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Transversal numbers of translates of a convex body

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Abstract

Let $\mathscr{F}$ be a family of translates of a fixed convex set M in $\mathbb{R}^n$. Let $\tau(\mathscr{F})$ and $v(\mathscr{F})$ denote the transversal number and the independence number of $\mathscr{F}$, respectively. We show that $v(\mathscr{F}) \leqslant \tau(\mathscr{F}) \leqslant 8v(\mathscr{F}) - 5$ for $n = 2$ and $\tau(\mathscr{F}) \leqslant 2^{n-1}n^n v(\mathscr{F})$ for $n \geqslant 3$. Furthermore, if M is centrally symmetric convex body in the plane, then $v(\mathscr{F}) \leqslant \tau(\mathscr{F}) \leqslant 6v(\mathscr{F}) - 3$.

Keywords: Transversal number; Intersection graph; Convex; Homothety

1. Introduction

For a family $\mathscr{F}$ of convex bodies in Euclidean space let $\tau(\mathscr{F})$ denote the minimum size of a set of points that intersects every member of $\mathscr{F}$ and let $v(\mathscr{F})$ denote the maximum size of a subfamily of pairwise disjoint members of $\mathscr{F}$. We call $\tau(\mathscr{F})$ the *transversal number* of $\mathscr{F}$ and $v(\mathscr{F})$ the *independence number* of $\mathscr{F}$. The *intersection graph* G of a family $\mathscr{F}$ of sets is the graph with vertex set $\mathscr{F}$ in which two members of $\mathscr{F}$ are adjacent if and only if they have nonempty intersection. We denote the complement of a graph G by $\overline{G}$. The chromatic number of the complement $\chi(\overline{G})$ of G is the minimum number of cliques that suffices to cover $V(G)$.

If every intersecting subfamily of a family $\mathscr{F}$ has a common point, then we say that the family $\mathscr{F}$ has the *Helly property*. Let $\mathscr{B}_n$ be a family of boxes in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geqslant 2$. Since the family $\mathscr{B}_n$ of boxes has the Helly property, $\tau(\mathscr{B}_n) = \chi(\overline{G})$, where G is the intersection graph of $\mathscr{B}_n$. The study of $\tau(\mathscr{B}_n)$ was initiated by Gyárfás and Lehel [7]. Put $k = v(\mathscr{B}_n)$. Gyárfás and Lehel [7] proved that $\tau(\mathscr{B}_n) \leqslant k(k-1)/2$ and Károlyi [9]

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improved this bound to $\tau(\mathscr{B}_n) \leqslant (1+o(1))k \log^{n-1} k$. Fon-Der-Flaass and Kostochka [5] gave a simple proof of a slight refinement of Károlyi's bound:

$$\tau(\mathscr{B}_n) \leqslant k \log_2^{n-1} k + n - 0.5k \log_2^{n-2} k.$$

However, families of convex sets of other types, such as disks, do not necessarily have the Helly property. Example 10 in Section 3 presents an intersecting family having no common point. Hence, it is interesting to study how much the transversal number of such a family can exceed its independence number.

In this paper, we consider families of convex bodies in $\mathbb{R}^n$ obtained by translations of a fixed convex body. We will show that

$$\nu(\mathscr{F}) \leqslant \tau(\mathscr{F}) \leqslant 8\nu(\mathscr{F}) - 5$$

for any family $\mathscr{F}$ of translates of a fixed convex set M in the plane. Furthermore, if M is centrally symmetric, then

$$\nu(\mathscr{F}) \leqslant \tau(\mathscr{F}) \leqslant 6\nu(\mathscr{F}) - 3.$$

Note that the smallest upper bound on $\tau(\mathscr{F})$ should be at least $3\nu(\mathscr{F})$ because there exists a family $\mathscr{F}$ for which $\tau(\mathscr{F}) = 3\nu(\mathscr{F})$ (see Example 10 in Section 3). We also obtain upper bounds on $\tau(\mathscr{F})$ in higher dimensions.

The paper is organized as follows: In Section 2, we bound the transversal number $\tau(\mathscr{F})$ in terms of $\nu(\mathscr{F})$. In Sections 3 and 4, we study $\tau(\mathscr{F})$ for special classes of families: in Section 3 for a family $\mathscr{F}$ of translates of a centrally symmetric convex set in the plane, and in Section 4 for a family $\mathscr{F}$ of homothetic copies of a fixed convex set in the plane. In Section 3 we also show that if $\mathscr{F}$ is a family of unit disks and $\tau(\mathscr{F}) = 2$, then $\nu(\mathscr{F}) \leqslant 6$, which is best possible.

2. Transversal numbers

Let $\mathscr{F}$ be a family of convex bodies obtained by translations of a fixed convex body in $\mathbb{R}^n$. Let G be the intersection graph of $\mathscr{F}$.

First we consider the case when $n = 2$. Grünbaum [4] proposed the following question: let $\mathscr{T}$ be a family of translates of a fixed convex set in the plane. If $\mathscr{T}$ is an intersecting family, does $\tau(\mathscr{T}) \leqslant 3$ hold? Karasev [8] showed that the answer to this question is yes.

Using Karasev's result, we have the following.

Proposition 1. *If $\mathscr{F}$ is a family of translates of a fixed convex body in the plane, then $\chi(\overline{G}) \leqslant \tau(\mathscr{F}) \leqslant 3\chi(\overline{G})$, where G is the intersection graph of $\mathscr{F}$.*

Kim and Nakprasit [11] showed that $\chi(\overline{G}) \leqslant 3\nu(\mathscr{F}) - 2$, where G is the intersection graph of $\mathscr{F}$. Using this result and Proposition 1, we obtain $\tau(\mathscr{F}) \leqslant 9\nu(\mathscr{F}) - 6$. Now we will improve the upper bounds on $\tau(\mathscr{F})$ in the plane and also obtain bounds on $\tau(\mathscr{F})$ in higher dimensions. The following lemma is folklore.

Lemma 2. *Let A be a centrally symmetric convex body in $\mathbb{R}^n$ and let B be a translation of A. If the center of B is in A, then B contains the center of A.*

Proof. Without loss of generality, we may assume that the center of A is the origin and the center of B is the point x. If $x \in A$, then $-x \in A$ because A is centrally symmetric. Hence $0 = x + (-x) \in x + A = B$. □

Before stating the next lemma, we need some definitions. For a convex body A in a family $\mathscr{F}$, let $N(A)$ be the set of all convex bodies in the family $\mathscr{F}$ that intersects A and let $N[A] = N(A) \cup \{A\}$. We also say that a convex body A is the *highest* in a family $\mathscr{F}$ if it has a point of the largest nth coordinate among all points in $\bigcup_{F \in \mathscr{F}} F$.

Definition 3. Let x be a point in $\mathbb{R}^n$ and λ be a real number. We set $x + \lambda D = \{x + \lambda w : w \in D\}$. Two convex bodies K and D in $\mathbb{R}^n$ are called *homothetic* if $K = x + \lambda D$ for some $x \in \mathbb{R}^n$ and $\lambda > 0$. Here we call λ the *homothety ratio* of K to D.

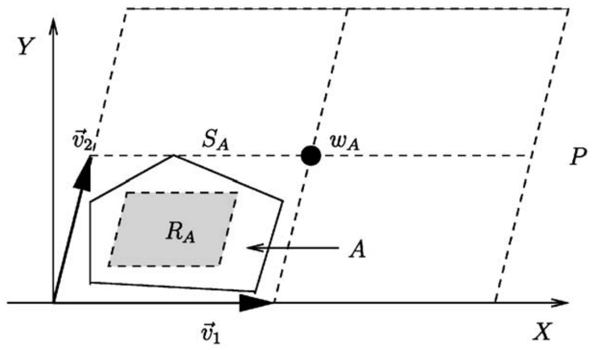

Fig. 1. When $n = 2$.

Lemma 4. *Let $\mathscr{F}$ be a family of translates of a fixed convex body in $\mathbb{R}^n$ and let A be any convex body in $\mathscr{F}$. If there exist parallelotopes R_A and S_A such that $R_A \subseteq A \subset S_A$ and R_A and S_A are homothetic with homothety ratio λ, then there is a set of $\lceil 2\lambda \rceil^n$ points that intersects every member of $N[A]$. Moreover, if A is the highest convex body in $\mathscr{F}$, then there is a set of $\lceil 2\lambda \rceil^{n-1} \lceil \lambda \rceil$ points intersecting every member of $N[A]$.*

Proof. Since $\mathscr{F}$ is a family of translates of a fixed convex body and A is a member in $\mathscr{F}$, we may assume that $\mathscr{F}$ is a family of translates of A. Let L be the set of points in $\mathbb{R}^n$ such that $\mathscr{F} = \{z + A: z \in L\}$. Since every $C \in \mathscr{F}$ has the form $z_c + A$ for some $z_c \in L$, we have $R_C \subseteq C \subseteq S_C$, where $R_C = z_c + R_A$ and $S_C = z_c + S_A$. By rotating and translating A, if necessary, we may assume that the origin is one of the vertices of S_A and a side of S_A is parallel to the hyperplane $x_n = 0$.

Then we can represent the parallelotope S_A using n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^n$ as follows:

$$S_A = \{\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n: \alpha_i \text{ is a real number and } 0 \leqslant \alpha_i \leqslant 1\}.$$

Here we assume that the initial point of every vector $\vec{v}$ is the origin.

Set $w_A = \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n$. We call w_A the "representing point" of A. The representing point of C is $w_C = z_c + w_A$. The representing point w_C of each convex body C in $N[A]$ is in the parallelotope

$$P = \{\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \cdots + \beta_n\vec{v}_n: 0 \leqslant \beta_j \leqslant 2, 1 \leqslant j \leqslant n\}.$$

(Fig. 1 illustrates this when $n = 2$.)

In Fig. 1, P is covered by 16 translates of R_A. In general, we consider $\lceil 2\lambda \rceil^n$ translates of R_A specified as follows. Every element of $N[A]$ has its representing point in one of the translates of R_A. For $1 \leqslant i \leqslant \lceil 2\lambda \rceil^n$, let M_i be the set of convex bodies in $N[A]$ whose representing points are in the ith translate of R_A. Let $\mathscr{R}_i = \{R_C: C \text{ is in } M_i\}$. The centers of all convex bodies in $\mathscr{R}_i$ are in a translate of R_A. The members of $\mathscr{R}_i$ have a common point since R_A is a centrally symmetric convex body, and so do the members of M_i. Therefore we have found a set of $\lceil 2\lambda \rceil^n$ points that intersects every convex body in $N[A]$.

If A is the highest convex body in $\mathscr{F}$, then w_C is in the parallelotope

$$P' = \{\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \cdots + \beta_n\vec{v}_n: 0 \leqslant \beta_j \leqslant 2, 1 \leqslant j \leqslant n-1, 0 \leqslant \beta_n \leqslant 1\}.$$

Thus, similarly as above, we can prove that there exists a set of $\lceil 2\lambda \rceil^{n-1} \lceil \lambda \rceil$ points that intersects every convex body in $N[A]$. □

Let $\mathscr{F}$ be a family of convex bodies obtained by translations of a fixed convex body in $\mathbb{R}^n$. We will decompose $\mathscr{F}$ into at most $v(\mathscr{F})$ subfamilies as follows. First, we pick the highest convex body A_1 and put $\mathscr{F}_1 = N[A_1]$. Next, we choose the highest convex A_2 in $\mathscr{F} \backslash \mathscr{F}_1$ and let $\mathscr{F}_2 = N[A_2] \cap (\mathscr{F} \backslash \mathscr{F}_1)$. We continue this process. At step i, we select the highest convex body A_i in $\mathscr{F} \backslash \bigcup_{s=1}^{i-1} \mathscr{F}_s$ and let $\mathscr{F}_i = N[A_i] \cap (\mathscr{F} \backslash (\bigcup_{s=1}^{i-1} \mathscr{F}_s))$. This process ends within $v(\mathscr{F})$ steps, since no more than $v(\mathscr{F})$ members of $\mathscr{F}$ are pairwise disjoint. Here $\mathscr{F}_{v(\mathscr{F})}$ is an empty set or an intersecting family.

Hence, $V(G)$ decomposes into $\nu(\mathscr{F})$ subsets of $\mathscr{F}$ such that $V(G)=\bigcup_{i=1}^{\nu(\mathscr{F})}\mathscr{F}_i$. We call the resulting decomposition process *greedy decomposition*.

Chakerian and Stein [3] proved the following theorem.

Theorem 5. *For every convex body C in $\mathbb{R}^n$ there exist parallelotopes R_C and S_C such that $R_C \subseteq C \subset S_C$, where R_C and S_C are homothetic with homothety ratio at most n.*

Combining this theorem with Lemma 4 yields:

Theorem 6. *If $\mathscr{F}$ is a family of convex bodies obtained by translations of a fixed convex body in $\mathbb{R}^n$, then $\nu(\mathscr{F}) \leqslant \tau(\mathscr{F}) \leqslant 2^{n-1}n^n\nu(\mathscr{F})$.*

Proof. By Theorem 5 and Lemma 4, if A is the highest convex body in $\mathscr{F}$, then $\tau(N[A]) \leqslant 2^{n-1}n^n$. Let $k=\nu(\mathscr{F})$. By the greedy decomposition above, $\mathscr{F}$ splits into $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$. Since for each $\mathscr{F}_i$, $1 \leqslant i \leqslant k$, we have $\mathscr{F}_i \subseteq N[A_j]$, where A_j is the highest convex body at step i in the greedy decomposition, there exists a set of $2^{n-1}n^n$ points that intersects all members of $\mathscr{F}_i$. Hence, $\tau(\mathscr{F}) \leqslant \sum_{i=1}^{k}\tau(\mathscr{F}_i) \leqslant \sum_{i=1}^{k} 2^{n-1}n^n = 2^{n-1}n^n\nu(\mathscr{F})$. □

When $n=2$, we modify the argument in Theorem 6 to strengthen the bound on $\tau(\mathscr{F})$.

Corollary 7. *If $\mathscr{F}$ is a family of translates of a fixed convex set in the plane, then $\nu(\mathscr{F}) \leqslant \tau(\mathscr{F}) \leqslant 8\nu(\mathscr{F})-5$.*

Proof. Let $k=\nu(\mathscr{F})$. Using the greedy decomposition, $\mathscr{F}$ decomposes into $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$, where $\mathscr{F}_k$ is an empty set or an intersecting family. Hence by Karasev's [8] result, $\tau(\mathscr{F}_k) \leqslant 3$. By Theorem 6, for each $\mathscr{F}_i$ with $1 \leqslant i \leqslant k-1$, there is a set of 8 points that intersects all convex sets of $\mathscr{F}_i$. Hence, $\tau(\mathscr{F}) \leqslant 8(k-1)+3 = 8\nu(\mathscr{F})-5$. □

3. Transversal numbers of a family of centrally symmetric convex bodies

When we consider centrally symmetric convex bodies in the plane, we have an upper bound on the homothety ratio guaranteed by Theorem 5.

Theorem 8 (*Asplund [2]*). *If C is a centrally symmetric convex set in the plane, there are parallelograms R and S such that $R \subseteq M \subseteq S$ where R and S are homothetic with homothety ratio at most $\frac{3}{2}$.*

Consequently, we have the following improvement of Corollary 7.

Theorem 9. *If $\mathscr{F}$ is a family of convex sets obtained by translations of a fixed centrally symmetric convex set M in the plane, then $\nu(\mathscr{F}) \leqslant \tau(\mathscr{F}) \leqslant 6\nu(\mathscr{F})-3$.*

Proof. By Theorem 8, there are parallelograms R and S such that $R \subseteq M \subseteq S$ and S is homothetic to R with homothety ratio at most $\frac{3}{2}$. By Lemma 4, if A is the highest convex set in a subfamily of $\mathscr{F}$, then $\tau(N[A]) \leqslant 6$. Let $k=\nu(\mathscr{F})$. Using the greedy decomposition, $\mathscr{F}$ splits into $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$, where $\mathscr{F}_k$ is either empty or an intersecting family. It follows from Karasev's [8] result that $\tau(\mathscr{F}_k) \leqslant 3$. Consequently, $\tau(\mathscr{F}) \leqslant \sum_{i=1}^{k}\tau(\mathscr{F}_i) \leqslant 6(k-1)+3 = 6\nu(\mathscr{F})-3$. □

The smallest upper bound on $\tau(\mathscr{F})$ must be at least $3\nu(\mathscr{F})$. We prove this by providing an arbitrarily large family $\mathscr{F}$ such that $\tau(\mathscr{F}) = 3\nu(\mathscr{F})$.

Example 10. Let T be the regular triangle with vertices x, y, and z of side length 2. Let D_x, D_y, and D_z be disks of radius 2 whose centers are x, y, and z, respectively. Let C be the curve consisting of the three arcs from x to y on D_z, from y to z on D_x, and from z to x on D_y. (See Fig. 2.) Let S be the set of points consisting of $\{x, y, z\}$ and 3×10^{10} additional points equally spaced on C. (Here the number 3×10^{10} is not important. Any large enough number of points works.) Let $\mathscr{F}'$ be the family of unit disks whose centers are the points in S, and let $\mathscr{F}$ be a family that is the union of k completely disjoint copies of $\mathscr{F}'$.

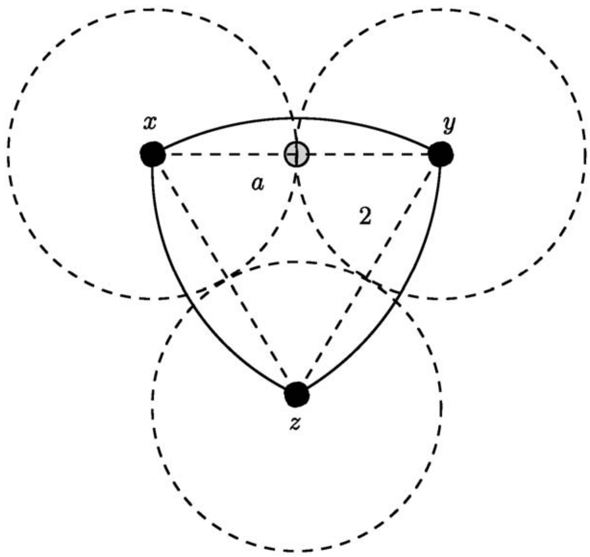

Fig. 2. $\tau(\mathscr{F}') = 3$.

First we will show that $\tau(\mathscr{F}')=3$. It is not difficult to show that $\mathscr{F}'$ is an intersecting family because C is of constant width 2. Hence $\nu(\mathscr{F}') = 1$. Since $\{x, y, z\}$ intersects all disks of $\mathscr{F}'$, we have $\tau(\mathscr{F}')\leqslant 3$. Suppose that two points are enough to cover all the disks in $\mathscr{F}'$. Since these two points must cover the three disks whose centers are x, y, and z, one of the points must cover two of them, so we may assume that one of the transversal points must be the middle point of the segment joining x and y. Since the arcs from y to z and from z to x are outside the unit disk with center a, the disks with centers on these arcs do not contain the point a. Therefore, the other point must cover all disks whose centers are on the two arcs from y to z and z to x. That is impossible, because three disks with centers at z, near x, and near y have no common point. Thus $\tau(\mathscr{F}')\geqslant 3$, and therefore $\tau(\mathscr{F}')=3$. Now since $\mathscr{F}$ consists of k disjoint copies of $\mathscr{F}'$, we have $\tau(\mathscr{F}) = 3k$ and $\nu(\mathscr{F}) = k$.

Now we find $\tau(\mathscr{F})$ when $\mathscr{F}$ is a family of unit disks in the plane with $\nu(\mathscr{F})=2$. By Theorem 9, $\tau(\mathscr{F})\leqslant 9$ if $\nu(\mathscr{F})=2$ and $\mathscr{F}$ is a family of translates of centrally symmetric convex set in the plane. We will show that $\tau(\mathscr{F})$ is at most 6 if $\mathscr{F}$ is a family of unit disks. Note that this is sharp by Example 10.

Theorem 11. *If $\mathscr{F}$ is a family of unit disks in the plane with $\nu(\mathscr{F}) = 2$, then $\tau(\mathscr{F})\leqslant 6$.*

Proof. Since $\nu(\mathscr{F}) = 2$, among every three unit disks, two of them intersect. Let U_x and U_y be unit disks in $\mathscr{F}$ whose centers x, y have the largest distance d in $\mathscr{F}$. Denote by D_x and D_y the disks of radius 2 whose centers are x and y, respectively. Furthermore, let W_x and W_y be disks of radius d also centered in x and y, respectively.

Then the center of each disk in $\mathscr{F}$ belongs to

$$T = (D_x \cap W_y) \cup (D_y \cap W_x)$$

because every disk in $\mathscr{F}$ must intersect either U_x or U_y.

Now we distinguish the following three cases.

Case 1 ($d\leqslant 2$): In this case, $\tau(\mathscr{F})\leqslant 3$, since $\mathscr{F}$ is an intersecting family.

Case 2 ($2 < d < 2\sqrt{3}$): In this case, we will cover the region T with six unit disks. Set $d = 2r$. Without loss of generality, we may assume that $x = (-r, 0)$ and $y = (r, 0)$. We place two unit disks centered at $(\pm r/2, 0)$ and four unit disks centered at $(\pm r/2, \pm 1)$. Denote by U_q the disk centered at $q = (r/2, 1)$ and by U_+ the disk centered at $(r/2, 0)$. By symmetry, it suffices to check that U_q contains the region T^+ consisting of $D_y \cap W_x \backslash U_+$ restricted to the first quadrant.

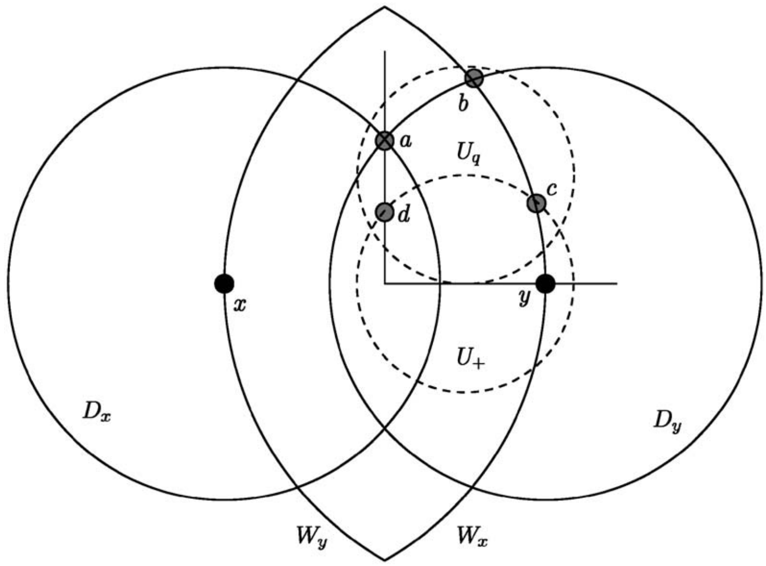

Fig. 3. The case when $1 < r < \sqrt{3}$.

Let a, b, c, d be the intersection points of the y-axis and D_y, W_x and D_y, W_x and U_+, U_+ and the y-axis, respectively (see Fig. 3). It is easy to observe that the coordinates of points a, b, c, and d are $a=(0, \sqrt{4-r^2})$, $b=(r-1/r, \sqrt{4-1/r^2})$, $c=(13r/12-1/(3r), \sqrt{1-(7r/12-1/(3r))^2})$, $d=(0, \sqrt{1-r^2/4})$. A short calculation shows that the distance of a, b, c, and d from q is smaller than 1 for every $1 < r < \sqrt{3}$. Thus, all these points are inside the disk U_q.

A moment of thought shows that T^+ is bounded by curve segments between a and b on D_y, b and c on W_x, c and d on U_+, and the line segment da. Since the curvature of each of these curves is at most the curvature of U_q, each curve lies entirely in U_q. By convexity, U_q contains T^+ as well.

Case 3 ($d \geqslant 2\sqrt{3}$): Let $S = D_x \cap D_y$, $L = (D_x \cap W_y)\backslash S$, and $R = (D_y \cap W_x)\backslash S$. Note that $T = L \cup S \cup R$ and this union is disjoint. Denote by $\mathscr{F}_L$, $\mathscr{F}_S$, and $\mathscr{F}_R$ the sets of disks whose centers are in L, S, and R, respectively. Hence $\mathscr{F} = \mathscr{F}_L \cup \mathscr{F}_S \cup \mathscr{F}_R$. Since $\nu(\mathscr{F}) = 2$, both $\mathscr{F}_L$ and $\mathscr{F}_R$ are intersecting families, as every disk in $\mathscr{F}_L$ is disjoint from U_y and every disk in $\mathscr{F}_R$ is disjoint from U_x.

It follows that $\tau(\mathscr{F}_L) \leqslant 3$ and $\tau(\mathscr{F}_R) \leqslant 3$, and if S is empty, six unit disks suffice to cover $\mathscr{F}$. Thus we may assume that $r \leqslant 2$.

Since $\mathscr{F}_L$ is an intersecting family, the distance between the centers of every two disks in $\mathscr{F}_L$ is at most 2. Hence, we may assume that there are two horizontal lines L_1 and L_2 in L of distance at most 2 such that the centers of disks in $\mathscr{F}_L$ are between the lines L_1 and L_2. Similarly, the centers of disks in $\mathscr{F}_R$ must lie between two horizontal lines L_3 and L_4 whose distance is at most 2. Let R_1 be the portion of L that lies between the lines L_1 and L_2, and let R_2 be the portion of R that lies between L_3 and L_4. (See Fig. 4.)

Let $T_r = S \cup R_1 \cup R_2$. Then the centers of disks in $\mathscr{F}$ are in T_r. We will cover T_r with six unit disks. First, place two unit disks U_1 and U_2 with centers $\left(0, \frac{1}{2}\right)$ and $\left(0, -\frac{1}{2}\right)$, respectively. Among the two intersection points of U_1 and U_2, let z be the one with smallest x-coordinate, and choose p_1 and p_2 similarly from the intersection points of D_x and U_1, and D_x and U_2, respectively. Then $z = \left(-\frac{\sqrt{3}}{2}, 0\right)$ and the x-coordinate of both p_1 and p_2 is

$$x_1 = \frac{22r - 8r^3 - \sqrt{-105 + 152r^2 - 16r^4}}{4(1+4r^2)}.$$

Note that since $\sqrt{3} \leqslant r \leqslant 2$, the region S is contained in $U_1 \cup U_2$. Let $R_1' = R_1\backslash(U_1 \cup U_2)$ and $R_2' = R_2\backslash(U_1 \cup U_2)$. Clearly, $T_r \subseteq U_1 \cup U_2 \cup R_1' \cup R_2'$. Hence, if we show that both R_1' and R_2' are covered by two unit disks each, then we have $\tau(\mathscr{F}) \leqslant 6$.

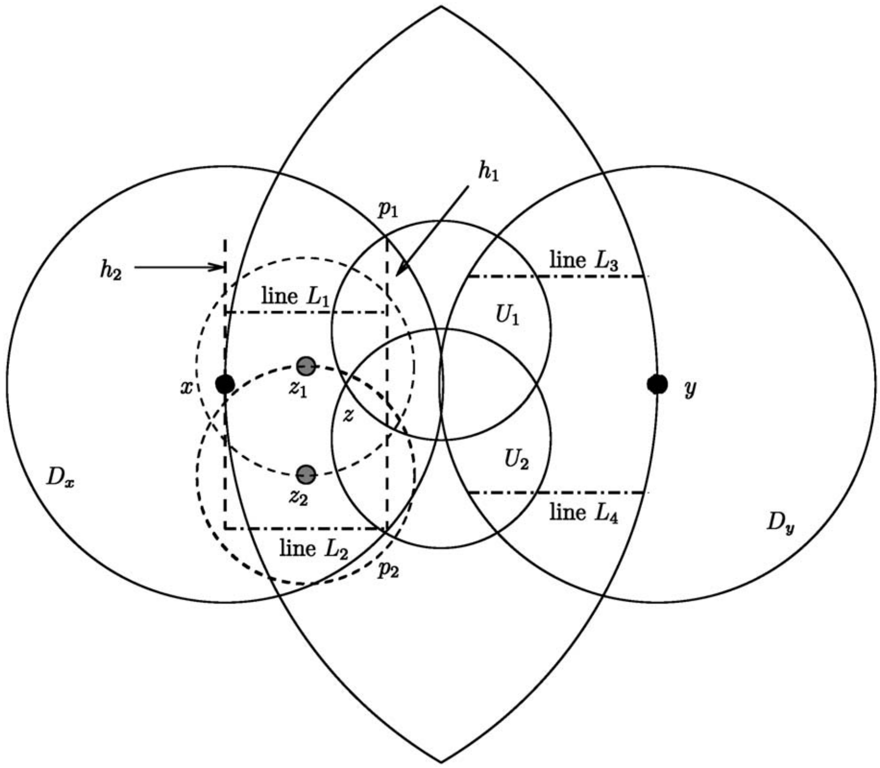

Fig. 4. The case when $\sqrt{3} \leqslant r \leqslant 2$.

Let h_1 be the vertical line connecting the points p_1 and p_2 and let h_2 be the vertical line passing through the point x. Observe that x_1 is greater than $-\frac{\sqrt{3}}{2}$ for $\sqrt{3} \leqslant r \leqslant 2$. This means that z is to the left of the line h_1. Consequently, the region R_1' lies in between the lines h_1 and h_2. Hence the region R_1' is contained in the rectangle $\overline{R_1}$ bounded by h_1, h_2, L_1, and L_2. Next we will show how to cover R_1' with two unit disks.

First observe that

$$|x_1 - (-r)| = \frac{22r - 8r^3 - \sqrt{-105 + 152r^2 - 16r^4}}{4(1 + 4r^2)} + r \leqslant 1.6 \tag{1}$$

for $\sqrt{3} \leqslant r \leqslant 2$. Suppose that L_1 is the line $y = t$ and L_2 is the line $y = t - 2$, where t is a constant. We cover R_1' with two unit disks Q_1 and Q_2 as follows: the center of Q_1 is $z_1 = \left((-r + x_1)/2, t - \frac{1}{2}\right)$ and the center of Q_2 is $z_2 = ((-r + x_1)/2, t - \frac{3}{2})$.

It follows from (1) that two vertices $(-r, t)$ and (x_1, t) of $\overline{R_1}$ belong to Q_1 and $(-r, t - 2)$ and $(x_1, t - 2)$ of $\overline{R_1}$ belong to Q_2. Furthermore, since the intersection points of Q_1 and Q_2 have distance $\sqrt{3}$ that is greater than $|x_1 - (-r)|$, we can conclude that $R_1' \subset \overline{R_1} \subset Q_1 \cup Q_2$.

We cover R_2' with two unit disks similarly. □

4. Transversal number of homothetic copies of a convex set

Lemma 12. *Given a convex set U, for each $v \in U$ and $0 \leqslant \lambda \leqslant 1$, the set $W(U, v, \lambda) = (1 - \lambda)v + \lambda U$ is contained in U and contains v.*

Proof. Let $w(u, v, \lambda) = v + \lambda(u - v)$. By the definition, $W(U, v, \lambda) = \{w(u, v, \lambda): u \in U\}$. Since $w(u, v, 0) = v \in U$ and $w(u, v, 1) = u \in U$, we have $w(u, v, \lambda) \in U$ for every $0 \leqslant \lambda \leqslant 1$. On the other hand, $v = w(v, v, \lambda) \in W(U, v, \lambda)$ for every $0 \leqslant \lambda \leqslant 1$. □

Lemma 13. *Let $\mathscr{D}$ be a family of homothetic copies of a convex set D in the plane. Let Z be the smallest convex set in $\mathscr{D}$, and denote by $N[Z]$ the set of all convex sets in $\mathscr{D}$ that intersects Z. Then there is a set of* 16 *points that intersects all convex sets in $N[Z]$, and there is a set of* 9 *points that intersects all convex sets in $N[Z]$ if D is a centrally symmetric convex set.*

Proof. For every $U \in N[Z]$, let λ_U be the positive real such that $Z = u + \lambda_U U$ for some u. For every $U \in N[Z]$, choose a point $z_U \in Z \cap U$ and denote $U^* = W(U, z_U, \lambda_U) = (1 - \lambda_U)z_U + \lambda_U U$. Note that U^* is a translate of Z and $U^* \subset U$. By Lemma 12, $\mathscr{D}^*(Z) = \{Z\} \cup \{U^*: U \in N[Z]\}$ is a set of convex sets that intersects Z.

By Theorem 5 and Lemma 4, there is a set of 16 points that intersects all convex sets in $\mathscr{D}^*(Z)$. Since $U^* \subset U$ for all U^* in $\mathscr{D}^*(Z)$, the set of 16 points also intersects all convex sets in $N[Z]$.

When D is a centrally symmetric convex set, by Theorem 8 and Lemma 4, there is a set of 9 points that intersects all convex sets of $\mathscr{D}^*(Z)$. Again since $U^* \subset U$ for all U^* in $\mathscr{D}^*(Z)$, the set of 9 points also intersects all convex sets in $N[Z]$. □

Theorem 14. *If $\mathscr{D}$ is a family of homothetic copies of a convex set D in the plane, then $v(\mathscr{D}) \leqslant \tau(\mathscr{D}) \leqslant 16v(\mathscr{D})$. In particular, $v(\mathscr{D}) \leqslant \tau(\mathscr{D}) \leqslant 9v(\mathscr{D})$ if D is a centrally symmetric convex set.*

Proof. Let $k = v(\mathscr{D})$. We decompose $\mathscr{D}$ into $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_k$ as follows. First, pick the smallest convex set A_1 and put $\mathscr{D}_1 = N[A_1]$. And next, pick the smallest convex A_2 in $\mathscr{D} \backslash \mathscr{D}_1$. Put $\mathscr{D}_2 = N[A_2] \cap (\mathscr{D} \backslash \mathscr{D}_1)$. We continue this process. At step i, pick the smallest convex set A_i in $\mathscr{D} \backslash \bigcup_{s=1}^{i-1} \mathscr{D}_s$. And then put $\mathscr{D}_i = N[A_i] \cap \left(\mathscr{D} \backslash \bigcup_{s=1}^{i-1} \mathscr{D}_s\right)$. This process ends within k steps.

By Lemma 13, for each $\mathscr{D}_i$, for $1 \leqslant i \leqslant k$, there is a set of 16 points that intersects all convex sets in $\mathscr{D}_i$, and there is a set of 9 points that intersects all convex sets in $\mathscr{D}_i$ if D is a centrally symmetric convex set. This completes the proof. □

5. Concluding remarks

We showed that if $\mathscr{F}$ is a family of unit disks in the plane with $v(\mathscr{F}) = 2$, then $\tau(\mathscr{F}) \leqslant 6$. It seems that every family of translates of centrally symmetric convex set in the plane has a similar property. Hence we would like to propose the following question.

Question 15. *If $\mathscr{F}$ is a family of translates of centrally symmetric convex set in the plane, is it true that $\tau(\mathscr{F}) \leqslant 3v(\mathscr{F})$?*

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