The Topology of the Universe

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Abstract

To overcome the problems involved with an infinite universe, we can use mathematical models of finite spaces without boundaries. In practically all cases, these imply a multiply-connected universe whereby the space we live in (and, therefore, make observations in) is really an infinite lattice of fundamental cells. This paper covers some of the basic mathematical theory and discusses some of the investigations made in recent years.

1 Introduction

When we gaze out upon the sky, our eyes are met with a vast black canvas apparently without end. There appears no physical boundary to our view, constrained only by the finite speed of light and the age of the universe. But is space truly infinite in extent? There are many problems with such a notion. Our immediate reality is purely a finite one – very rarely do practical notions of infinity come into our lives. The age of our universe, the dimensions of our galaxy, the number of atoms in the Sun; while large, these are still finite measurements of time, space and quantity. Is it plausible to jump then from our finite existence to a notion of infinite space? Scientific principles would make us question such a leap.[1]

Putting aside aesthetic reasons, there are physical problems with an open (infinite) universe. Consider the inertia of a body, a measure of its resistance to motion. One can ask how the body "knows" it is in motion? For example, how does a spinning bucket of water know it is spinning, exhibiting the centrifugal effects associated with rotational acceleration, while a stationary bucket shows no such behaviour? Mach postulated that the entire mass of the universe provided a stationary reference frame, against which a body could move relative to, and thus exhibit inertial effects. However, in an infinite universe with a uniform mass distribution (i.e. homogeneity), we would have an infinite quantity of mass, and hence infinite inertia. No motion would be possible and yet we are all capable of observing and undergoing it. Everyday experience would appear to require a finite space.

If we do live in a finite universe then we must consider possible boundary conditions – what happens at the edge? Furthermore, if our universe is truly bound by something, could there then be something on the "outside".[2] In §2 we shall cover some basic mathematical groundwork which will allow us to construct finite surfaces (two-dimensional spaces) without boundary which are not required to be embedded in a higher dimensional space. We can thus talk about a finite space without an "outside" to it. In §3 we shall then see what consequences these spaces will have on cosmological observations made in them, generalizing our two-dimensional results from the previous section to the realm of three-dimensional spaces. Section §4 then considers practical research in this fast developing field.

2 Bounding the Infinite

How do we achieve finiteness without boundary? Consider a triangle drawn on a square, both of which lie in a Euclidean space. The angles add up to 2π, and will continue to do so no matter the size of the triangle or its position in the square.

[1] Indeed Occam would positively cry out in alarm

[2] In fact, recent theoretical work suggests another quite interesting answer to what we shall consider here, and the reader is referred to "Inflation in a Low-Density Universe" by Martin A. Bucher and David N. Spergel; *Scientific American*, January 1999

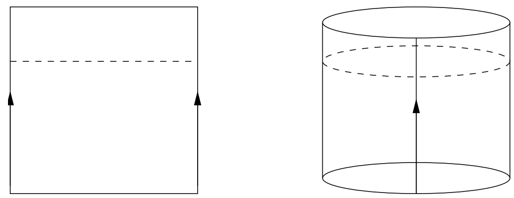

Figure 1: “Gluing” the sides of a square to form a cylinder

If we now glue two opposite sides of the square together to form a cylinder, see **Figure 1**, although lines not parallel to the axis are now curved, Euclidean geometry still reigns. The angle sum is unchanged, while the shortest distance between two points is still the length of a connecting Euclidean straight line on the unfurled square. *The* apparent *curvature has not affected the geometry.* Mathematically this is equivalent to noting that the **intrinsic curvature**, (that belonging to the surface alone), and not its embedding (representation) in space, is the same for both the cylinder and the plane.[3] In fact, the plane is of zero curvature, and hence so is the cylinder. However, fundamental differences *have* arisen: the cylinder is now finite transverse to its length. Any lines on the square connecting the two identified edges will wrap around the cylinder (E.g. the dashed line in Figure 1). By considering the tessellation of the Euclidean plane in **Figure 2**, we see that this identification of edges rolls the Euclidean plane around the cylinder. Inhabitants on the cylinder will see along lines of sight which appear to stretch out over infinite Euclidean space but in fact do so in only one direction, i.e. those along the cylinder's axis.

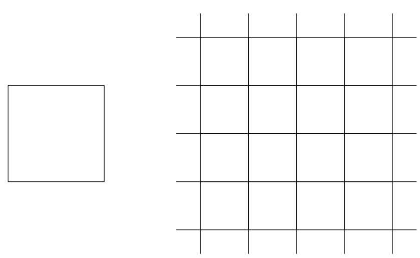

Figure 2: Our *fundamental* square can *cover* the Euclidean plane.

Finally, let us glue together the other two sides in like direction to obtain a torus, **Figure 3**. Once again, the intrinsic curvature is zero (we have merely identified sides) and Euclidean geometry holds on the surface.[4] However, the

[3]The curvature is an intrinsic measure of the deviation from flatness the surface represents that cannot be removed by flattening or rolling, (that is actions which do not involve stretching or tearing).

[4]Should you try to construct a torus in real life, effectively embedding it in $\mathbb{R}^3$, you will note that the surface is *not* Euclidean. Triangles are warped by the curving – compressed in the ring, stretched on the outside. This is a consequence of the embedding itself, not a property of our conceptual torus.

surface is now finite in all direction, but *without boundary.* Once more considering the tessellation in **Figure 2**, we see that the identifications in **Figure 3** wrap the Euclidean plane around the torus. Infinite lines (of sight) merely wrap around the torus continuously. Inhabitants on this surface would once again see along infinite lines of sight, but their universe is now finite without boundary. What this means for physical observation will be discussed in the next section.

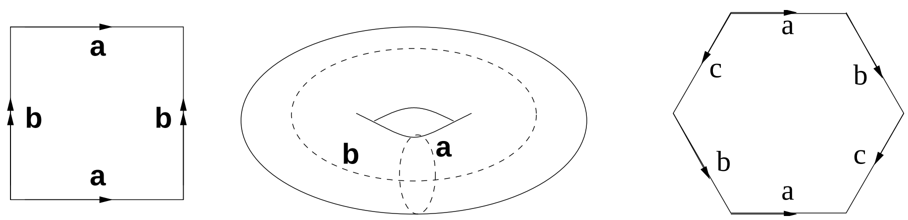

Figure 3: Constructing the torus. The joins are shown for the fundamental square.

Both the cylinder and torus, with their repeating universes, exhibit nontrivial topologies. Specifically, not all closed paths can be contracted to points – precisely those loops which circle the "hole" in the cylinder and torus. We can generalise this notion to surfaces with more holes (i.e a higher genus). To do so, first note that the Euclidean plane can be covered by tessellation with hexagons. By identifying opposing sides as indicated in **Figure 3** we are once again able to cover the torus with the plane. Mathematically, our torus is generated by a fundamental polygon, either the square or the hexagon. The covering spaces, in which the inhabitants live, is in this case the Euclidean plane. To generate a two-holed torus[5] we require an octagon, as in **Figure 4**. However, unlike the cases above, we are no longer able to tessellate the Euclidean plane, with this fundamental polygon – the angle sum at the join of octagons would be greater than 2π and we can no longer wrap the Euclidean plane around our 2-torus. Mathematically, the Euclidean plane is no longer the covering space of the 2-torus, so its inhabitants no longer see a Euclidean geometry. What geometry then do they see?

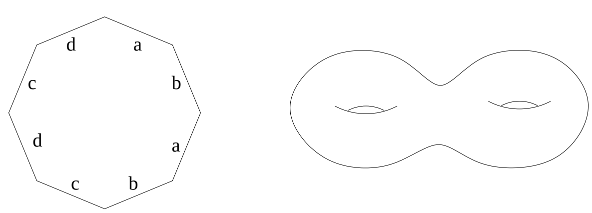

Figure 4: The fundamental polygon for a torus of genus 2.

In the Euclidean plane, angles are independent of the size of their parent

[5] A surface of genus 2.

shape. However, on a curved surface angle size *does* depend on this size. Consider our triangle now inscribed on a sphere. It should be a familiar fact that the angle sum is now *greater* than 2π. Furthermore the actual sum is dependent on the size of our triangle, relative to the sphere, in a manner which ultimately depends on the curvature of the sphere. For a sphere, the curvature k is constant and greater than zero. If you now try to flatten the triangle, i.e. embed it in the Euclidean plane, you will find that you do not have enough material to do so, the triangle is always curved; thus, areas on a spherical surface[6] are less than equivalent areas on a Euclidean surface. In a similar fashion, we can have negatively curved surfaces on which angle sums are now less than their Euclidean counterparts and every point behaves essentially like a saddle point. If you try to flatten a saddle, you will see that you now have too much material and you cannot "iron out" the shape; that is, areas are greater in negatively curved spaces, called **hyperbolic spaces**.

Moreover, as in the spherical case, by choosing an octagon of precisely the right size in hyperbolic space, we can shrink the angle sum down to 2π, so that the *natural geometry* for a 2-torus is that of its hyperbolic covering space, which implies a *negative* curvature. Such a tessellation is shown in **Figure 5**, where the hyperbolic plane is actually the Poincare circle as shown. Although the octagons appear in different sizes, the metric[7] means that they are all of the same size in the hyperbolic geometry.

Figure 5: The fundamental polygon for a 2-torus requires the hyperbolic field

Thus the natural geometry for a 2-torus (and, indeed, any n-hold torus, $nge2$) is hyperbolic.

[6] In fact, any surface of constant positive curvature is called a **spherical surface**

[7] Loosely speaking, this is the way the distance between two points is measured.

3 Physical Consequences

3.1 Infinitely many images

In section §2 we have laid most of the mathematical framework to consider physical models of the universe. Generalizing to three (or higher) dimensions, we summarize these results by noting that a **universal covering** space can be constructed as a tessellation of **fundamental polyhedra** by making identifications of their faces.[8] By altering the geometry of the underlying space we are then able to use the fundamental polyhedron to tessellate spaces with constant positive, zero, or negative curvature. Inhabitants in a fundamental polyhedron will see themselves living in the infinite universal covering, inheriting the underlying geometry of this space.

But, while the two-dimensional models we considered in §2 share these basic mathematical properties, they differ markedly in certain other respects. The cylinder, and the one-holed and many-holed tori all exhibit **multiply-connected topologies** – not every closed loop in these spaces can be contracted down to point, see lines a and b in Figure 3.[9] Physically, this is characterized as finiteness without boundary of the surface – lines can appear to extend towards infinity, but actually wrap eternally around the surface. Inhabitants on this surface will look along lines of sight which ultimately traverse the same region of space infinitely many times. There are thus, potentially, infinitely many copies of object images to be seen in our heavens, limited only by the time taken for their light to reach our observatories. Our universe now resembles a bizarre hall of mirrors with ghost images extending as far as the eye can see.

3.2 Infinitely many mirrors

To obtain multiply-connected topologies, we have seen (in the two-dimensional case) that it may be necessary to alter the curvature of the underlying covering space to accommodate certain tessellations of fundamental polyhedra (see **Figure 5**). Thus there is some interplay between geometry and topology. In fact, for compact two-dimensional surfaces there exists a precisely defined relationship between topology and geometry whereby knowledge of one is ultimately knowledge of the other.[10] One may hope for a similar relationship between topology and geometry in three dimensions.

At the turn of the century, Einstein forsook a gravitational theory of the universe for a geometric theory of space and time. The General Theory of Relativity. The field equations of his General Theory of Relativity encoded the local geometry of space-time into the metric tensor – space told matter how to move, and matter told space how to curve – thus allowing us to calculate the cur-

[8] In the case of two dimensions, fundamental polyhedra become the polygons for which 'identifying faces' requires gluing together the necessary edges of copies of this polygon, see §2.

[9] The infinite Euclidean plane is simply connected in that *any* closed loop *can* be continuously contracted down to a point. There are no holes acting as obstacles.

[10] The "Classification theorem" for (compact) surfaces was proved in the 1920s

vature (tensor) of space. So relativity theory offered us an insight into the curvature, and hence geometry, of space-time from which, it was thought, we could conclude the topology of the universe. According to Einstein, a negative or zero curvature implied a hyperbolic or Euclidean space, and hence an infinite space. To overcome the problems of an infinite Newtonian cosmology (see §1) necessarily required a positively curved, spherical space. Space was "thus" the three-dimensional "surface" of a four-dimensional hypersphere. Just as is the case with its two-dimensional counterpart, the surface of a sphere, any closed path or surface could be shrunk to a point in this space, thus implying a trivial topology – space was **simply connected**.

However, the almost unique relationship exhibited by surfaces is absent in three dimensions, geometry constrains but does not dictate topology, and so a far richer structure prevails. While the relativistic *partial differential* field equations provide us with *local* geometric solutions, they are not sufficient to tell us how to piece these neighbourhoods of space together to form the global topology of the space. In fact, *for any local geometry there are infinitely many topologically distinct spaces*. We are therefore faced with an infinite number of possible cosmologies to model our universe on.

4 Exploring the Infinite

However, this realization that precisely these topological notions were needed to determine finiteness of the universe has only just recently bore results. The easy elegance of Einstein's supposition of a three-dimensional hyperspherical surface soon became fixed in cosmology as fact, even though eminent physicists of the time[11] questioned both this 'conclusion' and the topological determinacy of the field equations themselves. It has only been in the previous five years that serious investigations into the topology of the universe have been undertaken. These have been motivated by two areas of research, old and new. Classical general relativity can now be seen to yield vastly different models when coupled with differing topological models of the universe. The young topic of quantum gravity also poses questions which are essentially topological in nature. Indeed recent work in quantum gravity suggests that negatively curved spaces are more probable than other geometries.

Investigations can be first roughly classified as either: the primarily theoretical , which seek to infer physical constraints on the universe from assumed topologies, and the observational, which seek to directly detect or constrain the underlying topology via new or existing data.

4.1 Splitting infinities

Given the infinite number of distinct topologies that any particular solution of the field equations allows, some theorists seek to reduce and tabulate the

[11] Notably the originators of the big-bang concept, Alexander Friedmann and Lemaître, as well as Felix Klein and Hermann Weyl. Indeed, Friedmann was the first to *explicitly* point out the topological indeterminacy of the solutions to the field equations of relativity.

possibilities under the assumption of certain basic constraints: notably those of symmetry, such as homogeneity and isotropy. An alternative response by some is to consider the physical consequences of different models of the universe. Thus already existing cosmological observations may help to invalidate or support different models.[12] We shall see in more detail how the **Cosmic Microwave Background**, CMB, may offer a way of determining topology. One particular method is to generate computer simulations of what the CMB should look like given a specific topology constrained by symmetry requirements (e.g. local homogeneity and isotropy).[13] To date, only two spaces have been constrained in this manner; the bulk of them remaining spaces do not admit analytical solutions, and hence require computationally burdensome statistics and mathematics. These theoretical investigations have yet to provide any definite results, although they do suggest some evidence for hyperbolic spaces.

4.2 Three-dimensional methods

Of more interest here is the observational work performed so far. If our universe does indeed have a non-trivial topology then it may lead to physically observable effects, see §3. Multiple-connectedness would mean that our universe repeats itself periodically, and the (covering) space we live in is an infinite crystal lattice. The simplest starting point for observational cosmology is then to consider the observable sphere of the universe and its relation to the covering space. If the fundamental polyhedron is smaller than the observable sphere, then we should be able to see in our heavens repeated images of astronomical objects. The success, or failure, of detecting these ghost images can impose bounds on the dimensions of the fundamental polyhedron with respect to the observable sphere. Although there are several methods proposed for this, they all share some basic properties. Clearly, astrophysical "objects" are required, upon observation of which we can draw conclusions. Ideal objects must satisfy certain properties which proves difficult to ensure in practice.

i. The farther afield we look, the further back in time we are effectively looking, due to the finite speed of light. Thus ideal objects should not evolve considerably with this lookback time. In addition it is clear that such an assumption also requires a *known* evolutionary process.

ii. In a similar vein, we of course require that the objects remain constant and visible over large distances and volumes of space.

iii. The behaviour of the object must have very little peculiarities, such as large intrinsic velocities, or anisotropic emission of light.

iv. Finally, observation of the object itself must be unimpeded, so the effects of galactic dust should be minimal.

[12] Topology and Fragility in Cosmology by M.J. Rebouças, R.K.Tavakol and A.F.F.Teixeira, preprint gr-qc/9711026

[13] The topology of the universe: the biggest manifold of the all by James Levin, Evan Scannapieco and Joseph Silk, preprint gr-qc/9803026; *Class. and Quant. Grav.*

Unfortunately, a trade-off between these requirements is necessary. Perhaps the most suitable galactic candles are clusters of galaxies. These are observable by X-ray emission from their hot gas, a process which varies little within the age of the universe. Furthermore, their emission should be isotropic compared to other candidates. However, these are only observable over small ranges, so at best they can provide a mere lower estimate to sizes of our fundamental polyhedron and will certainly not allow far-reaching conclusions to be made on multi-connectedness if the fundamental polyhedron exeeds this range. Regardless of these problems, due to the quality of the objects, any lower bounds obtained from observation are very strong ones. The lack of ghost images of the Coma cluster of galaxies or other examples of large scale structure (e.g. walls, filaments, voids) strongly suggest that the fundamental polyhedron must have sides of at least $60 \sim 150h^{-1}$Mpc in length.[14]

4.3 Gazing into our crystal lattice

We survey the various three-dimensional methods which have been proposed, whereby analysis is primarily focussed on the distribution of astrophysical objects throughout the observable sphere.

The simplest test to perform is to search for repeated arrangements of ideal objects, specifically galaxies, see §4.2. Put simply, if an arrangement of galaxies is observed equally spaced throughout the observable sphere, then the universe is a one-holed torus whose fundamental polyhedron is smaller than the observable universe. While this method allows us to accommodate more general multiply-connected topologies (for instance by observing the orientations of repeated arrangements and any underlying axes of symmetry), as we gradually look further and further back in time, the evolution of individual galaxies plays an increasingly discordant role. Investigations along these lines have been carried out over the past 25 years[15] revealing no repeated images of galaxies within one billion light years of the earth.

A possibility, in keeping with the ideas above but which has not yet been fully investigated, is observation of "large scale structure" in the universe, e.g. objects on scales of $50 \sim 150h^{-1}$Mpc. The evolution of these structural units is not expected to change much with the evolution of individual components. Effective considerations based upon this line of thinking has yet to be carried out.

As the distances concerned become larger, galaxies cease to become useful candles to use, and others have turned to more observable long range candles – quasars.[16] Boudewijn F. Roukema, of the Inter-University Center for Astronomy and Astrophysics, and others, have tried to look for repeated patterns amongst three-dimensional catalogues of quasars up to a distance of

[14]Gott J. R., 1980, MNRAS, 193, 153
1986, in Inner Space/Outer

[15]By researchers such as Dmitri Sokoloff (Moscow State University), Viktor Shvartsman (Soviet Academy of Sciences), L. Richard Gott III (Princeton University), and Helio V. Fagundes (Institute for Theoretical Physics in São Paulo)

[16]On Determining the Topology of the Observable Universe via 3-D Quasar Positions by Boudewijn F. Roukema, preprint `astro-ph/9603052` 12th March 1996

$3300h^{-1}$Mpc from the earth. Initially, Roukema sought repeated images of quintuplets, finding one pair of candidates; however, statistically simulated quasar catalogues reveal this number to be well below any significant quantities of matches that may occur at random, so his evidence was statistically insignificant. Further investigations have yet to reveal other candidates, but this could be due to the poor coverage of the skies of current quasar catalogues. Current research, such as at Imperial College in London, will produce expanded catalogues within a couple of years, so there is still a chance of detecting such quasar groups.

Using quasars is problematic in itself, since not enough is known about their properties. One particular method which utilizes the quasar catalogues is to search for repeated images of our own galaxy at an earlier stage of its evolution, as a quasar. Fagundes and Wichoski (1987)[17] have searched for such quasar images in directions separated by $90°$ and $180°$, but found no conclusive evidence of a non-trivial topology. An interesting aspects of this approach is that if indeed our galaxy was a quasar in its early development, and the fundamental polyhedron is significantly smaller than the observable universe, then ghost images from farther and farther copies will provide snapshots of quasar evolution in earlier and earlier stages of growth.

Of course, the properties that makes the use of quasars interesting also make them poor candles. Quasar lifetimes can be considerably shorter than the timespans neede to see past fundamental polyhedra above a certain size. Another substantial problem is that quasars may not be easily identifiable as such due to their angular aspect with respect to our point of view. Two alternative methods have been suggested to circumvent this problem.

To overcome identification problems of astrophysical objects, a team of French scientists have developed a statistical method called **cosmic crystallography**.[18] If the universe does repeat itself periodically (as in crystal lattices) then plots of object-pair separations should reveal spikes at certain distances corresponding to repeated images, i.e. to the lengths of the fundamental polyhedron. The researchers (see footnote 18) concluded that current quasar catalogues were insufficient to conclude multi-connectedness above a lower limit of $650h^{-1}$Mpc for the fundamental polyhedron, while their tests were negative for distances less than this. It is hoped that the ongoing American-Japanese Sloan Digital Sky Survey (SDSS) will provide a larger data set for this statistical method to be applied to.

An alternative method is to forsake quasars entirely, following previous work (see footnote 14) and use X-ray clusters as topology standard candles.[19] These candles satisfy many of the criteria outlined in §4.2: they are observable at scales of up to $1000h^{-1}$Mpc, a far wider range than other candidate objects, while their behaviour exhibits isotropic emission as well as small evolution over the timescales involved. No statistically significant discoveries (compared against

[17] H. V. Fagundes and U. F. Wichoski, 1987, Ap. J., 322, L5

[18] Cosmic Crystallography by Roland Lehoucq, Marc Lachièze-Rey and Jean-Pierre Luminet, preprint `gr-qc/9604050`; *Astronomy and Astrophysics*, 24th June 1998.

[19] Constraining Cosmological Topology via Highly Luminous X-ray Clusters by Boudewijn F. Roukema and Alastair C. Edge, preprint `astro-ph/9706166`, 16th June 1997.

purely chance coincidences arising from a random distribution) were made and researchers have conculded that the fundamental polyhedron must be of the order of at least $1000h^{-1}$Mpc.

4.4 Two-dimensional methods

One natural boundary to our observable universe does exists, namely the Cosmic Microwave Background (CMB), the last remnants of the Big Bang. This two-dimensional surface represents the last scattering surface (LSS), when the universe cooled enough to allow transmission of photons. Any of these microwave photons arriving at the Earth at the same time left the LSS at precisely the same time at equal distances from the Earth, so that the LSS is a sphere. This spherical surface exhibits a remarkable level of homogeneity, down to one part in 100,000 as measured by the recent Cosmic Background Explorer (COBE) satellite.

Some scientists believe that this surface may allow us to reconstruct the global topology of our universe.[20] [21] If the LSS is large enough to entirely contain the fundamental polyhedron, then our tessellation of the covering space will imply that the infinitely many copies of the LSS, one for each polyhedron, will intersect some of its neighbours. Illustrated in **Figure 6** is just such an example: the LSS, centred on the observation point, i.e. the earth, intersects with another LSS centred on a copy of the earth in another fundamental polyhedron. Because the LSS are all spheres, the intersection points form a circle whose points we see twice, one from each side of the circle.

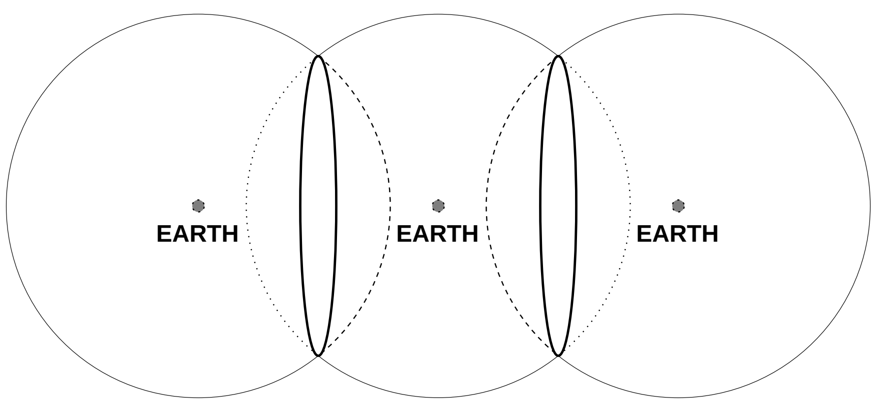

Figure 6: Self-intersection of the CMB particle horizon.

Thus if the LSS is large enough to cause intersections, the earthbound observers will see these common circles repeated throughout the sky, in fact at

[20] Circles in the Sky: Finding Topology with the Microwave Background Radiation by Neil J. Cornish, David N. Spergel and Glenn D. Starkman in *Classical and Quantum Gravity*, Vol. 25, No. 9, pages 2657-2670; September 1998. Preprint `astro-ph/9801212`.

[21] Reconstructing the global topology of the universe from the cosmic microwave background by Jeffrey R. Weeks in *Classical and Quantum Gravity*, Vol. 15, No. 9, pages 2599-2604; September 1998. Preprint `astro-ph/9802012`

antipodal positions. Moreover, these lines represent *precisely equal temperature distributions*, whose variations arise due to the low-level inhomogeneity of the CMB. Given precise enough data, and enough computational power, we may be able to statistically extract such distribution circles. The sizes of such would then give us a measure of the size of the fundamental polyhedron, and hence of our universe. Unfortunately, the data provided by COBE is not enough to extract such data, although much theoretical work has been done with the data to not only place constraints on the possible topologies of the universe, but also to test methods of topology detection. The COBE is not the end of research into the microwave sky. Underway are preparations for a year 2000 launch of its successor, the National Aeronautics and Space Administration's Microwave Anisotropy Probe (MAP), which will probe the background to a level of resolution almost thirty-fold higher than that of COBE. Following that will be the European Space Agency's PLANCK Surveyor mission in 2006, which is hoped to improve even on MAP's resolution.

The low resolution of the COBE data has not stopped research work involving it. Another approach by some theorists[22] is to compare the observed temperature distribution of the CMB to those predicted by computer models of different topological models of the universe, thus providing a more quantitative method of comparing possible topologies.

5 Results, Past and Future

So far, no positive evidence for non-trivial topology has been discovered. However, current research has put *lower bounds* on any dimensions of the fundamental polyhedron required of these topologies. Furthermore, more research is being planned to take advantage of higher quality data that is expected from a variety of sources. The Sloan Digital Sky Survey should provide a more comprehensive astrophysical object catalogue for statistical methods to be applied to. The next decade will see a number of other sky surveys come to fruition; those of MAP, PLANCK and the X-ray Multiple Mission (XMM) satellites. Theoretical work in cosmic topology and other subjects, most notably that of quantum cosmology, suggest increasingly that the underlying geometry of the universe is in fact hyperbolic, i.e. space has a negative curvature. Mathematically we know that most topologies require a hyperbolic geometry, while, on the observational side, astronomical data indicates that the average density of our observable universe[23] is less than the critical value needed for a zero or positive curvature.

[22] John D. Barrow and Janna H. Levin of the University of Sussex, Emory F. Bunn of Bates College and Evan Scannapieco and Joseph I. Silk of the University of California at Berkeley. Further research is also being carried out by J. Richard Bond, Dmitry Pogosyan and Tarun Souradeep of the Canadian Institute for Theoretical Astrophysics.

[23] We should be wary here. Matter alone is commonly accepted to be insufficient, in the quantities present, to obtain non-negative curvature, but other forms of energy may do so. A current candidate of intense interest is Einstein's cosmological constant. See "Cosmological Antigravity by Lawrence M. Krauss; *Scientific American*, January 1999"

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Preprints are available at `http://xxx.soton.ac.uk/abs` on the World Wide Web.

Other cosmological issues such as the determination of the **metric parameters**

$$\Omega + \lambda - 1 = \frac{kc^2}{H_0^2 R_0^2} \tag{1}$$

An independent method of determining the topology, and hence the curvature, of the universe would give us a way to *verify* any determined values of the other parameters