THE DISORDER PROBLEM FOR COMPOUND POISSON PROCESSES WITH EXPONENTIAL JUMPS

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The problem of disorder seeks to determine a stopping time which is as close as possible to the unknown time of 'disorder' when the observed process changes its probability characteristics. We give a partial answer to this question for some special cases of Lévy processes and present a complete solution of the Bayesian and variational problem for a compound Poisson process with exponential jumps. The method of proof is based on reducing the Bayesian problem to an integro-differential free-boundary problem where in some cases the smooth-fit principle breaks down and is replaced by the principle of continuous fit.

(Abbreviated Title: ON THE DISORDER PROBLEM)

1. Introduction. Assume that at time $t = 0$ we begin to observe a continuously updated process $X = (X_t)_{t \geq 0}$ whose probability characteristics change at some unknown time θ called the time of 'disorder' which cannot be observed directly. Throughout the paper the random time θ can take the value 0 with probability π, and under the condition that $\theta > 0$ it is exponentially distributed with parameter $\lambda > 0$. The disorder problem or the problem of quickest disorder detection is to decide by observing the process X at which time instant one should give an 'alarm' indicating the occurrence of 'disorder' as close as possible to the time θ in the sense that both the probability of 'false alarm' and the expectation of the time interval between the occurrence of 'disorder' and the 'alarm' when the latter is given correctly should be minimal.

The problem of detecting a change in drift of a Wiener process was formulated and solved by Shiryaev [12]-[15] (see also [16], [17; Ch. IV] and [17; p. 208] for historical notes and references). Some particular cases of the problem of detecting a change in the intensity of a Poisson process were considered in Gal'chuk and Rozovskii [6] and in Davis [4]. Peskir and Shiryaev [10] presented a complete solution of the disorder problem for a Poisson process in the Bayesian formulation. The main aim of this paper is to find

Mathematics Subject Classification 2000. Primary 60G40, 62M20, 34K10. Secondary 62C10, 62L15, 60J75.

Key words and phrases. Disorder (quickest detection) problem, Lévy process, compound Poisson process, optimal stopping, integro-differential free-boundary problem, principles of smooth and continuous fit, measure of jumps and its compensator, Girsanov's theorem for semimartingales, Itô's formula.

an explicit expression of the optimal stopping boundary for the a posteriori probability process in some special cases of the problem for Lévy processes and to present a complete solution to the problem for a compound Poisson process having exponentially distributed jumps. Actually, we give the next example of process for which the quickest disorder detection problem can be solved in an explicit form. Such processes are used, for example, in several models of stochastic finance and insurance (see e.g. Shiryaev [18]). For some other optimal stopping problems for such processes see e.g. Mordecki [9].

The paper is organized as follows. In Section 2 we give a formulation of the Bayesian and variational problem of quickest disorder detection for Lévy processes. In Section 3 by the examination of the sample-path behavior of the a posteriori probability process we find an optimal stopping boundary in some particular cases of the Bayesian problem. In Section 4 by means of solving the corresponding integro-differential free-boundary problem we derive a complete solution of the Bayesian problem for a compound Poisson process with exponential jumps, where we can observe the breakdown of the smooth-fit principle and its replacement by the principle of continuous fit. These effects can be explained both by the examination of the sample-path properties of the a posteriori probability process and by the existence of a singularity point of the integro-differential equation. Note that in models based on jump processes the situations when the continuous fit replaces the smooth fit were earlier observed, for example, in bandit problems (see e.g. Berry and Fristedt [2] for references). In Section 5 passing from the derived solution of the Bayesian problem we find an explicit expression for the optimal stopping boundary in the corresponding variational problem.

We should note here that the problem of quickest detection admits different formulations and appears in on-line quality control, radar-location, seismology, etc. (see e.g. Carlstein, Müller and Siegmund [3] and Kolmogorov, Prokhorov and Shiryaev [8]).

2. Formulation of the Bayesian and variational problem. For a precise probabilistic formulation of the quickest disorder detection problem for Lévy processes (see [17; Ch. IV] for the Wiener process case) let us suppose that on some measurable space $(\Omega, \mathcal{F})$ equipped with a family of probability measures $(P^s)_{s\geq 0}$ there exists a nonnegative random variable θ such that $P^s[\theta = s] = 1$ for all $s \geq 0$. It is assumed that we observe a continuously updated process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ and having under the measure P^s the following triplet:

$$((t \wedge s)b_0 + ((t-s) \vee 0)b_1, 0, dt\, [I_{\{t<s\}}\nu_0(dx) + I_{\{t\geq s\}}\nu_1(dx)]) \tag{2.1}$$

with respect to the function $h(x) = x$, $x \in \mathbb{R}$, for all $t, s \geq 0$, where $\nu_i(dx)$ is a Lévy measure on $\mathbb{R}$ such that $\nu_i(\{0\}) = 0$ and $\int (x^2 \wedge 1)\nu_i(dx) < \infty$ for $i = 0, 1$ (see e.g. [7; Ch. II.4] or [11; Ch. II.8]). Here θ and X are assumed to be stochastically independent under P^s for all $s \geq 0$. Let us fix $\lambda > 0$ and define the measures $P_\pi = \pi P^0 + (1-\pi)\int_0^\infty \lambda e^{-\lambda s} P^s ds$ for all $\pi \in [0, 1]$, so that we have $P_\pi[\theta = 0] = \pi$ and $P_\pi[\theta > t \,|\, \theta > 0] = e^{-\lambda t}$ for all $t \geq 0$.

Let τ be a stopping time with respect to the filtration $\mathbf{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$, where $\mathcal{F}_t^X = \sigma\{X_s \,|\, 0 \leq s \leq t\}$. We will interpret τ as the time at which the 'alarm' is sounded to signal the change in distribution of the observed process X. The *Bayesian disorder problem* is to minimize the risk function:

$$V(\pi) = \inf_{\tau} \left\{ P_\pi[\tau < \theta] + cE_\pi[\tau - \theta]^+ \right\}, \tag{2.2}$$

where the infimum is taken over all $\mathbf{F}^X$-stopping times τ, and to find an optimal stopping time τ_* at which the infimum in (2.2) is attained. Here $P_\pi[\tau < \theta]$ is the probability of 'false alarm', $E_\pi[\tau - \theta]^+$ is the 'average delay' in detecting of disorder correctly, and $c > 0$ is some constant.

It is easily shown (see [17; pp. 195-197]) that the value function $V(\pi)$ can be expressed in terms of the a posteriori probability process (π_t), where $\pi_t = P_\pi[\theta \leq t \,|\, \mathcal{F}_t^X]$ for all $t \geq 0$ and $P_\pi[\pi_0 = \pi] = 1$. Namely, we have:

$$V(\pi) = \inf_{\tau} E_\pi \left[1 - \pi_\tau + c \int_0^\tau \pi_t \, dt \right]. \tag{2.3}$$

Moreover, it is easily verified (see [17; p. 204]) that the infimum in (2.3) is actually taken over the class $\mathcal{M}(\pi)$ of stopping times τ such that $E_\pi[\tau] < \infty$.

In order to give the corresponding variational or fixed false-alarm probability formulation, let the number $\pi \in [0, 1)$ be fixed and let $\mathcal{M}(\pi, \alpha)$ denote the class of stopping times τ satisfying:

$$P_\pi[\tau < \theta] \leq \alpha, \tag{2.4}$$

where α is a given constant from the interval $[0, 1)$. The *variational disorder problem* is to find in the class $\mathcal{M}(\pi, \alpha)$ a stopping time $\widehat{\tau}$ such that:

$$E_\pi[\widehat{\tau} - \theta]^+ \leq E_\pi[\tau - \theta]^+ \tag{2.5}$$

for any other stopping time τ from $\mathcal{M}(\pi, \alpha)$.

3. Preliminary results and examples. Suppose that the filtration $\mathbf{F}^X$ is right-continuous and the following conditions are satisfied:

$$\int |x| \, \nu_i(dx) < \infty \quad (i = 0, 1), \tag{3.1}$$

$$b_1 = b_0 + \int x \, \nu_1(dx) - \int x \, \nu_0(dx), \tag{3.2}$$

$$\int \left(\sqrt{Y(x)} - 1 \right)^2 \nu_0(dx) < \infty, \tag{3.3}$$

where $Y(x) = \nu_1(dx)/\nu_0(dx)$ for all $x \in \mathbb{R}$. Then by means of Girsanov's theorem for semimartingales [7; Th. III.5.34] and Itô's formula [7; Th. I.4.57], using the schema of arguments in [17; p. 202] it can be verified that the process (π_t) solves the following stochastic differential equation:

$$d\pi_t = \lambda(1 - \pi_t) \, dt + \int \frac{\pi_{t-}(1 - \pi_{t-})(Y(x) - 1)}{1 + \pi_{t-}(Y(x) - 1)} \, (\mu^X - \nu^X)(dt, dx), \tag{3.4}$$

where μ^X is the measure of jumps of the process X and its $\mathbf{F}^X$-compensator ν^X is given by $\nu^X(dt,dx) = (\pi_{t-}\nu_1(dx) + (1-\pi_{t-})\nu_0(dx))\,dt$. From (3.4) it is easily seen that (π_t) is a time-homogeneous (strong) Markov process under P_π with respect to the natural filtration which clearly coincides with $\mathbf{F}^X$. The latter implies that the infimum in (2.3) can be taken over all stopping times of (π_t) playing the role of a sufficient statistic (see e.g. [17; Ch. II.15]).

It can be also verified (see [17; pp. 197-198] and [10]) that the value function $V(\pi)$ is decreasing and concave on $[0,1]$, and the optimal stopping time in (2.3) is given by:

$$\tau_* = \inf\{t \geq 0 \,|\, \pi_t \geq B_*\}, \tag{3.5}$$

where B_* is the smallest number π from $[0,1]$ such that $V(\pi) = 1-\pi$.

Using the arguments from [10] we now find an explicit expression for the optimal stopping boundary B_* in some particular cases of the problem.

LEMMA 3.1. *Assume in addition to (2.1) and (3.1)-(3.3) that we have:*

$$\nu_1(dx) \geq \nu_0(dx) \quad (x \in \mathbb{R}), \tag{3.6}$$

$$0 < \int x\,\nu_1(dx) - \int x\,\nu_0(dx) \leq c + \lambda. \tag{3.7}$$

Then in the Bayesian problem of quickest disorder detection (2.2)-(2.3) the stopping time τ_* from (3.5) is optimal with $B_* = \overline{B}$, where we set:

$$\overline{B} = \frac{\lambda}{\lambda + c}. \tag{3.8}$$

PROOF. The assumption (3.7) ensures that $\overline{B} \leq \widehat{B}$, where we set:

$$\widehat{B} = \lambda \Big/ \left(\int x\,\nu_1(dx) - \int x\,\nu_0(dx) \right). \tag{3.9}$$

From the equation (3.4) it is seen that if $\widehat{B} \geq 1$ then the process (π_t) is strictly increasing, and if $\widehat{B} < 1$ then the drift rate of the continuous part of (π_t) is positive on $[0,\widehat{B})$, negative on $(\widehat{B},1)$, and equal to zero at $\widehat{B}$. Thus, if (π_t) starts in $[0,\widehat{B})$ or in $(\widehat{B},1)$, then under the absence of jumps (π_t) will never reach $\widehat{B}$, because its drift tends to zero the same time with linear order. Therefore, by virtue of the fact that under the condition (3.6) the process (π_t) can have only positive jumps, it can leave $[0,\widehat{B})$ only by jumping and fluctuating in $(\widehat{B},1)$ cannot enter $[0,\widehat{B})$. If (π_t) starts or ends up at $\widehat{B}$, then it is trapped there ($P_\pi-$a.s.) until the next jump of X occurs.

From (3.4) it follows that the process (π_t) admits the representation:

$$\pi_t = \pi + \lambda \int_0^t (1-\pi_{s-})\,ds + M_t, \tag{3.10}$$

where (M_t) is a martingale under P_π with respect to $\mathbf{F}^X$. Hence, by means of the optional sampling theorem (see e.g. [7; Th. I.1.39]), from (3.10) together with (3.4) and according to (3.1) we obtain that $E_\pi[M_\tau]=0$ and hence:

$$E_\pi\left[1-\pi_\tau+c\int_0^\tau \pi_t\,dt\right]=1-\pi+(\lambda+c)\,E_\pi\int_0^\tau\left(\pi_t-\frac{\lambda}{\lambda+c}\right)dt \tag{3.11}$$

for all stopping times τ from $\mathcal{M}(\pi)$. Recalling that the process (π_t) is monotone increasing in $[\overline{B},\widehat{B})$ and after entering $[\widehat{B},1]$ cannot leave it anymore, from (3.11) we may therefore conclude that it is never optimal to stop (π_t) in $[0,\overline{B})$ as well as (π_t) must be stopped instantly after passing through $\overline{B}$. $\square$

EXAMPLE 3.2. Assume that in (2.1) we have $b_i=1/\lambda_i$ and $\nu_i(dx)=I_{\{x>0\}}e^{-\lambda_i x}dx/x$ with $\lambda_i>0$. Thus X is a gamma process with parameter changing from λ_0 to λ_1 (see e.g. [18; Ch. III.1]). In this case the integrals in (3.1) and (3.3) are equal to $1/\lambda_i$ and $\log[(\lambda_0+\lambda_1)^2/(4\lambda_0\lambda_1)]$, respectively. Therefore, by Lemma 3.1 we get that if $\lambda_0>\lambda_1>0$ and $\log(\lambda_0/\lambda_1)\le c+\lambda$, then the stopping time τ_* from (3.5) is optimal with $B_*=\lambda/(\lambda+c)$.

EXAMPLE 3.3. Suppose that in (2.1) we have $b_i=1/\gamma_i$ and $\nu_i(dx)=I_{\{x>0\}}e^{-\gamma_i^2x/2}dx/(2\pi x^3)^{1/2}$ with $\gamma_i>0$. Thus X is an inverse Gaussian process with parameter changing from γ_0 to γ_1 (see e.g. [1]). In this case the integrals in (3.1) and (3.3) are equal to $1/\gamma_i$ and $[2(\gamma_0^2+\gamma_1^2)]^{1/2}-\gamma_0-\gamma_1$, respectively. Therefore, by Lemma 3.1 we conclude that if $\gamma_0>\gamma_1>0$ and $\gamma_0-\gamma_1\le c+\lambda$, then τ_* from (3.5) is optimal with $B_*=\lambda/(\lambda+c)$.

REMARK 3.4. We note that from (3.11) it is seen that one should not stop (π_t) when it is in $[0,\overline{B}]$, so that for B_* from (3.5) we have $\overline{B}\le B_*\le 1$.

4. Solution of the Bayesian problem for a compound Poisson process with exponential jumps. In the rest of the paper it is assumed that the process X is defined by:

$$X_t=\int_0^t \theta_{s-}\,dX_s^1+\int_0^t(1-\theta_{s-})\,dX_s^0, \tag{4.1}$$

where $X_s^i=\sum_{j=1}^{N_s^i}\xi_j^i$ and $\theta_s=I_{\{s\ge\theta\}}$ for all $t,s\ge 0$, $N^i=(N_t^i)$ is a Poisson process with intensity $1/\lambda_i$, and $(\xi_j^i)_{j\in\mathbb{N}}$ is a sequence of independent random variables exponentially distributed with parameter λ_i (N^i, $(\xi_j^i)_{j\in\mathbb{N}}$ and θ are supposed to be independent) for $i=0,1$. Then in (2.1) we have $b_i=1/\lambda_i^2$ and $\nu_i(dx)=I_{\{x>0\}}e^{-\lambda_i x}dx$, and thus X is a compound Poisson process having exponentially distributed jumps with parameter changing from λ_0 to λ_1. In this case the integrals in (3.1) and (3.3) are equal to $1/\lambda_i^2$ and $(\lambda_0-\lambda_1)^2/[\lambda_0\lambda_1(\lambda_0+\lambda_1)]$, respectively, and (3.4) takes the form:

$$\begin{aligned} d\pi_t=\lambda(1-\pi_t)\,dt+\int_0^\infty &\frac{\pi_{t-}(1-\pi_{t-})(e^{-\lambda_1x}-e^{-\lambda_0x})}{\pi_{t-}e^{-\lambda_1x}+(1-\pi_{t-})e^{-\lambda_0x}} \\ &\times\left(\mu^X(dt,dx)-\left(\pi_{t-}e^{-\lambda_1x}+(1-\pi_{t-})e^{-\lambda_0x}\right)dt\,dx\right). \end{aligned} \tag{4.2}$$

Standard arguments imply that in this case the infinitesimal operator $\mathbb{L}$ of the process (π_t) acts on a function $f \in C^1([0,1])$ according to the rule:

$$(\mathbb{L}f)(\pi) = \left(\lambda - \left(\frac{\lambda_0 - \lambda_1}{\lambda_0 \lambda_1}\right)\pi\right)(1-\pi)f'(\pi) \tag{4.3}$$
$$+ \int_0^\infty \left[f\left(\frac{\pi e^{-\lambda_1 x}}{\pi e^{-\lambda_1 x} + (1-\pi)e^{-\lambda_0 x}}\right) - f(\pi)\right] \left(\pi e^{-\lambda_1 x} + (1-\pi)e^{-\lambda_0 x}\right) dx$$

for all $\pi \in [0,1]$. Using standard arguments based on the strong Markov property it follows that $V(\pi)$ is C^1 on $(0, B_*)$. Therefore, using the results from [17; Ch. III.8] we can formulate the following *integro-differential free-boundary problem* for the unknown function $V(\pi)$ from (2.3) and the unknown boundary B_* from (3.5):

$$(\mathbb{L}V)(\pi) = -c\pi \quad (0 < \pi < B_*), \tag{4.4}$$
$$V(\pi) = 1 - \pi \quad (B_* \le \pi \le 1), \tag{4.5}$$
$$V(B_*-) = 1 - B_* \quad (\textit{continuous fit}), \tag{4.6}$$

where the condition (4.6) is satisfied by virtue of the concavity argument above. Note that the superharmonic characterization of the value function (see [5] and [17]) implies that $V(\pi)$ is the largest function satisfying (4.4)-(4.6). Moreover, under some relations on the parameters of the model which are specified below, the following condition may be satisfied or break down:

$$V'(B_*) = -1 \quad (\textit{smooth fit}). \tag{4.7}$$

We also observe that in this case $\widehat{B}$ from (3.9) takes the form:

$$\widehat{B} = \frac{\lambda \lambda_0 \lambda_1}{\lambda_0 - \lambda_1} \tag{4.8}$$

and turns out to be a singularity point of the equation (4.4) when $\lambda_0 > \lambda_1$.

Using the schema of arguments in [10] we further show that the system (4.4)-(4.6) admits an explicit solution which turns out to be a solution of the initial optimal stopping problem (2.3). For this, let us consider a continuous function $f(\pi)$ satisfying (4.4) on $(0, B)$ and (4.5) on $[B, 1]$ for some $0 < B < 1$ given and fixed.

Let us first assume that $\lambda_0 > \lambda_1$. Then it follows that the function $\widetilde{f}(y) = f(\pi)$ with $\pi = e^y/(1+e^y)$ solves the following system:

$$\left(\frac{\lambda'(1+e^y)}{e^y} - \frac{1}{\gamma(\gamma-1)}\right)\widetilde{f}'(y) - \frac{\widetilde{f}(y)[\gamma(1+e^y)-1]}{\gamma(\gamma-1)(1+e^y)} \tag{4.9}$$
$$+\frac{e^{\gamma y}}{1+e^y}\left[\int_y^{\widetilde{B}} \frac{\widetilde{f}(z)(1+e^z)}{e^{\gamma z}}\, dz + \frac{e^{-\gamma \widetilde{B}}}{\gamma}\right] = -\frac{c(\lambda_0-\lambda_1)e^y}{1+e^y} \quad (y < \widetilde{B}),$$
$$\widetilde{f}(y) = 1/(1+e^y) \quad (y \ge \widetilde{B}), \tag{4.10}$$

where we set $\gamma = \lambda_0/(\lambda_0 - \lambda_1) > 1$, $\lambda' = \lambda(\lambda_0 - \lambda_1) > 0$ and $\widetilde{B} = \log[B/(1-B)]$. It can be easily shown that the system (4.9)-(4.10) has a unique solution which is given by:

$$\widetilde{f}(y;\widetilde{B}) = \frac{1}{1+e^{\widetilde{B}}} - \int_y^{\widetilde{B}} \frac{\gamma(\gamma-1)\widetilde{F}(z,\widetilde{B})e^{\gamma z}}{\gamma(1+e^z)-1}\, dz, \tag{4.11}$$

$$\widetilde{F}(y,\widetilde{B}) = \frac{1}{\widetilde{A}(y)} \left(\widetilde{C}(y,\widetilde{B}) - \int_y^{\widetilde{B}} \frac{\widetilde{C}(z,\widetilde{B})}{\widetilde{A}(z)} \frac{\widetilde{G}(z)}{\widetilde{G}(y)}\, dz \right), \tag{4.12}$$

$$\widetilde{A}(y) = \frac{1+e^y}{e^y} \left(\frac{\lambda'\gamma(\gamma-1)(1+e^y)-e^y}{\gamma(1+e^y)-1} \right), \tag{4.13}$$

$$\widetilde{C}(y,\widetilde{B}) = \frac{e^{-(\gamma-1)\widetilde{B}}}{\gamma(\gamma-1)(1+e^{\widetilde{B}})} - \frac{c\lambda_0 e^{-(\gamma-1)y}}{\gamma}, \tag{4.14}$$

$$\widetilde{G}(y) = \begin{cases} \left|e^y - \widehat{B}/(1-\widehat{B})\right|^a (1+e^y), & \text{if } \widehat{B} \neq 1, \\ \exp[-\gamma e^y](1+e^y), & \text{if } \widehat{B} = 1, \end{cases} \tag{4.15}$$

for $y \le \widetilde{B}$, and $a = (\widehat{B}+\gamma-1)/(1-\widehat{B})$ if $\widehat{B} \neq 1$. Using (4.11)-(4.15) we may thus conclude that the function $f(\pi;B) = \widetilde{f}(y;\widetilde{B})$ given by:

$$f(\pi;B) = 1 - B - \int_\pi^B \frac{\gamma F(x,B_*)(1-x)[x/(1-x)]^\gamma}{\lambda_1[\lambda_1 + (\lambda_0-\lambda_1)x]}\, dx, \tag{4.16}$$

$$F(\pi,B) = \frac{1}{A(\pi)\pi(1-\pi)} \left(C(\pi,B) - \int_\pi^B \frac{C(x,B)G(x)dx}{A(x)G(\pi)x(1-x)} \right), \tag{4.17}$$

$$A(\pi) = \frac{\lambda\lambda_0\lambda_1 - (\lambda_0-\lambda_1)\pi}{\pi[\lambda_1 + (\lambda_0-\lambda_1)\pi]}, \tag{4.18}$$

$$C(\pi,B) = \frac{1-B}{\gamma(\gamma-1)} \left(\frac{1-B}{B} \right)^{\gamma-1} - c(\lambda_0-\lambda_1) \left(\frac{1-\pi}{\pi} \right)^{\gamma-1}, \tag{4.19}$$

$$G(\pi) = \left| \frac{\lambda\lambda_0\lambda_1 - (\lambda_0-\lambda_1)\pi}{(\lambda_0-\lambda_1-\lambda\lambda_0\lambda_1)(1-\pi)} \right|^a \frac{1}{1-\pi}, \quad \text{if } \frac{\lambda\lambda_0\lambda_1}{\lambda_0-\lambda_1} \neq 1, \tag{4.20}$$

$$= \exp\left(\frac{\lambda_0\pi}{(\lambda_1-\lambda_0)(1-\pi)} \right) \frac{1}{1-\pi}, \quad \text{if } \frac{\lambda\lambda_0\lambda_1}{\lambda_0-\lambda_1} = 1,$$

$$a = \frac{\lambda_1(1+\lambda\lambda_0)}{\lambda_0-\lambda_1-\lambda\lambda_0\lambda_1}, \quad \text{if } \frac{\lambda\lambda_0\lambda_1}{\lambda_0-\lambda_1} \neq 1, \tag{4.21}$$

for $\pi \in (0,B]$ is a unique solution of the system (4.4)-(4.5).

Let us now assume that $\lambda_0 < \lambda_1$. In this case it follows that the function $\widetilde{f}(y) = f(\pi)$ with $\pi = e^y/(1+e^y)$ solves the equation:

$$\left(\frac{\lambda'(1+e^y)}{e^y} - \frac{1}{\gamma(\gamma-1)} \right) \widetilde{f}'(y) - \frac{\widetilde{f}(y)[\gamma(1+e^y)-1]}{\gamma(\gamma-1)(1+e^y)} \tag{4.22}$$

$$- \frac{e^{\gamma y}}{1+e^y} \int_{-\infty}^y \frac{\widetilde{f}(z)(1+e^z)}{e^{\gamma z}}\, dz = -\frac{c(\lambda_0-\lambda_1)e^y}{1+e^y} \quad (y < \widetilde{B})$$

and satisfies (4.10), where $\gamma = \lambda_0/(\lambda_0 - \lambda_1) < 0$, $\lambda' = \lambda(\lambda_0 - \lambda_1) < 0$ and $\widetilde{B} = \log[B/(1-B)]$. It can be easily verified that the system (4.22)+(4.10) has a unique solution which is given by:

$$\widetilde{f}(y) = \frac{1}{1+e^{\widetilde{B}}} + \int_{-\infty}^{y} \frac{\gamma(\gamma-1)\widetilde{F}(z)e^{\gamma z}}{\gamma(1+e^z)-1}\,dz, \tag{4.23}$$

$$\widetilde{F}(y) = -\frac{c(\lambda_0-\lambda_1)}{\widetilde{A}(y)}\left(e^{-(\gamma-1)y} + \int_{-\infty}^{y} \frac{e^{-(\gamma-1)z}}{\widetilde{A}(z)}\,\frac{\widetilde{G}(z)}{\widetilde{G}(y)}\,dz\right) \tag{4.24}$$

for $y \le \widetilde{B}$, where $\widetilde{A}(y)$ and $\widetilde{G}(y)$ are defined in (4.13) and (4.15), respectively. Using (4.23)-(4.24) and (4.13)+(4.15) we may therefore conclude that the function $f(\pi;B) = \widetilde{f}(y)$ given by (4.16) with:

$$F(\pi) = -\frac{c(\lambda_0-\lambda_1)}{A(\pi)\pi(1-\pi)}\left(\left(\frac{1-\pi}{\pi}\right)^{\gamma-1} + \int_0^{\pi} \frac{G(x)(1-x)^{\gamma-2}}{A(x)G(\pi)x^{\gamma}}\,dx\right) \tag{4.25}$$

for $\pi \in (0,B]$ is a unique solution of the system (4.4)-(4.5).

Taking into account the facts proved above we are now ready to formulate the main assertion of the section.

THEOREM 4.1. *Suppose that the observed process X is given by (4.1). Then in the Bayesian problem of quickest disorder detection (2.2)-(2.3) the value function $V(\pi)$ coincides with the function:*

$$V_*(\pi) = \begin{cases} f(\pi;B_*), & \pi \in (0,B_*), \\ 1-\pi, & \pi \in [B_*,1], \end{cases} \tag{4.26}$$

(with $V_*(0) = f(0+;B_*)$) and the optimal stopping time τ_* is explicitly given by (3.5), where $f(\pi;B)$ and the boundary B_* are specified as follows:

(i): if $\lambda_0 > \lambda_1$ and $c > 1/\lambda_1 - 1/\lambda_0 - \lambda$, then $f(\pi;B)$ is given by (4.16)-(4.17) and $B_* = \overline{B} \equiv \lambda/(\lambda+c)$;

(ii): if $\lambda_0 > \lambda_1$ and $c = 1/\lambda_1 - 1/\lambda_0 - \lambda$, then $f(\pi;B)$ is given by (4.16)-(4.17) and $B_* = \overline{B} = \widehat{B} \equiv \lambda\lambda_0\lambda_1/(\lambda_0-\lambda_1)$;

(iii): if $\lambda_0 > \lambda_1$ and $c < 1/\lambda_1 - 1/\lambda_0 - \lambda$, then $f(\pi;B)$ is given by (4.16)-(4.17) and $B_* > \overline{B}$ is a unique root of $H(B_*) = 0$, where we set:

$$H(B) = \int_{\widehat{B}}^{B} \frac{C(x,B)G(x)}{A(x)x(1-x)}\,dx; \tag{4.27}$$

(iv): if $\lambda_0 < \lambda_1$, then $f(\pi;B) = f(\pi)$ is given by (4.16)+(4.25) and B_* is uniquely determined from the equation:

$$f'(B_*) = -1. \tag{4.28}$$

PROOF. (i)+(ii) In these cases the conditions (3.6)-(3.7) are satisfied and thus $\overline{B} \le \widehat{B}$. Hence, by Lemma 3.1 we get that B_* coincides with $\overline{B}$ and, by means of the uniqueness arguments for solutions of the first-order ordinary differential equations, we may conclude that $V_*(\pi) = V(\pi)$ for all $\pi \in [0,1]$.

(iii) In this case we have $\widehat{B} < \overline{B}$, and thus, according to Remark 3.4 we see that the optimal boundary B_* is located to the right from $\widehat{B}$. Taking an arbitrary B from $(\widehat{B}, 1)$, by means of the arguments above we obtain that the function $f(\pi; B)$ from (4.16)-(4.17) is a unique solution of the system (4.4)-(4.6) for $\pi \in (\widehat{B}, B]$. Observe that in the given case there exists a unique point $B' \in (\widehat{B}, 1)$ such that $\lim_{\pi \downarrow \widehat{B}} f(\pi; B) = \pm\infty$ for $B \in (\widehat{B}, B') \cup (B', 1)$ and $\lim_{\pi \downarrow \widehat{B}} f(\pi; B')$ is finite. Hence $f(\pi; B)$ together with $F(\pi, B)$ from (4.17) can be uniquely extended to the interval $(0, \widehat{B}]$, where by the l'Hôpital's rule one may let $F(\widehat{B}, B') = F(\widehat{B}\pm, B')$ and thus $f'(\widehat{B}; B') = f'(\widehat{B}\pm; B') \equiv -c\lambda_1^2/(\lambda_0 - \lambda_1 - \lambda\lambda_0\lambda_1)$. Then from (4.16)-(4.17) it follows that B' can be characterized by means of $H(B') = 0$, where $H(B)$ is defined in (4.27). Since $H(\widehat{B}+) = +0$ and the derivative $H'(B) > 0$ for $B \in (\widehat{B}, \overline{B})$ and $H'(B) < 0$ for $B \in (\overline{B}, 1)$, the function $H(B)$ increases on $(\widehat{B}, \overline{B})$ and decreases on $(\overline{B}, 1)$. Thus, by virtue of the property $\lim_{B \uparrow \infty} H(B) = -\infty$, we get that B' belongs to the interval $(\overline{B}, 1)$ and $H(B') = 0$ has a unique solution.

Summarizing the facts proved above we see that the value function $V(\pi)$ and the optimal boundary B_* should necessarily solve the system (4.4)-(4.6) and there is only one point B' such that the solution $f(\pi; B')$ taken at $\pi = \widehat{B}$ is finite. We may therefore conclude that B_* coincides with B' and the uniqueness argument for solutions of first-order differential equations implies that $V_*(\pi) = V(\pi)$ for all $\pi \in [0,1]$, thus proving the claim.

(iv) Taking into account the fact that in this case the process (π_t) can increase only continuously, following the arguments in [17; Ch. IV.4] and [10] we may guess that the smooth-fit condition (4.7) is satisfied and thus the equation (4.28) holds. Using straightforward calculations it is shown that $f''(\pi) < 0$ for $\pi \in (0,1)$, hence the function $f(\pi)$ from (4.16)+(4.25) is concave on $[0,1]$ and its derivative $f'(\pi)$ is decreasing on $(0,1)$. Therefore, by virtue of the facts that $f'(0+) = 0$ and $f'(1-) = -\infty$, we may conclude that the equation (4.28) admits a unique solution.

Let us now show that the function $V_*(\pi)$ defined in (4.26)+(4.16)+(4.25) coincides with the value function $V(\pi)$ and B_* being a unique root of (4.28) is an optimal stopping boundary. For this, applying Itô's formula, we get:

$$V_*(\pi_t) = V_*(\pi) + \int_0^t (\mathbb{L}V_*)(\pi_{s-})\, ds + M_t^*, \tag{4.29}$$

where the process (M_t^*) defined by:

$$\begin{aligned} M_t^* = \int_0^t \int_0^\infty \bigg[V_*\left(\frac{\pi_{s-} e^{-\lambda_1 x}}{\pi_{s-} e^{-\lambda_1 x} + (1 - \pi_{s-}) e^{-\lambda_0 x}} \right) - V_*(\pi_{s-}) \bigg] \\ \times \big(\mu^X(ds, dx) - \big(\pi_{s-} e^{-\lambda_1 x} + (1 - \pi_{s-}) e^{-\lambda_0 x} \big)\, ds\, dx \big) \end{aligned} \tag{4.30}$$

is a martingale under P_π with respect to $\mathbf{F}^X$.

Since $V_*(\pi)$ is a bounded function, from (4.30) by means of the optional sampling theorem we get that $E_\pi[M^*_\tau] = 0$ for all τ from $\mathcal{M}(\pi)$. Thus, taking the expectation on both sides in (4.29) with τ instead of t and using the fact that a direct verification yields $(\mathbb{L}V_*)(\pi) \ge -c\pi$ and $V_*(\pi) \le 1-\pi$, we obtain:

$$V_*(\pi) \le E_\pi \left[1 - \pi_\tau + c \int_0^\tau \pi_t \, dt \right] \tag{4.31}$$

for all τ from the class $\mathcal{M}(\pi)$, and hence $V_*(\pi) \le V(\pi)$ for all $\pi \in [0,1]$.

Observe that straightforward calculations above imply that the function $V_*(\pi)$ and the boundary B_* solve the system (4.4)-(4.6), hence we have $V_*(\pi_{\tau_*}) = 1 - \pi_{\tau_*}$ and $(\mathbb{L}V_*)(\pi_t) = -c\pi_t$ for all $0 \le t \le \tau_*$. Therefore, taking the expectation on both sides in (4.29) with t replaced by τ_* and using the obvious fact that τ_* belongs to $\mathcal{M}(\pi)$, we see that the equality in (4.31) is attained at $\tau = \tau_*$. This implies that $V_*(\pi) = V(\pi)$ for all $\pi \in [0,1]$, and that B_* is an optimal stopping boundary, thus the proof is complete. □

Remark 4.2. We observe that in the case (i) of Theorem 4.1 it can be verified that $f'(B_*-; B_*) = -1$ and in the case (iv) we have proved that (4.28) holds, so that the smooth-fit condition (4.7) is satisfied. This can be explained by the both facts that the process (π_t) may pass through B_* continuously and the equation (4.4) has no singularity point.

On the other hand, in the case (ii) it is shown that $f'(B_*-; B_*) = -c\lambda_1^2/(\lambda_0 - \lambda_1 - \lambda\lambda_0\lambda_1) > -1$ and in the case (iii) it can be also proved that the smooth-fit condition (4.7) breaks down. This can be explained by means of the both facts that the process (π_t) may pass through B_* for the first time only by jumping and the equation (4.4) has a singularity point $\widehat{B}$.

Remark 4.3. We note that the function $f(\pi; B)$ for different $B \in (0,1)$ and the function $V_*(\pi)$ in the cases (i)-(iv) look the same as on [10; Fig. 2-5].

5. Solution of the variational problem for a compound Poisson process with exponential jumps. Let us first note that if $\alpha \ge 1-\pi$, then letting $\widehat{\tau} = 0$ we get $P_\pi[\widehat{\tau} < \theta] = P_\pi[\theta > 0] = 1 - \pi \le \alpha$ and $E_\pi[\widehat{\tau} - \theta]^+ = 0$, from where it is seen that $\widehat{\tau} = 0$ is optimal in the formulation (2.4)-(2.5).

Assuming that $0 < \alpha < 1 - \pi$ and following the arguments from [17; pp. 198-200], we further show that the solution of the variational problem (2.4)-(2.5) can be obtained using the solution of the Bayesian problem. For this, let us introduce the function:

$$u(\pi; B_*) = P_\pi[\tau_* < \theta] \; (= E_\pi[1 - \pi_{\tau_*}]). \tag{5.1}$$

In order to find an explicit expression for the function $u(\pi; B)$ in the case when $\lambda_0 > \lambda_1$, we observe that, by virtue of the strong Markov property, it should solve the following system:

$$(\mathbb{L}u)(\pi; B) = 0 \quad (0 < \pi < B), \tag{5.2}$$

$$u(\pi; B) = 1 - \pi \quad (B \le \pi \le 1). \tag{5.3}$$

By means of the same arguments as in the text accompanied by the formulas (4.9)-(4.21), it is shown that the system (5.2)-(5.3) admits the unique solution:

$$u(\pi;B) = 1 - B - \int_\pi^B \frac{\gamma\lambda_1 D(x,B)(1-x)}{\lambda_1 + (\lambda_0-\lambda_1)x}\left(\frac{x}{1-x}\right)^\gamma dx, \tag{5.4}$$

$$D(\pi,B) = \frac{1-B}{\gamma(\gamma-1)A(\pi)\pi(1-\pi)}\,\frac{G(B)}{G(\pi)}\left(\frac{1-B}{B}\right)^\gamma \tag{5.5}$$

for $\pi \in (0,B)$, $\pi \neq \widehat{B}$, where $\gamma = \lambda_0/(\lambda_0-\lambda_1) > 1$, the functions $A(\pi)$ and $G(\pi)$ are given by (4.18) and (4.20), respectively, and by the l'Hôpital's rule one may let $D(\widehat{B},B) = D(\widehat{B}\pm,B) \equiv 0$ as well as $u(0;B) = u(0+;B)$.

It is not difficult to verify that $\partial u(\pi;B)/(\partial B) < 0$ for $B \in (\pi,1)$, so that the function $u(\pi;B)$ is strictly decreasing on $(\pi,1)$ for $0<\pi<1-\alpha$ fixed. Therefore, by virtue of the obvious facts that $u(\pi;0) = 1-\pi$ and $u(\pi;1) = 0$, we may conclude that there exists a point $B(\alpha) \le 1-\alpha$ being a unique solution of the equation:

$$u(\pi;B(\alpha)) = \alpha. \tag{5.6}$$

Let us now formulate the main result of the section.

THEOREM 5.1. *Suppose that the observed process X is given by (4.1). Then in the variational problem of quickest disorder detection (2.4)-(2.5) the optimal stopping time $\widehat{\tau}$ is explicitly given by:*

$$\widehat{\tau} = \inf\{t \ge 0 \,|\, \pi_t \ge B(\alpha)\}, \tag{5.7}$$

where the boundary $B(\alpha) \le 1-\alpha$ is specified as follows:

(i): if $0<\alpha<1-\pi$ and $\lambda_0>\lambda_1$, then $B(\alpha)$ is a unique root of (5.6);

(ii): if $\alpha \ge 1-\pi$ or $\lambda_0<\lambda_1$, then $B(\alpha) = 1-\alpha$.

PROOF. (i) Let us consider the function $B_* = B_*(c)$ being an optimal boundary in the corresponding Bayesian problem which is uniquely determined from the parts (i)-(iii) of Theorem 4.1. It can be easily shown that $B_*(c)$ is continuous and strictly decreasing on $(0,\infty)$ and satisfies $\lim_{c\downarrow 0} B_*(c) = 1$ and $\lim_{c\uparrow\infty} B_*(c) = 0$. Then there exists a constant $c(\alpha)$ such that $B(\alpha) = B_*(c(\alpha))$, and by the definition (2.2) we have:

$$P_\pi[\widehat{\tau}<\theta] + c(\alpha)E_\pi[\widehat{\tau}-\theta]^+ \le P_\pi[\tau<\theta] + c(\alpha)E_\pi[\tau-\theta]^+ \tag{5.8}$$

for all stopping times τ. Since from (5.6) together with (5.1) and (3.5) it is seen that $P_\pi[\widehat{\tau}<\theta] = \alpha$, we may thus conclude that (5.8) directly yields:

$$c(\alpha)E_\pi[\widehat{\tau}-\theta]^+ \le c(\alpha)E_\pi[\tau-\theta]^+ \tag{5.9}$$

for all τ from $\mathcal{M}(\pi,\alpha)$. Therefore, by virtue of the obvious fact that $c(\alpha)>0$ for $0<\alpha<1-\pi$, we obtain that $\widehat{\tau}$ from (5.7) is optimal in (2.5).

(ii) Since whenever $\lambda_0 < \lambda_1$ the process (π_t) can increase only continuously, we get that $\{\pi_{\widehat{\tau}} \geq B(\alpha)\} = \{\pi_{\widehat{\tau}} = B(\alpha)\}$, and from (5.1) it thus follows that in this case we have $u(\pi; B) = 1 - B$. Hence, from (5.6) it is seen that $B(\alpha) = 1 - \alpha$, and the arguments from the previous part (i) complete the proof. $\square$

Acknowledgments. The author is grateful to A.N. Shiryaev and G. Peskir for the statement of the problem and for many helpful discussions. The author is thankful to the Editor for the encouragement to prepare the revised version. The author is obliged to an Associate Editor and a Referee for many useful suggestions which are incorporated into the final version of the paper.

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