Stochastics: An International Journal of Probability and Stochastic Processes **82**(4) (365–380)

An iterative procedure for solving integral equations related to optimal stopping problems*

Denis Belomestny[†] Pavel V. Gapeev[‡]

We present an iterative algorithm for computing values of optimal stopping problems for one-dimensional diffusions on finite time intervals. The method is based on a time discretisation of the initial model and a construction of discretised analogues of the associated integral equation for the value function. The proposed iterative procedure converges in a finite number of steps and delivers in each step a lower or an upper bound for the discretised value function on the whole time interval. We also give remarks on applications of the method for solving the integral equations related to several optimal stopping problems.

1 Introduction

Optimal stopping problems on finite time intervals play an important role in the recent literature on stochastic control (see, e.g. Peskir and Shiryaev [22] for general theory). A special interest to

*This research was supported by Deutsche Forschungsgemeinschaft through the SFB 649 Economic Risk.

[†]Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Mohrenstr. 39, D-10117 Berlin, Germany, e-mail: belomest@wias-berlin.de.

[‡]London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom; e-mail: p.v.gapeev@lse.ac.uk

Mathematics Subject Classification 2000: Primary 65D15, 60G40, 91B28. Secondary 65D30, 60J60, 60J65.

Key words and phrases: Optimal stopping, finite horizon, diffusion process, upper and lower bounds, Black-Merton-Scholes model, American put option, Asian option, Russian option, sequential testing problem, disorder detection problem.

such problems was attracted due to the requests of rational valuation of early exercise American options arising in the mathematical theory of modern finance. The latter problem was first studied by McKean [15] who derived a free-boundary problem for the value function and the optimal exercise boundary and obtained a countable system of nonlinear integral equations for the latter. Kim [13], Jacka [11] and Carr, Jarrow and Myneni [4] (see also Myneni [16]) have independently derived a nonlinear integral equation for the exercise boundary of the American put option, which follows from the more general *early exercise premium* (EEP) representation. The uniqueness of solution of that integral equation has been recently proven by Peskir [20].

Since the value function and the stopping boundary of a general optimal stopping problem with finite time horizon cannot be found in an explicit form, some different numerical procedures for computing the value and the boundary have been proposed. Carr [3] presented a method based on the randomisation of the time horizon using the Erlang distribution, which is equivalent to taking the Laplace transform of the initial American put option value. In that case, the solution of the associated free-boundary problem can be derived in a closed form. Hou, Little and Pant [10] established a new representation for the American put option value and proposed an efficient numerical algorithm for solving the appropriate nonlinear integral equation for the optimal exercise boundary. Pedersen and Peskir [18] (see also [6]-[7]) used the backward induction method and a simple time discretisation of the nonlinear integral equation to obtain the optimal stopping boundary. Kolodko and Schoenmakers [14] presented a policy iteration method for computing the appropriate optimal Bermudan stopping time. At the same time, Rogers [25] and Haugh and Kogan [9] developed Monte Carlo methods for computing rational values of early exercised American options (see also Glasserman [8] for an overview). Another iterative Monte Carlo procedure was proposed in [2], which was based on the EEP representation for American and Bermudan options. The method developed in [2] can be considered as an analogue of the classical Picard iteration scheme, which is applied for the proof of existence of solutions of integral equations (cf., e.g. Tricomi [30]), having the advantage that it allows for to obtain upper bounds for the value function from lower ones and lower bounds from upper ones. In this paper, we propose a modification of that method, which is based on an extension of the EEP representation, where the fixed maturity is replaced by a stopping time at which the value of the associated Snell envelope process is equal to the payoff. For example, such a stopping time can be constructed as the first time at which the underlying process hits the stopping boundary from the appropriate perpetual optimal stopping problem (cf., e.g. Novikov and Shiryaev [17] and Shiryaev [29; Chapter VIII]). Moreover, we prove the convergence of the method and determine the rate of convergence.

The paper is organised as follows. In Section 2, we give the setting of a finite horizon optimal stopping problem for a one-dimensional diffusion process and formulate the associated free-boundary problem. In Section 3, we describe a method for obtaining upper and lower bounds for the value of the optimal stopping problem and derive an extension of the EEP representation. In Section 4, we construct a time discretisation of the initial integral equation for the value function and propose a numerical algorithm for solving the resulting discretised equations. Such an iterative procedure provides in each step a lower or an upper bound for the discretised value function and arrives at the latter in a finite number of steps. We stress that, in contrast to the simple backward induction, this procedure delivers in each step an approximation for the initial value function on the whole time interval and not for a certain time interval before the maturity only. In Section 5, we prove the uniform convergence of the algorithm to the initial value function as the discretisation becomes finer and determine the rate of convergence. In Section 6, we give some remarks on applications of the method to the rational valuation of early exercise American put and Asian options in the Black-Merton-Scholes model as well as to solving the finite horizon versions of sequential testing and disorder detection problems for Wiener processes. The main results of the paper are stated in Lemma 4 and Theorem 5.

2 Preliminaries

In this section, we recall results of general theory from [22], [12] and [18] (see also [15], [13], [11], [4] and [20]) related to optimal stopping problems in one-dimensional diffusion models with finite time horizon and formulate the associated free-boundary problem.

2.1. (Formulation of the problem.) For a precise formulation of the finite horizon optimal stopping problem for a diffusion process, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{0 \le t \le T}$ started at zero. Suppose that there exists a process

$X = (X_t)_{0 \le t \le T}$ solving the stochastic differential equation:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \quad (X_0 = x) \tag{2.1}$$

where $x \in E$ is a given number from the connected state space $E \subseteq \mathbb{R}$ of the process X. Here, the local drift $\mu(x)$ and the diffusion coefficient $\sigma(x) > 0$ for $x \in E$ are assumed to be Lipschitz continuous (cf., e.g. (2.1) in [12]).

Let us consider the problem of computing the value function:

$$V(t,x) = \sup_{0 \le \tau \le T-t} E_{t,x}\big[e^{-\lambda\tau}\, G(t+\tau, X_{t+\tau})\big] \tag{2.2}$$

where the supremum is taken over all stopping times τ of the process X (i.e. stopping times with respect to the natural filtration of X). Here, $E_{t,x}$ denotes the expectation with respect to the initial measure $P_{t,x}$ when the process $(X_{t+u})_{0 \le u \le T-t}$ starts at $X_t = x$, for some $x \in E$, and $\lambda > 0$ is a discounting factor.

Throughout the paper, we assume that the gain function $G(t,x)$ satisfies some regularity conditions (cf. [31], [12] and [18]) implying the existence of a strictly decreasing continuous function $b(t)$ such that at the first passage time:

$$\begin{aligned} \tau_b &= \inf\{0 \le u \le T-t \,|\, X_{t+u} \ge b(t+u)\} \\ &\equiv \inf\{0 \le u \le T-t \,|\, V(t+u, X_{t+u}) \le G(t+u, X_{t+u})\} \end{aligned} \tag{2.3}$$

(the infimum of an empty set being equal to $T-t$) the supremum in (2.2) is attained. Among the regularity conditions mentioned above, we refer the following:

$$(t,x) \mapsto G(t,x) \quad \text{is} \quad C^{1,2} \quad \text{on} \quad [0,T] \times E \tag{2.4}$$

$$x \mapsto H(t,x) \quad \text{is decreasing on} \quad E \quad \text{for each} \quad 0 \le t \le T \tag{2.5}$$

$$t \mapsto H(t,x) \quad \text{is decreasing on} \quad [0,T] \quad \text{for each} \quad x \in E \tag{2.6}$$

where

$$H(t,x) = (G_t + \mu(x) G_x + (\sigma^2(x)/2) G_{xx} - \lambda G)(t,x) \tag{2.7}$$

for all $0 \le t \le T$ and $x \in E$ (cf. Theorem 4.3, Propositions 4.4 and 4.5 in [12]). Note that the problem (2.2) turns out to be non-trivial if there exists a continuous function $a(t)$ such that:

$$H(t,x) > 0 \quad \text{for} \quad x \in E \quad \text{such that} \quad x < a(t) \tag{2.8}$$

$$H(t,x) = 0 \quad \text{for} \quad x \in E \quad \text{such that} \quad x = a(t) \tag{2.9}$$

$$H(t,x) < 0 \quad \text{for} \quad x \in E \quad \text{such that} \quad x > a(t) \tag{2.10}$$

for any $0 \le t \le T$. The application of Itô's formula directly implies that $a(t) < b(t)$ for all $0 < t < T$. In the sequel, we assume that conditions (2.4)-(2.6) and (2.8)-(2.10) are satisfied. Further conditions on the functions $G(t,x)$ and $H(t,x)$ will be imposed below.

2.2. (Free-boundary problem.) By virtue of the regularity conditions in (2.4)-(2.6) and (2.8)-(2.10) and using the strong Markov property of the process X, we are thus naturally led to formulate the following *free-boundary problem* for the unknown value function $V(t,x)$ from (2.2) and the boundary $b(t)$ from (2.3) (cf. [12] and [18]):

$$(V_t + \mu(x)V_x + (\sigma^2(x)/2)V_{xx})(t,x) = \lambda V(t,x) \quad \text{for} \quad x \in E \quad \text{such that} \quad x < b(t) \tag{2.11}$$

$$V(t,x)\big|_{x=b(t)} = G(t,x)\big|_{x=b(t)} \quad \text{(instantaneous stopping)} \tag{2.12}$$

$$V_x(t,x)\big|_{x=b(t)} = G_x(t,x)\big|_{x=b(t)} \quad \text{(smooth fit)} \tag{2.13}$$

$$V(t,x) > G(t,x) \quad \text{for} \quad x \in E \quad \text{such that} \;\; x < b(t) \tag{2.14}$$

$$V(t,x) = G(t,x) \quad \text{for} \quad x \in E \quad \text{such that} \quad x > b(t) \tag{2.15}$$

$$(V_t + \mu(x)V_x + (\sigma^2(x)/2)V_{xx})(t,x) < \lambda V(t,x) \quad \text{for} \quad x \in E \quad \text{such that} \quad x > b(t) \tag{2.16}$$

where the conditions in (2.12) and (2.13) are satisfied for all $0 \le t < T$. Note that the superharmonic characterization of the value function (see [5], [28] and [22; Chapter IV, Section 9]) implies that $V(t,x)$ from (2.2) is the smallest functions satisfying the conditions in (2.11)-(2.12) and (2.14)-(2.15).

2.3. (Early exercise premium representation.) Taking into account the condition in (2.13), we may apply Itô's formula to the function $e^{-\lambda s}V(t+s, X_{t+s})$ and obtain:

$$\begin{aligned} e^{-\lambda s} V(t+s, X_{t+s}) &= V(t,x) \\ &+ \int_0^s e^{-\lambda u} \Big(V_t + \mu(X_{t+u})\, V_x + (\sigma^2(X_{t+u})/2)\, V_{xx} - \lambda V \Big)(t+u, X_{t+u})\, du + M_s \end{aligned} \tag{2.17}$$

where $M_s = \int_0^s e^{-\lambda u} V_x(t+u, X_{t+u}) dB_u$, for $0 \le s \le T-t$, is a continuous (local) martingale. It follows from the regularity conditions imposed above that the time spent by the process X at the boundary $b(t)$ is of Lebesgue measure zero, so that, the value $V_{xx}(t,x)$ can be set arbitrarily to $(t+u, b(t+u))$, for each $0 \le u \le T-t$. Hence, applying the formulas from (2.11) and (2.15) to the expression in (2.17), we get:

$$e^{-\lambda s} V(t+s, X_{t+s}) = V(t,x) + \int_0^s e^{-\lambda u} H(t+u, X_{t+u}) I(X_{t+u} \ge b(t+u)) \, du + M_s \tag{2.18}$$

where $I(\,\cdot\,)$ denotes the indicator function. Taking expectation from both sides of the expression in (2.18) with respect to the measure $P_{t,x}$, we obtain:

$$\begin{aligned} V(t,x) &= e^{-\lambda(T-t)} E_{t,x}\big[G(T, X_T)\big] - \int_0^{T-t} e^{-\lambda u} E_{t,x}\big[H(t+u, X_{t+u}) I(X_{t+u} \ge b(t+u))\big] \, du \\ &\equiv e^{-\lambda(T-t)} E_{t,x}\big[G(T, X_T)\big] \\ &\quad - \int_0^{T-t} e^{-\lambda u} E_{t,x}\big[H(t+u, X_{t+u}) I(V(t+u, X_{t+u}) \le G(t+u, X_{t+u}))\big] \, du \end{aligned} \tag{2.19}$$

for all $0 \le t \le T$ and $x \in E$. The expression in (2.19) is a general form of the *early exercise premium* representation for the value function of (2.2) derived in [13], [11] and [4] (see also [16]). Setting $x = b(t)$, it follows immediately from (2.19) that the stopping boundary $b(t)$ solves the nonlinear integral equation:

$$G(t, b(t)) = e^{-\lambda(T-t)} E_{t,b(t)}\big[G(T, X_T)\big] - \int_0^{T-t} e^{-\lambda u} E_{t,b(t)}\big[H(t+u, X_{t+u}) I(X_{t+u} \ge b(t+u))\big] \, du \tag{2.20}$$

for all $0 \le t \le T$ and $x \in E$ (see also [18], [13] and [11]-[12]). Applying the change-of-variable formula with local times on curves from [19], it was proven in [18] (see also [20]-[23], [6]-[7] and [22]) that the equation of (2.20) admits a unique solution. Note that the nonlinear integral equation in (2.19) is preferable over the equation in (2.20), which involves the optimal stopping boundary, since it allows a direct generalisation to the case of a multidimensional process X. In a general case, the equations in (2.19) and (2.20) cannot be solved in an explicit form, so that, numerical methods are required.

2.4. (Infinite horizon case.) Let us denote by $\overline{V}(t,x)$ and $\overline{b}(t)$ the value function and the stopping boundary of the infinite horizon optimal stopping problem associated with one of (2.2)-(2.3), by letting $T = \infty$. In the sequel, we will consider only the optimal stopping problems such that $\overline{V}(t,x) = \overline{V}(x)$ and $\overline{b}(t) = \overline{b}$ holds for all $0 \le t \le T$ and $x \in E$. Moreover, we will assume that the limit:

$$\overline{G}(x) = \lim_{T \to \infty} e^{-\lambda(T-t)} \, E_{t,x}\big[G(T, X_T)\big] \quad \text{exists and is finite.} \tag{2.21}$$

Then, letting T tend to infinity in (2.19) and (2.20) formally, we obtain:

$$\begin{aligned} \overline{V}(x) &= \overline{G}(x) - \int_0^\infty e^{-\lambda u} \, E_{t,x}\big[H(t+u, X_{t+u}) \, I(X_{t+u} \ge \overline{b})\big] \, du \\ &= \overline{G}(x) - \int_0^\infty e^{-\lambda u} \, E_{t,x}\big[H(t+u, X_{t+u}) \, I(\overline{V}(X_{t+u}) \le G(t+u, X_{t+u}))\big] \, du \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} G(t,\overline{b}) &= \overline{G}(\overline{b}) - \int_0^\infty e^{-\lambda u} \, E_{t,\overline{b}}\big[H(t+u, X_{t+u}) \, I(X_{t+u} \ge \overline{b})\big] \, du \\ &= \overline{G}(\overline{b}) - \int_0^\infty e^{-\lambda u} \, E_{t,\overline{b}}\big[H(t+u, X_{t+u}) \, I(\overline{V}(X_{t+u}) \le G(t+u, X_{t+u}))\big] \, du \end{aligned} \tag{2.23}$$

for all $0 \le t \le T$ and $x \in E$, where the function $\overline{V}(x)$ and the number $\overline{b}$ are uniquely determined by the equations in (2.22) and (2.23), respectively.

It follows from the formulas in (2.19) and (2.22) that:

$$\begin{aligned} V(t,x) &= \widetilde{V}(t,x) - \int_0^{T-t} e^{-\lambda u} \, E_{t,x}\big[H(t+u, X_{t+u}) \, I(b(t+u) \ge X_{t+u} > \overline{b})\big] \, du \\ &= \widetilde{V}(t,x) \\ &\quad - \int_0^{T-t} e^{-\lambda u} \, E_{t,x}\big[H(t+u, X_{t+u}) \, I(V(t+u, X_{t+u}) \le G(t+u, X_{t+u}) < \overline{V}(X_{t+u}))\big] \, du \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} \widetilde{V}(t,x) &= \overline{V}(x) + e^{-\lambda(T-t)} \, E_{t,x}\big[G(T, X_T)\big] - \overline{G}(x) \\ &\qquad + \int_{T-t}^\infty e^{-\lambda u} \, E_{t,x}\big[H(t+u, X_{t+u}) \, I(\overline{V}(X_{t+u}) \le G(t+u, X_{t+u}))\big] \, du \end{aligned} \tag{2.25}$$

for all $0 \le t \le T$ and $x \in E$. Note that the representation in (2.24) has an advantage over (2.19), since it contains integration of probabilities of events when X_t belongs to bounded intervals which are numerically easier to compute by using Monte Carlo methods than those ones for unbounded intervals.

3 Description of the method

In this section, we describe the procedure of obtaining an upper bound for the value function from a lower one and a lower bound from an upper one and present a generalisation of the early exercise premium representation.

3.1. Let σ be a stopping time for the process X such that:

$$V(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)}) = G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)}) \quad (P_{t,x}\text{-a.s.}) \tag{3.1}$$

For example, let $V'(t,x)$ be an upper bound for $V(t,x)$, that is, $V'(t,x) \geq V(t,x)$ for all $0 \leq t \leq T$ and $x \in E$. Then, by choosing $\sigma = \inf\{0 \leq u \leq T-t \,|\, V'(t+u, X_{t+u}) \leq G(t+u, X_{t+u})\}$, we obviously get that (3.1) is satisfied. Hence, applying the conditions from (2.12) and (2.15) to the expression in (2.18) and using the optional sampling theorem (see, e.g. [24; Chapter II, Theorem 3.2]), we obtain:

$$\begin{aligned} V(t,x) &= E_{t,x}\left[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)})\right] \\ &\quad - E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(X_{t+u} \geq b(t+u))\, du\right] \\ &\equiv E_{t,x}\left[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)})\right] \\ &\quad - E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(V(t+u, X_{t+u}) \leq G(t+u, X_{t+u}))\, du\right] \end{aligned} \tag{3.2}$$

for all $0 \leq t \leq T$ and $x \in E$. The expression in (3.2) is an extension of the early exercise premium representation from (2.19).

3.2. Let $V_l(t,x)$ be a lower bound and $V_u(t,x)$ be an upper bound for the value function $V(t,x)$, that is, $V_l(t,x) \leq V(t,x) \leq V_u(t,x)$ for all $0 \leq t \leq T$ and $x \in E$. For any stopping time σ such that the equation in (3.1) is satisfied, we can insert $V_l(t,x)$ and $V_u(t,x)$ instead of $V(t,x)$ into the right-hand side of the expression in (3.2) and define the functions $V_u'(t,x;\sigma)$ and $V_l'(t,x;\sigma)$ by:

$$\begin{aligned} V_u'(t,x;\sigma) &= E_{t,x}\left[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)})\right] \\ &- E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(V_l(t+u, X_{t+u}) \leq G(t+u, X_{t+u}))\, du\right] \end{aligned} \tag{3.3}$$

and

$$V_l'(t,x;\sigma) = E_{t,x}\left[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)})\right] \tag{3.4}$$
$$- E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(V_u(t+u, X_{t+u}) \le G(t+u, X_{t+u}))\, du\right].$$

Due to the assumptions in (2.8)-(2.10), the functions in (3.3) and (3.4) are an upper and a lower bound for $V(t,x)$, respectively, that is, $V_l'(t,x;\sigma) \le V(t,x) \le V_u'(t,x;\sigma)$ for all $0 \le t \le T$ and $x \in E$. For example, we can take $V_l(t,x) = G(t,x)$ or $V_l(t,x) = e^{-\lambda(T-t)} E_{t,x}[G(T, X_T)]$ as a lower bound and $V_u(t,x) = \sup_{\tau \ge 0} E_{t,x}[e^{-\lambda\tau} G(t+\tau, X_{t+\tau})]$ as an upper bound for the value function $V(t,x)$. Note that if there is a sequence of stopping times $\sigma_1, \ldots, \sigma_n$, satisfying (3.1), then one can consider the improved lower and upper bounds:

$$\min_{1\le k\le n} V_u'(t,x;\sigma_k) \quad \text{and} \quad \max_{1\le k\le n} V_l'(t,x;\sigma_k) \tag{3.5}$$

for all $0 \le t \le T$, $x \in E$, and some $n \in \mathbb{N}$ fixed.

3.3. Let $V^0(t,x)$ be a lower bound for $V(t,x)$ and σ be a stopping time satisfying (3.1). Let us define the function $V^1(t,x;\sigma)$ by the formula:

$$V^1(t,x;\sigma) = E_{t,x}\left[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)})\right] \tag{3.6}$$
$$- E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(V^0(t+u, X_{t+u}) \le G(t+u, X_{t+u}))\, du\right]$$

and the function $V^2(t,x;\sigma)$ by the formula:

$$V^2(t,x;\sigma) = E_{t,x}\left[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)})\right] \tag{3.7}$$
$$- E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(V^1(t+u, X_{t+u};\sigma) \le G(t+u, X_{t+u}))\, du\right]$$

for all $0 \le t \le T$, $x \in E$, such that $V^2(t,x;\sigma) \ge V^0(t,x)$, and $V^2(t,x;\sigma) = V^0(t,x)$ elsewhere. Let us now define sequentially the functions $V^m(t,x;\sigma)$, for every $m \in \mathbb{N}$, $m \ge 3$, by the formula:

$$V^m(t,x;\sigma) = E_{t,x}\left[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{t+\sigma\wedge(T-t)})\right] \tag{3.8}$$
$$- E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(V^{m-1}(t+u, X_{t+u};\sigma) \le G(t+u, X_{t+u}))\, du\right]$$

for all $0 \le t \le T$ and $x \in E$. Note that if the solution of the corresponding infinite horizon problem exists then the functions $V^m(t,x;\sigma)$ may be defined by the formula:

$$V^m(t,x;\sigma) = \widetilde{V}(t,x;\sigma) \tag{3.9}$$
$$- E_{t,x}\left[\int_0^{\sigma\wedge(T-t)} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(V^{m-1}(t+u, X_{t+u};\sigma) \le G(t+u, X_{t+u}) < \overline{V}(X_{t+u}))\, du\right]$$

where

$$\widetilde{V}(t,x;\sigma) = \overline{V}(x) + E_{t,x}\big[e^{-\lambda(\sigma\wedge(T-t))}\, G(t+\sigma\wedge(T-t), X_{\sigma\wedge(T-t)})\big] - \overline{G}(x) \tag{3.10}$$
$$+ E_{t,x}\left[\int_{\sigma\wedge(T-t)}^{\infty} e^{-\lambda u}\, H(t+u, X_{t+u})\, I(\overline{V}(X_{t+u}) \le G(t+u, X_{t+u}))\, du\right]$$

for all $0 \le t \le T$ and $x \in E$. The definition of $V^m(t,x;\sigma)$ as in (3.10) is more convenient for the use of Monte Carlo simulations than that in the formulas of (3.6) and (3.7)-(3.8) above.

Remark 1 *Observe that, by construction, we have:*

$$V^{2k-1}(t,x;\sigma) \ge V(t,x) \quad \textit{for} \quad 0 \le t \le T, \quad x \in E, \quad k \in \mathbb{N} \tag{3.11}$$

where the sequence $(V^{2k-1}(t,x;\sigma))_{k\in\mathbb{N}}$ *is monotone decreasing, and*

$$V^{2k}(t,x;\sigma) \le V(t,x) \quad \textit{for} \quad 0 \le t \le T, \quad x \in E, \quad k \in \mathbb{N} \tag{3.12}$$

where the sequence $(V^{2k}(t,x;\sigma))_{k\in\mathbb{N}}$ *is monotone increasing for each* $0 \le t \le T$, $x \in E$, *and every* $n \in \mathbb{N}$ *fixed. Moreover, any lower estimate* $V^{2k-2}(t,x;\sigma)$ *for* $V(t,x)$ *produces the upper one* $V^{2k-1}(t,x;\sigma)$, *and any upper estimate* $V^{2k-1}(t,x;\sigma)$ *produces the lower one* $V^{2k}(t,x;\sigma)$, *for each* $0 \le t \le T$, $x \in E$, *and every* $k \in \mathbb{N}$. *In the next section, we will consider the question of convergence of the sequence* $(V^m(t,x;\sigma))_{m\in\mathbb{N}}$ *to the value function* $V(t,x)$, *for a discretised version of* (2.19), *where we set* $\sigma = T - t$ *for simplicity of exposition.*

4 Discretisation and algorithm

In this section, we construct an approximation of the initial model based on the discretisation of the integral equation in (2.19). We propose an iterative procedure which solves the resulting discretised integral equation in a finite number of steps.

4.1. In order to construct a time-discretised analogue of the equation in (2.19), let us fix some arbitrary $0 \le t \le T$ and $n \in \mathbb{N}$ and introduce a partition of the time interval $[0, T-t]$. Let us set $u_0 = 0$ and $u_i = i\Delta_n$ with $\Delta_n = (T-t)/n$ implying that $u_i - u_{i-1} = \Delta_n$, for every $i = 1, \ldots, n$. Taking into account the structure of the expression (2.19), let us define the approximation $\widehat{V}_n(t+u, x)$ for the price $V(t+u, x)$ as a solution of the equation:

$$\begin{aligned}
\widehat{V}_n(t+u,x) &= e^{-\lambda(T-t-u)}\, E_{t+u,x}\big[G(T,X_T)\big] \\
&\quad - \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-\lambda u_i}\, E_{t+u,x}\big[H(t+u_i, X_{t+u_i})\, I(\widehat{b}_n(t+u_i) \ge X_{t+u_i})\big]\,\Delta_n \\
&\equiv e^{-\lambda(T-t-u)}\, E_{t+u,x}\big[G(T,X_T)\big] \\
&\quad - \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-\lambda u_i}\, E_{t+u,x}\big[H(t+u_i, X_{t+u_i})\, I(\widehat{V}_n(t+u_i, X_{t+u_i}) \le G(t+u_i, X_{t+u_i}))\big]\,\Delta_n
\end{aligned} \tag{4.1}$$

where the estimate $\widehat{b}_n(t+u)$ for the boundary $b(t+u)$ is defined as the intersection curve of the functions $\widehat{V}_n(t+u,x)$ and $G(t+u,x)$. Here, $\lceil z \rceil \equiv \inf\{k \in \mathbb{N} \,|\, k \ge z\}$ denotes the upper integer part of a given positive number $z > 0$. It is clear that the equation in (4.1) has a unique solution which can be obtained by means of backward induction in a finite number of steps. This implies that the (piecewise constant) function $\widehat{V}_n(t+u,x)$ is uniquely determined by the formula in (4.1), for all $0 \le u \le T-t$ and $x \in E$. Let us set $\widehat{V}_n^0(t+u,x) = G(t+u,x)$ and define the function $\widehat{V}_n^1(t+u,x)$ by the formula:

$$\begin{aligned}
\widehat{V}_n^1(t+u,x) &= e^{-\lambda(T-t-u)}\, E_{t+u,x}\big[G(T,X_T)\big] \\
&\quad - \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-\lambda u_i}\, E_{t+u,x}\big[H(t+u_i, X_{t+u_i})\, I(\widehat{V}_n^0(t+u_i, X_{t+u_i}) \le G(t+u_i, X_{t+u_i}))\big]\,\Delta_n
\end{aligned} \tag{4.2}$$

and the function $\widehat{V}_n^2(t+u,x)$ by the formula:

$$\begin{aligned}
\widehat{V}_n^2(t+u,x) &= e^{-\lambda(T-t-u)}\, E_{t+u,x}\big[G(T,X_T)\big] \\
&\quad - \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-\lambda u_i}\, E_{t+u,x}\big[H(t+u_i, X_{t+u_i})\, I(\widehat{V}_n^1(t+u_i, X_{t+u_i}) \le G(t+u_i, X_{t+u_i}))\big]\,\Delta_n
\end{aligned} \tag{4.3}$$

for all $0 \le u \le T-t$, $x \in E$, such that $\widehat{V}_n^2(t+u,x) \ge \widehat{V}_n^0(t+u,x)$, and $\widehat{V}_n^2(t+u,x) = \widehat{V}_n^0(t+u,x)$ elsewhere. Let us now define sequentially the functions $\widehat{V}_n^m(t+u,x)$, for every $m \in \mathbb{N}$, $m \ge 3$, by the formula:

$$\widehat{V}_n^m(t+u,x) = e^{-\lambda(T-t-u)}\, E_{t+u,x}\big[G(T,X_T)\big] - \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-\lambda u_i}\, E_{t+u,x}\big[H(t+u_i,X_{t+u_i}) \tag{4.4}$$

$$\times\, I(\widehat{V}_n^{m-1}(t+u_i,X_{t+u_i}) \le G(t+u_i,X_{t+u_i}) < \overline{V}(X_{t+u_i})\big]\,\Delta_n$$

for all $0 \le u \le T-t$ and $x \in E$.

Remark 2 *It is easily seen from (4.1) that, by the construction in (4.2)-(4.4), we have:*

$$\widehat{V}_n^{2k-1}(t+u,x) \ge \widehat{V}_n(t+u,x) \quad \textit{for} \quad 0 \le u \le T-t, \quad x \in E, \quad k \in \mathbb{N} \tag{4.5}$$

where the sequence $(\widehat{V}_n^{2k-1}(t+u,x))_{k\in\mathbb{N}}$ *is monotone decreasing, and*

$$\widehat{V}_n^{2k}(t+u,x) \le \widehat{V}_n(t+u,x) \quad \textit{for} \quad 0 \le u \le T-t, \quad x \in E, \quad k \in \mathbb{N} \tag{4.6}$$

where the sequence $(\widehat{V}_n^{2k}(t+u,x))_{k\in\mathbb{N}}$ *is monotone increasing, for each* $0 \le u \le T-t$, $x \in E$, *and every* $n \in \mathbb{N}$ *fixed. Moreover, any lower bound* $\widehat{V}_n^{2k-2}(t+u,x)$ *for* $\widehat{V}_n(t+u,x)$ *produces the upper one* $\widehat{V}_n^{2k-1}(t+u,x)$, *and any upper bound* $\widehat{V}_n^{2k-1}(t+u,x)$ *produces the lower one* $\widehat{V}_n^{2k}(t+u,x)$, *for each* $0 \le u \le T-t$, $x \in E$, *and every* $k \in \mathbb{N}$.

Remark 3 *For every* $m < n$, *the function* $u \mapsto \widehat{V}_n^m(t+u,x)$ *is an estimate for* $\widehat{V}_n(t+u,x)$ *on the whole interval* $[0, T-t]$, *for each* $0 \le t \le T$ *and* $x \in E$ *fixed. This fact shows the advantage of the proposed method over the standard backward induction.*

4.2. Let us now show that the sequence of functions $(\widehat{V}_n^m(t+u,x))_{k\in\mathbb{N}}$ from (4.2)-(4.4) converges to the function $\widehat{V}_n(t+u,x)$ in n steps, for all $0 \le u \le T-t$, $x \in E$, and every $n \in \mathbb{N}$.

Lemma 4 *For each* $0 \le t \le T$ *fixed, we have* $\widehat{V}_n^m(t+u,x) = \widehat{V}_n(t+u,x)$, *for all* $0 \le u \le T-t$ *and* $x \in E$, *and for every* $m \ge n$.

Proof. Let us fix some $0 \le t \le T$ and $n \in \mathbb{N}$. Then, by construction of $\widehat{V}_n^m(t+u,x)$ in (4.2)-(4.4), the equalities:

$$\widehat{V}_n^{2k+1}(t+u,x) - \widehat{V}_n^{2k}(t+u,x) = -\sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-\lambda u_i}\, E_{t+u,x}\big[H(t+u_i,X_{t+u_i})$$

$$\times\, I(\widehat{V}_n^{2k}(t+u_i,X_{t+u_i}) \le G(t+u_i,X_{t+u_i}) < \widehat{V}_n^{2k-1}(t+u_i,X_{t+u_i}))\big]\,\Delta_n \tag{4.7}$$

and

$$\widehat{V}_n^{2k+2}(t+u,x) - \widehat{V}_n^{2k+1}(t+u,x) = - \sum_{i=\lceil un/(T-t) \rceil}^{n} e^{-\lambda u_i} \, E_{t+u,x}\big[H(t+u_i, X_{t+u_i})$$

$$\times \, I(\widehat{V}_n^{2k}(t+u_i, X_{t+u_i}) \leq G(t+u_i, X_{t+u_i}) < \widehat{V}_n^{2k+1}(t+u_i, X_{t+u_i}))\big] \, \Delta_n \tag{4.8}$$

are satisfied for all $0 \leq u \leq T-t$ and $x \in E$, and for every $k \in \mathbb{N}$.

In order to prove the desired assertion, we should use the mathematical induction principle. First, we note that $\widehat{V}_n^m(T,x) = G(T,x)$ holds for all $x \in E$ and $m \in \mathbb{N}$. For checking the induction basis, it is enough to observe that if $m = 2k$ with $k = 0$ then the expression in (4.8) implies the equality:

$$\widehat{V}_n^2(t+u,x) - \widehat{V}_n^1(t+u,x) = -e^{-\lambda u_n} \, E_{t+u,x}\big[H(t+u_n, X_{t+u_n}) \tag{4.9}$$

$$\times \, I(\widehat{V}_n^0(t+u_n, X_{t+u_n}) \leq G(t+u_n, X_{t+u_n}) < \widehat{V}_n^1(t+u_n, X_{t+u_n}))\big] \, \Delta_n = 0$$

which is satisfied for all $(n-1)(T-t)/n \leq u \leq T-t$, where we have $t + u_n = T$, by definition of the partition. ■

5 Convergence of the algorithm

We now prove that the solution of the discretised equation in (4.1) converges to $V(t+u,x)$ uniformly on $[0, T-t]$ as n tends to infinity. For this, let us further denote:

$$F(t,x;t+u,y) = E_{t,x}\big[H(t+u, X_{t+u}) \, I(X_{t+u} \geq y)\big] \tag{5.1}$$

for all $0 \leq u \leq T-t$ and $x, y \in E$.

Theorem 5 *Suppose that the conditions in (2.4)-(2.6) and (2.8)-(2.10) are satisfied. Assume that the function:*

$$x \mapsto G(t,x) \quad \textit{is monotone and convex on} \quad E \quad \textit{with} \quad |G_x(t,x)| \geq \varepsilon \tag{5.2}$$

for some $\varepsilon > 0$ and the function:

$$y \mapsto F(t,x;t+u,y) \quad \textit{is} \quad C^1 \quad \textit{on} \quad E \tag{5.3}$$

and

$$\big|F_y(t,x;t+u,y)\big| \le \frac{C}{\sqrt{u}} \tag{5.4}$$

holds for all $0 \le t \le T$, $0 < u \le T-t$, *and* $x, y \in E$, *and some* $C > 0$ *fixed. Let* $\widehat{V}_n(t+u,x)$ *be a solution of the discretised equation in (4.1). Then, there exists some* $t \in [0,T]$ *close enough to* T *and such that the sequence* $(\widehat{V}_n(t+u,x))_{n\in\mathbb{N}}$ *converges to* $V(t+u,x)$ *uniformly, for* $0 \le u \le T-t$ *and* $x \in E$, *with the rate* $1/n$ *when* n *tends to infinity.*

Proof. First, we observe that the representations in (2.19) and (4.1) imply:

$$\begin{aligned}
&\Big|\widehat{V}_n(t,x) - V(t,x)\Big| \qquad (5.5)\\
&\le \left|\int_0^{T-t} e^{-\lambda u}\, F(t,x;t+u,b(t+u))\, du - \sum_{i=1}^{n} e^{-\lambda u_i}\, F(t,x;t+u_i,b(t+u_i))\,\Delta_n\right| \\
&\quad + \sum_{i=1}^{n} e^{-\lambda u_i}\, \Big|F(t,x;t+u_i,b(t+u_i)) - F(t,x;t+u_i,\widehat{b}(t+u_i))\Big|\,\Delta_n
\end{aligned}$$

for all $0 \le t \le T$ and $x \in E$. In order to deal with the first term on the right-hand side of the inequality in (5.5), we can use a standard estimate for the Riemann sum approximation and obtain:

$$\left|\int_0^{T-t} e^{-\lambda u}\, F(t,x;t+u,b(t+u))\, du - \sum_{i=1}^{n} e^{-\lambda u_i}\, F(t,x;t+u_i,b(t+u_i))\,\Delta_n\right| \le \frac{C_1}{n} \tag{5.6}$$

for all $n \ge N$ and some $C_1 > 0$ fixed. As to the second term in (5.5), we can make use of the mean value theorem and the inequality (5.4) to get:

$$\begin{aligned}
&\Big|F(t,x;t+u_i,b(t+u_i)) - F(t,x;t+u_i,\widehat{b}_n(t+u_i))\Big| \qquad (5.7)\\
&= \big|F_y(t,x;t+u_i,\xi_i)\big|\,\Big|\widehat{b}_n(t+u_i) - b(t+u_i)\Big| \le \frac{C}{\sqrt{u_i}}\,\Big|\widehat{b}_n(t+u_i) - b(t+u_i)\Big|
\end{aligned}$$

for some $\xi_i \in E$ and every $i = 1, \ldots, n$. From the assumption of (2.4), it follows by the mean value theorem that:

$$\Big|G(t+u_i,\widehat{b}(t+u_i)) - G(t+u_i,b(t+u_i))\Big| = \big|G_x(t+u_i,\eta_i)\big|\,\Big|\widehat{b}_n(t+u_i) - b(t+u_i)\Big| \tag{5.8}$$

for some $\eta_i \in E$ and every $i = 1, \ldots, n$. Then, using the expression in (5.8) and taking into account the assumption of (5.2), it follows from (5.7) that:

$$\begin{aligned}
&\Big|F(t,x;t+u_i,b(t+u_i)) - F(t,x;t+u_i,\widehat{b}(t+u_i))\Big| \\
&\le \frac{C}{\varepsilon\sqrt{u_i}}\Big|G(t+u_i,\widehat{b}_n(t+u_i)) - G(t+u_i,b(t+u_i))\Big| \\
&= \frac{C}{\varepsilon\sqrt{u_i}}\Big|\widehat{V}_n(t+u_i,\widehat{b}_n(t+u_i)) - V(t+u_i,b(t+u_i))\Big| \\
&\le \frac{C}{\varepsilon\sqrt{u_i}}\Big|\widehat{V}_n(t+u_i,x_i) - V(t+u_i,x_i)\Big|
\end{aligned} \tag{5.9}$$

for some $x_i \in E$ such that $x_i \in \big(\widehat{b}_n(t) \wedge b(t), \widehat{b}_n(t) \vee b(t)\big)$. Hence, combining the inequalities of (5.5)-(5.9), we get:

$$\Big|\widehat{V}_n(t+u_i,x_i) - V(t+u_i,x_i)\Big| \le \sup_{0 \le u_i \le T-t} \sup_{x_i \in E} \Big|\widehat{V}_n(t+u_i,x_i) - V(t+u_i,x_i)\Big| \tag{5.10}$$

for all $0 \le t \le T$ and every $i = 1, \ldots, n$. By virtue of the fact that the function $e^{-\lambda u}/\sqrt{u}$ is decreasing, straightforward calculations show that the inequalities:

$$\sum_{i=1}^{n} e^{-\lambda u_i} \frac{C}{\varepsilon\sqrt{u_i}} \Delta_n \le \frac{C}{\varepsilon} \int_0^{T-t} \frac{e^{-\lambda u}}{\sqrt{u}} \, du \le C_2 \sqrt{T-t} \tag{5.11}$$

hold for all $0 \le t \le T$ and some $C_2 > 0$ fixed. Therefore, combining the inequalities of (5.6)-(5.11), we obtain from (5.5) that:

$$\Big|\widehat{V}_n(t,x) - V(t,x)\Big| \le \frac{C_1}{n} + C_2\sqrt{T-t} \sup_{0 \le u_i \le T-t} \sup_{x_i \in E} \Big|\widehat{V}_n(t+u_i,x_i) - V(t+u_i,x_i)\Big| \tag{5.12}$$

for all $0 \le t \le T$ and $x \in E$. Hence, we have:

$$\begin{aligned}
&\sup_{0 \le u \le T-t} \sup_{x \in E} \Big|\widehat{V}_n(t+u,x) - V(t+u,x)\Big| \\
&\qquad \le \frac{C_1}{n} + C_2\sqrt{T-t} \sup_{0 \le u \le T-t} \sup_{x \in E} \Big|\widehat{V}_n(t+u,x) - V(t+u,x)\Big|
\end{aligned} \tag{5.13}$$

for all $0 \le t \le T$ and $x \in E$.

Let us finally choose some $t \in [0,T]$ such that $C_2\sqrt{T-t} \le 1/2$. It thus follows from (5.13) that:

$$\sup_{0 \le u \le T-t} \sup_{x \in E} \Big|\widehat{V}_n(t+u,x) - V(t+u,x)\Big| \le \frac{2C_1}{n} \tag{5.14}$$

for all $n \in \mathbb{N}$ such that $n \ge N$. This completes the proof of the theorem. ■

6 Examples

In this section, we give some remarks on the application of the iterative procedure introduced above to solving nonlinear integral equations arising from some optimal stopping problems with finite time horizon.

Example 6 *(Early exercise American put option [15], [20], [22; Chapter VII, Section 25].) Suppose that in (2.2) we have* $G(t,x) = (K-x)^+$ *and* $\lambda = r$, *for some* $K, r > 0$ *fixed. Assume that in (2.1) we have* $\mu(x) = rx$, $\sigma(x) = \theta x$, *for* $x \in E = (0, \infty)$ *and some* $\theta > 0$, *and hence,* $H(t,x) = -rK$ *in (2.7). In this case, as an analogue of the formula in (5.1), we have:*

$$F(0,x;t,y) = -rK\, P_{0,x}\big[X_t \le y\big] \tag{6.1}$$
$$= -rK\, \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\log\frac{y}{x} - \Big(r - \frac{\theta^2}{2}\Big)t\right)\right)$$

for all $t > 0$ *and* $x, y > 0$, *where* $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-y^2/2} dy$. *Thus, the conditions in (5.2)-(5.4) as well as the other essential assumptions of Theorem 5 are satisfied for almost all* $x > 0$. *We may also take* σ *as a first hitting of some curve which is strictly above the stopping boundary* b, *for example,* $h(t) = K - \beta K\sqrt{(T-t)|\log(T-t)|}$, *for some* $\beta > 0$ *(see [1]), and consider this bound as a lower for the price of an American option.*

Example 7 *(Early exercise Asian option [23], [22; Chapter VII, Section 27].) Suppose that in (2.2) we have* $G(t,x) = (1 - x/t)^+$ *and* $\lambda = 0$. *Assume that in (2.1) we have* $\mu(x) = (1 - rx)$, $\sigma(x) = \theta x$, *for all* $x \in E = (0,\infty)$ *and some* $r, \theta > 0$, *and hence,* $H(t,x) = ((1/t + r)x - 1)/t$ *in (2.7). In this case, as an analogue of the formula in (5.1), we have:*

$$F(0,x;t,y) = E_{0,x}\big[H(t,x)\, I(X_t \le y)\big] \tag{6.2}$$
$$= \int_0^\infty \int_0^\infty \frac{1}{t}\left(\Big(\frac{1}{t} + r\Big)\frac{x+a}{s} - 1\right) I\Big(\frac{x+a}{s} \le y\Big)\, f(t,s,a)\, ds\, da$$

for all $t > 0$ *and* $x, y > 0$, *where*

$$f(t,s,a) = \frac{2\sqrt{2}}{\pi^{3/2}\theta^3}\,\frac{s^{r/\theta^2}}{a^2\sqrt{t}}\,\exp\left(\frac{2\pi^2}{\theta^2 t} - \frac{(r+\theta^2/2)^2}{2\theta^2}t - \frac{2}{\theta^2 a}(1+s)\right) \tag{6.3}$$
$$\times \int_0^\infty \exp\left(-\frac{2z^2}{\theta^2 t} - \frac{4\sqrt{s}}{\theta^2 a}\cosh z\right)\sinh z\, \sin\left(\frac{4\pi z}{\theta^2 t}\right) dz$$

for all $t > 0$ *and* $s, a > 0$. *Thus, it can be verified that the conditions in (5.2)-(5.4) as well as the other essential assumptions of Theorem 5 are satisfied for almost all* $x > 0$.

Example 8 *(Wiener sequential testing problem with finite horizon [6], [22; Chapter VI, Section 21].) Suppose that in (2.2) we have* $G(t,x) = -t - ax \wedge b(1-x)$*, for some* $a, b > 0$ *fixed and* $\lambda = 0$*. Assume that in (2.1) we have* $\mu(x) = 0$*,* $\sigma(x) = \theta x(1-x)$*, for all* $x \in E = (0,1)$ *and some* $\theta > 0$*, and hence,* $H(t,x) = 1$ *in (2.7). In this case, as an analogue of the formula (5.1), we have:*

$$F(0,x;t,y) = P_{0,x}\big[X_t \le y\big] = x\,\Phi\left(\frac{1}{\theta\sqrt{t}}\log\left(\frac{y}{1-y}\,\frac{1-x}{x}\right) - \frac{\theta\sqrt{t}}{2}\right) + (1-x)\,\Phi\left(\frac{1}{\theta\sqrt{t}}\log\left(\frac{y}{1-y}\,\frac{1-x}{x}\right) + \frac{\theta\sqrt{t}}{2}\right) \tag{6.4}$$

for all $t > 0$ *and* $x, y \in (0,1)$*. Thus, it can be verified that the conditions (5.2)-(5.4) as well as the other essential assumptions of Theorem 5 are satisfied for almost all* $x \in (0,1)$*. In this case, the function* $x \mapsto G(t,x)$ *is monotone on the intervals* $(0,1/2)$ *and* $(1/2,1)$*, separately.*

Example 9 *(Wiener disorder detection problem with finite horizon [7], [22; Chapter VI, Section 22].) Suppose that in (2.2) we have* $G(t,x) = -(1-x)$ *and* $\lambda = 0$*. Assume that in (2.1) we have* $\mu(x) = \eta(1-x)$*,* $\sigma(x) = \theta x(1-x)$*, for all* $x \in E = (0,1)$ *and some* $\eta, \theta > 0$*. The reward of the related optimal stopping problem also contains an integral, and thus, as an analogue of the formula in (5.1), we have:*

$$F(0,x;t,y) = E_{0,x}\big[X_t\,I(X_t \le y) + (1-X_t)\,I(X_t \ge y)\big] = \int_0^y z\,p(x;t,z)\,dz + \int_y^1 (1-z)\,p(x;t,z)\,dz \tag{6.5}$$

for all $t > 0$ *and* $x, y \in (0,1)$*, where an explicit expression for the marginal density function* $p(x;t,z)$ *is derived in [7; Section 4] (see also [22; Chapter VI, Section 22]). It can be verified that the conditions in (5.2)-(5.4) as well as the other essential assumptions of Theorem 5 are satisfied (see [7]).*

Example 10 *(Early exercise Russian option [26]-[27], [21], [22; Chapter VII, Section 26].) Suppose that in (2.2) we have* $G(t,x) = x$*. Assume that in (2.1) we have:*

$$dX_t = -rX_t\,dt + \theta X_t\,dB_t + dR_t \quad (X_0 = x) \tag{6.6}$$

where

$$R_t = \int_0^t I(X_u = 1)\,\frac{d(\max_{0\le v\le u} S_u)}{S_u} \tag{6.7}$$

and $S_t = \exp(\theta B_t + (r+\theta^2/2)t)$, *for all* $t \geq 0$ *and some* $r, \theta > 0$, *and hence,* $H(t,x) = -(r+\lambda)x$ *for all* $x \in E = (0, \infty)$ *in (2.7). In this case, as an analogue of the formula in (5.1) we have:*

$$F(0,x;t,y) = E_{0,x}\big[H(t,x)\, I(X_t \geq y)\big] \tag{6.8}$$
$$= -\int_1^\infty \int_0^\infty \Big(\frac{m \vee x}{s}\Big)\, I\Big(\frac{m \vee x}{s} \geq y\Big)\, f(t,s,m)\, ds\, dm$$

for all $t > 0$ *and* $x, y > 0$, *where*

$$f(t,s,m) = \frac{2}{\theta^3 \sqrt{2\pi t^3}} \, \frac{\log(m^2/s)}{sm} \, \exp\left(-\frac{\log^2(m^2/s)}{2\theta^2 t} + \frac{\beta}{\theta}\log s - \frac{\beta^2}{2}\, t\right) \tag{6.9}$$

for $0 < s \leq m$ *and* $m \geq 1$ *with* $\beta = r/\theta + \theta/2$. *Thus, it can be shown that in this case the condition in (5.4) is not satisfied. This fact can be explained by the presence of a reflection term from (6.7) in the equation of (6.6). Therefore, one should find another arguments to prove the assertion of Theorem 5 for this case.*

Acknowledgments. This article was written during the time when the second author was a collaborator at the Weierstrass Institute for Applied Analysis and Stochastics (WIAS) in Berlin. He expresses his gratitude to Vladimir G. Spokoiny and other colleagues at the Research Group "Stochastic Algorithms and Nonparametric Statistics" for their constant support and warm atmosphere. This research was supported by Deutsche Forschungsgemeinschaft through the SFB 649 Economic Risk.

References

[1] Barles, G., Burdeau, J., Romano, M. and Samsoen, N. (1995). Critical stock price near expiration. *Mathematical Finance* **5** (77–95).

[2] Belomestny, D. and Milstein, G. N. (2006). Adaptive simulation algorithms for pricing American and Bermudan options by local analysis of the financial market. *International Journal of Theoretical and Applied Finance* **9** (455–481).

[3] Carr, P. (1998). Randomization and the American put. *Review of Financial Studies* **11** (597–626).

[4] Carr, P., Jarrow, R. and Myneni, R. (1992). Alternative characterization of American put options. *Mathematical Finance* **2** (78–106).

[5] Dynkin, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Soviet Mathematical Doklady* **4** (627–629).

[6] Gapeev, P. V. and Peskir, G. (2004). The Wiener sequential testing problem with finite horizon. *Stochastics and Stochastics Reports* **76** (59–75).

[7] Gapeev, P. V. and Peskir, G. (2006). The Wiener disorder problem with finite horizon. *Stochastic Processes and their Applications* **116**(12) (1770–1791).

[8] Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering.* Springer, New York.

[9] Haugh, M. B. and Kogan, L. (2004). Pricing American options: a duality approach. *Operations Research* **52** (258–270).

[10] Hou, C., Little, T. and Pant, V. (2000). A new integral representation of the early exercise bounday for American put options. *Journal of Computational Finance* **3** (73–96).

[11] Jacka, S. D. (1991). Optimal stopping and the American put. *Mathematical Finance* **1** (1–14).

[12] Jacka, S. D. and Lynn, J. R. (1992). Finite horizon optimal stopping, obstacle problems and the shape of the continution region. *Stochastics and Stochastics Reports* **39** (25–42).

[13] Kim, I. J. (1990). The analytic valuation of American options. *Review of Financial Studies* **3** (547–572).

[14] Kolodko, A. and Schoenmakers, J. (2005). Iterative construction of the optimal Bermudan stopping time. *Finance and Stochastics* **10**(1) (27–49).

[15] McKean, H. P. Jr. (1965). Appendix: A free boundary problem for the heat equation arising form a problem of mathematical economics. *Industrial Management Review* **6** (32–39).

[16] Myneni, R. (1992). The pricing of the American option. *Annals of Applied Probability* **2**(1) (1–23).

[17] Novikov, A. A. and Shiryaev, A. N. (2007). On solution of the optimal stopping problem for processes with independent increments. *Stochastics. An International Journal of Probability and Stochastic Processes* **79**(3-4) (393–406).

[18] Pedersen, J. L. and Peskir, G. (2002). On nonlinear integral equations arising in problems of optimal stopping. *Proceedings of Functional Analysis* VII (Dubrovnik 2001), *Various Publications Series* **46** (159–175).

[19] Peskir, G. (2005). A change-of-variable formula with local time on curves. *Journal of Theoretical Probability* **18**(3) (499–535).

[20] Peskir, G. (2005). On the American option problem. *Mathematical Finance* **15**(1) (169-181).

[21] Peskir, G. (2005). The Russian option: Finite horizon. *Finance and Stochastics* **9** (251-267).

[22] Peskir, G. and Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Birkhäuser, Basel.

[23] Peskir, G. and Uys, N. (2005). On Asian options of American type. In the Volume *Exotic Options and Advanced Levy Models.* Wiley, Chichester (217-235).

[24] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion.* (Third Edition) Springer, Berlin.

[25] Rogers, L. C. G. (2002). Monte Carlo valuation of American options. *Mathematical Finance* **12** (271–286).

[26] Shepp, L. A. and Shiryaev, A. N. (1993). The Russian option: reduced regret. *Annals of Applied Probability* **3**(3) (631–640).

[27] Shepp, L. A. and Shiryaev, A. N. (1994). A new look at the pricing of Russian options. *Theory of Probability and its Applications* **39**(1) (103–119).

[28] Shiryaev, A. N. (1978). *Optimal Stopping Rules.* Springer, Berlin.

[29] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance.* World Scientific, Singapore.

[30] Tricomi, F. G. (1957). *Integral Equations.* Interscience Publishers, London and New York.

[31] van Moerbeke, P. (1976). On optimal stopping and free-boundary problems. *Archive for Rational Mechanics and Analysis* **60** (101–148).