), we obtain

$$e(Z_W, N(W)) < |Z_W|(d-\delta)^{k+1} m \leq |Z_W|(d-\delta)|N(W)|$$

which is a contradiction. Thus, we proved $|Z_W| < \delta^{1/2} m$.

Inequality (3.3) is handled similarly. Since there are at most m^k k-tuples W, there exist at most $2\delta^{1/2} m \times m^k = 2\delta^{1/2} m^{k+1}$ $(k+1)$-tuples of this kind. Both cases together give the desired result. □

We will also use the following easy fact.

Fact 3.4. *Let $\mathcal{G}$ be a (δ, d)-regular $(s,2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, and let $V_1' \subseteq V_1$, $V_2' \subseteq V_2, \ldots, V_k' \subseteq V_k$, be subsets such that $|V_j'| \geq \delta^{1/4} m$ for all $j \in [k]$. Let $\mathcal{G}'$ be the subcylinder induced on $V_1' \cup \ldots \cup V_s'$. Then, $\mathcal{G}'$ is $(\delta^{1/2}, d)$-regular.*

Proof. It is an easy consequence of the definition of (δ, d)-regularity. □

Regular cylinders have the property that one can count the actual number of copies of small complete graphs. The precise statement is summarized in the following fact (see e.g. [NR99]):

Fact 3.5. *For any positive integer s and positive real numbers d, δ such that $\delta^{1/4} \leq (d-\delta)^{s-1}$, there exists a function $\theta_{s,d}(\delta)$, $\theta_{s,d}(\delta) \to 0$ as $\delta \to 0$, such that whenever $\mathcal{G}$ is a (δ, d)-regular $(s,2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$, then*

$$(1-\theta_{s,d}(\delta)) m^s d^{\binom{s}{2}} \leq |\mathcal{K}_s(\mathcal{G})| \leq (1+\theta_{s,d}(\delta)) m^s d^{\binom{s}{2}}. \tag{3.4}$$

We will frequently use the following easy corollary of Fact 3.5.

Corollary 3.6. *If δ is sufficiently small (i.e. $\delta \le \delta(s,d)$), then*

$$\frac{3}{4} m^s d^{\binom{s}{2}} \le |\mathcal{K}_s(\mathcal{G})| \le \frac{5}{4} m^s d^{\binom{s}{2}}. \tag{3.5}$$

We now define the notion of a *good vertex*.

Definition 3.7 (good vertex). *Let $\mathcal{G}$ be a (δ,d)-regular $(s,2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$. A vertex $x \in V_1$ is called* good *if it satisfies*

(i) $(d-\delta)m \le \deg_j(x) \le (d+\delta)m$ *for* $j = 2, \ldots, s$ *and* $(s-1,2)$*-cylinder* $\mathcal{G}[\mathcal{G}(x)]$ *is* $(\delta^{1/2}, d)$*-regular,*

(ii) x *extends to at most* $\delta^{1/4}m$ *pairs* $\{x,x'\} \subset V_1$ *not satisfying* $(d-\delta)^2 m \le \deg_j(x,x') \le (d+\delta)^2 m$ *for* $j = 2, \ldots, s$*, or for which* $(s-1,2)$*-cylinder* $\mathcal{G}[\mathcal{G}(x,x')]$ *is not* $(\delta^{1/2}, d)$*-regular,*

(iii) x *extends to at most* $\delta^{1/4}m^2$ *triples* $\{x,x',x''\} \subset V_1$ *not satisfying* $(d-\delta)^3 m \le \deg_j(x,x',x'') \le (d+\delta)^3 m$ *for* $j = 2, \ldots, s$*, or for which* $(s-1,2)$*-cylinder* $\mathcal{G}[\mathcal{G}(x,x',x'')]$ *is not* $(\delta^{1/2}, d)$*-regular.*

We denote by V_{good} the set of all good vertices in V_1.

Suppose that $(d-\delta)^3 \ge \delta^{1/4}$. Then for every vertex x (pair $\{x,x'\}$, triple $\{x,x',x''\}$, respectively) that satisfies condition (3.1), Fact 3.4 guarantees the regularity of $\mathcal{G}[\mathcal{G}(x)]$ ($\mathcal{G}[\mathcal{G}(x,x')]$, $\mathcal{G}[\mathcal{G}(x,x',x'')]$, respectively).

It follows from Fact 3.3 that at most $2(s-1)\delta^{1/2}m$ vertices x, at most $4(s-1)\delta^{1/4}m^2$ pairs $\{x,x'\}$, and at most $6(s-1)\delta^{1/4}m^3$ triples $\{x,x',x''\}$ violate condition (3.1). From this we can conclude that almost all vertices $x \in V_1$ are good.

Observation 3.8. *Let $\mathcal{G}$ be a (δ, d)-regular $(s, 2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$, and $(d-\delta)^3 \geq \delta^{1/4}$. Then*

$$|V_{\text{good}}| \geq \left(1 - 2(s-1)\delta^{1/2} - 10(s-1)\delta^{1/4}\right) m.$$

Now we are ready to prove Proposition 2.2.

Proof of Proposition 2.2. Choose any $x \in V_{\text{good}}$. Then x satisfies

$$(d_2 - \delta_2)n \leq \deg_j(x) \leq (d_2 + \delta_2)n$$

for $j = 2, 3, 4, 5$, and the $(4, 2)$-cylinder $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$ is $(\delta_2^{1/2}, d_2)$-regular. Since $\delta_2 \ll d_2$ by assumption (2.2), Fact 3.6 implies that

$$|\mathcal{K}_4(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)])| \leq \frac{5}{4} d_2^{\binom{4}{2}} \times (d_2 + \delta_2)^4 n^4 \leq 2 d_2^{\binom{5}{2}} n^4.$$

□

We next extend the notion of a good vertex to neighbors and pairs.

Definition 3.9 (good neighbor). *Let $\mathcal{G}$ be a (δ, d)-regular $(s, 2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$, and $x \in V_1$ be a good vertex. A vertex $y \in N_2(x)$ is called a* good neighbor *if it is a good vertex with respect to the $(s-1, 2)$-cylinder $\mathcal{G}[\mathcal{G}(x)]$. We also denote by $N(x)_{\text{good}}$ the set of all nice neighbors.*

Observe that for every good neighbor $y \in N(x)_{\text{good}}$ the link $\mathcal{G}[\mathcal{G}(x, y)]$ is $(\delta^{1/4}, d)$-regular and $\left(d - \delta^{1/2}\right)^2 m \leq \deg_j(x, y) \leq \left(d + \delta^{1/2}\right)^2 m$ holds for $j = 3, \ldots, s$.

One can observe the following:

Observation 3.10. *Let $\mathcal{G}$ be a (δ, d)-regular $(s, 2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$ and $\left(d - \delta^{1/2}\right)^4 \geq \delta^{1/4}$. Then for every good vertex $x \in V_1$ all but at most $12(s-2)\delta^{1/8}|N_2(x)|$ vertices $y \in N_2(x)$ are good neighbors.*

Proof. Let x be a good vertex. Then we know that $\mathcal{G}[\mathcal{G}(x)]$ is $(\delta^{1/2}, d)$-regular and $(d-\delta)m \leq \deg_j(x) \leq (d+\delta)m$ holds for $j = 2, \ldots, s$. We apply Observation 3.8 on $\mathcal{G}[\mathcal{G}(x)]$ and get that for all but at most $\left(2(s-2)\delta^{1/4} + 10(s-2)\delta^{1/8}\right)|N_2(x)| \leq 12(s-2)\delta^{1/8}|N_2(x)|$ vertices $y \in N_2(x)$ are good with respect to $\mathcal{G}[\mathcal{G}(x)]$, that is there are good neighbors. □

Definition 3.11 (good pair). *Let $\mathcal{G}$ be a (δ, d)-regular $(s,2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$. A pair of good vertices $\{x, x'\} \subset V_1$ is called* good *if it satisfies:*

(i) $(d-\delta)^2 m \leq \deg_j(x, x') \leq (d+\delta)^2 m$ *for* $j = 2, \ldots, s$,

(ii) $\mathcal{G}[\mathcal{G}(x, x')]$ *is* $(\delta^{1/2}, d)$*-regular,*

(iii) $\{x, x'\}$ *extends to at most* $\delta^{1/4} m$ *sets* $\{x, x', x''\} \subset V_1$ *not satisfying* $(d-\delta)^3 m \leq \deg_j(x, x', x'') \leq (d+\delta)^3 m$ *for* $j = 2, \ldots, s$, *or for which* $(s-1, 2)$*-cylinder* $\mathcal{G}[\mathcal{G}(x, x', x'')]$ *is not* $(\delta^{1/2}, d)$*-regular.*

Similar to the good vertex case, almost all pairs of good vertices are good.

Observation 3.12. *Let $0 < \delta < d$ be two real numbers such that $(d-\delta)^3 \geq \delta^{1/4}$ and $\mathcal{G}$ be a (δ, d)-regular $(s,2)$-cylinder with s-partition $V_1 \cup \ldots \cup V_s$, $|V_1| = \ldots = |V_s| = m$. Then all but $(4(s-1)\delta^{1/2} + 6(s-1)\delta^{1/4})m^2$ pairs in $[V_{\text{good}}]^2$ are good.*

The proof of this Observation is similar to the proof of Observation 3.8.

Proof of Proposition 2.3. The proof follows the lines of the proof of Proposition 2.2 where we replace a good vertex x with a good pair of vertices $\{x, x'\}$. □

Chapter 4

The l-graphs Lemma

The goal of this section is to develop the l-graphs Lemma which is the main tool in the proofs of Propositions 2.4, 2.5, and 2.6. We start with some definitions and technical observations.

4.1 Definitions and technical observations

It is convenient to introduce the following notation: for a sequence of positive real numbers $\{d_i\}$, we set

$$D_t = \prod_{i=1}^{t} d_i.$$

Observe that $D_{t+1} = d_{t+1} \times D_t$.

The next definition is crucial for this part of the paper.

Definition 4.1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two (k,k)-cylinders with k-partition $V_1 \cup \ldots \cup V_k$, and $\mathcal{H}_2 \subset \mathcal{H}_1$. We say that $\mathcal{H}_2$ is (ε, d, r)-regular with respect to $\mathcal{H}_1$ if the following is true: whenever $\mathcal{G}_1, \ldots, \mathcal{G}_r$ are $(k, k-1)$-cylinders with k-partition $V_1 \cup \ldots \cup V_k$*

such that

$$\left|\mathcal{H}_1 \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\right| \geq \varepsilon|\mathcal{H}_1|,$$

then

$$(d-\varepsilon)\left|\mathcal{H}_1 \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\right| \leq \left|\mathcal{H}_2 \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\right| \leq (d+\varepsilon)\left|\mathcal{H}_1 \cap \bigcup_{j=1}^{r} \mathcal{K}_k(\mathcal{G}_j)\right|. \quad (4.1)$$

If $\mathcal{H}_1 = K_k^{(k)}(V_1, \ldots, V_k)$*, then we simply say that* $\mathcal{H}_2$ *is* (ε, d, r)*-regular or, if* $r = 1$*,* (ε, d)*-regular.*

Observe that for $\mathcal{H}_1 = K_k^{(k)}(V_1, \ldots, V_k)$, this definition is equivalent to Definition 1.10 for $\mathcal{G} = K_k^{(k-1)}(V_1, \ldots, V_k)$, that is $\mathcal{H}_2$ is (ε, d, r)-regular with respect to the complete $(k, k-1)$-cylinder on $V_1 \cup \ldots \cup V_k$.

Note that if $\mathcal{H}_2$ is (ε, d, r)-regular with respect to $\mathcal{H}_1$, $\varepsilon' \geq \varepsilon$, and $r' \leq r$, then $\mathcal{H}_2$ is also (ε', d, r')-regular with respect to $\mathcal{H}_1$. We will use this observation many times without mentioning it explicitly.

One can observe that if $\mathcal{H}_2$ is (ε, d, r)-regular with respect to $\mathcal{H}_1$, then

$$(d-\varepsilon)|\mathcal{H}_1| \leq |\mathcal{H}_2| \leq (d-\varepsilon)|\mathcal{H}_1|,$$

and, more generally, that:

Observation 4.2. *Let* $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots \supset \mathcal{H}_l$ *be* (k, k)*-cylinders such that* $\mathcal{H}_i$ *is* (ε_i, d_i, r)*-regular with respect to* $\mathcal{H}_{i-1}$ *for all* $i \in [l] \setminus \{1\}$*. Then*

$$\prod_{j=2}^{l}(d_j - \varepsilon_j) \times |\mathcal{H}_1| \leq |\mathcal{H}_k| \leq \prod_{j=2}^{l}(d_j + \varepsilon_j) \times |\mathcal{H}_1|. \quad (4.2)$$

Moreover, if $\mathcal{H}_1$ *is* (ε_1, d_1)*-regular and* $|V_1| = \ldots = |V_k| = m$*, then*

$$\prod_{j=1}^{l}(d_j - \varepsilon_j) \times m^k \leq |\mathcal{H}_l| \leq \prod_{j=1}^{l}(d_j + \varepsilon_j) \times m^k. \quad (4.3)$$

We extend the above definition to the case of (s, k)-cylinders.

Definition 4.3. *Let $r \in \mathbb{N}$ and $\mathcal{H}_1 \supset \mathcal{H}_2$ two (s,k)-cylinders. We say that $\mathcal{H}_2$ is (δ, d, r)-regular with respect to $\mathcal{H}_1$ if $\mathcal{H}_2\left[\bigcup_{j\in I} V_j\right]$ is (δ, d, r)-regular with respect to $\mathcal{H}_1\left[\bigcup_{j\in I} V_j\right]$ for all $I \in [s]^k$.*

Armed with Definitions 4.1 and 4.3, we can present the statement of the l-graphs lemma. First, we describe the scenario we are going to work with.

Setup A. *Let $\mathcal{G}_1, \ldots, \mathcal{G}_l$ be $(s,2)$-cylinders with s-partition $V = V_1 \cup \ldots \cup V_s$, where $|V_1| = \ldots = |V_s| = m$, and such that the following conditions are satisfied:*

(i) $\mathcal{G}_i \subset \mathcal{G}_{i-1}$ *for all* $i \in [l] \setminus \{1\}$,

(ii) $0 < \varepsilon_i \ll d_i < 1$ *for all* $i \in [l]$,

(iii) $\varepsilon_{i-1} < \varepsilon_i$ *for all* $i \in [l] \setminus \{1\}$,

(iv) $\mathcal{G}_1$ *is* (ε_1, d_1)*-regular and* $\mathcal{G}_i$ *is* (ε_i, d_i, r)*-regular with respect to* $\mathcal{G}_{i-1}$ *for all* $i \in [l] \setminus \{1\}$,

(v) $r \geq 2\varepsilon_l^{1/2} \prod_{j=1}^{l} d_j^{-2} = 2\varepsilon_l^{1/2} D_l^{-2}$.

We want to prove the following statement.

Lemma 4.4 (l-graphs lemma). *Suppose that $s = 3$ and the above setup holds. Then*

$$\left(1 - 4l\varepsilon_l^{1/64}\right)^l D_l^3 \leq |\mathcal{K}_3(\mathcal{G}_l)| \leq \left(1 + 4l\varepsilon_l^{1/64}\right)^l D_l^3.$$

In the proof of the l-graphs Lemma, we will need the following technical observation.

Observation 4.5. *Let X be a set and $A_1, \ldots, A_t$ t of its arbitrary finite subsets. Then*

$$\left|\bigcup_{i=1}^{t} A_i\right| \geq \sum_{i=1}^{t} |A_i| - \sum_{1\leq i<j\leq t} |A_i \cap A_j|. \tag{4.4}$$

Furthermore, if $a \times \left(\sum_{i=1}^{t}|A_i|\right) - \sum_{1\le i<j\le t}|A_i \cap A_j| \ge 0$ *for some* $a \in (0,1)$*, then*

$$\left|\bigcup_{i=1}^{t} A_i\right| \ge (1-a)\sum_{i=1}^{t}|A_i|. \tag{4.5}$$

We split the proof of the l-graphs Lemma into two parts. In the first part, we prove certain auxiliary statements which are then used in an actual proof given in the second part.

4.2 Some facts about underlying 2-cylinders

To prove the above lemma, we will need three statements: the first two facts will show that almost all vertices and almost all pairs of vertices in V_1 have neighborhoods of "approximately the same size" in every $\mathcal{G}_i$. This can be viewed as an extension of Fact 3.3 to the case of a series of $(s,2)$-cylinders $\mathcal{G}_1 \supset \ldots \supset \mathcal{G}_l$.

The third claim will enable us to select a number of vertices from any sufficiently large subset of V_1 with the property that these vertices are involved in many triangles of $\mathcal{G}_l$. This claim will be then used to prove the l-graphs lemma. The proof of the claim is based on the first two facts.

Definition 4.6. *A vertex* $x \in V_1$ *is called* l-good *if*

$$\left(1-\varepsilon_i^{1/2}\right)^i D_i m \le \deg_{\mathcal{G}_{i,j}}(x) \le \left(1+\varepsilon_i^{1/2}\right)^i D_i m$$

for all $i \in [l]$ *and all* $j \in [s] \setminus \{1\}$*. We also denote by* $V_{l-\mathrm{good}}$ *the set of all* l*-good vertices in* V_1*. It is convinient to set* $V_{0-\mathrm{good}} = V_1$*.*

Observe that if x is l-good that it is also i-good for every $i \in [l]$.

Fact 4.7. *For every* $i \in [l]$*, all but most* $4(s-1)\left(\varepsilon_1^{1/2} + \ldots + \varepsilon_i^{1/2}\right) m$ *vertices* $x \in V_1$ *satisfy*

$$\left(1-\varepsilon_i^{1/2}\right)^i D_i m \le \deg_{\mathcal{G}_{i,j}}(x) \le \left(1+\varepsilon_i^{1/2}\right)^i D_i m \tag{4.6}$$

for all $j \in [s] \setminus \{1\}$.

Remark. Fact 4.7 can be rephrased as:

- all but at most $4(s-1)\left(\varepsilon_1^{1/2} + \ldots + \varepsilon_l^{1/2}\right) m$ vertices are l-good, or
- the size of $V_{(i-1)-\mathrm{good}} \setminus V_{i-\mathrm{good}}$ is bounded by $4(s-1)\varepsilon_i^{1/2} m$ for every $i \in [l]$.

Proof. As in Fact 3.3, we may assume that $s = 2$. We proceed by induction on l. For $l = 1$, Fact 4.7 follows from Fact 3.3. Now we prove the induction step.

Let $\mathcal{G}_1, \ldots, \mathcal{G}_{l+1}$ be $(2,2)$-cylinders satisfying (i)-(iv). By the induction assumption we know that inequality (4.6) holds for every $i \in [l]$ and for all but at most $4\left(\varepsilon_1^{1/2} + \ldots \varepsilon_i^{1/2}\right) m$ vertices $x \in V_1$. Our goal is to show that at most $4\varepsilon_{l+1}^{1/2} m$ of these vertices do not satisfy

$$\left(1 - \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m \leq \deg_{\mathcal{G}_{l+1}}(x) \leq \left(1 + \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m.$$

We recall that $D_{l+1} = \prod_{j=1}^{l+1} d_j$.

Denote by W the set of all vertices $x \in V_{l-\mathrm{good}}$ such that

$$\left(1 + \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m < \deg_{\mathcal{G}_{l+1}}(x) = |N_{\mathcal{G}_{l+1}}(x)|. \tag{4.7}$$

Suppose that $|W| \geq 2\varepsilon_{l+1}^{1/2} m$. Since for every vertex $x \in W$ we have $\deg_{\mathcal{G}_l}(x) \geq \left(1 - \varepsilon_l^{1/2}\right)^l D_l m$, the number of edges $e_{\mathcal{G}_l}(W, V_2)$ between W and V_2 in $\mathcal{G}_l$ can be bounded from below by:

$$\begin{aligned} e_{\mathcal{G}_l}(W, V_2) \geq |W| \times \left(1 - \varepsilon_l^{1/2}\right)^l D_l m &\geq 2\varepsilon_{l+1}^{1/2} m \times \left(1 - \varepsilon_l^{1/2}\right)^l D_l m \\ &\overset{(ii)}{\geq} 2\varepsilon_{l+1} D_l m^2. \end{aligned}$$

It follows from (4.3) and assumption (ii) that $\varepsilon_{l+1}|\mathcal{G}_l| \leq \varepsilon_{l+1} \times 2D_l m^2 \leq e_{\mathcal{G}_l}(W, V_2)$.

Since $\mathcal{G}_{l+1}$ is $(\varepsilon_{l+1}, d_{l+1}, r)$-regular with respect to $\mathcal{G}_l$, we obtain

$$\begin{aligned} e_{\mathcal{G}_{l+1}}(W, V_2) &\le (d_{l+1} + \varepsilon_{l+1}) e_{\mathcal{G}_l}(W, V_2) \\ &\overset{(ii)}{\le} d_{l+1}\left(1 + \varepsilon_{l+1}^{1/2}\right) \times |W| \times \left(1 + \varepsilon_l^{1/2}\right)^l D_l m \\ &\overset{(iii)}{\le} |W| \times \left(1 + \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m. \end{aligned}$$

On the other hand, from (4.7) we obtain that

$$e_{\mathcal{G}_{l+1}}(W, V_2) > |W| \times \left(1 + \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m,$$

which is a contradiction. Hence, we have $|W| < 2\varepsilon_{l+1}^{1/2} m$. Similarly, if we replace (4.7) with

$$\deg_{\mathcal{G}_{l+1}}(x) = |N_{\mathcal{G}_{l+1}}(x)| < \left(1 - \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m,$$

we get $|W| < 2\varepsilon_{l+1}^{1/2} m$ again. Consequently, the number of "bad" vertices is bounded by $4\varepsilon_{l+1}^{1/2} m$. □

We will also classify the pairs of vertices $\{x, x'\}$ according to their join neighborhood.

Definition 4.8. *A pair of vertices $\{x, x'\} \subset V_1$ is called* l-good *if the following to conditions are satisfied:*

(i) x and x' belong to $V_{l-\text{good}}$, and

(ii)

$$\left(1 - \varepsilon_i^{1/8}\right)^i D_i^2 m \le \deg_{\mathcal{G}_{i,j}}(x, x') \le \left(1 + \varepsilon_i^{1/8}\right)^i D_i^2 m \tag{4.8}$$

for all $i \in [l]$ and all $j \in [s] \setminus \{1\}$. We denote by $\Gamma_{l-\text{good}}$ the set of all l-good pairs in V_1. It is also convenient to set $\Gamma_{0-\text{good}} = [V_1]^2$.

Observe that $\Gamma_{l-\text{good}} \subset \Gamma_{(l-1)-\text{good}} \subset \ldots \subset \Gamma_{1-\text{good}} \subset \Gamma_{0-\text{good}}$.

Fact 4.9. *For every $i \in [l]$, $\left|\Gamma_{(i-1)-\text{good}} \setminus \Gamma_{i-\text{good}}\right| \le (s-1)\left(4\varepsilon_i^{1/2} + 8\varepsilon_i^{1/8}\right) m^2$.*

Remark. Fact 4.9 can be rephrased as all but at most $(s-1)\sum_{i=1}^{l}\left(4\varepsilon_i^{1/2}+8\varepsilon_i^{1/8}\right)m^2$ pairs are l-good.

Proof. We may assume that $s=2$ again. We proceed by induction on l. For $l=1$ our assertion follows from Fact 3.3.

Before we prove the induction step, remind that $\mathcal{G}_1,\dots,\mathcal{G}_{l+1}$ are $(2,2)$-cylinders satisfying the conditions of Setup A.

Since $\varepsilon_l^{1/2}d_l \le \varepsilon_l^{1/2} \le \varepsilon_{l+1}^{1/2}$, we have $r \ge \varepsilon_{l+1}^{1/2}D_l^{-1} \ge \varepsilon_l^{1/2}D_{l-1}^{-1}$. Thus, by the induction assumption, we know that condition $\left|\Gamma_{(i-1)-\text{good}}\setminus\Gamma_{i-\text{good}}\right| \le \left(4\varepsilon_i^{1/2}+8\varepsilon_i^{1/8}\right)m^2$ holds for every $i\in[l]$.

Our goal now is to show that $\left|\Gamma_{l-\text{good}}\setminus\Gamma_{(l+1)-\text{good}}\right| \le \left(4\varepsilon_{l+1}^{1/2}+8\varepsilon_{l+1}^{1/8}\right)m^2$. In other words, for not more than $4\varepsilon_{l+1}^{1/2}m^2+8\varepsilon_{l+1}^{1/8}m^2$ of pairs of vertices $\{x,x'\}\in\Gamma_{l-\text{good}}$, either one of x, x' is not $(l+1)$-good, or $\{x,x'\}$ does not satisfy

$$\left(1-\varepsilon_{l+1}^{1/8}\right)^{l+1}D_{l+1}^2 m \le \deg_{\mathcal{G}_{l+1}}(x,x') \le \left(1+\varepsilon_{l+1}^{1/8}\right)^{l+1}D_{l+1}^2 m. \tag{4.9}$$

It follows from Fact 4.7 that $V_{l-\text{good}}\setminus V_{(l+1)-\text{good}} \le 4\varepsilon_{l+1}^{1/2}m$, therefore, there are at most $4\varepsilon_{l+1}^{1/2}m^2$ pairs of l-good vertices that are not pairs of $(l+1)$-good vertices.

For an $(l+1)$-good vertex $x\in V_{(l+1)-\text{good}}$ denote by W_x the set of all $(l+1)$-good vertices $x'\in V_{(l+1)-\text{good}}$ such that the pair $\{x,x'\}$ satisfies (4.8) for every $i\in[l]$ (i.e. belongs to $\Gamma_{l-\text{good}}$) and

$$\deg_{\mathcal{G}_{l+1}}(x,x') < \left(1-\varepsilon_{l+1}^{1/8}\right)^{l+1}D_{l+1}^2 m. \tag{4.10}$$

Denote by X the set of all $(l+1)$-good vertices $x\in V_{(l+1)\text{good}}$, such that for each $x\in X$, $|W_x|\ge 3\varepsilon_{l+1}^{1/8}m$. We will show that

$$|X| \le \varepsilon_l^{1/32}m. \tag{4.11}$$

Then, for all but at most $\varepsilon_l^{1/32}m$ vertices $x\in V_{(l+1)-\text{good}}$, the size of W_x is not bigger than $3\varepsilon_{l+1}^{1/8}m$. So, there exist at most $\varepsilon_l^{1/32}m\times m + m\times 3\varepsilon_{l+1}^{1/8}m \le$

$4\varepsilon_{l+1}^{1/8}m^2$ pairs satisfying (4.10). Similarly, if we replace (4.10) with $\deg_{\mathcal{G}_{l+1}}(x,x') > \left(1+\varepsilon_{l+1}^{1/8}\right)^{l+1} D_{l+1}^2 m$, we obtain not more than $4\varepsilon_{l+1}^{1/8}m^2$ other "bad" pairs.

Altogether, we showed that

$$\left|\Gamma_{l-\text{good}} \setminus \Gamma_{(l+1)-\text{good}}\right| \leq \left(4\varepsilon_{l+1}^{1/2} + 8\varepsilon_{l+1}^{1/8}\right) m^2.$$

We prove (4.11) by contradiction. Suppose that (4.11) is not true, that is $|X| > \varepsilon_l^{1/32} m$.

For every $i \in [l]$, define a graph $\mathcal{P}_i$ with vertex set V_1 and edge set $E(\mathcal{P}_i) = \Gamma_{(i-1)-\text{good}} \setminus \Gamma_{i-\text{good}}$. Observe that the size of $\mathcal{P}_i$ is bounded by $4\varepsilon_i^{1/2}m^2 + 8\varepsilon_i^{1/8}m^2 \leq 12\varepsilon_i^{1/8}m^2$. We apply the Picking Lemma on X with parameters $\sigma_i = 12\varepsilon_i^{1/8}$, $t = \varepsilon_{l+1}^{1/2}D_l^{-1}$, $c = \varepsilon_l^{1/32}$, and obtain t vertices $x_1, \ldots, x_t \in X$ such that:

- all pairs $\{x_u, x_v\}$, $1 \leq u < v \leq t$, belongs to $\Gamma_{1-\text{good}}$, and
- all but $2l\varepsilon_i^{1/8}t^2\varepsilon_l^{-1/16} \leq \varepsilon_i^{1/16}t^2$ pairs $\{x_u, x_v\}$, $1 \leq u < v \leq t$, belongs to $\Gamma_{i-\text{good}}$.

Notice that condition (2.8) reduces to

$$\frac{2 \times 12\varepsilon_1^{1/8} \times \left(\varepsilon_{l+1}^{1/2}D_l^{-1}\right)}{\left(\varepsilon_l^{1/16}\right)^2} < \frac{1}{l}.$$

This inequality follows from the fact that $\varepsilon_1 \ll \varepsilon_i \ll d_i$ for all $i \in [l]$, more formally,

$$\frac{24\varepsilon_1^{1/8}\varepsilon_{l+1}D_l^{-2}}{\varepsilon_l^{1/16}} \overset{(iii)}{<} \varepsilon_1^{1/16}D_l^{-2} \overset{(iii)}{<} \prod_{j=1}^{l} \varepsilon_j^{1/16l} d_j^{-2} \overset{(ii)}{<} \prod_{j=1}^{l} \varepsilon_j^{1/32l} \overset{(ii)}{<} \frac{1}{l}.$$

For $j \in [t]$, let $\mathcal{B}_j = W_{x_j} \cup N_{\mathcal{G}_{l+1}}(x_j)$ be a $(2,1)$-cylinder. Notice that we have $\deg_{\mathcal{G}_{l+1}}(x_j, x') < \left(1 - \varepsilon_{l+1}^{1/8}\right)^{l+1} D_{l+1}^2 m$ for every $x' \in W_{x_j}$ from assumption (4.10). Therefore,

$$\left|\mathcal{G}_{l+1} \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)\right| \leq \sum_{j=1}^{t} |\mathcal{G}_{l+1} \cap \mathcal{K}_2(\mathcal{B}_j)| < \left(1 - \varepsilon_{l+1}^{1/8}\right)^{l+1} m D_{l+1}^2 \sum_{j=1}^{t} |W_{x_j}|. \quad (4.12)$$

We will show that

$$\left|\mathcal{G}_{l+1} \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)\right| \leq \sum_{j=1}^{t} |\mathcal{G}_{l+1} \cap \mathcal{K}_2(\mathcal{B}_j)| \geq \left(1 - \varepsilon_{l+1}^{1/8}\right)^{l+1} mD_{l+1}^2 \sum_{j=1}^{t} |W_{x_j}|, \quad (4.13)$$

which will be clearly a contradiction to (4.12).

In order to show (4.13), we will prove the following statements:

S1 $\sum_{j=1}^{t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| \geq (d_1 - \varepsilon_1) \times \sum_{j=1}^{t} |W_{x_j}||N_{\mathcal{G}_{l+1}}(x_j)| \geq \varepsilon_{l+1}^{5/8} d_1 d_{l+1} m^2$.

S2 $\sum_{1 \leq u < v \leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq 2\varepsilon_{l+1} d_1 m^2$.

S3 $\left|\mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right| \geq \left(1 - \varepsilon_{l+1}^{1/4}\right) \sum_{u=1}^{t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)|$.

S4 $\left|\mathcal{G}_{l+1} \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)\right| \geq \left[\prod_{j=2}^{l+1} (d_j - \varepsilon_j)\right] \left|\mathcal{G}_1 \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)\right|$.

Then the proof of (4.13) is straightforward: we combine S4, S3, and S1 in this order:

$$\begin{aligned}\left|\mathcal{G}_{l+1} \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)\right| &\overset{S4}{\geq} \left[\prod_{j=2}^{l+1} (d_j - \varepsilon_j)\right] \left|\mathcal{G}_1 \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)\right| \\ &\overset{S3}{\geq} \left(1 - \varepsilon_{l+1}^{1/4}\right) \left[\prod_{j=2}^{l+1} (d_j - \varepsilon_j)\right] \sum_{u=1}^{t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| \qquad (4.14) \\ &\overset{S1}{\geq} \left(1 - \varepsilon_{l+1}^{1/4}\right)^{l+1} D_{l+1} \sum_{j=1}^{t} |W_{x_j}||N_{\mathcal{G}_{l+1}}(x_j)|.\end{aligned}$$

Since every vertex x_j is $(l+1)$-good, we have $|N_{\mathcal{G}_{l+1}}(x_j)| \geq \left(1 - \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m$ and (4.13) follows because $\varepsilon_{l+1} \ll 1$.

Proof of S1: We use assumptions $\varepsilon_i \ll d_i$ and $(l+1)$-goodness of x_j to conclude that $|\mathcal{K}_2(\mathcal{B}_j)| = |W_{x_j}||N_{\mathcal{G}_{l+1}}(x_j)| \geq 3\varepsilon_{l+1}^{1/8} m \times \left(1 - \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m \geq \varepsilon_1 m^2$. Since $\mathcal{G}_1$ is (ε_1, d_1)-regular, we get

$$\begin{aligned}|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| &\geq (d_1 - \varepsilon_1)|W_{x_u}||N_{\mathcal{G}_{l+1}}(x_u)| \\ &\geq d_1 \left(1 - \varepsilon_1^{1/2}\right) \times 2\varepsilon_{l+1}^{1/8} \times \left(1 + \varepsilon_{l+1}^{1/2}\right)^{l+1} D_{l+1} m^2 \\ &\overset{(ii)}{\geq} \varepsilon_{l+1}^{1/8} d_1 D_{l+1} m^2.\end{aligned}$$

Thus, $\sum_{j=1}^{t}|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_j)| \geq t \times 2\varepsilon_{l+1}^{1/8} d_1 D_{l+1} m^2 \geq \varepsilon_{l+1}^{5/8} d_1 d_{l+1} m^2$.

Proof of S2: For every $j \in [l]$ denote by $\mathcal{I}_j$ the set of all pairs $\{u,v\} \in [t]^2$ for which $\{x_u, x_v\} \in \Gamma_{j-\text{good}}$. Then we know that $\mathcal{I}_1 = [t]^2$ (all pairs belong here) and $|\mathcal{I}_{j-1} \setminus \mathcal{I}_j| \leq \varepsilon_j^{1/16} t^2$ for every $j > 1$.

Note that $[t]^2 = \mathcal{I}_l \cup \bigcup_{j=2}^{l} (\mathcal{I}_{j-1} \setminus \mathcal{I}_j)$, and therefore

$$\sum_{1\leq u<v\leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq \sum_{\{u,v\}\in\mathcal{I}_l} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| + \sum_{j=2}^{l} \sum_{\{u,v\}\in\mathcal{I}_{j-1}\setminus\mathcal{I}_j} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)|. \quad (4.15)$$

Now we estimate the size of $\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)$ for a pair $\{x_u, x_v\} \in \mathcal{I}_j$. Note that this means: $|N_{\mathcal{G}_j}(x_u, x_v)| \leq \left(1 + \varepsilon_j^{1/8}\right)^j D_j^2 m$ (c.f. (4.10)). Since there is no information about the size of $W_{x_u} \cap W_{x_v}$, we must distinguish two cases:

Case 1 Suppose $|\mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| = |W_{x_u} \cap W_{x_v}||N_{\mathcal{G}_{l+1}}(x_u, x_v)| \geq \varepsilon_1 m^2$. Then, using the (ε_1, d_1)-regularity of $\mathcal{G}_1$, we obtain

$$\begin{aligned} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| &\leq (d_1 + \varepsilon_1)|W_{x_u} \cap W_{x_v}||N_{\mathcal{G}_j}(x_u, x_v)| \\ &\leq (d_1 + \varepsilon_1)\left(1 + \varepsilon_j^{1/8}\right)^j D_j^2 m \times m \leq 2d_1 D_j^2 m^2. \end{aligned}$$

Case 2 Suppose $|W_{x_u} \cap W_{x_v}||N_{\mathcal{G}_{l+1}}(x_u, x_v)| < \varepsilon_1 m^2$. Then

$$|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq |W_{x_u} \cap W_{x_v}||N_{\mathcal{G}_{l+1}}(x_u, x_v)| \leq \varepsilon_1 m^2 \leq 2d_1 D_j^2 m^2.$$

In any case, $|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq 2d_1 D_j^2 m^2$ holds for all pairs $\{u,v\} \in \mathcal{I}_j$.

Now we use (4.15), $\mathcal{I}_1 = [t]^2$, $|\mathcal{I}_{j-1} \setminus \mathcal{I}_j| \leq \varepsilon_j^{1/16} t^2$, and the above observation to conclude that

$$\sum_{1\leq u<v\leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq \binom{t}{2} \times 2d_1 D_l^2 m^2 + \sum_{i=2}^{l} \varepsilon_i^{1/16} t^2 \times 2D_{i-1}^2 d_1 m^2.$$

We further simplify the second term on the right-hand side:

$$\sum_{i=2}^{l} 2\varepsilon_i^{1/16} D_{i-1}^2 = \sum_{i=2}^{l} 2\varepsilon_i^{1/16} D_l^2 \prod_{j=i}^{l} d_j^{-2} \leq D_l^2 \sum_{i=2}^{l} 2 \prod_{j=i}^{l} \varepsilon_j^{1/16(l-i+1)} d_j^{-2} \leq D_l^2. \quad (4.16)$$

The last inequality in (4.16) follows from assumption $\varepsilon_j \ll d_j$. Hence

$$\sum_{1 \leq u < v \leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq t^2 \times d_1 D_l^2 m^2 + t^2 \times d_1 D_l^2 m^2 \leq 2\varepsilon_{l+1} d_1 m^2.$$

Proof of S3: To show this statement, we employ the second part of Observation 4.5. Indeed, it follows from S1 and S2 that:

$$\varepsilon_{l+1}^{1/4} \times \sum_{u=1}^{t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| - \sum_{1 \leq u < v \leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)|$$
$$\geq \varepsilon_{l+1}^{1/4} \times \varepsilon_{l+1}^{5/8} d_1 d_{l+1} m^2 - 2\varepsilon_{l+1} d_1 m^2 \geq 0$$

since we may assume that $d_{l+1} \geq 2\varepsilon_{l+1}^{1/8}$. Consequently, S3 follows from the second part of Observation 4.5 applied with $a = \varepsilon_{l+1}^{1/4}$.

Proof of S4: We will show that for every $i \in [l]$ the size of $\mathcal{G}_i \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)$ is large enough to apply the $(\varepsilon_{i+1}, d_{i+1}, r)$-regularity of $\mathcal{G}_{i+1}$.

Observe first that the size of $\mathcal{G}_1 \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j)$ can be bounded using Observation 4.5 as follows:

$$\left| \mathcal{G}_1 \cap \bigcup_{j=1}^{t} \mathcal{K}_2(\mathcal{B}_j) \right| \geq \sum_{j=1}^{t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_j)| - \sum_{1 \leq u < v \leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)|.$$

Then we use S1 and S2 to insist that

$$\left| \mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right| \geq \varepsilon_{l+1}^{5/8} d_1 d_{l+1} m^2 - 2\varepsilon_{l+1} d_1 m^2$$
$$\geq 2\varepsilon_{l+1} d_1 m^2 > \varepsilon_2 |\mathcal{G}_1|. \quad (4.17)$$

The last inequality follows from the fact that $|\mathcal{G}_1| \leq (d_1 + \varepsilon_1) m^2 \leq 2d_1 m^2$ and $\varepsilon_2 \ll \varepsilon_{l+1}$. Applying the (ε_2, d_2, r)-regularity of $\mathcal{G}_2$ with respect to $\mathcal{G}_1$ (recall that

$r \geq \varepsilon_{l+1}^{1/2} D_l^{-1} = t$) and $\varepsilon_2 \ll d_2 < 1$ yields

$$\left| \mathcal{G}_2 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right| \geq (d_2 - \varepsilon_2) \left| \mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right| \overset{(4.17)}{>} 2\varepsilon_{l+1} d_1 d_2 m^2.$$

We estimate the size of $\mathcal{G}_2$ using Observation 4.2: $|\mathcal{G}_2| \leq (d_2 + \varepsilon_2)|\mathcal{G}_1| \leq 2d_1 d_2 m^2$. Thus, $\varepsilon_3 |\mathcal{G}_2| \leq \varepsilon_3 \left| \mathcal{G}_2 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right|$ (because $\varepsilon_3 \leq \varepsilon_{l+1}$).

Then the (ε_3, d_3, r)-regularity of $\mathcal{G}_3$ with respect to $\mathcal{G}_2$ implies that

$$\left| \mathcal{G}_3 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right| \geq (d_3 - \varepsilon_3) \left| \mathcal{G}_2 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right|.$$

Repeating this argument l times (using (ε_i, d_i, r)-regularity of $\mathcal{G}_i$ with respect to $\mathcal{G}_{i-1}$) yields

$$\begin{aligned} \left| \mathcal{G}_{l+1} \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right| &\geq (d_{l+1} - \varepsilon_{l+1}) \left| \mathcal{G}_l \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right| \\ &\geq \prod_{j=2}^{l+1} (d_j - \varepsilon_j) \times \left| \mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u) \right|. \end{aligned}$$

□

For the remaining part of this section, we restrict ourself to the case of 3-partite graphs, i.e. $s = 3$.

Claim 4.10. *Let $s = 3$. Then for any subset $X \subset V_{l-\text{good}}$, $|X| > \varepsilon_l^{1/32} m$, there exist $t = \varepsilon_l^{1/2} / D_l^2$ vertices $x_1, \ldots, x_t \in X$ such that*

$$\left(1 - \varepsilon_l^{1/8}\right)^l t m^2 D_l^3 \leq \left| \bigcup_{u=1}^{t} \{x_u yz \colon x_u yz \in \mathcal{K}_3(\mathcal{G}_l)\} \right| \leq \left(1 + \varepsilon_l^{1/8}\right)^l t m^2 D_l^3. \qquad (4.18)$$

Proof. For every $i \in [l]$, define a graph $\mathcal{P}_i$ with vertex set V_1 and edge set

$$E(\mathcal{P}_i) = \Gamma_{(i-1)-\text{good}} \setminus \Gamma_{i-\text{good}}.$$

Fact 4.9 assures that $\left|\Gamma_{(i-1)-\text{good}} \setminus \Gamma_{i-\text{good}}\right| \leq 2\left(4\varepsilon_i^{1/2} + 8\varepsilon_i^{1/8}\right) m^2$ for every $i \in [l]$. Then, the size of $\mathcal{P}_i$ is bounded by $\left(8\varepsilon_i^{1/2} + 16\varepsilon_i^{1/8}\right) m^2 \leq 24\varepsilon_i^{1/8} m^2$. We apply the

Picking Lemma on $X \subset V_{l-\text{good}}$ with parameters $\sigma_i = 24\varepsilon_i^{1/8}$, $t = \varepsilon_l^{1/2} D_l^{-2}$, $c = \varepsilon_l^{1/32}$, and obtain t vertices $x_1, \ldots, x_t$ such that:

- every vertex x_u, $u \in [t]$ is l-good,
- all pairs $\{x_u, x_v\}$, $1 \leq u < v \leq t$, belong to $\Gamma_{1-\text{good}}$, and
- all but $2l \times 24\varepsilon_i^{1/8} t^2 \varepsilon_l^{-1/16} \leq 16\varepsilon_i^{1/16} t^2$ pairs $\{x_u, x_v\}$, $1 \leq u < v \leq t$, belong to $\Gamma_{i-\text{good}}$.

Moreover, note that condition (2.8) reduces to $2\times 24\varepsilon_1^{1/8} \times \left(\varepsilon_l^{1/2} D_l^{-2}\right)^2 / \left(\varepsilon_l^{1/32}\right)^2 < 1/l$. This condition is satisfied since

$$\frac{2 \times 24\varepsilon_1^{1/8} \times \left(\varepsilon_l^{1/2} D_l^{-2}\right)^2}{\left(\varepsilon_l^{1/32}\right)^2} = 48\varepsilon_1^{1/8} D_l^{-4} \varepsilon_l^{15/16} < 48 \prod_{j=1}^{l} \varepsilon_j^{1/8l} d_j^{-4} < \frac{1}{l}.$$

The last inequality follows from assumption $\varepsilon_j \ll d_j$.

For every x_u, we will define a $(2,1)$ cylinder $\mathcal{B}_u = N_{\mathcal{G}_l,2}(x_u) \cup N_{\mathcal{G}_l,3}(x_u)$. We will show that for every $i \in [l-1]$, $\left|\mathcal{G}_i \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right|$ is "big enough" to apply the $(\varepsilon_{i+1}, d_{i+1}, r)$-regularity of $\mathcal{G}_{i+1}$ with respect to $\mathcal{G}_i$. Using this argument and some estimates abot the size of $\mathcal{G}_1 \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)$, we will conclude that $\mathcal{G}_l \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)$ contains $\left(1 \pm \varepsilon_l^{1/8}\right)^l tm^2 D_l^3$ edges. This will conclude the proof since

$$\left|\mathcal{G}_l \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| = \left|\bigcup_{u=1}^t \{x_u yz \colon x_u yz \in \mathcal{K}_3(\mathcal{G}_l)\}\right|. \tag{4.19}$$

We will prove the following statements which, in combination with (4.19), will produce (4.18):

S1 $t \times \left(1 + \varepsilon_l^{1/2}\right)^{2l+1} d_1 D_l^2 m^2 \geq \sum_{u=1}^t |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| \geq t \times \left(1 - \varepsilon_l^{1/2}\right)^{2l+1} d_1 D_l^2 m^2$.

S2 $\sum_{1 \leq u < v \leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq 2t^2 d_1 D_l^4 m^2$.

S3 $\left|\mathcal{G}_1 \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| \geq \left(1 - \varepsilon_l^{1/4}\right) \sum_{u=1}^t |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)|$.

S4 $\left[\prod_{j=2}^l (d_j + \varepsilon_j)\right] \left|\mathcal{G}_1 \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| \geq \left|\mathcal{G}_l \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right|$, and
$\left|\mathcal{G}_l \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| \geq \left[\prod_{j=2}^l (d_j - \varepsilon_j)\right] \left|\mathcal{G}_1 \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right|$.

Then we get (4.18) almost right-away: to get the upper bound in (4.18), we combine (4.19) with S1 and S4. Indeed,

$$\begin{aligned}
\left|\bigcup_{u=1}^t \{x_u yz\colon x_u y \in \mathcal{K}_3(\mathcal{G}_l)\}\right| &\overset{(4.19)}{=} \left|\mathcal{G}_l \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| \\
&\overset{S4}{\leq} \prod_{j=2}^l (d_j + \varepsilon_j) \times \left|\mathcal{G}_1 \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| \\
&\leq \prod_{j=2}^l (d_j + \varepsilon_j) \times \sum_{u=1}^t |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| \\
&\overset{S1}{\leq} \prod_{j=2}^l d_j \left(1 + \varepsilon_j^{1/2}\right) \times t \times (1 + \varepsilon_l^{1/2})^{2k+1} d_1 D_l^2 m^2 \\
&\overset{\text{Setup A } (ii),(iii)}{\leq} \left(1 + \varepsilon_l^{1/8}\right)^l D_l^3 t m^2.
\end{aligned}$$

The lower bound is done similarly: we combine (4.19) with S1, S3, and S4:

$$\begin{aligned}
\left|\bigcup_{u=1}^t \{x_u yz\colon x_u y \in \mathcal{K}_3(\mathcal{G}_l)\}\right| &\overset{(4.19)}{=} \left|\mathcal{G}_l \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| \\
&\overset{S4}{\geq} \prod_{j=2}^l (d_j - \varepsilon_j) \times \left|\mathcal{G}_1 \cap \bigcup_{u=1}^t \mathcal{K}_2(\mathcal{B}_u)\right| \\
&\overset{S3}{\geq} \left(1 - \varepsilon_l^{1/4}\right) \prod_{j=2}^l (d_j - \varepsilon_j) \times \sum_{u=1}^t |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| \\
&\overset{S1}{\geq} \prod_{j=1}^l d_j \left(1 - \varepsilon_j^{1/4}\right) \times t \times \left(1 - \varepsilon_l^{1/2}\right)^{2l+1} D_l^2 m^2 \\
&\overset{\text{Setup A } (ii),(iii)}{\geq} \left(1 - \varepsilon_l^{1/8}\right)^l D_l^3 t m^2.
\end{aligned}$$

Now we have to prove statements S1-S4. This will be very similar to Fact 4.9.

Proof of S1: Observe first that since x_j is l-good, it satisfies

$$\left(1-\varepsilon_l^{1/2}\right)^l D_l m \leq \deg_{\mathcal{G}_l,2}(x_j), \deg_{\mathcal{G}_l,3}(x_j) \leq \left(1+\varepsilon_l^{1/2}\right)^l D_l m.$$

Therefore, we have $|\mathcal{K}_2(\mathcal{B}_j)| = |N_{\mathcal{G}_l,2}(x_j)||N_{\mathcal{G}_l,3}(x_j)| > \left(1-\varepsilon_l^{1/2}\right)^{2l} D_l^2 m^2 > \varepsilon_1 m^2$ because of the assumption $\varepsilon_1 \ll \varepsilon_j \ll d_j$. Since $\mathcal{G}_1$ is (ε_1, d_1)-regular, we obtain

$$\begin{aligned}|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_j)| &\geq (d_1-\varepsilon_1)|N_{\mathcal{G}_l,2}(x_j)||N_{\mathcal{G}_l,3}(x_j)| \geq (d_1-\varepsilon_1)\left(1-\varepsilon_l^{1/2}\right)^{2l} D_l^2 m^2 \\ &\geq \left(1-\varepsilon_l^{1/2}\right)^{2l+1} d_1 D_l^2 m^2.\end{aligned}$$

Thus, $\sum_{j=1}|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_j)| \geq t \times \left(1-\varepsilon_l^{1/2}\right)^{2l+1} d_1 D_l^2 m^2$. The upper bound follows from $|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_j)| \leq (d_1+\varepsilon_1)|N_{\mathcal{G}_l,2}(x_j)||N_{\mathcal{G}_l,3}(x_j)|$ in the same way.

Proof of S2: We borrow a large part from Fact 4.9. For every $j \in [l]$ denote by $\mathcal{I}_j$ the set of all pairs $\{u,v\} \in [t]^2$ for which $\{x_u, x_v\} \in \Gamma_{j-\text{good}}$. Then we know that $\mathcal{I}_1 = [t]^2$ (all pairs belong here) and $|\mathcal{I}_{j-1} \setminus \mathcal{I}_j| \leq 16\varepsilon_j^{1/16} t^2$ for every $j > 1$.

Note that $[t]^2 = \mathcal{I}_l \cup \bigcup_{j=2}^l (\mathcal{I}_{j-1} \setminus \mathcal{I}_j)$, and therefore

$$\begin{aligned}&\sum_{1\leq u<v\leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \\ &\qquad \leq \sum_{\{u,v\}\in\mathcal{I}_l} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| + \sum_{j=2}^{l} \sum_{\{u,v\}\in\mathcal{I}_{j-1}\setminus\mathcal{I}_j} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)|. \quad (4.20)\end{aligned}$$

Now we estimate the size of $\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)$ for a pair $\{x_u, x_v\} \in \mathcal{I}_j$. Note that this means that the pair $\{x_u, x_v\}$ is j-good, that is,

$$\left(1-\varepsilon_j^{1/8}\right)^j D_j^2 m \leq |N_{\mathcal{G}_j,2}(x_u,x_v)|,\ |N_{\mathcal{G}_j,3}(x_u,x_v)| \leq \left(1+\varepsilon_j^{1/8}\right)^j D_j^2 m$$

(c.f. Definition 4.8). Then we have

$$|N_{\mathcal{G}_j,2}(x_u,x_v)||N_{\mathcal{G}_j,3}(x_u,x_v)| \geq \left(1-\varepsilon_j^{1/8}\right)^{2j} D_j^4 m^2 \geq \varepsilon_1 m^2,$$

because of $\varepsilon_1 \ll \varepsilon_i \ll d_i$. Consequently,

$$|\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq (d_1 + \varepsilon_1)|N_{\mathcal{G}_j,2}(x_u, x_v)||N_{\mathcal{G}_j,3}(x_u, x_v)| \\ \leq (d_1 + \varepsilon_1)\left(1 + \varepsilon_j^{1/8}\right)^2 D_j^4 m^2 \leq 2d_1 D_j^4 m^2.$$

This is true for every $\{x_u, x_v\} \in \mathcal{I}_j$ and for all $j \in [l]$.

Then, we use (4.20), the above estimate, $\mathcal{I}_1 = [t]^2$, and $|\mathcal{I}_{i-1} \setminus \mathcal{I}_i| \leq 16\varepsilon_i^{1/16} t^2$, $i > 1$, to conclude that

$$\sum_{1 \leq u < v \leq r} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq \binom{t}{2} \times 2d_1 D_l^4 m^2 + \sum_{i=2}^{l} 16\varepsilon_i^{1/16} t^2 \times 2d_1 D_{i-1}^4 m^2. \quad (4.21)$$

We bound first the second term on the right-hand side in a similar way as in Fact 4.9:

$$\sum_{i=2}^{l} 32\varepsilon_i^{1/16} D_{i-1}^4 = \sum_{i=2}^{l} 32\varepsilon_i^{1/16} D_l^4 \prod_{j=i}^{l} d_j^{-4} \leq D_l^4 \sum_{i=2}^{l} 32 \prod_{j=i}^{l} \varepsilon_j^{1/16(l-i+1)} d_j^{-4} \leq D_l^4. \quad (4.22)$$

Here we used again $\varepsilon_j \ll d_j$. We combine (4.21) with (4.22) and obtain

$$\sum_{1 \leq u < v \leq r} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \leq t^2 \times d_1 D_l^4 m^2 + t^2 \times d_1 D_l^4 m^2 = 2t^2 d_1 D_l^4 m^2.$$

Proof of S3:

We use the second part of Observation 4.5, definition of $t = \varepsilon_l^{1/2} D_l^{-2}$, S1, and S2. Indeed, from S1 and S2 we have

$$\varepsilon_l^{1/4} \times \sum_{u=1}^{t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u)| - \sum_{1 \leq u < v \leq t} |\mathcal{G}_1 \cap \mathcal{K}_2(\mathcal{B}_u \cap \mathcal{B}_v)| \\ \geq \varepsilon_l^{1/4} \times t \times \left(1 - \varepsilon_l^{1/2}\right)^{2l+1} d_1 D_l^2 m^2 - 2t^2 d_1 D_l^4 m^2 \\ \geq \varepsilon_l^{1/4} \times \varepsilon_l^{1/2} \left(1 - \varepsilon_l^{1/2}\right)^{2l+1} d_1 m^2 - 4\varepsilon_l d_1 m^2 \geq 0$$

since we may assume that $\left(1 - \varepsilon_l^{1/2}\right)^{2l+1} \geq 2\varepsilon_l^{1/4}$. Using the second part of Observation 4.5 yields S3.

Proof of S4: We combine Observation 4.5 with statements S1 and S2:

$$\left|\mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right| \geq t \times \left(1-\varepsilon_l^{1/2}\right)^{2k+1} d_1 D_l^2 m^2 - 2t^2 d_1 D_l^4 m^2.$$

It is easy to observe that $|\mathcal{G}_1| \leq (d_1+\varepsilon_1)m^2 \leq 2d_1m^2$. We recall that $t = \varepsilon_l^{1/2} D_l^{-2}$ to get the following:

$$\begin{aligned}\left|\mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right| &\geq 2\varepsilon_l^{1/2}\left(1-\varepsilon_l^{1/2}\right)^{2l+1} d_1 m^2 - 8\varepsilon_l d_1 m^2 \\ &> 2\varepsilon_l d_1 m^2 \geq \varepsilon_2 |\mathcal{G}_1|.\end{aligned} \tag{4.23}$$

Since $\mathcal{G}_2$ is (ε_2, d_2, r)-regular with respect to $\mathcal{G}_1$ and $r \geq t$, we obtain that

$$\begin{aligned}\left|\mathcal{G}_2 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right| &\geq (d_2-\varepsilon_2)\left|\mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right| \\ &> (d_2-\varepsilon_2)2\varepsilon_l d_1 m^2 > \varepsilon_3|\mathcal{G}_2|.\end{aligned}$$

The last inequality follows from the observation that $2\varepsilon_l\left(1-\varepsilon_2^{1/2}\right) d_1 d_2 m^2 > \varepsilon_3|\mathcal{G}_2|$ (c.f. Fact 4.9). Moreover, since $\mathcal{G}_3$ is (ε_3, d_3, r)-regular with respect to $\mathcal{G}_2$, we have

$$\left|\mathcal{G}_3 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right| \geq (d_3-\varepsilon_3)\left|\mathcal{G}_2 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right|.$$

We repeat this process l times and after the last step, where we use the (ε_l, d_l, r)-regularity of $\mathcal{G}_l$ with respect to $\mathcal{G}_{l-1}$, we get:

$$\left|\mathcal{G}_l \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right| \geq \prod_{j=2}^{l}(d_j-\varepsilon_j) \times \left|\mathcal{G}_1 \cap \bigcup_{u=1}^{t} \mathcal{K}_2(\mathcal{B}_u)\right|. \tag{4.24}$$

The upper bound in S4 is proved in the same way.

□

4.3 The proof of the l-graphs Lemma

In this proof, our main tool will be Claim 4.10.

Proof. For each $i = l, l-1, \ldots, 1, 0$, we will define recursively sets X_i^f and X_i^l by the following algorithm:

Step 0 Set $X_l = X_l^f = V_{l-\mathrm{good}}$ and $i = l$.

Step 1 As long as $|X_i| > \varepsilon_i^{1/32} m$, apply Claim 4.10 on X_i (with $(3,2)$-cylinders $\mathcal{G}_1, \ldots, \mathcal{G}_i$) and obtain $t = t_i$ (we put index i here to stress the dependence of t on the number of 2-cylinders) vertices $x_1, \ldots, x_{t_i}$ with the property that

$$\left(1 - \varepsilon_i^{1/8}\right)^i D_i^3 t_i m^2 \leq \left| \bigcup_{u=1}^{t_i} \{x_u yz | x_u yz \in \mathcal{K}_3(\mathcal{G}_i)\} \right| \leq \left(1 + \varepsilon_i^{1/8}\right)^i D_i^3 t_i m^2. \tag{4.25}$$

We remove these vertices from X_i and repeat Step 1 again. This can bee done as long as $|X_i| > \varepsilon_i^{1/32} m$ (c.f. Claim 4.10).

Step 2 When $|X_i| \leq \varepsilon_i^{1/32} m$ (that is when we cannot apply Claim 4.10 on X_i anymore), we set

$$\begin{aligned} X_i^l &= X_i, \\ X_{i-1}^f &= X_i^l \cup \left(V_{(i-1)-\mathrm{good}} \setminus V_{i-\mathrm{good}}\right). \end{aligned} \tag{4.26}$$

If $i > 1$, we decrease i by 1 and go to Step 1, otherwise we set $Y_0 = \emptyset$ and stop the algorithm.

Note that during Step 1 we changed the set X_i from X_i^f to X_i^l. We will prove the following statements:

S1 For every $i \in [l]$, both sets X_i^f and X_i^l are subsets of $V_{i-\mathrm{good}}$ and $V_1 = \bigcup_{i=0}^l X_i^f$.

S2 For every $i \in [l]$, $|X_i^l| \leq \varepsilon_i^{1/32} m$.

S3 For every $i \in [l]$, $|X_{i-1}^f| \leq \left(\varepsilon_i^{1/32} + 8\varepsilon_i^{1/2}\right) m$.

S4 $\left(1-8l\varepsilon_l^{1/2}\right)m \le |X_l^f| \le m$.

S5 For every $i \in [l-1]$, vertices selected from X_i in Step 1 form at most $|X_i^f| \times \left(1+\varepsilon_i^{1/8}\right)^i D_i^3 m^2$ copies of K_3 in $\mathcal{G}_i \supset \mathcal{G}_l$ and vertices from X_0^f form at most $|X_0^f| \times m^2$ copies of K_3 in $\mathcal{G}_l$.

S6 The number of copies of K_3 in $\mathcal{G}_l$ produced by vertices selected from X_l is between $\left(|X_l^f| - \varepsilon_l^{1/32}m\right) \times \left(1-\varepsilon_l^{1/8}\right)^l D_l^3 m^2$ and $|X_l^f| \times \left(1+\varepsilon_l^{1/8}\right)^l D_l^3 m^2$.

The statement of the l-graphs Lemma then follows easily. Notice that by S1, the algorithm can always execute Step 1 (we need X_i^f to be a subset of $V_{i-\text{good}}$ to be able to apply Claim 4.10).

First we show the lower bound. The total number of K_3 in $\mathcal{G}_l$ is not bigger then the number of K_3 in $\mathcal{G}_l$ produced by vertices selected from X_l. By S6, this means that $|\mathcal{K}_3(\mathcal{G}_l)| \ge \left(|X_l^f| - \varepsilon_l^{1/32}m\right) \times \left(1-\varepsilon_l^{1/8}\right)^l D_l^3 m^2$. Moreover, we have from S4 that $\left(1-8l\varepsilon_l^{1/2}\right)m \le |X_l^f|$, and, therefore:

$$|\mathcal{K}_3(\mathcal{G}_l)| \ge \left(1-8l\varepsilon_l^{1/2}-\varepsilon_l^{1/32}\right)m \times \left(1-\varepsilon_l^{1/8}\right)^l D_l^3 m^2 \ge \left(1-4l\varepsilon_l^{1/64}\right)^l D_l^3 m^3.$$

In order to get the upper bound, we must estimate the number of K_3 produced by vertices in X_i for $i = 0, \ldots, l$. Combining S4 and S6 we have that vertices selected from X_l are in at most $|X_l^f| \times \left(1+\varepsilon_l^{1/8}\right)^l D_l^3 m^2 \le \left(1+\varepsilon_l^{1/8}\right)^l D_l^3 m^3$ copies of K_3.

Similarly, for $i = 0, 1, \ldots, l-1$, we estimate the contribution of vertices selected from X_i using S3 and S5. Vertices selected from X_i^f are in at most $|X_i^f| \times \left(1+\varepsilon_i^{1/8}\right)^i D_i^3 m^2 \le \left(\varepsilon_{i+1}^{1/32} + 8\varepsilon_{i+1}^{1/2}\right)m \times \left(1+\varepsilon_i^{1/8}\right)^i D_i^3 m^2 \le 2\varepsilon_{i+1}^{1/32} D_i^3 m^3$ copies of K_3 in $\mathcal{G}_i \supset \mathcal{G}_l$. Here we set $D_0 = 1$ and $\mathcal{G}_0 = K(V_1, V_2, V_3)$ (the complete 3-partite graph). Since $V_1 = \bigcup_{i=0}^{l} X_i^f$ (c.f. S1), we obtain

$$|\mathcal{K}_3(\mathcal{G}_l)| \le \left(1+\varepsilon_l^{1/8}\right)^l D_l^3 m^3 + \sum_{i=0}^{l-1} 2\varepsilon_{i+1}^{1/32} D_i^3 m^3.$$

We estimate the second term on the right-hand side in the following way:

$$\sum_{i=0}^{l-1} 2\varepsilon_{i+1}^{1/32} D_i^3 m^3 = D_l^3 m^3 \times \sum_{i=1}^{l} 2\varepsilon_i^{1/32} \prod_{j=i}^{l} d_j^{-3}$$
$$= D_l^3 m^3 \times \sum_{i=1}^{l} 2 \prod_{j=i}^{l} \varepsilon_i^{1/32(l-i+1)} d_j^{-3}$$
$$\overset{\text{SetupA }(iii)}{\leq} D_l^3 m^3 \times \sum_{i=1}^{l} 2 \prod_{j=i}^{l} \varepsilon_j^{1/32(l-i+1)} d_j^{-3}$$
$$\overset{(ii)}{\leq} D_l^3 m^3 \times \sum_{i=1}^{l} \varepsilon_l^{1/64} \leq l\varepsilon_l^{1/64} D_l^3 m^3$$

since $\varepsilon_j^{1/32(l-i+1)} d_j^{-3}$ can be made less than $\varepsilon_j^{1/64}$ by assumption $\varepsilon_j \ll d_j$. Then,

$$|\mathcal{K}_3(\mathcal{G}_l)| \leq \left(1+\varepsilon_l^{1/8}\right)^l D_l^3 m^3 + l\varepsilon_l^{1/64} D_l^3 m^3 \leq \left(1+4l\varepsilon_l^{1/64}\right)^l D_l^3 m^3.$$

Thus, it remains to prove statements S1-S6.

Proof of S1: We proceed by induction. From construction we have that X_i^l is a subset of X_i^f, hence we must show $X_i^f \subset V_{i-\text{good}}$. For $i = l$ it is obvious (see Step 0). Assume that $X_i^l \subset X_i^f \subset V_{i-\text{good}}$ for some $i \in [l]$. Since $V_{i-\text{good}} \subset V_{(i-1)-\text{good}}$ and $X_{i-1}^f = X_i^l \cup \left(V_{(i-1)-\text{good}} \setminus V_{i-\text{good}}\right)$, we immediately have that $X_{i-1}^f \subset V_{(i-1)-\text{good}}$.

$V_1 = \bigcup_{i=0}^l X_i^f$ follows from the definition of X_i^f. Inclusion $V_1 \supset \bigcup_{i=0}^l X_i^f$ is trivial. On the other hand, $X_l^f = V_{l-\text{good}}$ and $X_i^f \supset V_{i-\text{good}} \setminus V_{(i+1)-\text{good}}$ for $i = 0, 1, \ldots, l-1$. Hence, $V_1 = V_{l-\text{good}} \cup \left(\bigcup_{i=0}^{l-1} V_{i-\text{good}} \setminus V_{(i+1)-\text{good}}\right) \subset \bigcup_{i=0}^l X_i^f$.

Proof of S2: This trivially follows from Step 2.

Proof of S3: By Fact 4.7, we have $|V_{(i-1)-\text{good}} \setminus V_{i-\text{good}}| \leq 8\varepsilon_i^{1/2}$ and by S2 $|X_i^l| \leq \varepsilon_i^{1/32} m$ holds. Since $X_{i-1}^f = X_i^l \cup \left(V_{(i-1)-\text{good}} \setminus V_{i-\text{good}}\right)$, statement S3 follows immediately.

Proof of S4: Clearly $|X_l^f| \leq |V_1| \leq m$. On the other hand, it follows from Fact 4.7 that

$$|X_l^f| = |V_{l-\text{good}}| \geq \left(1 - 8\left(\varepsilon_1^{1/2} + \ldots + \varepsilon_l^{1/2}\right)\right) m \overset{SetupA\ (iii)}{\geq} \left(1 - 8l\varepsilon_l^{1/2}\right) m$$

Proof of S5: We can repeat Step 1 for X_i at most $|X_i^f| \times t_i^{-1}$ times. For $i > 0$, every t_i-touple selected from X_i forms at most $\left(1+\varepsilon_l^{1/8}\right)^i D_i^3 t_i m^2$ copies of K_3 in $\mathcal{G}_i$, therefore, Step 1 for X_i gives at most

$$|X_i^f| \times t_i^{-1} \times \left(1+\varepsilon_l^{1/8}\right)^i D_i^3 t_i m^2 = |X_i^f| \left(1+\varepsilon_i^{1/8}\right)^i D_i^3 m^2$$

triangles in $\mathcal{G}_i$.

For $i = 0$, each vertex in X_0^f is in at most m^2 triangles, therefore, these vertices contribute with at most $|X_0^f| \times m^2$ triangles.

Proof of S6: From the assumptions of Claim 4.10 we know we can repeat Step 1 for X_l as long as $|X_l| > \varepsilon_l^{1/32} m$, i.e. at least $\left(|X_l^f| - \varepsilon_l^{1/32} m\right) \times t_l^{-1}$ times and at most $|X_l^f| \times t_l^{-1}$ times. Each time we obtain at least $\left(1-\varepsilon_l^{1/8}\right)^l D_l^3 t_l m^2$ triangles in $\mathcal{G}_l$. Thus, in Step 1, vertices selected from X_l produce at least

$$\left(|X_l^f| - \varepsilon_l^{1/32} m\right) \times t_l^{-1} \times \left(1-\varepsilon_l^{1/8}\right)^l D_l^3 t_l m^2 = \left(1-\varepsilon_l^{1/8}\right)^l D_l^3 m^3$$

copies of K_3.

On the other hand, every t_l-touple forms at most $\left(1+\varepsilon_l^{1/8}\right)^l D_l^3 t_l m^2$ copies of K_3, therefore, Step 1 for X_l gives at most

$$|X_l^f| \times t_l^{-1} \times \left(1+\varepsilon_l^{1/8}\right)^l D_l^3 t_l m^2 = |X_l^f| \left(1+\varepsilon_l^{1/8}\right)^l D_l^3 m^2$$

triangles in $\mathcal{G}_l$. □

We will often use two special cases of l-graphs Lemma - for $l = 2$ and $l = 3$, therefore, we state them as two separate lemmas.

Lemma 4.11 (2-graphs lemma). *Suppose that $\varepsilon_1 \ll \varepsilon_2$, d_1, d_2 are positive real numbers such that $\varepsilon_1 \ll d_1,\ \varepsilon_2 \ll d_2$. If*

(i) $V = V_1 \cup V_2 \cup V_3$ is a partition with $|V_1| = |V_2| = |V_3| = m$,

(ii) $\mathcal{G}_1 = (V, E(\mathcal{G}_1))$ *is a* $(3,2)$*-cylinder that is* (ε_1, d_1)*-regular,*

(iii) $\mathcal{G}_2 = (V, E(\mathcal{G}_2))$ *is a* $(3,2)$*-cylinder that is* (ε_2, d_2, r)*-regular with respect to* $\mathcal{G}_1$*, and*

(iv) $r \geq \varepsilon_2^{1/2} d_1^{-2}$,

then

$$\left(1 - 8\varepsilon_2^{1/64}\right)^2 d_1^3 d_2^3 m^3 \leq |\mathcal{K}_3(\mathcal{G}_2)| \leq \left(1 + 8\varepsilon_2^{1/64}\right)^2 d_1^3 d_2^3 m^3.$$

Lemma 4.12 (3-graphs lemma). *Suppose that* $\varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3$*,* d_1*,* d_2*, and* d_3 *are positive real numbers such that* $\varepsilon_1 \ll d_1,\ \varepsilon_2 \ll d_2,\ \varepsilon_3 \ll d_3$*. If*

(i) $V = V_1 \cup V_2 \cup V_3$ *is a partition with* $|V_1| = |V_2| = |V_3| = m$*,*

(ii) $\mathcal{G}_1 = (V, E(\mathcal{G}_1))$ *is a* $(3,2)$*-cylinder that is* (ε_1, d_1)*-regular,*

(iii) $\mathcal{G}_2 = (V, E(\mathcal{G}_2))$ *is a* $(3,2)$*-cylinder that is* (ε_2, d_2, r)*-regular with respect to* $\mathcal{G}_1$*,*

(iv) $\mathcal{G}_3 = (V, E(\mathcal{G}_1))$ *is* $(3,2)$*-cylinder that is* (ε_3, d_3, r)*-regular with respect to* $\mathcal{G}_2$*, and*

(v) $r \geq \varepsilon_3^{1/2} (d_1 d_2)^{-2}$,

then

$$\left(1 - 12\varepsilon_3^{1/64}\right)^3 d_1^3 d_2^3 d_3^3 m^3 \leq |\mathcal{K}_3(\mathcal{G}_3)| \leq \left(1 + 12\varepsilon_3^{1/64}\right)^3 d_1^3 d_2^3 d_3^3 m^3.$$

Chapter 5

Properties of 3-cylinders

In this section, we investigate link properties of a regular $(s, 3)$-cylinder $\mathcal{H}$. We also prove Propositions 2.4 and 2.5.

5.1 Properties of links in the neighborhood of a single vertex

In a regular $(s, 2)$-cylinder $\mathcal{G}$, all good vertices have the property that their neighborhoods have almost the same size. Fact 3.4 shows that the restriction of $\mathcal{G}$ to such a neighborhood is regular as well. Moreover, we know that almost all vertices in V_1 are good.

In this sub-section, we show that if $\mathcal{G}$ underlies a regular $(s, 3)$-cylinder $\mathcal{H}$, then for almost all good vertices $x \in V_1$, the link $\mathcal{H}(x)$ and the restriction of $\mathcal{H}$ to the neighborhood of x "inherits" regularity. We consider the following scenario:

Setup B. *Let* $0 < \varepsilon_2 \ll d_2 \leq 1$ *and* $0 < \varepsilon_3 \ll d_3 \leq 1$ *be real numbers so that* $\varepsilon_2 \ll \varepsilon_3$*. Let* $V = V_1 \cup \ldots \cup V_s$ *be a partition, where* $|V_1| = \ldots = |V_s| = m$*,* $\mathcal{G} = (V, E(\mathcal{G}))$ *be an* $(s, 2)$*-cylinder that is* (ε_2, d_2)*-regular, and let* $\mathcal{H} = (V, E(\mathcal{H}))$ *be an* $(s, 3)$*-cylinder* *which is* (ε_3, d_3, r)*-regular with respect to* $\mathcal{G}$*.*

The next claim shows that the link $\mathcal{H}(x)$ is $(2\varepsilon_3^{1/2}, d_3, r)$-regular for almost all good vertices x.

Claim 5.1. *The link* $\mathcal{H}(x)$ *is* $(2\varepsilon_3^{1/2}, d_3, r)$*-regular with respect to* $\mathcal{G}[\mathcal{G}(x)]$ *for all but at most* $4\binom{s-1}{2}\varepsilon_3^{1/2}m$ *vertices* $x \in V_{\text{good}}$*.*

Proof. We may assume $s = 3$ since the validity of this special statement applied simultaneously to subcylinders of $\mathcal{G}$ and $\mathcal{H}$ induced on $V_1 \cup V_i \cup V_j$, $1 < i < j \le s$ yields the general result. Thus, $\mathcal{G}$ can be written as $\mathcal{G} = \mathcal{G}_{\hat{1}} \cup \mathcal{G}_{\hat{2}} \cup \mathcal{G}_{\hat{3}}$.

Suppose that one can find $t = 2\varepsilon_3^{1/2}m$ vertices $x_1, \ldots, x_t \in V_{\text{good}}$ such that for every $u \in [t]$ the link $\mathcal{H}(x_u)$ is irregular. Moreover, assume that for every x_u the first part of inequality (4.1) does not hold, i.e. there exist $(2,1)$-cylinders $\mathcal{B}_{ju} = Y_{ju} \cup W_{ju}$, where $Y_{ju} \subset \mathcal{G}(x_u) \cap V_2$, $W_{ju} \subset \mathcal{G}(x_u) \cap V_3$, $j \in [r]$, such that

$$\left| \mathcal{G}[\mathcal{G}(x_u)] \cap \bigcup_{j=1}^{r} \mathcal{K}_2(\mathcal{B}_{ju}) \right| \ge 2\varepsilon_3^{1/2} |\mathcal{G}[\mathcal{G}(x_u)]|, \tag{5.1}$$

but

$$\left| \mathcal{H}(x_u) \cap \bigcup_{j=1}^{r} \mathcal{K}_2(\mathcal{B}_{ju}) \right| < \left(d_3 - 2\varepsilon_3^{1/2}\right) \left| \mathcal{G}[\mathcal{G}(x_u)] \cap \bigcup_{j=1}^{r} \mathcal{K}_2(\mathcal{B}_{ju}) \right|. \tag{5.2}$$

Observe that since x_u is a good vertex, by the $(\varepsilon_2^{1/2}, d_2)$-regularity of $\mathcal{G}[\mathcal{G}(x_u)]$, we have

$$|\mathcal{G}[\mathcal{G}(x_u)]| \ge \left(d_2 - \varepsilon_2^{1/2}\right) |N_2(x_u)||N_3(x_u)| \ge \left(d_2 - \varepsilon_2^{1/2}\right)^3 m^2. \tag{5.3}$$

For every $j \in [r]$ define a $(3,2)$-cylinder $\mathcal{Q}_j = \mathcal{Q}_{j\hat{1}} \cup \mathcal{Q}_{j\hat{2}} \cup \mathcal{Q}_{j\hat{3}}$ by

$$\begin{aligned} \mathcal{Q}_{j\hat{1}} &= \mathcal{G}_{\hat{1}}, \\ \mathcal{Q}_{j\hat{2}} &= \bigcup_{u=1}^{t} \{x_u y \colon y \in W_{ju}\}, \text{ and} \\ \mathcal{Q}_{j\hat{3}} &= \bigcup_{u=1}^{t} \{x_u y \colon y \in Y_{ju}\}. \end{aligned}$$

Then we can estimate the size of $\bigcup_{j=1}^r \mathcal{K}_3(\mathcal{Q}_j)$ as follows:

$$\left|\bigcup_{j=1}^r \mathcal{K}_3(\mathcal{Q}_j)\right| = \sum_{u=1}^t \left|\mathcal{G}[\mathcal{G}(x_u)] \cap \bigcup_{j=1}^r \mathcal{K}_2(\mathcal{B}_{ju})\right| \overset{(5.1)}{\geq} t \times 2\varepsilon_3^{1/2} \times \left(d_2 - \varepsilon_2^{1/2}\right)^3 m^2$$
$$\geq 4\varepsilon_3 \left(1 - \varepsilon_2^{1/4}\right)^3 d_2^3 m^3 \geq \varepsilon_3 |\mathcal{K}_3(\mathcal{G})|.$$

The last inequality follows from Corollary 3.6:

$$\varepsilon_3 \left|\mathcal{K}_3(\mathcal{G})\right| \leq \varepsilon_3 \times \frac{5}{4} d_2^3 m^3 \leq 4\varepsilon_3 \left(1 - \varepsilon_2^{1/2}\right)^3 d_2^3 m^3.$$

Thus, the (ε_3, d_3, r)-regularity of $\mathcal{H}$ with respect to $\mathcal{G}$ implies that

$$\begin{aligned} \left|\mathcal{H} \cap \bigcup_{j=1}^r \mathcal{K}_3(\mathcal{Q}_j)\right| &\geq (d_3 - \varepsilon_3) \left|\bigcup_{j=1}^r \mathcal{K}_3(\mathcal{Q}_j)\right| \\ &= (d_3 - \varepsilon_3) \sum_{u=1}^t \left|\mathcal{G}[\mathcal{G}(x_u)] \cap \bigcup_{j=1}^r \mathcal{K}_2(\mathcal{B}_{ju})\right|. \end{aligned} \tag{5.4}$$

On the other hand, every x_u is contained in $\left|\mathcal{H}(x_u) \cap \bigcup_{j=1}^r \mathcal{K}_2(\mathcal{B}_{ju})\right|$ triples (this follows from the definition of $\mathcal{Q}_j$). We use (5.2) to conclude that

$$\begin{aligned} \left|\mathcal{H} \cap \bigcup_{j=1}^r \mathcal{K}_3(\mathcal{Q}_j)\right| &= \sum_{u=1}^t \left|\mathcal{H}(x_u) \cap \bigcup_{j=1}^r \mathcal{K}_2(\mathcal{B}_{ju})\right| \\ &\overset{(5.2)}{<} \left(d_3 - 2\varepsilon_3^{1/2}\right) \sum_{u=1}^t \left|\mathcal{G}[\mathcal{G}(x_u)] \cap \bigcup_{j=1}^r \mathcal{K}_2(\mathcal{B}_{ju})\right|. \end{aligned} \tag{5.5}$$

Comparing (5.4) with (5.5) we get a contradiction. Thus, there are at most $2\varepsilon_3^{1/2} m$ vertices satisfying (5.1) and (5.2).

The case when the second part of inequality (4.1) is not true, i.e. (5.2) is replaced by $\left|\mathcal{H}(x_u) \cap \bigcup_{j=1}^r \mathcal{K}_2(\mathcal{B}_{ju})\right| > \left(d_3 + 2\varepsilon_3^{1/2}\right) \left|\bigcup_{j=1}^r \mathcal{K}_2(\mathcal{B}_{ju})\right|$, is handled similarly. □

Claim 5.2. *Let $t = \varepsilon_3^{1/2} d_2^{-3}$ and $r' = r/t$. Then $(s-1, 3)$-cylinder $\mathcal{H}$ is $(2\varepsilon_3^{1/4}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$ for all but at most $2\binom{s-1}{3}\varepsilon_2^{1/16} m$ vertices $x \in V_{\text{good}}$.*

Proof. We may assume $s = 4$ for a similar reason as in Claim 5.1. Denote by W the set of all vertices $x \in V_{\text{good}}$ for which $\mathcal{H}$ is not $(2\varepsilon_3^{1/4}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$ and the first part of inequality (1.1) in Definition 1.10 is not satisfied. Moreover, suppose that $|W| \geq \varepsilon_2^{1/16} m$.

We define an auxiliary graph $\Gamma = (V, E)$ where a pair of vertices $\{x, x'\} \subset V$ form an edge if the pair $\{x, x'\}$ is not good. It follows from Observations 3.8 and 3.12 applied with $s = 4$ that $|E| \leq 66\varepsilon_2^{1/4} m^2$.

Using the Picking Lemma with parameters $k = 1$, $\sigma_1 = 66\varepsilon_2^{1/4}$, $c = \varepsilon_2^{1/16}$, and $t = \varepsilon_3^{1/2} d_2^{-3}$, one can choose t vertices $x_1, \ldots, x_t \in W$, such that all pairs $\{x_u, x_v\}$, $1 \leq u < v \leq t$, are good, as long as

$$\frac{2 \times 66\varepsilon_2^{1/4} \times \left(\varepsilon_3^{1/2} d_2^{-3}\right)^2}{\left(\varepsilon_2^{1/16}\right)^2} < 1.$$

This condition is satisfied since

$$\frac{2 \times 66\varepsilon_2^{1/4} \times \left(\varepsilon_3^{1/2} d_2^{-3}\right)^2}{\left(\varepsilon_2^{1/16}\right)^2} = 132\varepsilon_2^{1/8}\varepsilon_3 d_2^{-6} < 132\varepsilon_2^{1/8} d_2^{-6} < 1.$$

The last inequality follows from $0 < \varepsilon_2 \ll d_2$ and $\varepsilon_3 < 1$.

The set W contains precisely those vertices x for which the link $\mathcal{H}(x)$ does not satisfy the first part of inequality (1.1). Thus, for every x_u there exist r' $(3, 2)$-cylinders $\mathcal{B}_{ju} \subset \mathcal{G}[\mathcal{G}(x_u)]$, $j \in [r']$, so that

$$\left| \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{B}_{ju}) \right| \geq 2\varepsilon_3^{1/4} |\mathcal{K}_3(\mathcal{G}[\mathcal{G}(x_u)])|, \tag{5.6}$$

but

$$\left| \mathcal{H} \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{B}_{ju}) \right| \leq \left(d_3 - 2\varepsilon_3^{1/4}\right) \left| \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{B}_{ju}) \right|. \tag{5.7}$$

We find a lower bound on the size of $\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})$ using Observation 4.5:

$$\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \geq \sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| - \sum_{1\leq u<v\leq t}\left|\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{B}_{iu})\cap\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{jv})\right|.$$

Now we estimate both terms on the right size. To do this, notice that $\mathcal{G}[\mathcal{G}(x_u)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular (because x_u is a good vertex) for all vertices x_u and so, by Corollary 3.6, we have

$$\frac{3}{4}d_2^3 \times (d_2-\varepsilon_2)^3 m^3 \leq |\mathcal{K}_3(\mathcal{G}[\mathcal{G}(x_u)])| \leq \frac{5}{4}d_2^3 \times (d_2+\varepsilon_2)^3 m^3.$$

This can be further simplified as

$$\frac{1}{2}d_2^6 m^3 \leq |\mathcal{K}_3(\mathcal{G}[\mathcal{G}(x_u)])| \leq 2d_2^6 m^3. \tag{5.8}$$

Hence, we can estimate $\sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right|$ as follows:

$$\sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \overset{(5.6)}{\geq} \sum_{u=1}^{t} 2\varepsilon_3^{1/4}|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(x_u)])| \overset{(5.8)}{\geq} t \times 2\varepsilon_3^{1/4} \times \frac{1}{2}d_2^6 m^3 \geq \varepsilon_3^{3/4} d_2^3 m^3.$$

In order to estimate the second term $\sum_{1\leq u<v\leq t}\left|\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{B}_{iu})\cap\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{jv})\right|$, we observe two facts:

- we have $\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{B}_{iu})\cap\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{jv}) \subset \mathcal{K}_3(\mathcal{G}[\mathcal{G}(x_u, x_v)])$ because $\mathcal{B}_{iu}\cap\mathcal{B}_{jv} \subset \mathcal{G}[\mathcal{G}(x_u, x_v)]$, and
- $\mathcal{G}[\mathcal{G}(x_u, x_v)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular for all pairs $\{x_u, x_v\}$ (because $\{x_u, x_v\}$ is a good pair).

Thus, using Fact 3.6, we have $|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(x_u, x_v)])| \leq 2d_2^9 m^3$ (c.f. (5.8)) and, therefore, the second term can be bounded in the following way:

$$\sum_{1\leq u<v\leq t}\left|\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{B}_{iu})\cap\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{jv})\right| \leq \binom{t}{2} \times 2d_2^9 m^3 \leq t^2 \times d_2^9 m^3 = \varepsilon_3 d_2^3 m^3.$$

Consequently,

$$\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \geq \varepsilon_3^{3/4} d_2^3 m^3 - \varepsilon_3 d_2^3 m^3 \geq 2\varepsilon_3 d_2^3 m^3. \tag{5.9}$$

It follows from Fact 3.6 that the size of $\mathcal{K}_3(\mathcal{G}_{\hat{1}})$ is bounded by $2d_2^3 m^3$ provided that ε_2 is sufficiently small. Hence, we get

$$\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \geq 2\varepsilon_3 d_2^3 m^3 \geq \varepsilon_3 |\mathcal{K}_3(\mathcal{G}_{\hat{1}})|.$$

We apply the regularity of $\mathcal{H}$ with respect to $\mathcal{G}$ (note that $r \geq t \times r'$) to obtain

$$\left|\mathcal{H} \cap \bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \geq (d_3 - \varepsilon_3)\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right|.$$

Moreover, one can see (c.f. (5.9)) that

$$\varepsilon_3^{1/4}\sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| - \sum_{1\leq u<v\leq t}\left|\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{B}_{iu}) \cap \bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{jv})\right|$$
$$\geq \varepsilon_3^{1/4} \times \varepsilon_3^{3/4} d_2^3 m^3 - \varepsilon_3 d_2^3 m^3 = 0.$$

Therefore, the second part of Observation 4.5 yields

$$\left|\mathcal{H} \cap \bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \geq (d_3 - \varepsilon_3)\left(1 - \varepsilon_3^{1/4}\right)\sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right|. \tag{5.10}$$

On the other hand, it follows from the assumption (5.7) that:

$$\left|\mathcal{H} \cap \bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \leq \sum_{u=1}^{t}\left|\mathcal{H} \cap \bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right| \leq \left(d_3 - 2\varepsilon_3^{1/4}\right)\sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{B}_{ju})\right|.$$

This is a contradiction to (5.10) since $(d_3 - \varepsilon_3)\left(1 - \varepsilon_3^{1/4}\right) > d_3 - \varepsilon_3^{1/4} - \varepsilon^{1/2} > d_3 - 2\varepsilon_3^{1/4}$. Hence, $|W| < \varepsilon_2^{1/16} m$.

Similarly, if we consider the set W of all vertices for which the second part of inequality (4.1) does not hold, i.e. we replace (5.7) with

$$\left|\mathcal{H} \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{B}_{ju})\right| \geq \left(d_3 + 2\varepsilon_3^{1/4}\right) \left|\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{B}_{ju})\right|,$$

we obtain $|W| < \varepsilon_2^{1/16} m$ again. □

Definition 5.3 (nice vertex). *A vertex $x \in V_{\text{good}}$ is called* nice *if it satisfies the following conditions:*

(i) $\mathcal{H}(x)$ is $(2\varepsilon_3^{1/2}, d_3, r)$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$

(ii) $\mathcal{H}$ is $(2\varepsilon_3^{1/4}, d_3, r/\left(\varepsilon_3^{1/2} d_2^{-3}\right))$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$.

We denote by V_{nice} the set of all nice vertices in V_1.

The two previous claims imply the following observation:

Observation 5.4. *All but $4\binom{s-1}{3}\varepsilon_3^{1/2} m + 2\binom{s-1}{2}\varepsilon_2^{1/16} m$ good vertices are nice, i.e.*

$$|V_{\text{nice}}| \geq |V_{\text{good}}| - 4\binom{s-1}{3}\varepsilon_3^{1/2} m - 2\binom{s-1}{2}\varepsilon_2^{1/16} m.$$

Remark. Based on the above claims we can conclude that for all nice vertices $x \in V_{\text{nice}}$:

- $(4,2)$-cylinder $\mathcal{H}^{(3)}(x)$ is regular with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$, and
- $(4,3)$-cylinder $\mathcal{H}^{(3)}$ is regular with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$.

Furthermore, the goodness of x implies:

- $(4,2)$-cylinder $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$ is regular.

This situation resembles the set-up of Theorem 1.13 and therefore it is tempting to try to prove Proposition 2.4 (i.e. bounds on $\mathcal{K}_4(\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x))))$ using this theorem. However, Theorem 1.13 can only count copies of $K_4^{(3)}$ in the restriction $\mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x)]$ because it considers only a $(4,3)$-cylinder underlied by one sparse $(4,2)$-cylinder. In our case, we have two underlying $(4,2)$-cylinders instead. To overcome this difficulty, we reach into the original proof [FR00] and together with the 2-graphs lemma we prove the lower and upper bounds (2.5). This is done in the next section.

5.2 Counting

This section provides some technical facts necessary in sections 5.3 and 5.4. We consider the following scenario:

Setup C. *Let $0 < \varepsilon_2 \ll d_2 \leq 1$, $0 < \varepsilon_3 \ll d_3 \leq 1$, and $0 < \varepsilon_3 \ll \tilde{d}_3 \leq 1$ be real numbers so that $\varepsilon_2 \ll \varepsilon_3$. Let $V = V_1 \cup \ldots \cup V_s$ be a partition, where $|V_1| = \ldots = |V_s| = m$, $\mathcal{G}_2 = (V, E(\mathcal{G}_2))$ be an $(s,2)$-cylinder that is (ε_2, d_2)-regular, $\mathcal{G}_3 = (V, E(\mathcal{G}_3))$ be an $(s,2)$-cylinder that is $(\varepsilon_3, \tilde{d}_3, r)$-regular with respect to $\mathcal{G}_2$, and let $\mathcal{H} = (V, E(\mathcal{H}))$ be an $(s,3)$-cylinder which is (ε_3, d_3, r)-regular with respect to $\mathcal{G}_2$.*

We start with an easy consequence of Claim 5.2:

Corollary 5.5. *Let $t = \varepsilon_3^{1/2} d_2^{-3}$ and $r' = r/t$. Then $\mathcal{H}$ is $(2\varepsilon_3^{1/4}, d_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$ for all but at most $2\binom{s-1}{3}\varepsilon_2^{1/16} m$ vertices $x \in V_{\text{good}}$.*

Remark. Observe that in the proof we do not need edges of $\mathcal{H}$ which contain a vertex from V_1. Therefore, the Claim 5.5 remains true if $\mathcal{H}$ is a $(s-1,3)$-cylinder defined on $V_2 \cup \ldots \cup V_s$ that is (ε_3, d_3, r)-regular with respect to $\mathcal{G}_{2\hat{1}}$.

Now we prove that $\mathcal{G}_3$ is regular in the neighborhood $\mathcal{G}_2(x)$ for almost all vertices x.

Claim 5.6. *Let* $t = \varepsilon_3^{1/2} d_2^{-2}$ *and* $r' = r/t$. *Then,* $(s-1,2)$*-cylinder* $\mathcal{G}_3[\mathcal{G}_2(x)]$ *is* $(2\varepsilon_3^{1/4}, \tilde{d}_3, r')$*-regular with respect to* $\mathcal{G}_2[\mathcal{G}_2(x)]$ *for all but at most* $2\binom{s-1}{2}\varepsilon_2^{1/8} m$ *good vertices* $x \in V_{\text{good}}$.

Proof. Consider the case $s = 3$. Suppose there exists a set of good vertices $W \subset V_{\text{good}}$, $|W| \geq \varepsilon_2^{1/8} m$, such that for every $x \in W$, $\mathcal{G}_3[\mathcal{G}_2(x)]$ is not $(2\varepsilon_3^{1/4}, \tilde{d}_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$. Moreover, assume that the first part of inequality (4.1) is not satisfied, i.e. there exist $(2,1)$-cylinders $\mathcal{B}_j = Y_j \cup Z_j$, $Y_j \subset \mathcal{G}_2(x) \cap V_2$, $Z_j \subset \mathcal{G}_2(x) \cap V_3$, $j \in [r']$, such that

$$\left| \mathcal{G}_2[\mathcal{G}_2(x)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j) \right| \geq 2\varepsilon_3^{1/4} |\mathcal{G}_2[\mathcal{G}_2(x)]|, \tag{5.11}$$

but

$$\left| \mathcal{G}_3[\mathcal{G}_2(x)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j) \right| < \left(\tilde{d}_3 - 2\varepsilon_3^{1/4} \right) \left| \mathcal{G}_2[\mathcal{G}_2(x)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j) \right|. \tag{5.12}$$

We define an auxiliary graph $\Gamma = (V_1, E)$ where a pair of vertices $\{x, x'\}$ is an edge if either $(d_2 - \varepsilon_2)^2 m > \deg_j(x, x')$ or $(d_2 + \varepsilon_2)^2 m < \deg_j(x, x')$ for $j = 2$ or $j = 3$.

Since $\mathcal{G}_2$ is (ε_2, d_2)-regular, by Fact 3.3 applied with $k = 2$, $s = 3$, and $\delta = \varepsilon_2$, the size of E is bounded by $8\varepsilon_2^{1/2} m^2$. Using the Picking Lemma with $\sigma_1 = 8\varepsilon_2^{1/2}$, $c = \varepsilon_2^{1/8}$, and $t = \varepsilon_3^{1/2} d_2^{-2}$, we choose t vertices $x_1, \ldots, x_t \in W$ satisfying

$$(d_2 - \varepsilon_2)^2 m \leq \deg_j(x_u, x_v) \leq (d_2 + \varepsilon_2)^2 m \tag{5.13}$$

for all $1 \leq u < v \leq t$. Condition (2.8) is satisfied since

$$\frac{2 \times 8\varepsilon_2^{1/2} \times t^2}{\left(\varepsilon_2^{1/8}\right)^2} = 16\varepsilon_2^{1/4} d_2^{-4} \varepsilon_3 < 1,$$

where we used the fact that $\varepsilon_2 \ll d_2$ and $\varepsilon_3 < 1$.

For every x_u, denote by $\mathcal{B}_{ju}$ the $(2,1)$-cylinders satisfying (5.11) and (5.12). We will show that the following two statements hold:

S1

$$\sum_{u=1}^{t}\left|\mathcal{G}_2[\mathcal{G}_2(x_u)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|\geq\varepsilon_3^{3/4}d_2m^2, \tag{5.14}$$

and

$$\sum_{1\leq u<v\leq t}\left|\mathcal{G}_2[\mathcal{G}_2(x_u,x_v)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\cap\bigcup_{i=1}^{r'}\mathcal{K}_2(\mathcal{B}_{iv})\right|\leq\varepsilon_3d_2m^2. \tag{5.15}$$

S2

$$\varepsilon_3d_2m^2>\left(2\varepsilon_3^{1/4}-\varepsilon_3\right)\sum_{u=1}^{t}\left|\mathcal{G}_2[\mathcal{G}_2(x_u)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|.$$

Then, however, we use (5.14) to infer that

$$\left(2\varepsilon_3^{1/4}-\varepsilon_3\right)\sum_{u=1}^{t}\left|\mathcal{G}[\mathcal{G}(x_u)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|\overset{(5.14)}{\geq}\varepsilon_3^{1/4}\times\varepsilon_3^{3/4}d_2m^2=\varepsilon_3d_2m^2.$$

This is a contradiction to statement S2, thus $|W|<\varepsilon_2^{1/8}m$.

The situation when we assume that there is a set of good vertices $W\subset V_{\text{good}}$ for which the second part of inequality (4.1) does not hold is handled similarly.

If $s>3$, then we apply the result for $s=3$ simultaneously to $\binom{s-1}{2}$ restrictions of $\mathcal{G}_2$ and $\mathcal{G}_3$ on $V_1\cup V_i\cup V_j$, $2\leq i<j\leq s$.

Hence, it only remains to prove S1 and S2.

S1: Since x_u is a good vertex, $\mathcal{G}_2[\mathcal{G}_2(x_u)]$ is $(\varepsilon_2^{1/2},d_2)$-regular and, therefore,

$$\left(d_2-\varepsilon_2^{1/2}\right)^3m^2\leq|\mathcal{G}[\mathcal{G}(x_u)]|\leq\left(d_2+\varepsilon_2^{1/2}\right)^3m^2$$

for all $u\in[t]$. Furthermore, observe that the $(2,2)$-cylinder $\mathcal{G}[\mathcal{G}(x_u,x_v)]$ is $(\varepsilon_2^{1/2},d_2)$-regular due to (5.13), $\left(d_2-\varepsilon_2^{1/2}\right)^2\geq\varepsilon_2^{1/4}$, and Fact 3.4. Thus, Corollary 3.6 yields

$$|\mathcal{G}_2[\mathcal{G}_2(x_u,x_v)]|\leq\frac{5}{4}d_2\times\left(d_2+\varepsilon_2^{1/2}\right)^4m^2\leq2d_2^5m^2$$

for all $1\leq u<v\leq t$.

Notice that then the above inequalities imply

$$\sum_{u=1}^{t}\left|\mathcal{G}_2[\mathcal{G}_2(x_u)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right| \overset{(5.11)}{\geq} 2\varepsilon_3^{1/4}\times t\times\left(d_2-\varepsilon_2^{1/2}\right)^3 m^2\geq \varepsilon_3^{3/4}d_2m^2,$$

and

$$\sum_{1\leq u<v\leq t}\left|\mathcal{G}_2[\mathcal{G}_2(x_u,x_v)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\cap\bigcup_{i=1}^{r'}\mathcal{K}_2(\mathcal{B}_{iv})\right|$$
$$\leq \sum_{1\leq u<v\leq t}|\mathcal{G}_2[\mathcal{G}_2(x_u,x_v)]|\leq\binom{t}{2}\times 2d_2^5m^2\leq\varepsilon_3d_2m^2.$$

However, this is precisely what statement S1 claims.

S2: We show that the size of $\mathcal{G}_2\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})$ is at least $\varepsilon_3|\mathcal{G}_{2\hat{1}}|$, so we can apply the $(\varepsilon_3,\tilde{d}_3,r)$-regularity of $\mathcal{G}_3$ with respect to $\mathcal{G}_2$. Then we deduce S2 from this and assumption (5.12).

To estimate the size of $\mathcal{G}_2\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})$, we use Observation 4.2:

$$\left|\mathcal{G}_2\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|\geq\sum_{u=1}^{t}\left|\mathcal{G}_2\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|$$
$$-\sum_{1\leq u<v\leq t}\left|\mathcal{G}_2\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\cap\bigcup_{i=1}^{r'}\mathcal{K}_2(\mathcal{B}_{iv})\right|.$$

Since $\mathcal{B}_{ju}\subset\mathcal{G}_2(x_u)$, we have $\mathcal{G}_2\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})=\mathcal{G}_2[\mathcal{G}_2(x_u)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})$, and we may use bounds (5.14) and (5.15) to obtain

$$\left|\mathcal{G}_2\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|\geq\varepsilon_3^{3/4}d_2m^2-\varepsilon_3d_2m^2\geq 2\varepsilon_3d_2m^2\geq\varepsilon_3|\mathcal{G}_{2\hat{1}}|.$$

Since $t\times r'<r$, the $(\varepsilon_3,\tilde{d}_3,r)$-regularity of $\mathcal{G}_3$ with respect to $\mathcal{G}_2$ implies

$$\left|\mathcal{G}_3\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|\geq(\tilde{d}_3-\varepsilon_3)\left|\mathcal{G}_2\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|$$
$$\geq(\tilde{d}_3-\varepsilon_3)\sum_{u=1}^{t}\left|\mathcal{G}_2\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_{ju})\right|-\varepsilon_3d_2m^2.$$

Here we used Observation 4.5 and estimate (5.15) again. We have already observed that we have $\mathcal{G}_2 \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju}) = \mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})$ therefore,

$$\left|\mathcal{G}_3 \cap \bigcup_{u=1}^{t} \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \geq (\tilde{d}_3 - \varepsilon_3) \sum_{u=1}^{t} \left|\mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| - \varepsilon_3 d_2 m^2. \quad (5.16)$$

On the other hand, we use assumption (5.12) and fact that $\mathcal{G}_3 \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju}) = \mathcal{G}_3[\mathcal{G}_2(x)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})$ to get

$$\begin{aligned}\left|\mathcal{G}_3 \cap \bigcup_{u=1}^{t} \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| &\leq \sum_{u=1}^{t} \left|\mathcal{G}_3 \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \\ &< \left(\tilde{d}_3 - 2\varepsilon_3^{1/4}\right) \sum_{u=1}^{t} \left|\mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right|. \quad (5.17)\end{aligned}$$

Finally, comparing (5.16) and (5.17) yields statement S2. □

Remark. Similarly to the previous claim, observe that in the proof we do not need edges of $\mathcal{G}_3$ which contain a vertex from V_1. Therefore, Claim 5.6 holds also if $\mathcal{G}_3$ is a $(s-1, 2)$-cylinder defined on $V_2 \cup \ldots \cup V_s$ that is $(\varepsilon_3, \tilde{d}_3, r)$-regular with respect to $\mathcal{G}_{2\hat{1}}$.

The next claim shows that the 2-cylinder $\mathcal{G}_3 \cap \mathcal{H}(x)$ is regular with respect to $\mathcal{G}_2$ for almost all good vertices x.

Claim 5.7. *Let* $r' = r/\left(\varepsilon_3^{1/2} d_2^{-2}\right)$. *Then,*

$$\left(1 - \varepsilon_3^{1/2}\right)^2 d_2 \tilde{d}_3 m \leq \deg_{\mathcal{G}_{3,j}}(x) \leq \left(1 + \varepsilon_3^{1/2}\right)^2 d_2 \tilde{d}_3 m$$

for $j \in [s] \setminus \{1\}$, $\mathcal{G}_2[\mathcal{G}_3(x)]$ *is* $(\varepsilon_2^{1/2}, d_2)$*-regular, and* $(s-1, 2)$*-cylinder* $\mathcal{G}_3[\mathcal{G}_3(x)] \cap \mathcal{H}(x)$ *is* $(\varepsilon_3^{1/8}, d_3\tilde{d}_3, r')$*-regular with respect to* $\mathcal{G}_2[\mathcal{G}_3(x)]$ *for all but at most* $2\binom{s-1}{2}(4\varepsilon_2^{1/2} + 4\varepsilon_3^{1/2} + \varepsilon_2^{1/8} + \varepsilon_3^{1/8})m$ *good vertices* $x \in V_{\text{good}}$.

Proof. We will restrict ourselves to the case $s = 3$ (the case $s > 3$ is handled in the same way as in Claim 5.6).

Observe first that the (ε_2, d_2)-regularity of $\mathcal{G}_2$, the $(\varepsilon_3, \tilde{d}_3, r)$-regularity of $\mathcal{G}_3$ with respect to $\mathcal{G}_2$, and Claim 4.7 implies that for all but at most $8\left(\varepsilon_2^{1/2} + \varepsilon_3^{1/2}\right) m$ vertices $x \in V_1$ we have

$$\left(1 - \varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m \leq \deg_{\mathcal{G}_{3,j}}(x) \leq \left(1 + \varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m \tag{5.18}$$

for $j = 2, 3$. Moreover, $\left(1 - \varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 \geq \varepsilon_2^{1/4}$ holds due to our assumptions $d_2 \gg \varepsilon_2$ and $\tilde{d}_3 \gg \varepsilon_3 \gg \varepsilon_2$. Thus, by Fact 3.4, $\mathcal{G}_2[\mathcal{G}_3(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular and

$$\left(1 - \varepsilon_3^{1/2}\right)^5 d_2^3\tilde{d}_3^2 m^2 \leq |\mathcal{G}_2[\mathcal{G}_3(x)]| \leq \left(1 + \varepsilon_3^{1/2}\right)^5 d_2^3\tilde{d}_3^2 m^2. \tag{5.19}$$

Furthermore, it follows from Claim 5.6 that there are at most $2\varepsilon_2^{1/8} m$ good vertices $x \in V_{\text{good}}$ for which the $(2,2)$-cylinder $\mathcal{G}_3[\mathcal{G}_2(x)]$ is not $(2\varepsilon_3^{1/4}, \tilde{d}_3, r/\left(\varepsilon_3^{1/2} d_2^{-2}\right))$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$.

Suppose there are $t \geq \varepsilon_3^{1/8} m$ good vertices $x_1, \ldots, x_t \in V_{\text{good}}$ satisfying (5.18) for which

- $\mathcal{G}_3[\mathcal{G}_2(x_u)]$ is $(2\varepsilon_3^{1/4}, \tilde{d}_3, r/\left(\varepsilon_3^{1/2} d_2^{-2}\right))$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x_u)]$, $u \in [t]$, and
- $\mathcal{G}_3[\mathcal{G}_3(x_u)] \cap \mathcal{H}(x_u)$ is not $(\varepsilon_3^{1/8}, d_3\tilde{d}_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x_u)]$.

Assume also that the first part of inequality (4.1) is not satisfied, i.e. for every x_u there exist $(2,1)$-cylinders $\mathcal{B}_{ju} = Y_{ju} \cup Z_{ju}$, where Y_{ju} is a subset of $\mathcal{G}_3(x_u) \cap V_2$ and $Z_{ju} \subset \mathcal{G}_3(x_u) \cap V_3$, $j \in [r']$, such that

$$\left|\mathcal{G}_2[\mathcal{G}_3(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \geq \varepsilon_3^{1/8} |\mathcal{G}_2[\mathcal{G}_3(x_u)]|, \tag{5.20}$$

but

$$\left|\mathcal{G}_3[\mathcal{G}_3(x_u)] \cap \mathcal{H}(x_u) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| < \left(d_3\tilde{d}_3 - \varepsilon_3^{1/8}\right) \left|\mathcal{G}_2[\mathcal{G}_3(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right|. \tag{5.21}$$

Notice that the size of $\mathcal{G}_2[\mathcal{G}_2(x_u)]$ is bounded by $2d_2^3m^2$ (because of Corollary 3.6 and goodness of x_u) and, therefore,

$$2\varepsilon_3^{1/4}|\mathcal{G}_2[\mathcal{G}_2(x_u)]| \leq 4\varepsilon_3^{1/4}d_2^3m^2 \leq \varepsilon_3^{1/8}\left(1-\varepsilon_3^{1/2}\right)^5 d_2^3\tilde{d}_3^2m^2 \overset{(5.19)}{\leq} \varepsilon_3^{1/8}|\mathcal{G}_2[\mathcal{G}_3(x_u)]|.$$

Here we used assumption $\varepsilon_3 \ll \tilde{d}_3$.

Moreover, $\mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju}) = \mathcal{G}_2[\mathcal{G}_3(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})$ holds because $\mathcal{B}_{ju} \subset \mathcal{G}_3(x_u)$. Then it follows from (5.20) and the above inequality that

$$\left|\mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \geq 2\varepsilon_3^{1/4}|\mathcal{G}_2[\mathcal{G}_2(x_u)]|,$$

Since $\mathcal{G}_3[\mathcal{G}_2(x_u)]$ is $(2\varepsilon_3^{1/4}, \tilde{d}_3, r/\left(\varepsilon_3^{1/2}d_2^{-2}\right))$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x_u)]$ and $r' = r/\left(\varepsilon_3^{1/2}d_2^{-2}\right)$, we conclude that

$$\left(\tilde{d}_3 - 2\varepsilon_3^{1/4}\right)\left|\mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| < \left|\mathcal{G}_3[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right|. \tag{5.22}$$

Now we define $(3,2)$-cylinders $\mathcal{Q}_1, \ldots, \mathcal{Q}_{r'}$ on $V_1 \cup V_2 \cup V_3$ by $\mathcal{Q}_j = \mathcal{Q}_{j\hat{1}} \cup \mathcal{Q}_{j\hat{2}} \cup \mathcal{Q}_{j\hat{3}}$, where, for $j \in [r']$,

$$\begin{aligned}
\mathcal{Q}_{j\hat{1}} &= \mathcal{G}_{3\hat{1}}, \\
\mathcal{Q}_{j\hat{2}} &= \bigcup_{u=1}^{t}\{x_uy \colon y \in Z_{ju}\}, \text{ and} \\
\mathcal{Q}_{j\hat{3}} &= \bigcup_{u=1}^{t}\{x_uy \colon y \in Y_{ju}\}.
\end{aligned}$$

We will show that the size of $\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)$ is bounded from below by $\varepsilon_3|\mathcal{K}_3(\mathcal{G}_2)|$, so we can apply the (ε_3, d_3, r)-regularity of $\mathcal{H}$. This will give a lower bound on the size of $\mathcal{H} \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)$. Then, we will use assumption (5.21) and get an upper bound on the size of $\mathcal{H} \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)$. The comparison of both bounds will yield a contradiction. Indeed,

$$\begin{aligned} \left|\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)\right| &= \sum_{u=1}^{t} \left|\mathcal{G}_3[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \\ &\overset{(5.22)}{\geq} \sum_{u=1}^{t} \left(\tilde{d}_3 - 2\varepsilon_3^{1/4}\right) \left|\mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right|. \end{aligned} \tag{5.23}$$

We recall that $\mathcal{G}_2[\mathcal{G}_2(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju}) = \mathcal{G}_2[\mathcal{G}_3(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})$ holds. Then we use (5.19), (5.20), and $\varepsilon_3 \ll \tilde{d}_3$ to get

$$\begin{aligned} \left|\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)\right| &\geq \sum_{u=1}^{t} \left(\tilde{d}_3 - 2\varepsilon_3^{1/4}\right) \left|\mathcal{G}_2[\mathcal{G}_3(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \\ &\overset{(5.20)}{\geq} t \times \left(\tilde{d}_3 - 2\varepsilon_3^{1/4}\right) \times \varepsilon_3^{1/8} \times \left(1 - \varepsilon_3^{1/2}\right)^5 d_2^3 \tilde{d}_3^2 m^2 \\ &\geq 2\varepsilon_3 d_2^3 m^3 \geq \varepsilon_3 \left|\mathcal{K}_3(\mathcal{G}_2)\right|. \end{aligned}$$

Applying the (ε_3, d_3, r)-regularity of $\mathcal{H}$ with respect to $\mathcal{G}_2$ we get

$$\begin{aligned} \left|\mathcal{H} \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)\right| &\geq (d_3 - \varepsilon_3) \left|\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)\right| \\ &\geq (d_3 - \varepsilon_3) \left(\tilde{d}_3 - 2\varepsilon_3^{1/4}\right) \sum_{u=1}^{t} \left|\mathcal{G}_2[\mathcal{G}_3(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \end{aligned} \tag{5.24}$$

On the other hand, we use (5.23) and assumption (5.21) to obtain the following upper bound on $\left|\mathcal{H} \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)\right|$:

$$\begin{aligned} \left|\mathcal{H} \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j)\right| &= \sum_{u=1}^{t} \left|\mathcal{G}_3[\mathcal{G}_3(x_u)] \cap \mathcal{H}(x_u) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right| \\ &\overset{(5.21)}{<} \left(d_3\tilde{d}_3 - \varepsilon_3^{1/8}\right) \sum_{u=1}^{t} \left|\mathcal{G}_2[\mathcal{G}_3(x_u)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_{ju})\right|. \end{aligned} \tag{5.25}$$

Comparing equations (5.24) and (5.25) yields $\varepsilon_3^{1/8} < 2\varepsilon_3^{1/4} + \varepsilon_3 < 3\varepsilon_3^{1/4}$, which is a contradiction to $\varepsilon_3 \ll 1$. The case when there exist $t \geq \varepsilon_2^{1/8} m$ vertices for which the second part of inequality (4.1) is not satisfied is handled in the same way. $\square$

Now we concentrate on the situation when $s = 4$. We will show that the number of copies of $K_4^{(3)}$ in $\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)$ is between $(9/16) d_2^6 \tilde{d}_3^6 d_3^4 m^4$ and $(15/8) d_2^6 \tilde{d}_3^6 d_3^4 m^4$. From this we later deduce Propositions 2.4 and 2.5.

Claim 5.8. *Let* $s = 4$. *Then*

$$\frac{9}{16} d_2^6 \tilde{d}_3^6 d_3^4 m^4 \leq \mathcal{K}_4(\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)) \leq \frac{15}{8} d_2^6 \tilde{d}_3^6 d_3^4 m^4$$

Proof. Set $r' = r / \left(\varepsilon_3^{1/2} d_2^{-3} \right)$ and denote by W the set of all good vertices $x \in V_{\text{good}}$ for which

(i) $\mathcal{H}[\mathcal{G}_2(x)]$ is $(2\varepsilon_3^{1/4}, d_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$,

(ii)
$$\left(1 - \varepsilon_3^{1/2}\right)^2 d_2 \tilde{d}_3 m \leq \deg_{\mathcal{G}_{3,j}}(x) \leq \left(1 + \varepsilon_3^{1/2}\right)^2 d_2 \tilde{d}_3 m$$
for $j = 2, 3, 4$ and

(iii) $\mathcal{G}_3[\mathcal{G}_2(x)] \cap \mathcal{H}(x)$ is $(\varepsilon_3^{1/8}, d_3 \tilde{d}_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x)]$.

Furthermore, the goodness of every vertex $x \in W$ implies

(iv) $\mathcal{G}_2[\mathcal{G}_2(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular and $(d_2 - \varepsilon_2) m \leq \deg_{\mathcal{G}_{2,j}}(x) \leq (d_2 + \varepsilon_2) m$ for all $j = 2, 3, 4$.

It follows from Corollary 5.5 and Claim 5.7 that

- all but at most $2\varepsilon_2^{1/16} m$ vertices $x \in V_{\text{good}}$ satisfy (i) (c.f. Corollary 5.5 applied with $s = 4$),

- all but at most $6(4\varepsilon_2^{1/2} + 4\varepsilon_3^{1/2} + \varepsilon_2^{1/8} + \varepsilon_3^{1/8})m$ vertices $x \in V_{\text{good}}$ satisfy (ii) and (iii) (c.f. Claim 5.7 applied with $s = 4$ and $r' < r/\left(\varepsilon_3^{1/2} d_2^{-2}\right)$).

We use assumptions $\varepsilon_2 \ll \varepsilon_2 \ll 1$ and obtain

$$|W| \geq |V_{\text{good}}| - 10\varepsilon_3^{1/8} m.$$

Moreover, the size of V_{good} is bounded from below by Observation 3.8 applied with $s = 4$ and $\delta = \varepsilon_3$:

$$|V_{\text{good}}| \geq \left(1 - 36\varepsilon_3^{1/4}\right) m.$$

We will do the following: for every vertex $x \in W$ we apply the 2-graphs Lemma on $(3, 2)$-cylinders $\mathcal{G}_2[\mathcal{G}_3(x)]$ and $\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)]$. We will show that the number of copies of K_3 in $\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)]$ is sufficiently large to apply the regularity of $\mathcal{H}[\mathcal{G}_2(x)]$ with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$. This way we will be able to count the number of edges in $\mathcal{H}$ which are also copies of K_3 in $\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)]$. Notice that every such an edge together with x form a copy of $K_4^{(3)}$ in $\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)$ that uses x as a vertex. Then we add these numbers through all $x \in W$. Finally, we estimate the number of copies of $K_4^{(3)}$ in $\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)$ that use vertices not belonging to W.

Consider arbitrary $x \in W$. We apply the 2-graphs lemma with

- $(3, 2)$-cylinder G_1 played by $\mathcal{G}_2[\mathcal{G}_3(x)]$ which is $(\varepsilon_2^{1/2}, d_2)$-regular (c.f. (iv));
- $(3, 2)$-cylinder G_2 played by $\mathcal{G}_3[\mathcal{G}_3(x)] \cap \mathcal{H}(x)$ which is $(\varepsilon_3^{1/8}, d_3\tilde{d}_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x)]$ (c.f. (iii));
- V_i replaced by $\mathcal{G}_3(x) \cap V_i$;

and obtain

$$\left(1 - 8\varepsilon_3^{1/512}\right)^2 (d_2 d_3 \tilde{d}_3)^3 \left(\left(1 - \varepsilon_3^{1/2}\right)^2 d_2 \tilde{d}_3 m\right)^3 \leq |\mathcal{K}_3(\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)])|$$
$$\leq \left(1 + 8\varepsilon_3^{1/512}\right)^2 (d_2 d_3 \tilde{d}_3)^3 \left(\left(1 + \varepsilon_3^{1/2}\right)^2 d_2 \tilde{d}_3 m\right)^3.$$

This can be further simplified using assumptions $\varepsilon_2 \ll \varepsilon_2 \ll 1$:

$$\frac{3}{4} d_2^6 \tilde{d}_3^6 d_3^3 m^3 \leq |\mathcal{K}_3(\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)])| \leq \frac{3}{2} d_2^6 \tilde{d}_3^6 d_3^3 m^3. \tag{5.26}$$

Then observe the following: $\mathcal{G}_2[\mathcal{G}_2(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular, therefore, by Corollary 3.6, $|\mathcal{K}_3(\mathcal{G}_2[\mathcal{G}_2(x)])| \leq (5/4)d_2^3 \times (d_2 + \varepsilon_2)^3 m^3 \leq (3/2) d_2^6 m^3$. Since $d_3, \tilde{d}_3 \gg \varepsilon_3$, we obtain

$$2\varepsilon_3^{1/4} |\mathcal{K}_3(\mathcal{G}_2[\mathcal{G}_2(x)])| \leq 2\varepsilon_3^{1/4} \times \frac{3}{2} d_2^6 m^3 \leq \frac{3}{4} d_2^6 \tilde{d}_3^6 d_3^3 m^3 \leq |\mathcal{K}_3(\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)])|.$$

Thus, we can use the $(2\varepsilon_3^{1/4}, d_3, r')$-regularity of $\mathcal{H}[\mathcal{G}_2(x)]$ with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$ (c.f. (i)) and obtain

$$\begin{aligned}\left(d_3 - 2\varepsilon_3^{1/4}\right) |\mathcal{K}_3(\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)])| &\leq |\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)])| \\ &\leq \left(d_3 + 2\varepsilon_3^{1/4}\right) |\mathcal{K}_3(\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)])|. \end{aligned} \tag{5.27}$$

Combining (5.26) and (5.27) yields

$$\frac{5}{8} d_2^6 \tilde{d}_3^6 d_3^4 m^3 \leq |\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)])| \leq \frac{7}{4} d_2^6 \tilde{d}_3^6 d_3^4 m^3.$$

As it was mentioned before, every edge in $\mathcal{H}$ which is also a copy of K_3 in $\mathcal{H}(x) \cap \mathcal{G}_3[\mathcal{G}_3(x)]$ forms together with x a copy of $K_4^{(3)}$ in $\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)$ that uses x as a vertex. Therefore, there is at least $|W| \times (5/8) d_2^6 \tilde{d}_3^6 d_3^4 m^3$ copies of $K_4^{(3)}$ in $\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)$.

Since the size of W is at least $|V_{\text{good}}| - 10\varepsilon_3^{1/8} m \geq \left(1 - 36\varepsilon_3^{1/4} - 10\varepsilon_3^{1/8}\right) m \geq \left(1 - 11\varepsilon_3^{1/8}\right) m$, the following lower bound holds:

$$|\mathcal{K}_4(\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3))| \geq \left(1 - 11\varepsilon_3^{1/8}\right) m \times \frac{5}{8} d_2^6 \tilde{d}_3^6 d_3^4 m^3 \geq \frac{9}{16} d_2^6 \tilde{d}_3^6 d_3^4 m^4.$$

For the upper bound we must count not only

(a) the contribution of vertices taken from W, but also

(b) contribution of vertices in $V_1 \setminus W$.

We will handle each of these cases separately.

(a) The upper bound on the number of copies of $K_4^{(3)}$ in $\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)$ that use as a vertex members of W can be found similarly to the lower bound: every vertex $x \in W$ is in at most $\frac{7}{4} d_2^6 \tilde{d}_3^6 d_3^4 m^3$ copies of $K_4^{(3)}$, therefore, we have at most $|W| \times \frac{7}{4} d_2^6 \tilde{d}_3^6 d_3^4 m^3 \leq \frac{7}{4} d_2^6 \tilde{d}_3^6 d_3^4 m^4$ copies of $K_4^{(3)}$ altogether.

(b) There are two kind of vertices in $V_1 \setminus W$: good and not good.

   (i) The number of vertices which are not good is at most $36\varepsilon_2^{1/4} m$ (c.f. Observation 3.8). Each such vertex can be in at most m^3 copies of $K_4^{(3)}$, thus, these vertices are involved in at most $36\varepsilon_2^{1/4} m^4 \leq \varepsilon_2^{1/8} d_2^6 m^4 \leq \varepsilon_3^{1/8} d_2^6 m^4 \leq \varepsilon_3^{1/32} d_2^6 \tilde{d}_3^6 d_3^4 m^4$ copies of $K_4^{(3)}$.

   (ii) For every good vertex $x \in V_1 \setminus W$, we use Corollary 3.6 to insist that $|\mathcal{K}_3(\mathcal{G}_2[\mathcal{G}_2(x)])| \leq (5/4) d_2^3 \times \left(d_2 + \varepsilon_2^{1/2}\right)^3 m^3 \leq 2 d_2^6 m^3$. Since $|V_1 \setminus W| \leq 11\varepsilon_3^{1/8} m$, the good vertices in $V_1 \setminus W$ can produce at most $11\varepsilon_3^{1/8} m \times 2 d_2^6 m^3 \leq \varepsilon_3^{1/16} d_2^6 \tilde{d}_3^6 d_3^4 m^4$ copies of $K_4^{(3)}$. Note that we used $\varepsilon_3 \ll d_3, \tilde{d}_3$ again.

We add the contributions from (a) and (b) to get

$$|\mathcal{K}_4(\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3))| \leq \left(\frac{7}{4} + \varepsilon_3^{1/32} + \varepsilon_3^{1/16}\right) d_2^6 \tilde{d}_3^6 d_3^4 m^4 \leq \frac{15}{8} d_2^6 \tilde{d}_3^6 d_3^4 m^4.$$

□

5.3 Proof of Proposition 2.4

In this part, we use the properties of nice vertices and Claim 5.8 to show Proposition 2.4.

Proof. Consider a (δ_2, d_2)-regular $(5,2)$-cylinder $\mathcal{H}^{(2)}$ and a $(5,3)$-cylinder $\mathcal{H}^{(3)}$ which is (δ_3, d_3, r)-regular with respect to $\mathcal{H}^{(2)}$.

Let $x \in V_1$ be a nice vertex (c.f. Definition 5.3), i.e. a good vertex for which we have:

(i) the link $\mathcal{H}^{(3)}(x)$ is $(2\delta_3^{1/2}, d_3, r)$-regular with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$.

(ii) $\mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x)]$ is $(2\delta_3^{1/4}, d_3, r/\left(\delta_3^{1/2}d_2^{-3}\right))$-regular with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$,

Also remind that the goodness of x implies

(iii) $(d_2 - \delta_2)n \le \deg_j(x) \le (d_2 + \delta_2)n$ for $j = 2, \ldots, 5$,

(iv) the $(4,2)$-cylinder $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$ is $(\delta_2^{1/2}, d_2)$-regular.

We apply Claim 5.8 with $\mathcal{G}_2 = \mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$, $\mathcal{G}_3 = \mathcal{H}^{(3)}(x)$, $\mathcal{H} = \mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x)]$, $(d_3 - \delta_2)n \le m \le (d_3 + \delta_2)n$, $\varepsilon_2 = \delta_2^{1/2}$, $\varepsilon_3 = 2\delta_3^{1/4}$, $\tilde{d}_3 = d_3$, and r replaced by $r/\left(\delta_3^{1/2}d_2^{-3}\right)$. Observe that (i)-(iv) verify conditions of the Setup C. More precisely,

- (iv) verifies that $\mathcal{G}_2$ is (ε_2, d_2)-regular,
- (i) verifies that $\mathcal{G}_3$ is $(\varepsilon_3, \tilde{d}_3, r)$-regular with respect to $\mathcal{G}_2$, and
- (ii) verifies that $\mathcal{H}$ is (ε_3, d_3, r)-regular with respect to $\mathcal{G}_2$.

Hence, we obtain

$$\frac{9}{16}d_2^6 d_3^6 d_3^4 (d_2 - \delta_2)^4 n^4 \le \mathcal{K}_4(\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)) \le \frac{15}{8} d_2^6 d_3^6 d_3^4 (d_2 + \delta_2)^4 n^4.$$

Estimate (2.5) follows from this inequality since $d_2 \gg \delta_2$. This is true for every nice vertex. By Observation 5.4 applied with $s = 4$, at most $10\delta_3^{1/2}n$ good vertices are not nice, and from Observation 3.8 we have that at most $36\delta_2^{1/4}n$ vertices are not good. From this we conclude that (2.5) holds for all but at most $46\delta_3^{1/2}n$ vertices $x \in V_1$.

□

Remark. It is possible to replace constants $1/2$ and 2 in the lower and upper bound by $1-\psi(\delta_3)$ and $1+\psi(\delta_3)$, where $\psi(\delta_3)\to 0$ as $\delta_3\to 0$. This can be done by more precise estimation in Claim 5.8.

5.4 Properties of links in the neighborhood of a pair of vertices

In this section, we prove that the restriction of $\mathcal{H}$ to the neighborhood $\mathcal{G}(x,x')$ and the link $\mathcal{H}(x,x')$ are regular for almost all pairs of vertices $\{x,x'\}\subset V_1$. These two claims play the same role for the pair $\{x,x'\}\subset V_1$ as Claims 5.2 and 5.1 do for a single vertex. Thus, for every such pair $\{x,x'\}$ we can mimick the proof of Proposition 2.4, that is apply Claim 5.8 on $\mathcal{H}[\mathcal{G}(x,x')]$ and the link $\mathcal{H}(x,x')$, and prove Proposition 2.5.

We consider the scenario given by the Setup B from Section 5.1.

Setup B. *Let $0<\varepsilon_2\ll d_2\le 1$ and $0<\varepsilon_3\ll d_3\le 1$ be real numbers so that $\varepsilon_2\ll\varepsilon_3$. Let $V=V_1\cup\ldots\cup V_s$ be a partition, where $|V_1|=\ldots=|V_s|=m$, $\mathcal{G}=(V,E(\mathcal{G}))$ be an $(s,2)$-cylinder that is (ε_2,d_2)-regular, and let $\mathcal{H}=(V,E(\mathcal{H}))$ be an $(s,3)$-cylinder which is (ε_3,d_3,r)-regular with respect to $\mathcal{G}$.*

Claim 5.9. *Let $r'=r/\left(\varepsilon_3^{1/2}d_2^{-3}\right)\left(2\varepsilon_3^{1/8}d_2^{-3}\right)$. Then, for every nice vertex $x\in V_{\mathrm{nice}}$, the restriction of $\mathcal{H}$ to $\mathcal{G}(x,x')$ is $(4\varepsilon_3^{1/16},d_3,r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x,x')]$ for all but at most $3\binom{s-1}{3}\varepsilon_2^{1/32}m$ good vertices $x'\in V_{\mathrm{good}}$.*

Proof. Let x be a nice vertex (c.f. Definition 5.3), i.e. a good vertex satisfying

(i) $\mathcal{H}(x)$ is $(2\varepsilon_3^{1/2},d_3,r)$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$, and

(ii) $\mathcal{H}$ is $(2\varepsilon_3^{1/4},d_3,r/\left(\varepsilon_3^{1/2}d_2^{-3}\right))$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$.

Moreover, since x is also a good vertex, we have

(iii) $\mathcal{G}[\mathcal{G}(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular, and

(iv) for all but at most $\varepsilon_2^{1/4}m$ vertices x'

$$(d_2 - \varepsilon_2)^2 m \le \deg_j(x, x') \le (d_2 + \varepsilon_2)^2 m \tag{5.28}$$

holds for every $j \in [s] \setminus \{1\}$.

Denote by W' the set of all $x' \in V_1$ satisfying (5.28) for which $\mathcal{H}$ is $(4\varepsilon_3^{1/16}, d_3, r')$-irregular with respect to $\mathcal{G}[\mathcal{G}(x, x')]$.

Let W be a set, $W' \subset W \subset V_1$, such that $(d_2 - \varepsilon_2)m \le |W| \le (d_2 + \varepsilon_2)m$. We can achieve this by throwing out some vertices from W' or by adding some vertices from V_1 to W'.

Set $V_1' = W$, $V_j' = \mathcal{G}(x) \cap V_j$ for $j \in [s] \setminus \{1\}$, and $V' = V_1' \cup \ldots \cup V_s'$. Notice that due to $d_2 - \varepsilon_2 \ge \varepsilon_2^{1/4}$ and Fact 3.4,

(iii') the restriction $\mathcal{G}[V']$ is an $(\varepsilon_2^{1/2}, d_2)$-regular $(s, 2)$-cylinder.

Set $\mathcal{G}_2' = \mathcal{G}[V']$, $\mathcal{G}_3' = \mathcal{H}(x)$, and $\mathcal{H}' = \mathcal{H}[\mathcal{G}(x)]$. Note that $\mathcal{G}_2'$ is an $(s, 2)$-cylinder, $\mathcal{G}_3'$ is an $(s-1, 2)$-cylinder, and $\mathcal{H}'$ is an $(s-1, 3)$-cylinder. Moreover, in view of the remark following Corrolary 5.5, $\mathcal{G}_2'$, $\mathcal{G}_3'$, $\mathcal{H}'$ satisfy the assumptions of Corollary 5.5 with ε_2 replaced by $\varepsilon_2^{1/2}$, ε_3 by $2\varepsilon_3^{1/4}$, $\tilde{d}_3 = d_3$, and r replaced by $r/\left(\varepsilon_3^{1/2} d_2^{-3}\right)$ (c.f. (i), (ii), (iii')).

We apply Corrolary 5.5 and obtain that the restriction $\mathcal{H}'[\mathcal{G}_2'(x')]$ is $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}_2'[\mathcal{G}_2'(x')]$ for all but at most $2\binom{s-1}{2}\varepsilon_2^{1/32}|W|$ vertices $x' \in W$. Note that $\mathcal{G}_2'[\mathcal{G}_2'(x')] = \mathcal{G}[\mathcal{G}(x, x')]$ and $\mathcal{H}'[\mathcal{G}_2'(x')] = \mathcal{H}[\mathcal{G}(x, x')]$.

However, all such vertices are contained in W', therefore

$$|W'| \le 2\binom{s-1}{3}\varepsilon_2^{1/32}|W| \le 2\binom{s-1}{3}\varepsilon_2^{1/32} m.$$

Since (5.28) is satisfied for all but at most $\varepsilon_2^{1/4}m$ good vertices x' and W' contains all good vertices x' satisfying (5.28) for which $\mathcal{H}[\mathcal{G}(x,x')]$ is irregular, we infer that the total number of vertices x' for which the restriction of $\mathcal{H}$ to $\mathcal{G}(x,x')$ fails to be $(4\varepsilon_3^{1/16}, d_3, r')$-regular does not exceed $2\binom{s-1}{3}\varepsilon_2^{1/32}m + \varepsilon_2^{1/4}m \leq 3\binom{s-1}{3}\varepsilon_2^{1/32}m$. □

The proof of the next claim is based the same idea except that it uses Claim 5.6 instead of Corrolary 5.5.

Claim 5.10. *Let $x \in V_{\text{nice}}$ be a nice vertex and set $r' = r/\left(\varepsilon_3^{1/2}d_2^{-3}\right)\left(2\varepsilon_3^{1/8}d_2^{-2}\right)$. Then the link $\mathcal{H}(x)$ is $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x,x')]$ for all but at most $3\binom{s-1}{2}\varepsilon_2^{1/16}m$ good vertices x'.*

Proof. Let x be a nice vertex. Then we know (c.f. Definition 5.3):

(i) $\mathcal{H}(x)$ is $(2\varepsilon_3^{1/2}, d_3, r)$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$, and

(ii) $\mathcal{H}$ is $(2\varepsilon_3^{1/4}, d_3, r/\left(\varepsilon_3^{1/2}d_2^{-3}\right))$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$.

Since every nice vertex is also good, we have (c.f. Definition 3.7):

(iii) $\mathcal{G}[\mathcal{G}(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular, and

(iv) for all but at most $\varepsilon_2^{1/4}m$ vertices x'

$$(d_2 - \varepsilon_2)^2 m \leq \deg_j(x,x') \leq (d_2 + \varepsilon_2)^2 m \tag{5.29}$$

holds for every $j \in [s] \setminus \{1\}$.

Denote by W' the set of all $x' \in V_1$ satisfying (5.29) for which $\mathcal{H}(x)$ is not $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x,x')]$.

Let W be a set, $W' \subset W \subset V_1$, such that $(d_2 - \varepsilon_2)m \leq |W| \leq (d_2 + \varepsilon_2)m$. This can be achieved by throwing out some vertices from W' or by adding some vertices from V_1 to W'. Set $V_1' = W$, $V_j' = \mathcal{G}(x) \cap V_j$ for $j \in [s] \setminus \{1\}$, and $V' = V_1' \cup \ldots \cup V_s'$. Notice that (iii') $\mathcal{G}[V']$ is an $(\varepsilon_2^{1/2}, d_2)$-regular $(s,2)$-cylinder

due to $d_2 - \varepsilon_2 \geq \varepsilon_2^{1/4}$ and Fact 3.4.

As in the previous claim, set $\mathcal{G}_2' = \mathcal{G}[V']$, $\mathcal{G}_3' = \mathcal{H}(x)$, and $\mathcal{H}' = \mathcal{H}[\mathcal{G}(x)]$ and notice that $\mathcal{G}_2'$, $\mathcal{G}_3'$, $\mathcal{H}'$ satisfy the assumptions of Claim 5.6 (which are the same as the assumptions of Corollary 5.5 and these assumptions were checked in the previous claim).

We apply Claim 5.6 on $\mathcal{G}_2'$ and $\mathcal{G}_3'$. By this claim, the restriction of $\mathcal{G}_3'$ to $\mathcal{G}_2'[\mathcal{G}_2'(x')]$ is $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}_2'[\mathcal{G}_2'(x')]$ for all but at most $2\binom{s-1}{2}\varepsilon_2^{1/16}|W|$ vertices $x' \in W$. Recall that $\mathcal{G}_3' = \mathcal{H}(x)$ and $\mathcal{G}_2'[\mathcal{G}_2'(x')] = \mathcal{G}[\mathcal{G}(x,x')]$, therefore, all such vertices are contained in W'.

Thus,

$$|W'| \leq 2\binom{s-1}{2}\varepsilon_2^{1/16}|W| \leq 2\binom{s-1}{2}\varepsilon_2^{1/16}m.$$

Since (5.29) is satisfied for all but at most $\varepsilon_2^{1/4}m$ good vertices x' and W' contains all good vertices x' satisfying (5.29) for which $\mathcal{H}(x)$ is irregular, we infer that the total number of vertices x' for which the restriction of $\mathcal{H}(x)$ to $\mathcal{G}(x,x')$ fails to be $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x,x')]$ does not exceed $2\binom{s-1}{2}\varepsilon_2^{1/16}m + \varepsilon_2^{1/4}m \leq 3\binom{s-1}{2}\varepsilon_2^{1/16}m$. □

We use the previous claim to prove an analogy of Claim 5.1 for pairs of vertices. Note that the same proof in a slightly different setting is given in [DHNR00].

Claim 5.11. *Let* $r' = r/\left(\varepsilon_3^{1/2}d_2^{-3}\right)\left(2\varepsilon_3^{1/8}d_2^{-2}\right)$. *Then for all but* $2\binom{s-1}{2}\varepsilon_2^{1/16}m$ *nice vertices* $x \in V_{\text{nice}}$ *the following statement is true: the link* $\mathcal{H}(x,x')$ *is* $(\varepsilon_3^{1/32}, d_3^2, r')$-*regular with respect to* $\mathcal{G}[\mathcal{G}(x,x')]$ *for at most* $5\binom{s-1}{2}\varepsilon_3^{1/16}m$ *good pairs* $\{x,x'\}$, *where* $x' \in V_{\text{nice}}$.

Proof. We will restrict ourselves to the case $s = 3$ (the case $s > 3$ is handled in the same way as in Claim 5.6).

For a nice vertex $x \in V_{\text{nice}}$, denote by W_x the set of nice vertices w for which

(i) pair $\{x, w\}$ is good,

(ii) $\mathcal{H}(x)$ is $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x,w)]$, and

(iii) $\mathcal{H}(x,w)$ is not $(\varepsilon_3^{1/32}, d_3^2, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x,w)]$ and the second part of inequality (4.1) is not satisfied, i.e. there exist $(2,1)$-cylinders $\mathcal{B}_j^{xw} = Y_j^{xw} \cup Z_j^{xw}$, where Y_j^{xw} is a subset of $\mathcal{G}(x,w) \cap V_2$ and $Z_j^{xw} \subset \mathcal{G}(x,w) \cap V_3$, such that

$$\left| \mathcal{G}[\mathcal{G}(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| \geq \varepsilon_3^{1/32} |\mathcal{G}[\mathcal{G}(x,w)]|, \tag{5.30}$$

but

$$\left| \mathcal{H}(x,w) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| > \left(d_3^2 + \varepsilon_3^{1/32} \right) \left| \mathcal{G}[\mathcal{G}(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right|. \tag{5.31}$$

From the previous claim, for every nice vertex x we have at most $3\varepsilon_2^{1/16} m \leq \varepsilon_3^{1/16} m$ good pairs $\{x, x'\}$, $x' \in V_{\text{nice}}$, for which the link $\mathcal{H}(x)$ is $(4\varepsilon_3^{1/16}, d_3, r')$-irregular with respect to $\mathcal{G}[\mathcal{G}(x, x')]$.

Moreover, denote by X, the set of those nice vertices x for which $|W_x| \geq 2\varepsilon_3^{1/16} m$ and suppose $|X| \geq \varepsilon_2^{1/16} m$. We also make an assumption $|W_x| = 2\varepsilon_3^{1/16} m$ for all $x \in X$. This can be achieved by possible deletion of some vertices from W_x.

Since by (ii) the link $\mathcal{H}(x)$ is $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x,w)]$ for every $x \in X$ and $w \in W_x$, and (5.30) holds (c.f. (iii)), we conclude that

$$\left(d_3 - 4\varepsilon_3^{1/16} \right) \left| \mathcal{G}[\mathcal{G}(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| < \left| \mathcal{H}(x) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| \tag{5.32}$$

and

$$\left| \mathcal{H}(x) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| < \left(d_3 + 4\varepsilon_3^{1/16} \right) \left| \mathcal{G}[\mathcal{G}(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right|. \tag{5.33}$$

For every vertex $x \in X$, we define $(3,2)$-cylinders $\mathcal{Q}_1^x, \dots, \mathcal{Q}_{r'}^x$ by $\mathcal{Q}_j^x = \mathcal{Q}_{j\hat{1}}^x \cup \mathcal{Q}_{j\hat{2}}^x \cup \mathcal{Q}_{j\hat{3}}^x$, where

$$\begin{aligned} \mathcal{Q}_{j\hat{1}}^x &= \mathcal{H}(x), \\ \mathcal{Q}_{j\hat{2}}^x &= \bigcup_{w \in W_x} \{wz \colon z \in Z_j^{xw}\}, \text{ and} \\ \mathcal{Q}_{j\hat{3}}^x &= \bigcup_{w \in W_x} \{wy \colon y \in Y_j^{xw}\}. \end{aligned} \tag{5.34}$$

We show that we can choose $t = \varepsilon_3^{1/8} d_2^{-2}$ vertices $x_1, \dots, x_t \in X$ so that the union $\bigcup_{u=1}^t \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})$ is sufficiently large to apply the (ε_3, d_3, r)-regularity of $\mathcal{H}$ with respect to $\mathcal{G}$. This will be, however, in a contradiction with with the assumption (5.31).

First, we give a lower bound on the size of $\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^x)$ for an arbitrary vertex $x \in X$. Indeed, from the construction (5.34), we have

$$\begin{aligned} \left| \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^x) \right| &= \sum_{w \in W_x} \left| \mathcal{H}(x) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| \\ &\overset{(5.32)}{\geq} \sum_{w \in W_x} \left(d_3 - 4\varepsilon_3^{1/16} \right) \left| \mathcal{G}[\mathcal{G}(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right|. \end{aligned} \tag{5.35}$$

We recall that $\mathcal{G}[\mathcal{G}(x, x_u)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular (from the goodness of the pair $\{x, w\}$ (c.f. Definition 3.11)) and, thus, its size is bounded from below by $\left(d_2 - \varepsilon_2^{1/2}\right)^5 m^2$ (c.f. Fact 3.6). Then we use inequality (5.30) and get

$$\begin{aligned} \left| \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^x) \right| &\geq \sum_{w \in W_x} \left(d_3 - 4\varepsilon_3^{1/16} \right) \left| \mathcal{G}[\mathcal{G}(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| \\ &\overset{(5.30)}{\geq} |W_x| \left(d_3 - 4\varepsilon_3^{1/16} \right) \times \varepsilon_3^{1/32} \left(d_2 - \varepsilon_2^{1/2} \right)^5 m^2 \geq \varepsilon_3^{1/8} d_2^5 m^3. \end{aligned} \tag{5.36}$$

Second, we also need an upper bound on the size of $\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^x) \cap \bigcup_{i=1}^{r'} \mathcal{K}_3(\mathcal{Q}_i^{x'})$ for a good pair of nice vertices $\{x, x'\}$. Notice, that any triangle $z_1 z_2 z_3$, $z_j \in V_j$, belonging to this intersection must satisfy

$$z_1 \in W_x \cap W_{x'}$$

and

$$z_2z_3 \in \left(\mathcal{G}[\mathcal{G}(x,w)] \cap \mathcal{K}_2(\mathcal{B}_j^{xw})\right) \cap \left(\mathcal{G}[\mathcal{G}(x',w')] \cap \mathcal{K}_2(\mathcal{B}_i^{x'w'})\right)$$

for some $i, j \in [r']$.

The above intersection can be clearly overestimated by $\mathcal{G}[\mathcal{G}(x,w)] \cap \mathcal{G}[\mathcal{G}(x',w')] = \mathcal{G}[\mathcal{G}(x,w,x',w')]$. Moreover, regardless which $w' \in W_{x'}$ we take, vertex z_1 is always in W_x and edge z_2z_3 is in $\mathcal{G}[\mathcal{G}(x,w,x')]$. Hence

$$\left|\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^x) \cap \bigcup_{i=1}^{r'} \mathcal{K}_3(\mathcal{Q}_i^{x'})\right| \le \sum_{w \in W_x} |\mathcal{G}[\mathcal{G}(x,w,x')]|.$$

Since $\{x, x'\}$ is a good pair, there are at most $\varepsilon_2^{1/4}m$ vertices $x'' \in V_1$ for which either $\mathcal{G}[\mathcal{G}(x,x',x'')]$ is not $(\varepsilon_2^{1/2}, d_2)$-regular or $(d_2 - \varepsilon_2)^3 m \le \deg_j(x,x',x'') \le (d_2 + \varepsilon_2)^3 m$ does not hold for $j = 2$ or $j = 3$.

Thus, for triples $\{x, x', w\}$ satisfying both conditions we have $|\mathcal{G}[\mathcal{G}(x,w,x')]| \le 2d_2^7 m^2$ (c.f. Fact 3.6), and for the remaining $\varepsilon_2^{1/4}m$ triples we have $|\mathcal{G}[\mathcal{G}(x,w,x')]| \le |\mathcal{G}[\mathcal{G}(x,x')]| \le 2d_2^5 m^2$ (again using Fact 3.6). Then,

$$\sum_{w \in W_x} |\mathcal{G}[\mathcal{G}(x,w,x')]| \le |W_x| \times 2d_2^7 m^2 + \varepsilon_2^{1/4} m \times 2d_2^5 m^2 \le 4\varepsilon_3^{1/16} d_2^7 m^3.$$

The last inequality follows from $\varepsilon_2^{1/4} \le \varepsilon_2^{1/8} d_2^2 \le 2\varepsilon_3^{1/16} d_2^2$ and $|W_x| = \varepsilon_3^{1/16} m$. Consequently,

$$\left|\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^x) \cap \bigcup_{i=1}^{r'} \mathcal{K}_3(\mathcal{Q}_i^{x'})\right| \le 4\varepsilon_3^{1/16} d_2^7 m^3 \tag{5.37}$$

for every pair of good vertices $\{x, x'\} \subset X$.

Now we define an auxiliary graph $\Gamma = (V, E)$, where a pair of vertices $\{x', x''\}$ is an edge whenever it is not a good pair. It follows from Observation 3.12 that the size of E is bounded by $20\varepsilon_2^{1/4} m^2$. Using the Picking Lemma with $\sigma_1 = 20\varepsilon_2^{1/4}$, $c = \varepsilon_2^{1/16}$, and $t = \varepsilon_3^{1/8} d_2^{-2}$, we choose t vertices $x_1, \ldots, x_t \in X$ so that all pairs $\{x_u, x_v\}$ are good.

Then, we can estimate $\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right|$ using Observation 4.5 in the following way:

$$\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \geq \sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| - \sum_{1\leq u<v\leq t}\left|\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{Q}_i^{x_u})\cap\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_v})\right|$$

This can be further simplified with the use of (5.36) and (5.37).

$$\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \;\geq\; t\times\varepsilon_3^{1/8}d_2^5m^3 - \binom{t}{2}4\varepsilon_3^{1/16}d_2^7m^3 \;\geq\; 2\varepsilon_3 d_2^3m^3 \;\geq\; \varepsilon_3|\mathcal{K}_3(\mathcal{G})|$$

The last part of this inequality follows from Fact 3.6. Applying the (ε_3, d_3, r)-regularity of $\mathcal{H}$ with respect to $\mathcal{G}$, we get

$$\left|\mathcal{H}\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \leq (d_3+\varepsilon_3)\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right|$$

Using the first part (equality) of (5.35) and inequality (5.33), we conclude that

$$\begin{aligned}(d_3+\varepsilon_3)\sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| &= (d_3+\varepsilon_3)\sum_{u=1}^{t}\sum_{w\in W_{x_u}}\left|\mathcal{H}(x_u)\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^{x_uw})\right| \\ &\leq (d_3+\varepsilon_3)\left(d_3+4\varepsilon_3^{1/16}\right)\sum_{u=1}^{t}\sum_{w\in W_{x_u}}\left|\mathcal{G}[\mathcal{G}(x_u,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^{x_uw})\right|\end{aligned}$$

Combining the previous two inequalities yields:

$$\left|\mathcal{H}\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \leq \left(d_3^2+5\varepsilon_3^{1/16}\right)\sum_{u=1}^{t}\sum_{w\in W_{x_u}}\left|\mathcal{G}[\mathcal{G}(x_u,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^{x_uw})\right|$$

On the other hand, we use assumption (5.31) and obtain the following lower bound on $\left|\mathcal{H}\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right|$:

$$\begin{aligned}\left|\mathcal{H}\cap\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| &= \sum_{u=1}^{t}\sum_{w\in W_{x_u}}\left|\mathcal{H}(x_u,w)\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^{x_uw})\right| \\ &\geq \left(d_3^2+\varepsilon_3^{1/32}\right)\sum_{u=1}^{t}\sum_{w\in W_{x_u}}\left|\mathcal{G}[\mathcal{G}(x_u,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^{x_uw})\right|.\end{aligned}$$

Comparing the last two inequalities yields $d_3^2 + 5\varepsilon_3^{1/16} > d_3^2 + \varepsilon_3^{1/32}$, which is a contradiction with our assumption that $|X| \geq \varepsilon_2^{1/16} m$. Therefore $|X| < \varepsilon_2^{1/16} m$.

The case when W_x is the set of vertices w for which the second part of inequality (4.1) is not satisfied, i.e. for which

$$\left| \mathcal{H}(x,w) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right| < \left(d_3^2 - \varepsilon_3^{1/32} \right) \left| \mathcal{G}[\mathcal{G}(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^{xw}) \right|,$$

can be handled in the very same way.

Hence, there are at most $2 \times \varepsilon_2^{1/16} m \times m$ nice vertices x (more precisely, all $x \in X$) for which the link $\mathcal{H}(x,x')$ is $(\varepsilon_3^{1/32}, d_3^2, r')$-irregular for at least $2\varepsilon_3^{1/16} m$ good pairs $\{x, x'\}$, $x' \in V_{\text{nice}}$.

Moreover, for every other nice vertex (that is $x \in V_{\text{nice}} \setminus X$, there at most $4\varepsilon_3^{1/16} m$ good pairs $\{x, x'\}$, $x' \in V_{\text{nice}}$ for which the link $\mathcal{H}(x,x')$ is $(\varepsilon_3^{1/32}, d_3^2, r')$-irregular (c.f. definition of W_x) and at most $3\varepsilon_2^{1/16} m \leq \varepsilon_3^{1/16} m$ good pairs $\{x, x'\}$, $x' \in V_{\text{nice}}$, for which the link $\mathcal{H}(x)$ is $(4\varepsilon_3^{1/16}, d_3, r')$-irregular with respect to $\mathcal{G}[\mathcal{G}(x,x')]$.

Thus, for all but at most $2\varepsilon_2^{1/16} m^2$ nice vertices x, the link $\mathcal{H}(x,x')$ is $(\varepsilon_3^{1/32}, d_3^2, r')$-regular for all but at most $5\varepsilon_3^{1/16} m$ good pairs $\{x, x'\}$, $x' \in V_{\text{nice}}$. □

Now we define a nice pair of vertices $\{x, x'\}$.

Definition 5.12 (nice pair). *Set* $r' = r / \left(\varepsilon_3^{1/2} d_2^{-3} \right) \left(2\varepsilon_3^{1/8} d_2^{-3} \right)$. *A pair of vertices* $\{x, x'\}$ *is called* nice *if it satisfies the following conditions:*

(i) both x *and* x' *are nice vertices,*

(ii) the pair $\{x, x'\}$ *is good,*

(iii) the link $\mathcal{H}(x,x')$ *is* $(\varepsilon_3^{1/32}, d_3^2, r')$*-regular with respect to* $\mathcal{G}[\mathcal{G}(x,x')]$*, and*

(iv) $\mathcal{H}$ *is* $(4\varepsilon_3^{1/16}, d_2, r')$*-regular with respect to* $\mathcal{G}[\mathcal{G}(x,x')]$*.*

Observations 3.8, 3.12, and 5.4, and Claims 5.9 and 5.11 imply the following:

- all but at most $\left(2(s-1)\varepsilon_2^{1/2}+10(s-1)\varepsilon_2^{1/4}\right)m^2$ pairs $\{x,x'\}$ are pairs of good vertices (c.f. Observation 3.8);
- all but at most $\left(4(s-1)\varepsilon_2^{1/2}+6(s-1)\varepsilon_2^{1/4}\right)m^2$ pairs $\{x,x'\}$ are good pairs in $[V_{\text{good}}]^2$ (c.f. Observation 3.12);
- all but at most $\left(4\binom{s-1}{3}\varepsilon_3^{1/2}+2\binom{s-1}{2}\varepsilon_3^{1/4}\right)m^2$ good pairs $\{x,x'\}$ are good pairs of nice vertices (c.f. Observation 5.4);
- for all but at most $3\binom{s-1}{3}\varepsilon_2^{1/32}m^2$ of the above pairs, the restriction $\mathcal{H}[\mathcal{G}(x,x')]$ is not $(4\varepsilon_3^{1/16},d_3,r')$-regular with respect to $\mathcal{G}(x,x')$ (c.f. Claim 5.9);
- for all but at most $6\binom{s-1}{2}\varepsilon_3^{1/16}m^2$ of the above pairs, the link $\mathcal{H}(x,x')$ is not $(\varepsilon_3^{1/32},d_3^2,r')$-regular with respect to $\mathcal{G}(x,x')$ (c.f. Claim 5.11).

We can summarize these facts into the following observation (we use $\varepsilon_2 \ll \varepsilon_3$ to simplify this result).

Observation 5.13. *All but* $10\binom{s-1}{2}\varepsilon_3^{1/16}m^2$ *pairs in* $[V_1]^2$ *are nice.*

5.5 Proof of Proposition 2.5

In this section, we use the properties of nice pairs and Claim 5.8 to show Proposition 2.5.

Proof. Set $r' = r/\left(\delta_3^{1/2}d_2^{-3}\right)\left(2\delta_3^{1/8}d_2^{-3}\right)$ and let $\{x,x'\}\in V_1$ be a nice pair, i.e. a good pair for which we have (c.f. Definition 5.12):

(i) the link $\mathcal{H}^{(3)}(x,x')$ is $(\delta_3^{1/32},d_3^2,r')$-regular with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,x')]$,

(ii) $\mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x,x')]$ is $(4\delta_3^{1/16},d_3,r')$-regular with respect to $[\mathcal{H}^{(2)}(x,x')]$.

Since $\{x,x'\}$ is also a good pair, we have (c.f. Definition 3.11)

(iii) $(d_2 - \delta_2)^2 n \le \deg_j(x, x') \le (d_2 + \delta_2)^2 n$ for $j = 2, \ldots, 5$, and

(iv) the $(4, 2)$-cylinder $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x, x')]$ is $(\delta_2^{1/2}, d_2)$-regular.

We apply Claim 5.8 with cylinders $\mathcal{G}_2 = \mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x, x')]$, $\mathcal{G}_3 = \mathcal{H}^{(3)}(x, x')$, $\mathcal{H} = \mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x, x')]$, and $\varepsilon_2 = \delta_2^{1/2}$, $\varepsilon_3 = \delta_3^{1/32}$, $\tilde{d}_3 = d_3^2$, $(d_2 - \delta_2)^2 n \le m \le (d_2 + \delta_2)^2 n$, and r replaced by r'. Observe that (i)-(iv) verify conditions of the Setup C. More precisely,

- (iv) verifies that $\mathcal{G}_2$ is (ε_2, d_2)-regular,
- (i) verifies that $\mathcal{G}_3$ is $(\varepsilon_3, \tilde{d}_3, r)$-regular with respect to $\mathcal{G}_2$, and
- (ii) verifies that $\mathcal{H}$ is (ε_3, d_3, r)-regular with respect to $\mathcal{G}_2$.

Hence, we obtain

$$\frac{9}{16} d_2^6 \left(d_3^2\right)^6 d_3^4 (d_2 - \delta_2)^8 n^4 \le \mathcal{K}_4(\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3)) \le \frac{15}{8} d_2^6 \left(d_3^2\right)^6 d_3^4 (d_2 + \delta_2)^8 n^4.$$

Estimate (2.6) follows from this inequality since $d_2 \gg \delta_2$. This is true for every nice pair $\{x, x\}$. By Observation 5.13 applied with $s = 5$, all but at most $60\delta_3^{1/16} n^2$ pairs $\{x, x'\}$ are nice. Therefore, (2.6) holds for all but at most $60\delta_3^{1/16} n^2$ pairs $\{x, x'\} \subset V_1$.

□

5.6 Counting II

In this section, we will expand statements for a single vertex from Section 5.2 to pairs. We consider the scenario given by Setup C:

Setup C. *Let* $0 < \varepsilon_2 \ll d_2 \le 1$, $0 < \varepsilon_3 \ll d_3 \le 1$, *and* $0 < \varepsilon_3 \ll \tilde{d}_3 \le 1$ *be real numbers so that* $\varepsilon_2 \ll \varepsilon_3$. *Let* $V = V_1 \cup \ldots \cup V_s$ *be a partition, where* $|V_1| = \ldots = |V_s| = m$, $\mathcal{G}_2 = (V, E(\mathcal{G}_2))$ *be an* $(s, 2)$*-cylinder that is* (ε_2, d_2)*-regular,*

$\mathcal{G}_3 = (V, E(\mathcal{G}_3))$ be an $(s,2)$-cylinder that is $(\varepsilon_3, \tilde{d}_3, r)$-regular with respect to $\mathcal{G}_2$, and let $\mathcal{H} = (V, E(\mathcal{H}))$ be an $(s,3)$-cylinder which is (ε_3, d_3, r)-regular with respect to $\mathcal{G}_2$.

Our objective is to prove the following technical claim.

Claim 5.14. *Let $s = 4$ and $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{H}$ are as in Setup C. Then, for all but at most $10\varepsilon_3^{1/64}m^2$ pairs $\{x, x'\} \subset V_1$, the following is true:*

$$|\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x,x')[\mathcal{G}_3(x,x')])| \leq \frac{3}{2} d_2^9 \tilde{d}_3^9 d_3^7 m^3.$$

This claim will be used in the next section to prove a claim necessary for proving Proposition 2.6.

We start with some technical observation. The first one is a consequence of Fact 4.9 and shows that almost all pairs have approximately the same joint degree in $\mathcal{G}_3$.

Fact 5.15. *For all but at most $10(s-1)\varepsilon_3^{1/8}m^2$ pairs $\{x, x'\} \subset V_1$ we have:*

$$\left(1 - \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m \leq \deg_{\mathcal{G}_{3,j}}(x, x') \leq \left(1 + \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m \tag{5.38}$$

for every $j \in [s] \setminus \{1\}$. Consequently, $\mathcal{G}_2[\mathcal{G}_3(x,x')]$ is $(\varepsilon_2^{1/2}, d_2)$-regular.

Proof. Since $\mathcal{G}_2$ is (ε_2, d_2)-regular and $\mathcal{G}_3$ is $(\varepsilon_3, \tilde{d}_3, r)$-regular with respect to $\mathcal{G}_2$, $\mathcal{G}_2$ and $\mathcal{G}_3$ satisfy Setup A. Consequently, we can apply the remark following Fact 4.9 with $l = 2$ to infer that all but at most

$$(s-1)\left(4\varepsilon_2^{1/2} + 8\varepsilon_2^{1/8} + 4\varepsilon_3^{1/2} + 8\varepsilon_3^{1/8}\right) m^2 \leq 10(s-1)\varepsilon_3^{1/8} m^2$$

pairs $\{x, x'\}$ satisfy (5.38).

Using assumptions $\varepsilon_2 \ll \varepsilon_3 \ll \tilde{d}_3$ and $\varepsilon_2 \ll d_2$ yields $\left(1 - \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 > \varepsilon_2^{1/4}$. Thus, by Fact 3.4, $\mathcal{G}_2[\mathcal{G}_3(x,x')]$ is $(\varepsilon_2^{1/2}, d_2)$-regular. ∎

The second fact proves that the restriction of $\mathcal{H}$ to the joint neighborhood inherits regularity for almost all pairs.

Fact 5.16. *Set* $r' = r/\left(\varepsilon_3^{1/2} d_2^{-3}\right)\left(2\varepsilon_3^{1/8} d_2^{-3}\right)$. *Then,*

(i) the pair $\{x, x'\}$ *is good, and*

(ii) the restriction $\mathcal{H}[\mathcal{G}_2(x, x')]$ *is* $(2\varepsilon_3^{1/16}, d_3, r')$*-regular with respect to* $\mathcal{G}_2[\mathcal{G}_2(x, x')]$

for all but at most $5\binom{s-1}{3}\varepsilon_3^{1/2} m^2$ *pairs* $\{x, x'\} \subset V_1$.

Proof. Observe first that due to Observations 3.8 and 3.12:

- all but at most $\left(2(s-1)\varepsilon_2^{1/2} + 10(s-1)\varepsilon_2^{1/4}\right) m^2$ pairs $\{x, x'\}$ are pairs of good vertices (c.f. Observation 3.8), and
- all but at most $\left(4(s-1)\varepsilon_2^{1/2} + 6(s-1)\varepsilon_2^{1/4}\right) m^2$ pairs $\{x, x'\}$ are good pairs in $[V_{\text{good}}]^2$ (c.f. Observation 3.12).

Also notice that by Observation 5.4

- all but at most $\left(4\binom{s-1}{3}\varepsilon_3^{1/2} + 2\binom{s-1}{2}\varepsilon_2^{1/16}\right) m^2$ good pairs $\{x, x'\}$ are good pairs of nice vertices.

Furthermore, it follows from Claim 5.9 that for every nice vertex $x \in V_{\text{nice}}$ there are at most $3\binom{s-1}{3}\varepsilon_2^{1/32} m$ good vertices $x' \in V_{\text{good}}$ for which $\mathcal{H}$ is not $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x, x')]$.

We use the assumption $\varepsilon_2 \ll \varepsilon_3$ and conclude that all but at most $5\binom{s-1}{2}\varepsilon_3^{1/2} m^2$ pairs satisfy conditions (i) and (ii). □

The next two facts will show that the $(s-1, 2)$-cylinder $\mathcal{G}_3[\mathcal{G}_3(x, x')] \cap \mathcal{H}(x, x')$ is regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x, x')]$ for almost all pairs $\{x, x'\}$. Fact 5.17 is of a technical nature and it will be later used in Fact 5.18 which actually proves the regularity of $\mathcal{G}_3[\mathcal{G}_3(x, x')] \cap \mathcal{H}(x, x')$.

Fact 5.17. *Let $r'' = r/\left(\varepsilon_3^{1/2}d_2^{-2}\right)\left(\varepsilon_3^{1/16}d_2^{-2}\right)$. Then, for all but at most $4\binom{s-1}{2}\varepsilon_3^{1/8}m$ good vertices $x \in V_{\text{good}}$ the following statement holds: all but at most $3\binom{s-1}{2}\varepsilon_2^{1/16}m$ good vertices x' have the following properties*

(i) $\left(1-\varepsilon_3^{1/2}\right)^3 d_2^2\tilde{d}_3 m \le |N_{\mathcal{G}_{3,j}}(x) \cap N_{\mathcal{G}_{2,j}}(x')| \le \left(1+\varepsilon_3^{1/2}\right)^3 d_2^2\tilde{d}_3 m$,

(ii) $\mathcal{G}_2[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')]$ *is* $(\varepsilon^{1/2}, d_2)$*-regular,*

(iii) the $(s-1,2)$*-cylinder* $\mathcal{G}_3[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')] \cap \mathcal{H}(x)$ *is* $(2\varepsilon_3^{1/32}, d_3\tilde{d}_3, r'')$*-regular with respect to* $\mathcal{G}_2[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')]$.

Proof. Observe first that (ii) follows from (i). Since $\varepsilon_2 \ll \varepsilon_3 \ll \tilde{d}_3$ and $\varepsilon_2 \ll d_2$, we have $\left(1-\varepsilon_3^{1/2}\right)^3 d_2^2\tilde{d}_3 > \delta_2^{1/4}$. Then, the $(\varepsilon_2^{1/2}, d_2)$-regularity of $\mathcal{G}_2[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')]$ follows from Fact 3.4. Hence, we will concentrate on properties (i) and (iii).

Note that this is claim is being proved under Setup C. Then, Claim 5.7 implies that for all but at most $2\binom{s-1}{2}(4\varepsilon_2^{1/2} + 4\varepsilon_3^{1/2} + \varepsilon_2^{1/8} + \varepsilon_3^{1/8})m \le 4\binom{s-1}{2}\varepsilon_3^{1/8}m$ good vertices $x \in V_{\text{good}}$ the following conditions are satisfied:

(a) $\left(1-\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m \le \deg_{\mathcal{G}_{3,j}}(x) \le \left(1+\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m$ for $j \in [s]\backslash\{1\}$, and

(b) $(s-1,2)$-cylinder $\mathcal{G}_3[\mathcal{G}_3(x)] \cap \mathcal{H}(x)$ is $(\varepsilon_3^{1/8}, d_3\tilde{d}_3, r/\left(\varepsilon_3^{1/2}d_2^{-2}\right))$-regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x)]$.

Fix a good vertex $x \in V_{\text{good}}$ for which (a) and (b) are satisfied and let W' be the set of vertices x' which violate

$$\left(1-\varepsilon_3^{1/2}\right)^3 d_2^2\tilde{d}_3 m \le |N_{\mathcal{G}_{3,j}}(x) \cap N_{\mathcal{G}_{2,j}}(x')| \le \left(1+\varepsilon_3^{1/2}\right)^3 d_2^2\tilde{d}_3 m \tag{5.39}$$

for some $j \in [s] \setminus \{1\}$ such that $|W'| \le \left(1+\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m$.

Furthermore, let W'', $|W''| \le \left(1+\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m$, be the set of vertices x' satisfying (5.39) for which the $(s-1,2)$-cylinder $\mathcal{G}_3[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')] \cap \mathcal{H}(x)$ is $(2\varepsilon_3^{1/32}, d_3\tilde{d}_3, r'')$-irregular with respect to $\mathcal{G}_2[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')]$.

We will show the following two statements.

S1 $|W'| \le 2(s-1)\varepsilon_2^{1/4}m$.

S2 $|W''| \le 2\binom{s-1}{2}\varepsilon_2^{1/16}m$.

Note that every vertex that does not belong to $W' \cup W''$ satisfies (i), (ii), and (iii). Finally, using S1 and S2 yields $|W' \cup W''| \le 2(s-1)\varepsilon_2^{1/4}m + 2\binom{s-1}{2}\varepsilon_2^{1/16}m \le 3\binom{s-1}{2}\varepsilon_2^{1/16}m$.

Remark. Observe that S1 and S2 indeed show that W' and W'' contains all vertices which violate (i) or (iii).

Proof of S1: Let W, $W' \subset W \subset V_1$, be a set for which $\left(1-\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m \le |W| \le \left(1+\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m$. This can be achieved by adding some vertices to W'. Set $V_1' = W$, $V_j' = V_j \cap \mathcal{G}_3(x)$ for $j \in [s] \setminus \{1\}$, $V' = V_1' \cup \ldots \cup V_s'$, and $\mathcal{G}_2' = \mathcal{G}_2[V']$.

Since $\varepsilon_2 \ll \varepsilon_3$, $\varepsilon_2 \ll d_2$, and $\varepsilon_3 \ll \tilde{d}_3$, we have $\left(1-\varepsilon_3^{1/2}\right)^3 d_2^2\tilde{d}_3 > \varepsilon_2^{1/4}$. Thus, by Observation 3.4, $\mathcal{G}_2'$ is $(\varepsilon_2^{1/2}, d_2)$-regular. Applying Fact 3.3 with $\delta = \varepsilon_2^{1/2}$ and $k = 1$ on $\mathcal{G}_2'$ yields that all but at most $2(s-1)\varepsilon_2^{1/4}|V_1'| \le 2(s-1)\varepsilon_2^{1/4}m$ vertices $x \in V_1'$ satisfy

$$\left(d_2 - \varepsilon_2^{1/2}\right)|V_j'| \le |N_{\mathcal{G}_2',j}(x')| \le \left(d_2 + \varepsilon_2^{1/2}\right)|V_j'|$$

for every $j \in [s] \setminus \{1\}$. Since $\left(1-\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m \le |V_j'| \le \left(1+\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m$ due to (a) and the definition of V_i' and $N_{\mathcal{G}_2',j}(x') = N_{\mathcal{G}_3,j}(x) \cap N_{\mathcal{G}_2,j}(x')$, one easily gets that all but at most $2(s-1)\varepsilon_2^{1/4}m$ vertices $x' \in V_1$ satisfy (5.39). Since W' contains all vertices that does not satisfy (5.39), we have $|W'| \le 2(s-1)\varepsilon_2^{1/4}m$.

Proof of S2: Let W, $W'' \subset W \subset V_1$, be a set for which $\left(1-\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m \le |W| \le \left(1+\varepsilon_3^{1/2}\right)^2 d_2\tilde{d}_3 m$. Such a choice can be made by adding (if necessary) some vertices to W''. Set $V_1'' = W$, $V_j'' = V_j \cap \mathcal{G}_3(x)$ for $j \in [s] \setminus \{1\}$, $V'' = V_1'' \cup \ldots \cup V_s''$, and $\mathcal{G}_2'' = \mathcal{G}_2[V'']$. Using the same argument as for $\mathcal{G}_2'$ above, we get that $\mathcal{G}_2''$ is $(\varepsilon_2^{1/2}, d_2)$-regular.

We also set $\mathcal{G}_3'' = \mathcal{G}_3[\mathcal{G}_3(x)] \cap \mathcal{H}(x)$. By our choice of x (c.f. condition (b)), $\mathcal{G}_3''$ is $(\varepsilon_3^{1/8}, d_3\tilde{d}_3, r/\left(\varepsilon_3^{1/2} d_2^{-2}\right))$-regular with respect to $\mathcal{G}_2''$. Hence, in a view of the remark after Claim 5.6, $\mathcal{G}_2''$ and $\mathcal{G}_3''$ satisfy the assumptions of Claim 5.6 with ε_2 replaced by $\varepsilon_2^{1/2}$, ε_3 replaced by $\varepsilon_3^{1/8}$, $\tilde{d}_3$ by $d_3\tilde{d}_3$, r by $r/\left(\varepsilon_3^{1/2} d_2^{-2}\right)$, and V_j by V_j'.

Using this claim, $\mathcal{G}_3''[\mathcal{G}_2''(x')]$ is $(2\varepsilon_3^{1/32}, d_3\tilde{d}_3, r'')$-irregular with respect to $\mathcal{G}_2''[\mathcal{G}_2''(x')]$ for at most $2\binom{s-1}{2}\varepsilon_2^{1/16}|V_1'| \leq 2\binom{s-1}{2}\varepsilon_2^{1/16} m$ good vertices $x' \in V_{\text{good}}$.

Observe that all such vertices are contained in W'' since $\mathcal{G}_2''[\mathcal{G}_2''(x')] = \mathcal{G}_2[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')]$, and $\mathcal{G}_3''[\mathcal{G}_2''(x')] = \mathcal{G}_3[\mathcal{G}_3(x) \cap \mathcal{G}_2(x')] \cap \mathcal{H}(x)$. Hence, $|W''| \leq 2\binom{s-1}{2}\varepsilon_2^{1/16} m$. □

Fact 5.18. *Let* $r' = r/\left(\varepsilon_3^{1/2} d_2^{-2}\right)\left(\varepsilon_3^{1/16} d_2^{-2}\right)$. *Then for all but at most* $3\binom{s-1}{2}\varepsilon_3^{1/64} m^2$ *pairs of good vertices* $\{x, x'\} \subset V_{\text{good}}$ *the following statements hold.*

(a)

$$\left(1 - \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m \leq \deg_{\mathcal{G}_{3,j}}(x, x') \leq \left(1 + \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m, \tag{5.40}$$

for every $j \in [s] \setminus \{1\}$, *and* $\mathcal{G}_2[\mathcal{G}_3(x, x')]$ *is* $(\varepsilon_2^{1/2}, d_2)$*-regular, and*

(b) $\mathcal{G}_3[\mathcal{G}_3(x, x')] \cap \mathcal{H}(x, x')$ *is* $(\varepsilon_3^{1/64}, \tilde{d}_3 d_3^2, r')$*-regular with respect to* $\mathcal{G}_2[\mathcal{G}_3(x, x')]$.

Proof. Suppose $s = 3$ and for a pair of good vertices $\{x, w\}$ consider the following properties:

(i) $\{x, w\}$ is a good pair,

(ii) $\left(1 - \varepsilon_3^{1/2}\right)^3 d_2^2 \tilde{d}_3 m \leq |N_{\mathcal{G}_{3,j}}(x) \cap N_{\mathcal{G}_{2,j}}(w)| \leq \left(1 + \varepsilon_3^{1/2}\right)^3 d_2^2 \tilde{d}_3 m$ for $j = 2, 3$, and $\mathcal{G}_2[\mathcal{G}_3(x) \cap \mathcal{G}_2(w)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular,

(iii) the $(s-1, 2)$-cylinder $\mathcal{G}_3[\mathcal{G}_3(x) \cap \mathcal{G}_2(w)] \cap \mathcal{H}(x)$ is $(2\varepsilon_3^{1/32}, d_3\tilde{d}_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x) \cap \mathcal{G}_2(w)]$,

Observe that

- there are at most $\left(8\varepsilon_2^{1/2} + 12\varepsilon_2^{1/4}\right) m^2 \leq 14\varepsilon_2^{1/4} m^2$ pairs of good vertices $\{x, w\}$ which are not good pairs, that is they violate (i) (c.f. Observation 3.12),

- there are at most $4\varepsilon_3^{1/8} m$ good vertices x for which there are more than $3\varepsilon_2^{1/16} m$ good vertices w violating (ii) or (iii) (c.f. Fact 5.17), and

- for every remaining vertex x there are at most $3\varepsilon_2^{1/16} m$ good vertices w violating (ii) or (iii) (c.f. Fact 5.17).

Thus, all but at most

$$14\varepsilon_2^{1/4} m^2 + 4\varepsilon_3^{1/8} m \times m + m \times 3\varepsilon_2^{1/16} m \leq 5\varepsilon_3^{1/8} m^2$$

pairs of good vertices satisfy conditions (i)-(iii).

Furthermore, Fact 5.15 implies that there are at most $20\varepsilon_3^{1/8} m^2$ pairs $\{x, w\}$ that violate (a), that is

(iv) $\left(1 - \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m \leq \deg_{\mathcal{G}_{3,j}}(x, w) \leq \left(1 + \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m$ for $j = 2, 3$, and cylinder $\mathcal{G}_2[\mathcal{G}_3(x, w)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular.

For every good vertex $x \in V_{\text{good}}$ denote by W_x the set of all good vertices $w \in V_{\text{good}}$ which satisfy conditions (i)-(iv) and violate (b), that is

(v) $\mathcal{G}'_3(x, w) = \mathcal{G}_3[\mathcal{G}_3(x, w)] \cap \mathcal{H}(x, w)$ is not $(\varepsilon_3^{1/64}, \tilde{d}_3 d_3^2, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x, w)]$.

Denote by X the set of all good vertices $x \in V_{\text{good}}$ for which $|W_x| \geq 2\varepsilon_3^{1/64} m$. We will show that $|X| < 2\varepsilon_2^{1/16} m$.

Then we can easily finish the proof:

- all but at most $5\varepsilon_3^{1/8} m^2 + 20\varepsilon_3^{1/8} m^2$ pairs of good vertices satisfies (i)-(iv),

- for every vertex $x \in X$ there are more than $2\varepsilon_3^{1/64} m$ good vertices w satisfying (i)-(iv) and violating (v) (c.f. definition of X and W_x), and

- for every vertex $x \notin X$ there are at most $2\varepsilon_3^{1/64}m$ good vertices w satisfying (i)-(iv) and violating (v) (c.f. definition of X and W_x).

Thus, all but at most

$$5\varepsilon_3^{1/8}m^2 + 20\varepsilon_3^{1/8}m^2 + 2\varepsilon_2^{1/16}m \times m + m \times 2\varepsilon_3^{1/64}m \leq 3\varepsilon_3^{1/64}m^2$$

pairs $\{x, w\}$ of good vertices satisfy conditions (a) and (b).

If $s > 3$, we apply the result for $s = 3$ simultaneously to $\binom{s-1}{2}$ restrictions of $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{H}$ to $V_1 \cup V_i \cup V_j$, $1 < i < j \leq s$.

Now we show that $|X| < 2\varepsilon_2^{1/16}m$. Suppose in contrary that $|X| \geq 2\varepsilon_2^{1/16}m$. We write $W_x = W_x^+ \cup W_x^-$, where W_x^- (W_x^+ respectively) is the set of all vertices $w \in W_x$ for which the first (respectively second) part of inequality (4.1) is not satisfied. We also define X^+ (respectively X^-) to be the set of all vertices $x \in X$ for which $|W_x^+| \geq \varepsilon_3^{1/64}m$ (respectively $|W_x^-| \geq \varepsilon_3^{1/64}m$). Clearly $X = X^+ \cup X^-$ and, thus, either $|X^+|$ or $|X^-|$ is at least $\varepsilon_2^{1/16}m$. Assume that $|X^+| \geq \varepsilon_2^{1/16}m$ (we can repeat the same for X^-).

Let $x \in X^+$. Then, for every $w \in W_x$ there exist $(2,1)$-cylinders $\mathcal{B}_j^w = Y_j^w \cup Z_j^w$, where Y_j^w is a subset of $\mathcal{G}_3(x, w) \cap V_2$ and $Z_j^w \subset \mathcal{G}_3(x, w) \cap V_3$, $j \in [r']$, such that

$$\left| \mathcal{G}_2[\mathcal{G}_3(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w) \right| \geq \varepsilon_3^{1/64} |\mathcal{G}_2[\mathcal{G}_3(x,w)]|, \tag{5.41}$$

but

$$\left| \mathcal{G}_3'(x,w) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w) \right| > \left(\tilde{d}_3 d_3^2 + \varepsilon_3^{1/64} \right) \left| \mathcal{G}_2[\mathcal{G}_3(x,w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w) \right|. \tag{5.42}$$

Recall that $\mathcal{G}_3'(x,w) = \mathcal{G}_3[\mathcal{G}_3(x,w)] \cap \mathcal{H}(x,w)$. Notice that due to assumptions (ii)-(iv), $\varepsilon_2 \ll d_2$, $\varepsilon_3 \ll \tilde{d}_3$ and $\varepsilon_2 \ll \varepsilon_3$, we have

$$|\mathcal{G}_2[\mathcal{G}_3(x,w)]| \overset{(iv)}{\geq} \left(d_2 - \varepsilon_2^{1/2} \right) \left(\left(1 - \varepsilon_3^{1/8} \right)^2 d_2^2 \tilde{d}_3^2 m \right)^2 \geq (1/2) d_2^5 \tilde{d}_3^4 m^2 \tag{5.43}$$

and

$$|\mathcal{G}_2[\mathcal{G}_3(x)\cap\mathcal{G}_2(w)]| \overset{(ii)}{\leq} \left(d_2+\varepsilon_2^{1/2}\right)\left(\left(1+\varepsilon_3^{1/2}\right)^3 d_2^2\tilde{d}_3 m\right)^2 \leq 2d_2^5\tilde{d}_3^2 m^2.$$

Subsequently,

$$\varepsilon_3^{1/64}|\mathcal{G}_2[\mathcal{G}_3(x,w)]| \geq \varepsilon_3^{1/64}(1/2)d_2^5\tilde{d}_3^4 m^2$$
$$\geq 2\varepsilon_3^{1/32}2d_2^5\tilde{d}_3^2 m^2 \geq 2\varepsilon_3^{1/32}|\mathcal{G}_2[\mathcal{G}_3(x)\cap\mathcal{G}_2(w)]|.$$

We combine the above inequality with (5.41) and obtain

$$\left|\mathcal{G}_2[\mathcal{G}_3(x,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w)\right| \geq 2\varepsilon_3^{1/32}|\mathcal{G}_2[\mathcal{G}_3(x)\cap\mathcal{G}_2(w)]|.$$

Since $\mathcal{B}_j^w \subset \mathcal{G}_3(x,w) \subset \mathcal{G}_3(x)\cap\mathcal{G}_2(w)$ for every $w \in W_x$ and $j \in [s]\setminus\{1\}$, we have

$$\mathcal{G}_2[\mathcal{G}_3(x,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w) = \mathcal{G}_2[\mathcal{G}_3(x)\cap\mathcal{G}_2(w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w) \tag{5.44}$$

In a view of (5.44), $\left|\mathcal{G}_2[\mathcal{G}_3(x)\cap\mathcal{G}_2(w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w)\right| \geq 2\varepsilon_3^{1/32}|\mathcal{G}_2[\mathcal{G}_3(x)\cap\mathcal{G}_2(w)]|$, therefore, we can apply the $(2\varepsilon_3^{1/32}, d_3\tilde{d}_3, r')$-regularity of $\mathcal{G}_3[\mathcal{G}_3(x)]\cap\mathcal{H}(x)$ (c.f. (ii)) and (5.44) to conclude that

$$\left(d_3\tilde{d}_3 - 2\varepsilon_3^{1/32}\right)\left|\mathcal{G}_2[\mathcal{G}_3(x,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w)\right|$$
$$\leq \left|\mathcal{G}_3[\mathcal{G}_3(x)\cap\mathcal{G}_2(w)]\cap\mathcal{H}(x)\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w)\right|$$
$$\leq \left(d_3\tilde{d}_3 + 2\varepsilon_3^{1/32}\right)\left|\mathcal{G}_2[\mathcal{G}_3(x,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w)\right| \tag{5.45}$$

For every vertex $x \in X$, we define $(3,2)$-cylinders $\mathcal{Q}_1^x, \ldots, \mathcal{Q}_{r'}^x$ by $\mathcal{Q}_j^x = \mathcal{Q}_{j\hat{1}}^x \cup$

$\mathcal{Q}^x_{j\hat{2}} \cup \mathcal{Q}^x_{j\hat{3}}$, where

$$\begin{aligned} \mathcal{Q}^x_{j\hat{1}} &= \mathcal{G}_3[\mathcal{G}_3(x)] \cap \mathcal{H}(x), \\ \mathcal{Q}^x_{j\hat{2}} &= \bigcup_{w \in W_x} \{wz \colon z \in Z^w_j\}, \text{ and} \\ \mathcal{Q}^x_{j\hat{3}} &= \bigcup_{w \in W_x} \{wy \colon y \in Y^w_j\}. \end{aligned} \tag{5.46}$$

Observe that the union $\bigcup_{u=1}^{t} \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}^{x_u}_j)$ can be written in the following way

$$\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}^{x_u}_j) = \bigcup_{w \in W_{x_u}} \left(\mathcal{G}_3[\mathcal{G}_3(x_u)] \cap \mathcal{H}(x_u) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}^w_j) \right). \tag{5.47}$$

Moreover, since $\mathcal{B}^w_j \subset \mathcal{G}_3(x_u, w) \subset \mathcal{G}_3(x_u) \cap \mathcal{G}_2(w) \subset \mathcal{G}_3(x_u)$ for every $w \in W_{x_u}$ and $j \in [s] \setminus \{1\}$, we also have

$$\begin{aligned} \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}^{x_u}_j) &= \bigcup_{w \in W_{x_u}} \left(\mathcal{G}_3[\mathcal{G}_3(x_u) \cap \mathcal{G}_2(w)] \cap \mathcal{H}(x_u) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}^w_j) \right) \\ &= \bigcup_{w \in W_{x_u}} \left(\mathcal{G}_3[\mathcal{G}_3(x_u, w)] \cap \mathcal{H}(x_u) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}^w_j) \right). \end{aligned} \tag{5.48}$$

We show that we can choose $t = \varepsilon_3^{1/8} d_2^{-2}$ vertices $x_1, \ldots, x_t \in X$ so that the union $\bigcup_{u=1}^{t} \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}^{x_u}_j)$ is sufficiently large to apply the (ε_3, d_3, r)-regularity of $\mathcal{H}$ with respect to $\mathcal{G}_2$. This will be, however, in a contradiction with with the assumption (5.42).

First, we need a lower bound on the size of $\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}^x_j)$ for an arbitrary vertex $x \in X$. In a view of (5.48), we have

$$\begin{aligned} \left| \bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}^x_j) \right| &= \sum_{w \in W_x} \left| \mathcal{G}_3[\mathcal{G}_3(x) \cap \mathcal{G}_2(w)] \cap \mathcal{H}(x) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}^w_j) \right| \\ &\overset{(5.45)}{\geq} \sum_{w \in W_x} \left(d_3 \tilde{d}_3 - 2\varepsilon_3^{1/32} \right) \left| \mathcal{G}_2[\mathcal{G}_3(x, w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}^w_j) \right|. \end{aligned} \tag{5.49}$$

We recall that $|\mathcal{G}_3[\mathcal{G}_2(x,w)]| \geq (1/2)d_2^5\tilde{d}_3^4m^2$ (c.f. 5.43). Then we use assumption (5.41) and get

$$\begin{aligned}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^x)\right| &\geq \sum_{w\in W_x}\left(d_3\tilde{d}_3 - 2\varepsilon_3^{1/32}\right)\left|\mathcal{G}_2[\mathcal{G}_3(x,w)]\cap\bigcup_{j=1}^{r'}\mathcal{K}_2(\mathcal{B}_j^w)\right| \\ &\overset{(5.41)}{\geq} |W_x|\left(d_3\tilde{d}_3 - 2\varepsilon_3^{1/32}\right)\times\varepsilon_3^{1/64}\times(1/2)d_2^5\tilde{d}_3^4m^2 \\ &\geq (1/4)\varepsilon_3^{1/32}d_3\tilde{d}_3^5d_2^5m^3.\end{aligned}\tag{5.50}$$

Second, we also need an upper bound on the size of $\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^x)\cap\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{Q}_i^{x'})$ for a good pair $\{x,x'\}$. We can repeat the argument from Claim 5.11 with $|W_x| = \varepsilon_3^{1/64}m$ (rather than $\varepsilon_3^{1/16}m$, c.f. (5.37)) and conclude that for every pair of good vertices $\{x,x'\}\subset X$ we have

$$\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^x)\cap\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{Q}_i^{x'})\right| \leq 4\varepsilon_3^{1/64}d_2^7m^3 \tag{5.51}$$

Now we define an auxiliary graph $\Gamma = (V,E)$, where a pair of vertices $\{x',x''\}$ is an edge whenever it is not a good pair. It follows from Observations 3.8 and 3.12 that the size of E is bounded by $34\varepsilon_2^{1/4}m^2$. Using the Picking Lemma with $\sigma_1 = 34\varepsilon_2^{1/4}$, $c = \varepsilon_2^{1/16}$, and $t = \varepsilon_3^{1/8}d_2^{-2}$, we choose t vertices $x_1,\ldots,x_t \in X$ so that all pairs $\{x_u,x_v\}$ are good.

Note that condition (2.8) is satisfied, because

$$\frac{2\times 34\varepsilon_2^{1/4}\times t^2}{\left(\varepsilon_2^{1/16}\right)^2} < 68\varepsilon_2^{1/8}d_2^{-2} < \frac{1}{2}.$$

Then, we can estimate $\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right|$ using Observation 4.5 in the following way:

$$\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \geq \sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| - \sum_{1\leq u<v\leq t}\left|\bigcup_{i=1}^{r'}\mathcal{K}_3(\mathcal{Q}_i^{x_u})\cap\bigcup_{j=1}^{r'}\mathcal{K}_3(\mathcal{Q}_j^{x_v})\right|$$

This can be further simplified with the use of (5.50), (5.51), $t = \varepsilon_3^{1/8} d_2^{-2}$, and $\varepsilon_3 \ll d_3$, $\varepsilon_3 \ll \tilde{d}_3$.

$$\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \geq t \times (1/4)\varepsilon_3^{1/32} d_3 \tilde{d}_3^5 d_2^5 m^3 - \binom{t}{2} \times 4\varepsilon_3^{1/64} d_2^7 m^3$$
$$\geq 2\varepsilon_3 d_2^3 m^3 \geq \varepsilon_3 |\mathcal{K}_3(\mathcal{G}_3)|.$$

The last part of this inequality follows from Corollary 3.6. Applying the (ε_3, d_3, r)-regularity of $\mathcal{H}$ with respect to $\mathcal{G}_2$ (note that $t \times r' \leq r$) yields

$$\left|\mathcal{H} \cap \bigcup_{u=1}^{t}\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \leq (d_3 + \varepsilon_3)\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right|$$

Using the first part (equality) of (5.49) and inequality (5.45), we conclude that

$$(d_3 + \varepsilon_3)\left|\bigcup_{u=1}^{t}\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \leq (d_3 + \varepsilon_3)\sum_{u=1}^{t}\left|\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right|$$
$$\overset{(5.49)}{=} (d_3 + \varepsilon_3)\sum_{u=1}^{t}\sum_{w \in W_{x_u}}\left|\mathcal{G}_3[\mathcal{G}_3(x_u) \cap \mathcal{G}_2(w)] \cap \mathcal{H}(x_u) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w)\right|$$
$$\overset{(5.45)}{\leq} (d_3 + \varepsilon_3)\left(d_3\tilde{d}_3 + 2\varepsilon_3^{1/32}\right)\sum_{u=1}^{t}\sum_{w \in W_{x_u}}\left|\mathcal{G}_2[\mathcal{G}_3(x_u, w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w)\right|.$$

Combining the previous two inequalities, we obtain:

$$\left|\mathcal{H} \cap \bigcup_{u=1}^{t}\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| \leq \left(\tilde{d}_3 d_3^2 + 3\varepsilon_3^{1/32}\right)\sum_{u=1}^{t}\sum_{w \in W_{x_u}}\left|\mathcal{G}_2[\mathcal{G}_3(x_u, w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w)\right|.$$

On the other hand, we use assumption (5.42) and obtain the following lower bound on $\left|\mathcal{H} \cap \bigcup_{u=1}^{t}\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right|$ (recall that $\mathcal{G}_3'(x_u, w) = \mathcal{G}_3[\mathcal{G}_3(x_u, w)] \cap \mathcal{H}(x_u, w)$):

$$\left|\mathcal{H} \cap \bigcup_{u=1}^{t}\bigcup_{j=1}^{r'} \mathcal{K}_3(\mathcal{Q}_j^{x_u})\right| = \sum_{u=1}^{t}\sum_{w \in W_{x_u}}\left|\mathcal{G}_3'(x_u, w) \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w)\right|$$
$$\overset{(5.42)}{>} \left(\tilde{d}_3 d_3^2 + \varepsilon_3^{1/64}\right)\sum_{u=1}^{t}\sum_{w \in W_{x_u}}\left|\mathcal{G}_2[\mathcal{G}_3(x_u, w)] \cap \bigcup_{j=1}^{r'} \mathcal{K}_2(\mathcal{B}_j^w)\right|.$$

Comparing the last two inequalities yields $\tilde{d}_3 d_3^2 + 3\varepsilon_3^{1/32} > \tilde{d}_3 d_3^2 + \varepsilon_3^{1/64}$, which is a contradiction. Thus, our assumption that $|X^+| \geq \varepsilon_2^{1/32} m$ is incorrect. Since this was a consequence of $|X| \geq 2\varepsilon_2^{1/32} m$, we must have $|X| < 2\varepsilon_2^{1/32} m$. □

Now we are ready to prove Claim 5.14.

Proof of Claim 5.14. Set $r' = r/\left(\varepsilon_3^{1/2} d_2^{-3}\right)\left(\varepsilon_3^{1/16} d_2^{-3}\right)$ and let $\{x, x'\}$ be a pair of vertices such that

(i) $\{x, x'\}$ is a good pair,

(ii) the restriction of $\mathcal{H}$ to $\mathcal{G}_2[\mathcal{G}_2(x, x')]$ is $(2\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x, x')]$,

(iii) $\left(1 - \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m \leq \deg_{\mathcal{G}_{3,j}}(x, x') \leq \left(1 + \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m$ for every $j \in [s] \setminus \{1\}$, $\mathcal{G}_2[\mathcal{G}_3(x, x')]$ is $(\varepsilon_2^{1/2}, d_2)$-regular, and

(iv) $\mathcal{G}_3[\mathcal{G}_3(x, x')] \cap \mathcal{H}(x, x')$ is $(\varepsilon_3^{1/64}, \tilde{d}_3 d_3^2, r')$-regular with respect to $\mathcal{G}_2[\mathcal{G}_3(x, x')]$.

It follows from Fact 5.16 that all but at most $5\varepsilon_3^{1/2} m^2$ pairs $\{x, x'\}$ satisfy (i) and (ii) (note that $r' \leq r/\left(\varepsilon_3^{1/2} d_2^{-3}\right)\left(2\varepsilon_3^{1/8} d_2^{-3}\right)$).

Furthermore, Fact 5.18 impies that all but at most $9\varepsilon_3^{1/64} m^2$ good pairs $\{x, x'\}$ satisfy (iii) and (iv) (note that in this case $r' \leq r/\left(\varepsilon_3^{1/2} d_2^{-2}\right)\left(\varepsilon_3^{1/16} d_2^{-2}\right)$).

We define two $(3, 2)$-cylinders $\mathcal{G}_1'$, $\mathcal{G}_2'$ by $\mathcal{G}_1' = \mathcal{G}_2[\mathcal{G}_3(x, x')]$ and $\mathcal{G}_2' = \mathcal{G}_3[\mathcal{G}_3(x, x')] \cap \mathcal{H}(x, x')$.

Then, $\mathcal{G}_1'$ and $\mathcal{G}_2'$ satisfy the assumptions of the 2-graphs Lemma. More precisely, setting $\varepsilon_1' = \varepsilon_2^{1/2}$, $\varepsilon_2' = \varepsilon_3^{1/64}$, $d_1' = d_2$, $d_2' = \tilde{d}_3 d_3^2$, and $\left(1 - \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m \leq m' \leq \left(1 + \varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m$, we have that $\mathcal{G}_1'$ is (ε_1', d_1')-regular and $\mathcal{G}_2'$ is $(\varepsilon_2', d_2', r')$-regular with respect to $\mathcal{G}_1'$ (c.f. (iii) and (iv)).

We apply the 2-graphs Lemma and obtain the following

$$|\mathcal{K}_3(\mathcal{G}_3[\mathcal{G}_3(x, x')] \cap \mathcal{H}(x, x'))| = |\mathcal{K}_3(\mathcal{G}_2')| \geq \left(1 - 8(\varepsilon_2')^{1/64}\right)^2 (d_1' d_2')^3 (m')^3$$

Then we use the definitions of ε_1', ε_2' and m' and assumption $\varepsilon_2 \ll \varepsilon_3 \ll d_3 \le 1$ to obtain

$$\begin{aligned}|\mathcal{K}_3(\mathcal{G}_3[\mathcal{G}_3(x,x')] \cap \mathcal{H}(x,x'))| &\ge \left(1-\varepsilon_2^{1/4096}\right)^2 d_2^3 \tilde{d}_3^3 d_3^6 \left(\left(1-\varepsilon_3^{1/8}\right)^2 d_2^2 \tilde{d}_3^2 m\right)^3 \\ &\ge \frac{3}{4} d_2^9 \tilde{d}_3^9 d_3^6 m^3.\end{aligned}$$

In a similar fashion we get

$$|\mathcal{K}_3(\mathcal{G}_3[\mathcal{G}_3(x,x')] \cap \mathcal{H}(x,x'))| \le \frac{5}{4} d_2^9 \tilde{d}_3^9 d_3^6 m^3. \tag{5.52}$$

Since the pair $\{x, x'\}$ is good (c.f. (i)), the $(3,2)$-cylinder $\mathcal{G}_2[\mathcal{G}_2(x,x')]$ is $(\varepsilon_2^{1/2}, d_2)$-regular. It follows from Corollary 3.6 that

$$|\mathcal{K}_3(\mathcal{G}_2[\mathcal{G}_2(x,x')])| \le (5/4) d_2^3 \left(d_2 + \varepsilon_2^{1/2}\right)^6 m^3 \le 2 d_2^9 m^3.$$

Since $2\varepsilon_3^{1/16} \times 2d_2^9 m^3 \le (3/4) d_2^9 \tilde{d}_3^9 d_3^6$ because of $\varepsilon_3 \ll d_3$ and $\varepsilon_3 \ll \tilde{d}_3$, we obtain

$$|\mathcal{K}_3(\mathcal{G}_3[\mathcal{G}_3(x,x')] \cap \mathcal{H}(x,x'))| \ge 2\varepsilon_3^{1/16} |\mathcal{K}_3(\mathcal{G}_2[\mathcal{G}_2(x,x')])|.$$

We apply the $(2\varepsilon_3^{1/16}, d_3, r')$-regularity of $\mathcal{H}[\mathcal{G}_2(x,x')]$ with respect to $\mathcal{G}_2[\mathcal{G}_2(x,x')]$ and obtain

$$\begin{aligned}|\mathcal{H} \cap \mathcal{K}_3(\mathcal{G}_3[\mathcal{G}_3(x,x')] \cap \mathcal{H}(x,x'))| &\le \left(d_3 + 2\varepsilon_3^{1/16}\right) |\mathcal{K}_3(\mathcal{G}_3[\mathcal{G}_3(x,x')] \cap \mathcal{H}(x,x'))| \\ &\overset{(5.52)}{\le} \left(d_3 + 2\varepsilon_3^{1/16}\right) \frac{5}{4} d_2^9 \tilde{d}_3^9 d_3^6 m^3 \le \frac{3}{2} d_2^9 \tilde{d}_3^9 d_3^7 m^3.\end{aligned}$$

This is, however, what we wanted to prove. □

5.7 Additional claims

The motivation for this subsection is twofold. First, we need to define the notion of a *nice neighbor* and prove that almost all neighbors of a nice vertex are nice. Second, in the proof of Proposition 2.6, we will need an upper bound on the number of copies of K_3 in the joint neigborhood of two nice neighbors. We will provide this estimate at the end of this section. We start with the definition of a nice neighbor.

Suppose that $\mathcal{G}$ and $\mathcal{H}$ are as in Setup B, i.e. $\mathcal{G} = (V, E(\mathcal{G}))$ is an (ε_2, d_2)-regular $(s, 2)$-cylinder and $\mathcal{H} = (V, E(\mathcal{H}))$ an $(s, 3)$-cylinder which is (ε_3, d_3, r)-regular with respect to $\mathcal{G}$.

Definition 5.19 (nice neighbor). *Let $x \in V_{\text{nice}}$ be a nice vertex and set $r' = r/\left(\varepsilon_3^{1/2} d_2^{-3}\right)\left(2\varepsilon_3^{1/8} d_2^{-3}\right)$. A good neighbor $y \in N_2(x)_{\text{good}}$ is called* nice *if*

(i) $\left(1 - 2\varepsilon_3^{1/8}\right)^2 d_2^2 d_3 m \leq \deg_{\mathcal{H}(x),j}(y) \leq \left(1 + 2\varepsilon_3^{1/8}\right)^2 d_2^2 d_3 m$ *for every $j = 3, \ldots, s$;*

(ii) $\mathcal{G}[\mathcal{H}(x)(y)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular, and the $(s-2, 2)$-cylinder $\mathcal{H}(x, y)[\mathcal{H}(x)(y)]$ is $(2\varepsilon_3^{1/32}, d_3^2, r')$-regular with respect to $\mathcal{G}[\mathcal{H}(x)(y)]$; and

(iii) the restriction $\mathcal{H}[\mathcal{G}(x, y)]$ is $(4\varepsilon_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{G}[\mathcal{G}(x, y)]$.

We denote by $N_2(x)_{\text{nice}}$ the set of all nice neighbors in $N_2(x) = \mathcal{G}(x)$.

Remark. Recall that $\mathcal{H}(x, y) = \mathcal{H}(x) \cap \mathcal{H}(y)$ stands for the joint link of x and y, whereas $\mathcal{H}(x)(y)$ is the neighborhood of y in the graph $\mathcal{H}(x)$.

The following observation shows that almost all good neighbors are nice.

Observation 5.20. *All but $4\binom{s-2}{2}\varepsilon_3^{1/32} d_2 m$ good neighbors in $N_2(x)_{\text{good}}$ are nice neighbors, i.e.*

$$|N_2(x)_{\text{nice}}| \geq |N_2(x)_{\text{good}}| - 4\binom{s-2}{2}\varepsilon_3^{1/32} d_2 m.$$

Proof. Let $x \in V_{\text{nice}}$ be a nice vertex (c.f. Definition 5.3), that is a good vertex satisfying:

(i) $\mathcal{H}(x)$ is $(2\varepsilon_3^{1/2}, d_3, r)$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$, and

(ii) $\mathcal{H}$ is $(2\varepsilon_3^{1/4}, d_3, r/\left(\varepsilon_3^{1/2} d_2^{-3}\right))$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$.

Since every nice vertex is also good, we have (c.f. Definition 3.7):

(iii) $\mathcal{G}[\mathcal{G}(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular, and

(iv) $(d_2 - \varepsilon_2)m \leq \deg_j(x) \leq (d_2 + \varepsilon_2)m$ holds for every $j \in [s] \setminus \{1\}$.

Set $\mathcal{G}'_2 = \mathcal{G}[\mathcal{G}(x)]$, $\mathcal{G}'_3 = \mathcal{H}(x)$, $\mathcal{H}' = \mathcal{H}[\mathcal{G}(x)]$, $\varepsilon'_2 = \varepsilon_2^{1/2}$, $d'_2 = d_2$, $\varepsilon'_3 = 2\varepsilon_3^{1/4}$, $d'_3 = \tilde{d}'_3 = d_3$, $s' = s-1$, $r' = r/\left(\varepsilon_3^{1/2} d_2^{-3}\right)\left(2\varepsilon_3^{1/8} d_2^{-2}\right)$, and $(d_2-\varepsilon_2)m \leq m' \leq (d_2+\varepsilon_2)m$.

Then, $\mathcal{G}'_2$ is an $(s', 2)$-cylinder, $\mathcal{G}'_3$ is an $(s', 2)$-cylinder, and $\mathcal{H}'$ is an $(s', 3)$-cylinder. Moreover, $\mathcal{G}'_2$, $\mathcal{G}'_3$, and $\mathcal{H}'$ satisfy the Setup C. More precisely,

- $\mathcal{G}'_2$ is (ε'_2, d'_2)-regular (c.f. (iii)),
- $\mathcal{G}'_3$ is $(\varepsilon'_3, \tilde{d}'_3, r/\left(\varepsilon_3^{1/2} d_2^{-3}\right))$-regular with respect to $\mathcal{G}'_2$ (c.f. (i)), and
- $\mathcal{H}'$ is $(\varepsilon'_3, d'_3, r/\left(\varepsilon_3^{1/2} d_2^{-3}\right))$-regular with respect to $\mathcal{G}'_2$ (c.f. (ii)).

We apply Claim 5.7 and obtain that for all but at most

$$2\binom{s'-1}{2}\left(4(\varepsilon'_2)^{1/2} + 4(\varepsilon'_3)^{1/2} + (\varepsilon'_2)^{1/8} + (\varepsilon'_3)^{1/8}\right)m'$$

vertices $y \in N_2(x)_{\text{good}}$ the following is true:

(a) $\left(1-(\varepsilon'_3)^{1/2}\right)^2 d'_2 \tilde{d}'_3 m' \leq \deg_{\mathcal{G}'_3,j}(y) \leq \left(1+(\varepsilon'_3)^{1/2}\right)^2 d'_2 \tilde{d}'_3 m'$ for $j = 3, \ldots, s$;

(b) $\mathcal{G}'_2[\mathcal{G}'_3(y)]$ is $((\varepsilon'_2)^{1/2}, d'_2)$-regular and the $(s-2,2)$-cylinder $\mathcal{G}'_3[\mathcal{G}'_3(y)] \cap \mathcal{H}'(y)$ is $((\varepsilon'_3)^{1/8}, d'_3\tilde{d}'_3, r')$-regular with respect to $\mathcal{G}'_2[\mathcal{G}'_3(y)]$, where .

We use the assumption $\varepsilon_2 \ll \varepsilon_3$ and the definitions of ε'_2, ε'_3, m', $\mathcal{G}'_2$, $\mathcal{G}'_3$, and $\mathcal{H}'$ to conclude that

- $\mathcal{G}'_2[\mathcal{G}'_3(y)] = \mathcal{G}[\mathcal{H}(x)(y)]$,
- $\mathcal{G}'_3[\mathcal{G}'_3(y)] \cap \mathcal{H}'(y) = \mathcal{H}(x,y)[\mathcal{H}(x)(y)]$,

- $2\binom{s'-1}{2}\left(4(\varepsilon_2')^{1/2}+4(\varepsilon_3')^{1/2}+(\varepsilon_2')^{1/8}+(\varepsilon_3')^{1/8}\right)m' \le 3\binom{s-2}{2}\varepsilon_3^{1/32}d_2m$,
- $\left(1-2\varepsilon_3^{1/8}\right)^2 d_2^2d_3m \le \deg_{\mathcal{H}(x),j}(y) \le \left(1+2\varepsilon_3^{1/8}\right)^2 d_2^2d_3m$ for $j \in [s]\setminus\{1,2\}$ (c.f. (a)),
- $\mathcal{G}[\mathcal{H}(x)(y)]$ is $(\varepsilon_2^{1/4}, d_2)$-regular (c.f. (b)),
- $\mathcal{H}(x,y)[\mathcal{H}(x)(y)]$ is $(2\varepsilon_3^{1/32}, d_3^2, r')$-regular with respect to $\mathcal{G}[\mathcal{H}(x)(y)]$ (c.f. (b)).

Therefore, all but $4\binom{s-2}{2}\varepsilon_3^{1/32}d_2m$ vertices $y \in N_2(x)_{\text{good}}$ satisfy (i) and (ii) in Definition 5.19.

It remains to show that almost all neighbors y satisfy (iii). We apply Corollary 5.5 and obtain that for all but at most $2\binom{s-2}{3}(\varepsilon_2')^{1/16}m'$ vertices $y \in N_2(x)_{\text{good}}$, the $(s-2,3)$-cylinder $\mathcal{H}'[\mathcal{G}_2'(y)]$ is $(2(\varepsilon_3')^{1/4}, d_3', r')$-regular with respect to $\mathcal{G}_2'[\mathcal{G}_2'(y)]$. Since

- $\mathcal{G}_2'[\mathcal{G}_2'(y)] = \mathcal{G}[\mathcal{G}(x,y)]$,
- $2(\varepsilon_3')^{1/4} \le 4\varepsilon_3^{1/16}$,
- $\mathcal{H}'[\mathcal{G}_2'(y)] = \mathcal{H}[\mathcal{G}(x,y)]$, and
- $2\binom{s'-1}{3}(\varepsilon_2')^{1/16}m' \le 3\binom{s-2}{3}\varepsilon_2^{1/32}d_2m$,

we conclude that all but at most $3\binom{s-2}{3}\varepsilon_2^{1/32}d_2m$ vertices $y \in N_2(x)_{\text{good}}$ satisfy (iii) in Definition 5.19.

Hence, all but at most $3\binom{s-2}{2}\varepsilon_3^{1/32}d_2m+3\binom{s-2}{3}\varepsilon_2^{1/32}d_2m \le 4\binom{s-2}{2}\varepsilon_3^{1/32}d_2m$ vertices $y \in N_2(x)_{\text{good}}$ are nice. □

We will prove now that for every nice vertex x we have control over the number of copies of K_3 in the joint neighborhood $\mathcal{H}(x)(y,y')$ for almost all pairs of vertices $\{y,y'\} \subset N_2(x)$. Recall that $\mathcal{H}(x)(y,y')$ stands for the joint neighborhood of $\{y,y'\}$ in $\mathcal{H}(x)$.

Claim 5.21. *Suppose that $s = 5$. Then for every nice vertex $x \in V_{\text{nice}}$, the following is true: all but at most $20\varepsilon_3^{1/256} d_2^2 m^2$ pairs $\{y, y'\} \subset N_2(x)$ satisfy*

$$|\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x, y, y')[\mathcal{H}(x)(y, y')])| \le 2d_2^{12} d_3^{16} m^3.$$

Proof. Let $x \in V_{\text{nice}}$ be a nice vertex (c.f. Definition 5.3), that is a good vertex for which we have

(i) the link $\mathcal{H}(x)$ is $(2\varepsilon_3^{1/2}, d_3, r)$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$.

(ii) $\mathcal{H}[\mathcal{G}_2(x)]$ is $(2\varepsilon_3^{1/4}, d_3, r/\left(\varepsilon_3^{1/2} d_2^{-3}\right))$-regular with respect to $\mathcal{G}_2[\mathcal{G}_2(x)]$,

Also remind that the goodness of x implies

(iii) $(d_2 - \varepsilon_2)m \le \deg_j(x) \le (d_2 + \varepsilon_2)m$ for $j = 2, \ldots, 5$,

(iv) the $(4, 2)$-cylinder $\mathcal{G}_2[\mathcal{G}_2(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular.

We apply Claim 5.14 with $\mathcal{G}_2' = \mathcal{G}_2[\mathcal{G}_2(x)]$, $\mathcal{G}_3' = \mathcal{H}(x)$, and $\mathcal{H}' = \mathcal{H}[\mathcal{G}_2(x)]$. Observe that if we set $\varepsilon_2' = \varepsilon_2^{1/2}$, $\varepsilon_3' = 2\varepsilon_3^{1/4}$, $(d_2 - \varepsilon_2)m \le m' \le (d_2 + \varepsilon_2)m$, $\tilde{d}_3' = d_3' = d_3$, and $r' = r/\left(\delta_3^{1/2} d_2^{-3}\right)$, then (i)-(iv) verify conditions of the Setup C. More precisely,

- (iv) verifies that $\mathcal{G}_2'$ is (ε_2', d_2')-regular,
- (i) verifies that $\mathcal{G}_3'$ is $(\varepsilon_3', \tilde{d}_3', r')$-regular with respect to $\mathcal{G}_2'$, and
- (ii) verifies that $\mathcal{H}'$ is $(\varepsilon_3', d_3', r')$-regular with respect to $\mathcal{G}_2'$.

By Claim 5.14, all but $10(\varepsilon_3')^{1/64}(m')^2$ pairs $\{y, y'\} \subset N_2(x)$ satisfy

$$|\mathcal{H}' \cap \mathcal{K}_3(\mathcal{H}'(y, y')[\mathcal{G}_3'(y, y')])| \le \frac{3}{2}(d_2')^9 (\tilde{d}_3')^9 (d_3')^7 (m')^3.$$

This concludes the proof because

- $\mathcal{H}' \cap \mathcal{K}_3(\mathcal{H}'(y, y')[\mathcal{G}_3'(y, y')]) = \mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x, y, y')[\mathcal{H}(x)(y, y')])$,
- $10(\varepsilon_3')^{1/64}(m')^2 \leq 20\varepsilon_3^{1/256} d_2^2 m^2$, and
- $(3/2)(d_2')^9(\tilde{d}_3')^9(d_3')^7(m')^3 \leq 2d_2^{12} d_3^{16} m^3$.

Here we used the definitions of ε_2', ε_3', m', and $\varepsilon_2 \ll d_2$. □

Chapter 6

Properties of 4-cylinders

This section has two parts. In the first part, we derive the two basics properties of the links of an $(s,4)$-cylinder $\mathcal{F}$: the regularity of the link $\mathcal{F}(x)$ and the regularity of $\mathcal{F}(x,y)$, where y is a neighbor of x. The second part provides a proof of Proposition 2.6.

6.1 Regularity of the links of 4-cylinders

In this section, we investigate link properties of a regular $(s,4)$-cylinder $\mathcal{F}$. First, we describe our situation.

Setup D. *Let $0 < \varepsilon_2 \ll d_2 \leq 1$, $0 < \varepsilon_3 \ll d_3 \leq 1$, and $0 < \varepsilon_4 \ll d_4 \leq 1$ be real numbers so that $\varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4$. Let $V = V_1 \cup \ldots \cup V_s$ be a partition, where $|V_1| = \ldots = |V_s| = m$, $\mathcal{G} = (V, E(\mathcal{G}))$ be an $(s,2)$-cylinder that is (ε_2, d_2)-regular, $\mathcal{H} = (V, E(\mathcal{H}))$ be an $(s,3)$-cylinder which is (ε_3, d_3, r)-regular with respect to $\mathcal{G}$, and let $\mathcal{F}$ be a $(s,4)$-cylinder which is (ε_4, d_4, r)-regular with respect to $\mathcal{H}$.*

The following claim shows that the link $\mathcal{F}(x)$ "inherits" regularity from $\mathcal{F}$. It can be viewed as an analogy to Claim 5.1.

Claim 6.1. *The $(s-1,3)$-cylinder $\mathcal{F}(x)$ is $(2\varepsilon_4^{1/2}, d_4, r)$-regular with respect to $\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x))$ for all but at most $4\binom{s-1}{3}\varepsilon_4^{1/2}m$ vertices $x \in V_{\text{nice}}$.*

Proof. We may assume $s = 4$ because we can apply this result simultaneously to subcylinders of $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{F}$ induced on $V_1 \cup V_i \cup V_j \cup V_k$, $1 < i < j < k \le s$.

Let x be a nice vertex. By Definition 5.3 we know that x satisfies:

(i) the link $\mathcal{H}(x)$ is $(2\varepsilon_3^{1/2}, d_3, r)$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$,

(ii) $\mathcal{H}[\mathcal{G}(x)]$ is $(2\varepsilon_3^{1/4}, d_3, r/\left(\varepsilon_3^{1/2}d_2^{-3}\right))$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$.

Moreover, since x must be also a good vertex, we have (c.f. Definition 3.7):

(iii) $(d_2 - \varepsilon_2)m \le \deg_j(x) \le (d_2 + \varepsilon_2)m$ for $j = 2, 3, 4$,

(iv) the $(3,2)$-cylinder $\mathcal{G}[\mathcal{G}(x)]$ is $(\varepsilon_2^{1/2}, d_2)$-regular.

We apply the 2-graphs Lemma (Lemma 4.11) with

- $(3,2)$-cylinder $\mathcal{G}_1$ played by $\mathcal{G}[\mathcal{G}(x)]$ which is $(\varepsilon_2^{1/2}, d_2)$-regular (c.f. (iv));
- $(3,2)$-cylinder $\mathcal{G}_2$ played by $\mathcal{H}(x)$ which is $(2\varepsilon_3^{1/2}, d_3, r)$-regular with respect to $\mathcal{G}[\mathcal{G}(x)]$ (c.f. (i));
- V_i replaced by $\mathcal{G}(x) \cap V_i$ for $i = 2, 3, 4$;

and obtain

$$\left(1 - 8\left(2\varepsilon_3^{1/2}\right)^{1/64}\right)^2 d_2^3 d_3^3 \times \left((d_2 - \varepsilon_2)\, m\right)^3 \le |\mathcal{K}_3(\mathcal{H}(x))| \le \left(1 + 8\left(2\varepsilon_3^{1/2}\right)^{1/64}\right)^2 d_2^3 d_3^3 \times \left((d_2 + \varepsilon_2)\, m\right)^3.$$

This can be further simplified using $\varepsilon_2 \ll d_2$ and $\varepsilon_3 \ll d_3$ to

$$\frac{3}{4} d_2^6 d_3^3 m^3 \le |\mathcal{K}_3(\mathcal{H}(x))| \le \frac{5}{4} d_2^6 d_3^3 m^3. \tag{6.1}$$

It follows from Fact 3.6 applied with $s = 3$ and $\varepsilon_3 \ll d_3$ that $2\varepsilon_3^{1/4}|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(x)])| \leq 2\varepsilon_3^{1/4} \times (5/4)d_2^3(d_2+\varepsilon_2)^3m^3 \leq (3/4)d_2^6d_3^3m^3$. Hence we have

$$|\mathcal{K}_3(\mathcal{H}(x))| \geq 2\varepsilon_3^{1/4}|\mathcal{K}_3(\mathcal{G}[\mathcal{G}(x)])|.$$

Applying the $(2\varepsilon_3^{1/4}, d_3, r/\left(\varepsilon_3^{1/2}d_2^{-3}\right))$-regularity of $\mathcal{H}$ with respect to $\mathcal{G}[\mathcal{G}(x)]$ yields

$$\left(d_3 - 2\varepsilon_3^{1/4}\right)|\mathcal{K}_3(\mathcal{H}(x))| \leq |\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x))| \leq \left(d_3 + 2\varepsilon_3^{1/4}\right)|\mathcal{K}_3(\mathcal{H}(x))|.$$

We combine this inequality with (6.1) to conclude that for every nice vertex x we have

$$\frac{1}{2}d_2^6d_3^4m^3 \leq |\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x))| \leq 2d_2^6d_3^4m^3. \tag{6.2}$$

Suppose that one can find $t = 2\varepsilon_4^{1/2}m$ nice vertices $x_1, \ldots, x_t \in V_{\text{nice}}$ such that for every $u \in [t]$ the link $\mathcal{F}(x_u)$ is $(2\varepsilon_4^{1/4}, d_4, r)$-irregular with respect to $\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x_u))$. Moreover, assume that for every x_u the second part of inequality (4.1) does not hold, that is there exist $(3,2)$-cylinders $\mathcal{B}_{ju} \subset \mathcal{H}(x_u)$, $j \in [r]$, such that

$$\left|\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x_u)) \cap \bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{B}_{ju})\right| \geq 2\varepsilon_4^{1/2}|\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x_u))|, \tag{6.3}$$

but

$$\left|\mathcal{F}(x_u) \cap \bigcup_{j=1}^{r} \mathcal{K}_2(\mathcal{B}_{ju})\right| > \left(d_4 + 2\varepsilon_4^{1/2}\right)\left|\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x_u)) \cap \bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{B}_{ju})\right|. \tag{6.4}$$

For every $j \in [r]$ define a $(4,3)$-cylinder $Q_j = Q_{j\hat{1}} \cup Q_{j\hat{2}} \cup Q_{j\hat{3}} \cup Q_{j\hat{4}}$ by

$$\begin{aligned}
Q_{j\hat{1}} &= \mathcal{H}_{\hat{1}}, \\
Q_{j\hat{2}} &= \bigcup_{u=1}^{t} \{x_u yz \colon yz \in \mathcal{B}_{ju} \cap K(V_3, V_4)\}, \\
Q_{j\hat{3}} &= \bigcup_{u=1}^{t} \{x_u yz \colon yz \in \mathcal{B}_{ju} \cap K(V_2, V_4)\}, \\
Q_{j\hat{4}} &= \bigcup_{u=1}^{t} \{x_u yz \colon yz \in \mathcal{B}_{ju} \cap K(V_2, V_3)\}.
\end{aligned}$$

We will show using (δ_4, d_4, r)-regularity of $\mathcal{F}$ that $|\mathcal{F} \cap \bigcup_{u=1}^r \mathcal{K}_4(Q_u)| \le (d_3 + \varepsilon_4) \sum_{j=1}^t |\mathcal{H} \cap \bigcup_{u=1}^r \mathcal{K}_3(\mathcal{B}_{ju})|$ and then we use assumption (6.4) to show a contradiction.

Observe that since $\mathcal{B}_{ju} \subset \mathcal{H}(x_u)$ for every $j \in [r]$, we have

$$\sum_{u=1}^{t} \left| \mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x_u)) \cap \bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{B}_{ju}) \right| = \sum_{u=1}^{t} \left| \mathcal{H} \cap \bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{B}_{ju}) \right|. \tag{6.5}$$

We estimate the size of $\bigcup_{j=1}^r \mathcal{K}_4(Q_j)$ as follows:

$$\begin{aligned}\left| \bigcup_{j=1}^{r} \mathcal{K}_4(Q_j) \right| &= \sum_{u=1}^{t} \left| \mathcal{H} \cap \bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{B}_{ju}) \right| \overset{(6.5)}{=} \sum_{u=1}^{t} \left| \mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x_u)) \cap \bigcup_{j=1}^{r} \mathcal{K}_3(\mathcal{B}_{ju}) \right| \\ &\overset{(6.3)}{\ge} \sum_{u=1}^{t} 2\varepsilon_4^{1/2} |\mathcal{H} \cap \mathcal{K}_3(\mathcal{H}(x_u))| \overset{(6.2)}{\ge} t \times 2\varepsilon_4^{1/2} \times \frac{1}{2} d_2^6 d_3^4 m^3 \\ &\ge 2\varepsilon_4 d_2^6 d_3^4 m^4 \ge \varepsilon_4 |\mathcal{K}_4(\mathcal{H})|.\end{aligned}$$

The last inequality follows from the Theorem 1.13: $\mathcal{G}$ is a (δ_2, d_2)-regular $(4,2)$-cylinder, $\mathcal{H}$ is a $(4,2)$-cylinder that is (δ_3, d_3, r)-regular with respect to $\mathcal{G}$, and we can choose ε_2 and ε_3 so that the assumptions of Theorem 1.13 are satisfied. Thus, $\varepsilon_4 |\mathcal{K}_4(\mathcal{H})| \le \varepsilon_4 (1 \pm \nu) d_2^6 d_3^4 m^4 \le 2\varepsilon_4 d_2^6 d_3^4 m^4$.

Subsequently, the (ε_4, d_4, r)-regularity of $\mathcal{F}$ with respect to $\mathcal{H}$ implies that

$$\begin{aligned}\left| \mathcal{F} \cap \bigcup_{j=1}^{r} \mathcal{K}_4(Q_j) \right| &\le (d_3 + \varepsilon_4) \left| \bigcup_{j=1}^{r} \mathcal{K}_4(Q_j) \right| \\ &= (d_3 + \varepsilon_4) \sum_{j=1}^{t} \left| \mathcal{H} \cap \bigcup_{u=1}^{r} \mathcal{K}_3(\mathcal{B}_{ju}) \right|.\end{aligned} \tag{6.6}$$

On the other hand, every x_u is contained in $\left| \mathcal{F}(x_u) \cap \bigcup_{j=1}^r \mathcal{K}_3(\mathcal{B}_{ju}) \right|$ triples (this follows from the definition of Q_j). We use (6.4) to conclude that

$$\left|\mathcal{F}\cap\bigcup_{u=1}^{r}\mathcal{K}_3(Q_u)\right| = \sum_{u=1}^{t}\left|\mathcal{F}(x_u)\cap\bigcup_{j=1}^{r}\mathcal{K}_3(Q_{ju})\right|$$
$$\overset{(6.4)}{>} \left(d_4+2\varepsilon_4^{1/2}\right)\sum_{u=1}^{t}\left|\mathcal{H}\cap\mathcal{K}_3(\mathcal{H}(x_u))\cap\bigcup_{j=1}^{r}\mathcal{K}_3(\mathcal{B}_{ju})\right|. \quad (6.7)$$

Comparing (6.7) with (6.6) we get a contradiction. Thus, there are at most $2\varepsilon_4^{1/2}m$ vertices satisfying (6.3) and (6.4).

The case when the second part of inequality (4.1) is not true, i.e. (6.4) is replaced by $\left|\mathcal{F}(x_j)\cap\bigcup_{j=1}^{r}\mathcal{K}_3(Q_{ju})\right| < \left(d_4-2\varepsilon_4^{1/2}\right)\left|\bigcup_{j=1}^{r}\mathcal{K}_3(Q_{ju})\right|$, is handled similarly. $\square$

The next claims shows that majority of nice vertices in V_{nice} have the property that the link $\mathcal{F}(x,y)$ is regular for almost all nice neighbors y of x.

Claim 6.2. *For all but at most $2\binom{s-2}{2}\varepsilon_4^{1/4}m$ nice vertices $x\in V_1$ the following statement is true.*

There are at most $2\binom{s-2}{2}\varepsilon_4^{1/4}d_2m$ nice neighbors $y\in N_2(x)$ for which the link $\mathcal{F}(x,y)$ is not $(\varepsilon_4^{1/4},d_4,r)$-regular with respect to $\mathcal{H}(x,y)[\mathcal{H}(x)(y)]$.

It is sufficient to consider the case $s=4$ only because we can treat the case $s>4$ by applying the result for $s=4$ to $\binom{s-2}{2}$ sets of cylinders induced on $V_1\cup V_2\cup V_i\cup V_j$, $2<i<j\le s$.

Proof. Set $r' = r/\left(\varepsilon_3^{1/2}d_2^{-3}\right)\left(2\varepsilon_3^{1/8}d_2^{-3}\right)$ and let x be arbitrary nice vertex and y be its nice neighbor (c.f. Definition 5.19). Then y satisfies the following conditions:

(i) $\left(1-2\varepsilon_3^{1/8}\right)^3 d_2^2d_3m \le \deg_{\mathcal{H}(x),j}(y) \le \left(1+2\varepsilon_3^{1/8}\right)^3 d_2^2d_3m$ for $j=3,4$, and $\mathcal{G}[\mathcal{H}(x)(y)]$ is $(\varepsilon_2^{1/2},d_2)$-regular,

(ii) the link $\mathcal{H}(x,y)[\mathcal{H}(x)(y)]$ is $(2\varepsilon_3^{1/32},d_3^2,r')$-regular with respect to $\mathcal{G}[\mathcal{H}(x)(y)]$.

Observe that the $(2\varepsilon_3^{1/32}, d_3^2, r')$-regularity of the link $\mathcal{H}(x,y)[\mathcal{H}(x)(y)]$ with respect to $\mathcal{G}[\mathcal{H}(x)(y)]$ is a sufficient condition to apply Observation 4.2. It a view of (i), this observation implies

$$\frac{1}{2}d_2^5 d_3^4 m^2 \le |\mathcal{H}(x,y)[\mathcal{H}(x)(y)]| \le 2d_2^5 d_3^4 m^2. \tag{6.8}$$

Suppose there exist $t_1 = \varepsilon_4^{1/4} m$ nice vertices $x_1, \ldots, x_{t_1} \in V_{\text{nice}}$ so that for every x_u, $u \in [t_1]$, there are at least $t_2 = \varepsilon_4^{1/4} d_2 m$ nice neighbors $y_{1u}, \ldots, y_{t_2 u} \in N_2(x_u)_{\text{nice}}$ for which the link $\mathcal{F}(x_u, y_{vu})$, $v \in [t_2]$, is not $(\varepsilon_4^{1/4}, d_4, r)$-regular with respect to $\mathcal{H}(x_u, y_{vu})[\mathcal{H}(x_u)(y_{vu})]$.

We further assume that the second part of inequality (4.1) is not satisfied, that is for every x_u and y_{vu} there exist $(2,1)$-cylinders $\mathcal{B}_{jvu} = Y_{jvu} \cup W_{jvu}$, $j \in [r]$, where $Y_{jvu} \subset \mathcal{H}(x_u)(y_{vu}) \cap V_3$ and $W_{jvu} \subset \mathcal{H}(x_u)(y_{vu}) \cap V_4$, such that

$$\left| \mathcal{H}(x_u, y_{vu})[\mathcal{H}(x_u)(y_{vu})] \cap \bigcup_{j=1}^{r} \mathcal{K}_2(\mathcal{B}_{jvu}) \right| \ge \varepsilon_4^{1/4} |\mathcal{H}(x_u, y_{vu})[\mathcal{H}(x_u)(y_{vu})]|, \tag{6.9}$$

but

$$\left| \mathcal{F}(x_u, y_{vu}) \cap \bigcup_{j=1}^{r} \mathcal{K}_2(\mathcal{B}_{jvu}) \right| > \left(d_4 + \varepsilon_4^{1/4}\right) \left| \mathcal{H}(x_u, y_{vu})[\mathcal{H}(x_u)(y_{vu})] \cap \bigcup_{j=1}^{r} \mathcal{K}_2(\mathcal{B}_{jvu}) \right|. \tag{6.10}$$

For every $j \in [r]$ define a $(4,3)$-cylinder $Q_j = Q_{j\hat{1}} \cup Q_{j\hat{2}} \cup Q_{j\hat{3}} \cup Q_{j\hat{4}}$ by

$$\begin{aligned} Q_{j\hat{1}} &= \mathcal{H}_{\hat{1}} \\ Q_{j\hat{2}} &= \mathcal{H}_{\hat{2}} \\ Q_{j\hat{3}} &= \bigcup_{u=1}^{t_1} \bigcup_{v=1}^{t_2} \{x_u y_{vu} z \colon z \in W_{jvu}\}, \\ Q_{j\hat{4}} &= \bigcup_{u=1}^{t_1} \bigcup_{v=1}^{t_2} \{x_u y_{vu} z \colon z \in Y_{jvu}\}. \end{aligned}$$

It follows from the above construction that

$$\left|\bigcup_{j=1}^{r}\mathcal{K}_4(Q_j)\right| = \sum_{u=1}^{t_1}\sum_{v=1}^{t_2}\left|\mathcal{H}(x_u,y_{vu})[\mathcal{H}(x_u)(y_{vu})]\cap\bigcup_{j=1}^{r}\mathcal{K}_2(\mathcal{B}_{jvu})\right|. \tag{6.11}$$

We use this equation together with the assumption (6.9) and estimate (6.8) to conclude that $\left|\bigcup_{j=1}^{r}\mathcal{K}_4(Q_j)\right| \geq \varepsilon_4|\mathcal{K}_4(\mathcal{H})|$. Indeed,

$$\begin{aligned}\sum_{u=1}^{t_1}\sum_{v=1}^{t_2}&\left|\mathcal{H}(x_u,y_{vu})[\mathcal{H}(x_u)(y_{vu})]\cap\bigcup_{j=1}^{r}\mathcal{K}_2(\mathcal{B}_{jvu})\right|\\ &\overset{(6.9)}{\geq} \sum_{u=1}^{t_1}\sum_{v=1}^{t_2}\varepsilon_4^{1/4}|\mathcal{H}(x_u,y_{vu})[\mathcal{H}(x_u)(y_{vu})]|\\ &\overset{(6.8)}{\geq} t_1\times t_2\times\varepsilon_4^{1/4}\times\frac{1}{2}d_2^5d_3^4m^2 = \frac{1}{2}\varepsilon_4^{3/4}d_2^6d_3^4m^4 > 2\varepsilon_4 d_2^6d_3^4m^4.\end{aligned}$$

Since $\mathcal{G}$ is a (ε_2, d_2)-regular $(4,2)$-cylinder, $\mathcal{H}$ is a $(4,3)$-cylinder that is (ε_3, d_3, r)-regular with respect to $\mathcal{G}$, and we can choose ε_2 and ε_3 so that the assumptions of Theorem 1.13 are satisfied, we conclude that $|\mathcal{K}_4(\mathcal{H})| \leq 2d_2^6d_3^4m^4$.

Hence, $\varepsilon_4\,|\mathcal{K}_4(\mathcal{H})| \leq \varepsilon_4 2d_2^6d_3^4m^4 \leq \left|\bigcup_{j=1}^{r}\mathcal{K}_4(Q_j)\right|$, and we can apply the (ε_4, d_4, r)-regularity of $\mathcal{F}$ with respect to $\mathcal{H}$. Then,

$$\begin{aligned}\left|\mathcal{F}\cap\bigcup_{j=1}^{r}\mathcal{K}_4(Q_j)\right| &\leq (d_4+\varepsilon_4)\left|\bigcup_{j=1}^{r}\mathcal{K}_4(Q_j)\right|\\ &\overset{(6.11)}{=} (d_4+\varepsilon_4)\sum_{u=1}^{t_1}\sum_{v=1}^{t_2}\left|\mathcal{H}(x_u,y_{vu})[\mathcal{H}(x_u)(y_{vu})]\cap\bigcup_{j=1}^{r}\mathcal{K}_2(\mathcal{B}_{jvu})\right|.\end{aligned} \tag{6.12}$$

On the other hand, assumption (6.10) yields

$$\begin{aligned}\left|\mathcal{F}\cap\bigcup_{j=1}^{r}\mathcal{K}_4(Q_j)\right| &= \sum_{u=1}^{t_1}\sum_{v=1}^{t_2}\left|\mathcal{F}(x_u,y_{vu})\cap\bigcup_{j=1}^{r}\mathcal{K}_2(\mathcal{B}_{jvu})\right|\\ &\overset{(6.10)}{>} \left(d_4+\varepsilon_4^{1/4}\right)\sum_{u=1}^{t_1}\sum_{v=1}^{t_2}\left|\mathcal{H}(x_u,y_{vu})[\mathcal{H}(x_u)(y_{vu})]\cap\bigcup_{j=1}^{r}\mathcal{K}_2(\mathcal{B}_{jvu})\right|.\end{aligned} \tag{6.13}$$

Comparing inequalities (6.12) and (6.13) we get a contradiction. Hence $t_1 < \varepsilon_4^{1/4} m$. If we assume, that the first part of inequality (4.1) is not satisfied, we obtain contradiction in exactly the same way. Thus, there for all but at most $2\varepsilon_4^{1/4} m$ nice vertices $x \in V_{\text{nice}}$ there are at most $2\varepsilon_4^{1/4} d_2 m$ nice neighbors $y \in N_2(x)_{\text{nice}}$ such that the link $\mathcal{F}(x,y)$ is not $(\varepsilon_4^{1/4}, d_4, r)$-regular with respect to $\mathcal{H}(x,y)[\mathcal{H}(x)(y)]$.

□

Definition 6.3 (fine vertex). *A nice vertex $x \in V_1$ is called* fine *if it satisfies the following conditions:*

(i) $\mathcal{F}(x)$ is $(2\varepsilon_4^{1/2}, d_4, r)$-regular with respect to $\mathcal{H} \cap \mathcal{K}_3([\mathcal{H}(x)])$, and

(ii) $\mathcal{F}(x,y)$ is not $(\varepsilon_4^{1/4}, d_4, r)$-regular with respect to $\mathcal{H}(x,y)[\mathcal{H}(x)(y)])$ for at most $2\binom{s-2}{2}\varepsilon_4^{1/4} d_2 m$ nice neighbors $y \in N_2(x)_{\text{nice}}$.

We denote by V_{fine} the set of all nice vertices in V_1.

Definition 6.4 (fine neighbor). *Let $x \in V_1$ be a fine vertex. A nice neighbor $y \in N_2(x)$ is called* fine *if the link $\mathcal{F}(x,y)$ is $(\varepsilon_4^{1/4}, d_4, r)$-regular with respect to $\mathcal{H}(x,y)[\mathcal{H}(x)(y)]$. We denote by $N_2(x)_{\text{fine}}$ the set of all fine neighbors in $N_2(x)$.*

Observe that $V_{\text{fine}} \subset V_{\text{nice}} \subset V_{\text{good}} \subset V_1$ and $N_2(x)_{\text{fine}} \subset N_2(x)_{\text{nice}} \subset N_2(x)_{\text{good}} \subset N_2(x)$ for every (fine) vertex $x \in V_1$. Moreover, the following two observations are an easy consequence of Claims 6.2 and 6.1.

Observation 6.5. *All but at most $4\binom{s-1}{3}\varepsilon_4^{1/2} m + 2\binom{s-2}{2}\varepsilon_4^{1/4} m$ nice vertices are fine, that is*

$$|V_{\text{fine}}| \geq |V_{\text{nice}}| - 4\binom{s-1}{3}\varepsilon_4^{1/2} m - 2\binom{s-2}{2}\varepsilon_4^{1/4} m.$$

Observation 6.6. *Let x be a fine vertex. Then all but at most $2\binom{s-2}{2}\varepsilon_4^{1/4} d_2 m$ nice neighbors in $N_2(x)_{\text{nice}}$ are fine, that is*

$$|N_2(x)_{\text{fine}}| \geq |N_2(x)_{\text{nice}}| - 2\binom{s-2}{2}\varepsilon_4^{1/4} d_2 m.$$

6.2 Proof of Proposition 2.6

We structure the proof into five parts.

Part A In this part we show that

(a) $|\mathcal{K}_3(\mathcal{H}^{(4)}(x,y))| = (1 \pm \nu/6)d_2^9 d_3^9 d_4^3 n^3$ for every $x \in V_{\text{fine}}$ and $y \in N_2(x)_{\text{fine}}$, and

(b) $(1/2)d_2^9 d_3^9 n^3 \le |\mathcal{K}_3(\mathcal{H}^{(3)}(x,y))| \le 2d_2^9 d_3^9 n^3$ for every $x \in V_{\text{fine}}$ and $y \in N_2(x)_{\text{nice}}$.

Part B Here we prove that

(a) $|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x,y))| = (1 \pm \nu/5)d_2^9 d_3^{10} d_4^3 n^3$ for every $x \in V_{\text{fine}}$ and $y \in N_2(x)_{\text{fine}}$, and

(b) $|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x,y))| \le 3d_2^9 d_3^{10} n^3$ for every $x \in V_{\text{fine}}$ and $y \in N_2(x)_{\text{nice}}$.

Part C We show that if W is a subset of $N_2(x)_{\text{fine}}$ such that $|W| \ge 2\delta_4^{1/4} d_2 m$. Then, there exist $t = \delta_4^{1/4}/(d_2^3 d_3^6)$ fine neighbors $y_1, \dots, y_t \in N_2(x)_{\text{fine}}$ such that

$$\left| \mathcal{H}^{(4)}(x) \cap \bigcup_{u=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x,y_u)) \right| = \left(d_4 \pm 2\delta_4^{1/2} \right) \left| \mathcal{H}^{(3)} \cap \bigcup_{u=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x,y_u)) \right|.$$

Part D The lower bound $|\mathcal{K}_4(\mathcal{H}^{(4)}(x))| \ge (1 - \nu/2)d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ is proved here for every fine vertex $x \in V_{\text{fine}}$.

Part E We show the upper bound $|\mathcal{K}_4(\mathcal{H}^{(4)}(x))| \le (1 + \nu/2)d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ for every fine vertex $x \in V_{\text{fine}}$.

Since the lower and upper bounds are valid for fine vertices, it remains to show how vertices are not fine.

It follows from Observation 6.5 that all but at most $8\delta_4^{1/4}m$ nice vertices are fine. Moreover, Observation 5.4 gives that all but at most $20\delta_3^{1/2}m$ good vertices are nice. Finally, from Observation 3.8 we have that all but at most $48\delta_2^{1/4}m$ vertices are good. Altogether we obtain that all but at most $10\delta_4^{1/4}m$ vertices $x \in V_1$ are fine.

Now we show Parts A-E.

Part A(a). Fix a fine vertex $x \in V_{\text{fine}}$ and its arbitrary fine neighbor $y \in N_2(x)_{\text{fine}}$ and set $r' = r/\left(\delta_3^{1/2}d_2^{-3}\right)\left(2\delta_3^{1/8}d_2^{-3}\right)$. Then y satisfies the following condition (c.f. Definition 6.4)

(i) the link $\mathcal{H}^{(4)}(x,y)$ is $(\delta_4^{1/4}, d_4, r)$-regular with respect to $\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)]$.

Since y is also a nice neigbor of x, it satisfies (c.f. Definition 5.19)

(ii) $\left(1-2\delta_3^{1/8}\right)^2 d_2^2 d_3 n \leq \deg_{\mathcal{H}^{(3)}(x),j}(y) \leq \left(1+2\delta_3^{1/8}\right)^2 d_2^2 d_3 n$ for every $j=3,4,5$;

(iii) $\mathcal{H}^{(2)}[\mathcal{H}^{(3)}(x)(y)]$ is $(\delta_2^{1/4}, d_2)$-regular, $\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)]$ is $(2\delta_3^{1/32}, d_3^2, r')$-regular with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(3)}(x)(y)]$, and

(iv) $\mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x,y)]$ is $(4\delta_3^{1/16}, d_3, r')$-regular with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)]$.

Moreover, y is also a good neigbor of x, thus we have

(v) the link $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)]$ is $(\delta_2^{1/4}, d_2)$-regular, and

(vi) $\left(d_2-\delta_2^{1/2}\right)^2 m \leq \deg_j(x,y) \leq \left(d_2+\delta_2^{1/2}\right)^2 m$ holds for $j=3,4,5$.

Then we set

- $\mathcal{G}_1' = \mathcal{H}^{(2)}[\mathcal{H}^{(3)}(x)(y)]$, $\varepsilon_1' = \delta_2^{1/4}$, $d_1' = d_2$,
- $\mathcal{G}_2' = \mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)]$, $\varepsilon_2' = 2\delta_3^{1/32}$, $d_2' = d_3^2$,
- $\mathcal{G}_3' = \mathcal{H}^{(4)}(x,y)$, $\varepsilon_3' = \delta_4^{1/4}$, $d_3' = d_4$, and

- $V_i' = V_{i+2} \cap \mathcal{H}^{(3)}(x)(y)$, $i = 1, 2, 3$.

Observe that $\mathcal{G}_1'$, $\mathcal{G}_2'$, and $\mathcal{G}_3'$ are $(3,2)$-cylinders which satisfy Setup A. Indeed,

- $\mathcal{G}_1'$ is (ε_1', d_1')-regular (c.f. (v)),
- $\mathcal{G}_2'$ is $(\varepsilon_2', d_2', r')$-regular with respect to $\mathcal{G}_1'$ (c.f. (iii)),
- $\mathcal{G}_2'$ is $(\varepsilon_3', d_3', r')$-regular with respect to $\mathcal{G}_2'$ (c.f. (i)),
- $\left(1 - 2\delta_3^{1/8}\right)^2 d_2^2 d_3 n \leq m' \leq \left(1 + 2\delta_3^{1/8}\right)^2 d_2^2 d_3 n$ (c.f. (ii)),
- $\varepsilon_1' \ll d_1'$, $\varepsilon_2' \ll d_2'$, $\varepsilon_3' \ll d_3'$, and $\varepsilon_1' \ll \varepsilon_2'\varepsilon_3'$.

Thus, we can apply the 3-graphs Lemma and obtain that

$$\left(1 - 12(\varepsilon_3')^{1/64}\right)^3 (d_1'd_2'd_3')^3 (m')^3 \leq |\mathcal{K}_3(\mathcal{G}_3')| \leq \left(1 + 12(\varepsilon_3')^{1/64}\right)^3 (d_1'd_2'd_3')^3 (m')^3.$$

We use the definition of d_1', d_2', d_3', ε_3', m', and $\mathcal{G}_3'$ to conclude that

$$\begin{aligned}\left(1 - 12\delta_4^{1/256}\right)^3 d_2^3 d_3^6 d_4^3 \left(1 - 2\delta_3^{1/8}\right)^6 d_2^6 d_3^3 n^3 &\leq |\mathcal{K}_3(\mathcal{H}^{(4)}(x,y))| \\ &\leq \left(1 + 12\delta_4^{1/256}\right)^3 d_2^3 d_3^6 d_4^3 \left(1 + 2\delta_3^{1/8}\right)^6 d_2^6 d_3^3 n^3 \quad (6.14)\end{aligned}$$

Since by (2.2) $\delta_3 \ll \delta_4 \ll \nu$, we can conclude that

$$(1 - \nu/6) d_2^9 d_3^9 d_4^3 n^3 \leq |\mathcal{K}_3(\mathcal{H}^{(4)}(x,y))| \leq (1 + \nu/6) d_2^9 d_3^9 d_4^3 n^3.$$

Part A (b). Fix a fine vertex $x \in V_{\text{fine}}$ and its arbitrary nice neighbor $y \in N_2(x)_{\text{nice}}$. Then y satisfies conditions (ii)-(vi) (c.f. Part A(a)). We set

- $\mathcal{G}_1' = \mathcal{H}^{(2)}[\mathcal{H}^{(3)}(x)(y)]$, $\varepsilon_1' = \delta_2^{1/4}$, $d_1' = d_2$,
- $\mathcal{G}_2' = \mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)]$, $\varepsilon_2' = 2\delta_3^{1/32}$, $d_2' = d_3^2$.

In the same was as in Part A(a) we can observe that $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are $(3,2)$-cylinders which satisfy Setup A. We apply the 2-graphs Lemma and obtain that

$$\left(1-8(\varepsilon_2')^{1/64}\right)^3 (d_1'd_2')^3(m')^3. \le |\mathcal{K}_3(\mathcal{G}_3')| \le \left(1+8(\varepsilon_2')^{1/64}\right)^3 (d_1'd_2')^3(m')^3.$$

We use the definition of d_1', d_2', ε_2', m', and $\mathcal{G}_2'$ to conclude that

$$\begin{aligned}\left(1-10\delta_3^{1/2048}\right)^3 d_2^3d_3^6\left(1-2\delta_3^{1/8}\right)^6 d_2^6d_3^3n^3 &\le |\mathcal{K}_3(\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)])| \\ &\le \left(1+10\delta_3^{1/2048}\right)^3 d_2^3d_3^6\left(1+2\delta_3^{1/8}\right)^6 d_2^6d_3^3n^3\end{aligned}$$

Since $\delta_3 \ll \delta_4 \ll \nu$, we can conclude that

$$\frac{1}{2}d_2^9d_3^9n^3 \le |\mathcal{K}_3(\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)])| \le 2d_2^9d_3^9n^3.$$

Part B(a). For the proof of this part, we use the estimate from Part A(a) and the $(4\delta_3^{1/16}, d_3, r')$-regularity of $\mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x,y)]$ with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)]$.

Set $r' = r/\left(\delta_3^{1/2}d_2^{-3}\right)\left(\delta_3^{1/8}d_2^{-3}\right)$ and fix a fine vertex $x \in V_{\text{fine}}$ and its arbitrary fine neighbor $y \in N_2(x)_{\text{fine}}$. Recall that y satisfies conditions (i)-(vi) (c.f. Part A(a)). It follows from part A(a) that

$$(1-\nu/6)d_2^9d_3^9d_4^3n^3 \le |\mathcal{K}_3(\mathcal{H}^{(4)}(x,y))| \le (1+\nu/6)d_2^9d_3^9d_4^3n^3. \tag{6.15}$$

Since y satisfies (v) and (vi), we know that $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)]$ is $(\delta_2^{1/2}, d_2)$-regular and $\left(d_2-\delta_2^{1/2}\right)^2 n \le \deg_j(x,y) \le \left(d_2+\delta_2^{1/2}\right)^2 n$ holds for $j=3,4,5$. We apply Corollary 3.6 and obtain $|\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)])| \le (5/4)d_2^3\left(d_2+\delta_2^{1/2}\right)^6 n^3 \le (3/2)d_2^9n^3$ (we used $\delta_2 \ll d_2$).

Furthermore, since $\delta_2 \ll \delta_3 \ll d_3 \ll \nu$, we have

$$4\delta_3^{1/16}|\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)])| \le 4\delta_3^{1/16}\times\frac{3}{2}d_2^9n^3 \le (1-\nu/6)d_2^9d_3^9d_4^3n^3 \le |\mathcal{K}_3(\mathcal{H}^{(4)}(x,y))|.$$

Then we apply the $(4\delta_3^{1/16}, d_3, r')$-regularity of $\mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x,y)]$ with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)]$, and obtain

$$\left(d_3 - 4\delta_3^{1/16}\right)|\mathcal{K}_3(\mathcal{H}^{(4)}(x,y))| \leq \left|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x,y))\right| \leq \left(d_3 + 4\delta_3^{1/16}\right)|\mathcal{K}_3(\mathcal{H}^{(4)}(x,y))|. \quad (6.16)$$

We combine (6.15), (6.16), and assumption $\delta_3 \ll d_3 \ll \nu$ to get

$$(1-\nu/5)d_2^9 d_3^{10} d_4^3 n^3 \leq |\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x,y))| \leq (1+\nu/5)d_2^9 d_3^{10} d_4^3 n^3. \quad (6.17)$$

Part B(b). Now y is a nice neighbor of a fine vertex x, that is, a vertex satisfying (ii)-(vi). Then, from Part A(b), we have

$$\frac{1}{2}d_2^9 d_3^9 n^3 \leq |\mathcal{K}_3(\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)])| \leq 2d_2^9 d_3^9 n^3. \quad (6.18)$$

Similarly to Part B(a), since $\delta_2 \ll \delta_3 \ll d_3 \ll \nu$, we have

$$4\delta_3^{1/16}|\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)])| \leq 4\delta_3^{1/16} \times \frac{3}{2}d_2^9 n^3 \leq \frac{1}{2}d_2^9 d_3^9 n^3 \leq |\mathcal{K}_3(\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)])|.$$

Then we apply the $(4\delta_3^{1/16}, d_3, r')$-regularity of $\mathcal{H}^{(3)}[\mathcal{H}^{(2)}(x,y)]$ with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y)]$, and obtain

$$|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)])| \leq (d_3 + 4\delta_3^{1/16})|\mathcal{K}_3(\mathcal{H}^{(3)}(x,y)[\mathcal{H}^{(3)}(x)(y)])| \overset{(6.18)}{\leq} 3d_2^9 d_3^{10} n^3.$$

Part C. Let $x \in V_{\text{fine}}$ be a fine vertex and W is a subset of $N_2(x)_{\text{fine}}$ such that $|W| \geq 2\delta_4^{1/4} d_2 m$.

We define two graphs $\mathcal{P}_1$ and $\mathcal{P}_2$, both with vertex set $N_2(x)$ and edge sets defined by:

$$E(\mathcal{P}_1) = \left\{yy' \colon |\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(y,y')])| > 2d_2^{12} n^3\right\},$$
$$E(\mathcal{P}_2) = \left\{yy' \colon |\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x,y,y')[\mathcal{H}^{(3)}(x)(y,y')])| > 2d_3^{16} d_2^{12} n^4\right\}.$$

Now we estimate the sizes of $E(\mathcal{P}_1)$ and $E(\mathcal{P}_1)$. Since x is also a good vertex (recall $V_{\text{fine}} \subset V_{\text{nice}} \subset V_{\text{good}} \subset V_1$), the $(4,2)$-cylinder $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$ is $(\delta_2^{1/2}, d_2)$-regular. We apply Observations 3.8 and 3.12 on $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$ and obtain that all but $66\delta_2^{1/8}|N_2(x)|^2$ pairs $\{y, y'\}$ are good with respect to $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x)]$.

Thus (c.f. Definition 3.11), $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x, y, y')]$ is $(\delta_2^{1/4}, d_2)$-regular and

$$\left(d_2 - \delta_2^{1/4}\right)^3 n \leq \deg_j(x, y, y') \leq \left(d_2 + \delta_2^{1/4}\right)^3 n$$

holds for $j = 3, 4, 5$. We apply Corollary 3.6 and obtain $|\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x, y)])| \leq (5/4)d_2^3 \left(d_2 + \delta_2^{1/4}\right)^9 n^3 \leq 2d_2^{12}n^3$ (we used $\delta_2 \ll d_2$). Consequently,

$$|E(\mathcal{P}_1)| \leq 66\delta_2^{1/8}|N_2(x)|^2.$$

It follows from Claim 5.21 that for all but $20\delta_3^{1/256}d_2^2n^2$ pairs $\{y, y'\} \subset N_2(x)$ we have

$$|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x, y, y')[\mathcal{H}^{(3)}(x)(y, y')])| \leq 2d_2^{12}d_3^{16}n^3.$$

Therefore,

$$|E(\mathcal{P}_2)| \leq 20\delta_3^{1/256}d_2^2n^2 \leq 21\delta_3^{1/256}|N_2(x)|^2.$$

We apply the Picking Lemma on W with parameters $\sigma_1 = 66\delta_2^{1/8}$, $\sigma_2 = 21\delta_3^{1/256}$, $t = \delta_4^{1/4}/(d_2^3d_3^6)$, $c = \delta_4^{1/4}$, and obtain t nice neighbors $y_1, \ldots, y_t \in W$ such that all pairs $\{y_i, y_j\}$ satisfy

$$|\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x, y_i, y_j)])| \leq 2d_2^{12}n^3, \tag{6.19}$$

and all but $\left(2 \times 2 \times 21\delta_3^{1/256}/\delta_4^{1/2}\right)t^2 \leq \delta_3^{1/512}t^2$ pairs $\{y_i, y_j\}$ satisfy

$$|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x, y, y')[\mathcal{H}^{(3)}(x)(y_i, y_j)])| \leq 2d_3^{16}d_2^{12}n^3. \tag{6.20}$$

This is possible as long as $|W| \geq 2\delta_4^{1/4}d_2m \geq c \times |N_2(x)|$ and condition (2.8) is satisfied, in other words, if

$$\frac{2 \times 66\delta_2^{1/8} \times t^2}{\left(\delta_4^{1/4}\right)^2} < \frac{1}{2} \tag{6.21}$$

holds. This is true because

$$\frac{2\times 66\delta_2^{1/8}\times t^2}{\left(\delta_4^{1/4}\right)^2}=\frac{132\delta_2^{1/8}}{d_2^6 d_3^{12}}\le 132\times\frac{\delta_2^{1/16}}{d_2^8}\times\frac{\delta_3^{1/16}}{d_3^{12}}\le 132\times\delta_2^{1/32}\times\delta_3^{1/32}<\frac{1}{2}.$$

Here we used assumption (2.2): $\delta_2 \ll d_2 \le 1$, $\delta_3 \ll d_3 \le 1$, and $\delta_2 \ll \delta_3$.

Now we estimate the size of $\mathcal{H}^{(3)} \cap \bigcup_{j=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x,y_j))$. We first apply Observation 4.5:

$$\left|\bigcup_{j=1}^{t}\mathcal{H}^{(3)}\cap\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_j))\right|\ge\sum_{j=1}^{t}\left|\mathcal{H}^{(3)}\cap\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_j))\right|$$
$$-\sum_{1\le i<j\le t}\left|\mathcal{H}^{(3)}\cap\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_i))\cap\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_j))\right|.$$

The next step is to estimate both terms on the right-hand side. The first term is easier to handle. We use (6.17) to conclude that:

$$\sum_{j=1}^{t}\left|\mathcal{H}^{(3)}\cap\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_j))\right|\ge t\times(1-\nu/5)d_2^9 d_3^{10} d_4^3 n^3. \tag{6.22}$$

To get an estimate for the second term, we must observe several facts:

- $\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_i))\cap\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_j))=\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_i,y_j))$ for every $1\le i<j\le t$.
- Every copy of K_3 in $\mathcal{H}^{(4)}(x,y_i,y_j)$ is a copy of K_3 in $\mathcal{H}^{(3)}\cap\mathcal{K}_3(\mathcal{H}^{(3)}(x,y_i,y_j))$ as well. This follows from the fact that $\mathcal{H}^{(4)}(x)\subset\mathcal{K}_4(\mathcal{H}^{(3)}(x))$ and $\mathcal{H}^{(4)}(x)\subset\mathcal{H}^{(3)}$.
- Every copy of K_3 in $\mathcal{H}^{(4)}(x,y_i,y_j)$ is also a copy of K_3 in $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y_i,y_j)]$). This follows from the fact that $\mathcal{H}^{(4)}\subset\mathcal{K}_4(\mathcal{H}^{(3)})$ and $\mathcal{H}^{(3)}\subset\mathcal{K}_3(\mathcal{H}^{(2)})$.

Since we know that all but at most $\delta_3^{1/512}t^2$ pairs $\{y_i,y_j\}$ satisfy (6.20), for these pairs we use the estimate

$$\left|\mathcal{H}^{(3)}\cap\mathcal{K}_3(\mathcal{H}^{(4)}(x,y_i,y_j))\right|\le\left|\mathcal{H}^{(3)}\cap\mathcal{K}_3(\mathcal{H}^{(3)}(x,y_i,y_j))\right|\overset{(6.20)}{\le}2d_3^{16}d_2^{12}n^3. \tag{6.23}$$

Remaining $\delta_3^{1/512}t^2$ pairs $\{y_i, y_j\}$ satisfy (6.19). For these pairs we use the estimate

$$\left|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_i, y_j))\right| \leq |\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x, y_i, y_j)])| \overset{(6.19)}{\leq} 2d_2^{12}n^3. \tag{6.24}$$

Now we combine (6.23) and (6.24) to obtain

$$\sum_{1 \leq i < j \leq t} \left|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_i) \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| \leq \binom{t}{2} \times 2d_3^{16}d_2^{12}n^3 + \delta_3^{1/512}t^2 \times 2d_2^{12}n^3.$$

We use the assumption $\delta_3 \ll d_3$ and $t = \delta_4^{1/4}/(d_2^3 d_3^6)$ to conclude that $\delta_3^{1/512}t^2 \times 2d_2^{12}n^3 \leq t^2 \times d_3^{16}d_2^{12}n^3$. Then,

$$\sum_{1 \leq i < j \leq t} \left|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_i) \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| \leq 2t^2 d_3^{16}d_2^{12}n^3. \tag{6.25}$$

Using (6.22), (6.25), and the definition of t (recall $t = \delta_4^{1/4}/(d_2^3 d_3^6)$), we obtain that

$$\begin{aligned}\left|\bigcup_{j=1}^{t} \mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| &\geq t \times (1 - \nu/5)d_2^9 d_3^{10} d_4^3 n^3 - 2t^2 d_2^{12} d_3^{16} n^3 \\ &\geq \frac{1}{2}\delta_4^{1/4} d_2^6 d_3^4 d_4^3 n^3 - 2\delta_4^{1/2} d_2^6 d_3^4 n^3 \\ &\overset{(2.2)}{\geq} 2\delta_4^{1/2} \times 2d_2^6 d_3^4 n^4.\end{aligned} \tag{6.26}$$

Since $x \in V_{\text{fine}}$ is a fine vertex (c.f. Definition 6.3), the link $\mathcal{H}^{(4)}(x)$ is $(2\delta_4^{1/2}, d_4, r)$-regular with respect to $\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x))$. Moreover, we know from (6.2)

$$\frac{1}{2}d_2^6 d_3^4 n^3 \leq |\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x))| \leq 2d_2^6 d_3^4 n^3. \tag{6.27}$$

We combine (6.26) and (6.27) and obtain

$$\left|\bigcup_{j=1}^{t} \mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| \geq 2\delta_4^{1/2}|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x))|.$$

The $(2\delta_4^{1/2}, d_4, r)$-regularity of $\mathcal{H}^{(4)}(x)$ with respect to $\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x))$ yields (note that we can choose $r \geq t$ upfront (c.f. (2.2))

$$\left(d_4 - 2\delta_4^{1/2}\right) \left|\mathcal{H}^{(3)} \cap \bigcup_{j=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| \leq \left|\mathcal{H}^{(4)}(x) \cap \bigcup_{j=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right|$$
$$\leq \left(d_4 + 2\delta_4^{1/2}\right) \left|\mathcal{H}^{(3)} \cap \bigcup_{j=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right|,$$

which is what we wanted to show.

Part D. In Part C, we proved that whenever W is a subset of $N_2(x)_{\text{fine}}$ such that $|W| \geq 2\delta_4^{1/4} d_2 m$, we can choose $t = \delta_4^{1/4}/(d_2^3 d_3^6)$ nice neighbors $y_1, \ldots, y_t \in W$ such that

$$\left|\mathcal{H}^{(4)}(x) \cap \bigcup_{j=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| = \left(d_4 \pm 2\delta_4^{1/2}\right) \left|\mathcal{H}^{(3)} \cap \bigcup_{j=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right|, \quad (6.28)$$

Moreover, $y_1, \ldots, y_t$ also satisfy (6.22) and (6.25). Using these two equations and $t = \delta_4^{1/4}/(d_2^3 d_3^6)$, we obtain

$$\delta_4^{1/8} \sum_{j=1}^{t} \left|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| - \sum_{1 \leq i < j \leq t} \left|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_i) \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right|$$
$$\geq \delta_4^{1/8} \times t \times (1 - \nu/5) d_2^9 d_3^{10} d_4^3 n^3 - 2t^2 d_3^{16} d_2^{12} n^3 \geq 0.$$

We apply the second part of Observation 4.5:

$$\left|\bigcup_{j=1}^{t} \mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| \geq \left(1 - \delta_4^{1/8}\right) \sum_{j=1}^{t} \mathcal{H}^{(3)} \cap \left|\mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right|. \quad (6.29)$$

We combine (6.28), (6.29), and (6.22) and get

$$\left|\mathcal{H}^{(4)}(x) \cap \bigcup_{j=1}^{t} \mathcal{K}_3(\mathcal{H}^{(4)}(x, y_j))\right| \geq \left(d_4 - 2\delta_4^{1/2}\right) \left(1 - \delta_4^{1/8}\right) t \times (1 - \nu/5) d_2^9 d_3^{10} d_4^3 n^3. \quad (6.30)$$

We set $W = N_2(x)_{\text{fine}}$ and find vertices $y_1, \dots, y_t$ as described above. Then we remove $y_1, \dots, y_t$ from W and iterate the whole process again. We can repeat this process as long as (c.f. Part C)

$$|W| > 2\delta_4^{1/4} n. \tag{6.31}$$

This way we produce a sequence of at least $\left(|N_2(x)_{\text{fine}}| - 2\delta_4^{1/4} n\right)/t$ t-tuples $Y^{(1)} = \{y_1, \dots, y_t\} = \left\{y_1^{(1)}, \dots, y_t^{(1)}\right\}$, $Y^{(2)} = \left\{y_1^{(2)}, \dots, y_t^{(2)}\right\}$, etc.

Analoguously to (6.30), each iteration produces at least

$$\left(d_4 - 2\delta_4^{1/2}\right)\left(1 - \delta_4^{1/8}\right) t \times (1 - \nu/5) d_2^9 d_3^{10} d_4^3 n^3 \geq (1 - \nu/4) d_2^9 d_3^{10} d_4^4 t n^3$$

copies of $K_4^{(3)}$ in $\mathcal{H}^{(4)}(x)$. Each such a copy uses exactly one vertex from $Y^{(i)} = \left\{y_1^{(i)}, \dots, y_t^{(i)}\right\}$.

Notice that since x is a fine vertex,

- $|N_2(x)_{\text{fine}}| \geq |N_2(x)_{\text{nice}}| - 6\delta_4^{1/4} d_2 n$ (c.f. Observation 6.6),
- $|N_2(x)_{\text{nice}}| \geq |N_2(x)_{\text{good}}| - 12\delta_3^{1/32} d_2 n$ (c.f. Observation 5.20),
- $|N_2(x)_{\text{good}}| \geq |N_2(x)| - 36\delta_2^{1/2} d_2 n$ (c.f. Observation 3.10), and
- $|N_2(x)| \geq (d_2 - \delta_2) n$ (c.f. Definition 3.7).

Consequently, $\left(|N_2(x)_{\text{fine}}| - 2\delta_4^{1/4} n\right)/t \geq \left(1 - 7\delta_4^{1/4}\right) d_2 n/t$. Therefore, the sequence of t-touples $Y^{(i)}$ produces at least $\left(1 - 7\delta_4^{1/4}\right) d_2 n/t \times (1 - \nu/4) d_2^9 d_3^{10} d_4^4 t n^3$ copies of $K_4^{(3)}$ in $\mathcal{H}^{(4)}(x)$. Hence

$$\begin{aligned} |\mathcal{K}_4(\mathcal{H}^{(4)}(x))| &\geq \left(1 - 7\delta_4^{1/4}\right) d_2 n/t \times (1 - \nu/4) d_2^9 d_3^{10} d_4^4 t n^3 \\ &\overset{(2.2)}{\geq} (1 - \nu/2) d_2^{\binom{5}{3}} d_3^{\binom{5}{2}} d_4^4 n^4. \end{aligned}$$

Part E. The upper bound causes some extra difficulties - we must count not only

(i) the contribution of t-tuples of neighbors taken from W,

but also

(ii) contribution of neighbors left in W, and

(iii) neighbors which are not fine.

We will handle each of these categories of vertices separately:

(i) An upper bound on number of copies of $K_4^{(3)}$ produced by taking t-tuples from W can be obtained in a way similar to the lower bound in Part D: every t-tuple is in at most $\left(d_4 + 2\delta_4^{1/4}\right) \times t(1+\nu/5)d_4^3 d_3^{10} d_2^9 n^3$ copies of $K_4^{(3)}$ in $\mathcal{H}^{(4)}(x)$. There are at most $|N_2(x)_{\text{fine}}|/t \le |N_2(x)|/t \le (d_2 + \delta_2)n/t$ such t-tuples, together producing at most

$$\left(d_4 + 2\delta_4^{1/4}\right) \times t(1+\nu/4)d_4^3 d_3^{10} d_2^9 n^3 \times (d_2 + \delta_2)n/t \le (1+\nu/5) d_2^{\binom{5}{3}} d_3^{\binom{5}{2}} d_4^4 n^4$$

copies of $K_4^{(3)}$.

(ii) The number of vertices left in W is at most $2\delta_4^{1/4} d_2 n$ (c.f. (6.31)). Each such vertex satisfies (6.17) and, consequently, is involved in not more than $|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(4)}(x,y)))| \le (1+\nu/5) d_2^9 d_3^{10} d_4^3 n^3$ copies of $K_4^{(3)}$.

Therefore, this group of vertices contributes at most

$$2\delta_4^{1/4} d_2 n \times (1+\nu/5) d_2^9 d_3^{10} d_4^3 n^3 < \delta_4^{1/8} d_2^{\binom{5}{2}} d_3^{\binom{5}{3}} d_4^4 n^4$$

copies of $K_4^{(3)}$. We used again the assumption $\delta_4 \ll d_4$.

(iii) Now we must estimate the contribution of neighbors y which are not fine, that is $y \in N_2(x) \setminus N_2(x)_{\text{fine}}$. Since

$$N_2(x) \setminus N_2(x)_{\text{fine}} = (N_2(x)_{\text{nice}} \setminus N_2(x)_{\text{fine}}) \cup (N_2(x)_{\text{good}} \setminus N_2(x)_{\text{nice}}) \\ \cup (N_2(x) \setminus N_2(x)_{\text{good}}),$$

we distinguish three categories of these vertices:

(a) Consider vertices $y \in N_2(x)_{\text{nice}} \setminus N_2(x)_{\text{fine}}$. We know from Observation 6.6 that $|N_2(x)_{\text{nice}} \setminus N_2(x)_{\text{fine}}| \le 6\delta_4^{1/4} d_2 n$.

Due to Part B(b), we estimate contribution of every such vertex y by

$$|\mathcal{H}^{(3)} \cap \mathcal{K}_3(\mathcal{H}^{(3)}(x,y))| \le 3d_2^9 d_3^{10} n^3.$$

Therefore, vertices from $N_2(x)_{\text{nice}} \setminus N_2(x)_{\text{fine}}$ can contribute by at most $6\delta_4^{1/4} d_2 n \times 3d_3^{10} d_2^9 n^3 \le \delta_4^{1/8} d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ copies of $K_5^{(4)}$.

(b) Consider vertices $y \in N_2(x)_{\text{good}} \setminus N_2(x)_{\text{nice}}$. Observation 5.20 implies that $|N_2(x)_{\text{good}} \setminus N_2(x)_{\text{nice}}| \le 12\delta_3^{1/32} d_2 n$.

Then, each such neighbor y is in at most $|\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y))]|$ copies of $K_4^{(3)}$. Since $y \in N_2(x)_{\text{good}}$, $\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y))]$ is $(\delta_2^{1/2}, d_2)$-regular (c.f. Definition 5.19). Consequently, $|\mathcal{K}_3(\mathcal{H}^{(2)}[\mathcal{H}^{(2)}(x,y))]| \le 2d_2^9 n^3$ (c.f. Corollary 3.6).

The total contribution of these vertices is then bounded by $12\delta_3^{1/32} d_2 n \times 2d_2^9 n^3 \le \delta_3^{1/64} d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \le \delta_4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \le \delta_4^{1/2} d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$. Here we used assumptions (2.2).

(c) The remaining neighbors y belongs to $N_2(x) \setminus N_2(x)_{\text{good}}$. It follows from Observation 3.10 that $N_2(x) \setminus N_2(x)_{\text{good}} \le 36\delta_2^{1/8} d_2 n$.

In this case, we use a rough estimate that every vertex is in at most n^3 copies of $K_4^{(3)}$ and, thus, the contribution of these vertices is at most $36\delta_2^{1/8} d_2 n \times n^3 \le 36\delta_2^{1/16} d_2^{\binom{5}{2}} n^4 \le \delta_3 d_2^{\binom{5}{2}} n^4 \le \delta_3^{1/2} d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \le \delta_4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$ $\le \delta_4^{1/2} d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4$.

At this point we are ready to derive the upper bound. We add the contributions of all vertices considered in (i), (ii) and (a), (b), (c) of (iii) to infer that

$$|\mathcal{K}_4(\mathcal{H}^{(4)}(x))| \le \left(1 + \nu/4 + \delta_4^{1/8} + 2\delta_4^{1/2}\right) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4 \le (1 + \nu/2) d_4^4 d_3^{\binom{5}{3}} d_2^{\binom{5}{2}} n^4.$$

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