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The spread option optimal stopping game

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We present a solution to an optimal stopping game for geometric Brownian motion with gain functions having the form of payoff functions of spread options. The method of proof is based on reducing the initial problem to a free-boundary problem and solving the latter by means of the smooth-fit principle. The derived result can be interpreted as pricing the (perpetual) spread game option in the Black-Merton-Scholes model.

1. Introduction

Optimal stopping games (usually called *Dynkin's games*) were introduced and studied by Dynkin [7]. The purely probabilistic theory of such games was developed in Frid [9], Kiefer [18]-[19], Neveau [27], Elbakidze [8], Krylov [21], Bismut [3], Stettner [33], Alario-Nazaret, Lepeltier and Marchal [1], Morimoto [26], Lepeltier and Mainguenau [25] and others. This approach was based on applying the martingale theory for solving a generalization of the optimal stopping problem introduced by Snell [32]. The analytical theory of stochastic differential games with stopping times in Markov diffusion models was developed in Bensoussan and Friedman [4]-[5] and Friedman [10] (see also Friedman [11; Chapter XVI]). This approach for studying the value functions and saddle points of such games was based on using the theory of variational inequalities and free-boundary problems for partial differential equations. Cvitanić and Karatzas [6] established a connection between the values of optimal stopping games and the solutions of backward stochastic differential equations with reflection and provided a pathwise approach to these games. Karatzas and Wang [17] studied such games in a more general non-Markovian setting and brought them into connection with bounded-variation optimal control problems.

Recently Kifer [20] introduced the concept of a *game* (or *Israeli*) *option* generalizing the concept of an American option by also allowing the seller to cancel the option prematurely, but at the expense of some penalty. It was shown that the problem of pricing and hedging such options reduces to solving an associated optimal stopping game. Kyprianou [24] obtained explicit expressions for the value functions of two classes of perpetual game option problems. Kühn and Kyprianou [22]-[23] characterized the value functions of the finite expiry versions of these classes of options via mixtures of other exotic options using martingale arguments and then

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produced the same for a more general class of finite expiry game options via a pathwise pricing formulae. Kallsen and Kühn [15]-[16] applied the neutral valuation approach to American and game options in incomplete markets and introduced a mathematically rigorous dynamic concept to define no-arbitrage prices for game contingent claims. Further calculations for game options were done by Baurdoux and Kyprianou [2]. In the present paper we introduce the perpetual *spread game option* problem and find sufficient conditions for the existence of a (nontrivial) closed form solution to the problem.

The paper is organized as follows. In Section 2 we give a formulation of the spread option optimal stopping game in the Black-Merton-Scholes model and discuss its economic interpretation. In Section 3 we formulate the corresponding free-boundary problem for the infinitesimal operator of geometric Brownian motion and derive sufficient conditions for the existence of a unique solution to the problem. In Section 4 we verify that under certain relations on the parameters of the model the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping game. In Section 5 we give some remarks and mention another question arising from the spread game option problem.

2. Formulation of the problem

For a precise probabilistic formulation of the problem let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$ started at zero. It is assumed that the price of a risky asset (e.g. a stock) on a financial market is described by a geometric Brownian motion $X = (X_t)_{t \geq 0}$ defined by:

$$X_t = x \, \exp\Big(\big(r - \theta^2/2 \big) \, t + \theta \, B_t \Big) \tag{2.1}$$

and hence solving the stochastic differential equation:

$$dX_t = rX_t \, dt + \theta X_t \, dB_t \quad (X_0 = x) \tag{2.2}$$

where $r > 0$ is the interest rate, $\theta > 0$ is the volatility coefficient, and $x > 0$ is given and fixed. The main purpose of the present paper is to find a solution to the following optimal stopping game for the time-homogeneous (strong) Markov process X having the value function:

$$\begin{aligned} V_*(x) &= \inf_{\sigma} \sup_{\tau} E_x \Big[e^{-(\lambda + r)(\sigma \wedge \tau)} \Big(G_1(X_\sigma) \, I(\sigma < \tau) + G_2(X_\tau) \, I(\tau \leq \sigma) \Big) \Big] \\ &= \sup_{\tau} \inf_{\sigma} E_x \Big[e^{-(\lambda + r)(\sigma \wedge \tau)} \Big(G_1(X_\sigma) \, I(\sigma < \tau) + G_2(X_\tau) \, I(\tau \leq \sigma) \Big) \Big] \end{aligned} \tag{2.3}$$

where P_x is a probability measure under which the process X defined in (2.1)-(2.2) starts at some $x > 0$, the infimum and supremum are taken over all finite stopping times σ and τ of the process X (i.e. stopping times with respect to $(\mathcal{F}_t^X)_{t \geq 0}$ denoting the natural filtration of X: $\mathcal{F}_t^X = \sigma\{X_u \,|\, 0 \leq u \leq t\}$, $t \geq 0$), $\lambda > 0$ is a discounting rate, and the functions $G_i(x)$ are defined by:

$$G_i(x) = (x - L_i) \, I(L_i \leq x < K_i) + (K_i - L_i) \, I(x \geq K_i) \tag{2.4}$$

for all $x > 0$ with some constants L_i and K_i such that $0 < L_i < K_i$, $i = 1, 2$, as well as $L_1 < L_2$, $K_1 < K_2$ and $K_1 - L_1 = K_2 - L_2$. We will derive sufficient conditions for the existence of a nontrivial closed form solution to the problem (2.3). Note that the existence of a unique value (2.3) was proved in [25] and [20]. This fact will be reproved in Theorem 4.1 below under certain conditions on the parameters of the model. It also follows from (2.3) that the inequalities $G_2(x) \le V_*(x) \le G_1(x)$ hold for all $x > 0$.

We will search for optimal stopping times in the problem (2.3) of the following form:

$$\sigma_* = \inf\{t \ge 0 \,|\, X_t \le A_*\} \tag{2.5}$$

$$\tau_* = \inf\{t \ge 0 \,|\, X_t \ge B_*\} \tag{2.6}$$

for some numbers A_* and B_* such that $L_1 \le A_* \le D_1$ and $D_2 \le B_* \le K_2$ hold with $D_i = L_i(\lambda + r)/\lambda$, $i = 1, 2$ (for an explanation of the latter inequalities see the text following (4.5) below). In this connection the points A_* and B_* are called *optimal stopping boundaries*. Note that in this case A_* is the largest number from $L_1 \le x \le D_1$ such that $V_*(x) = G_1(x)$, and B_* is the smallest number from $D_2 \le x \le K_2$ such that $V_*(x) = G_2(x)$. The pair of stopping times (σ_*, τ_*) is usually called a *saddle point* of the optimal stopping game.

On a financial market there are investors speculating for a rise of stock prices (so-called "bulls" playing on the *increase*) and investors speculating for a fall of stock prices (so-called "bears" playing on the *decrease*), and their strategies on the market are asymmetric (see e.g. Shiryaev [31; Chapter I, Section 1c]). In order to restrict their losses and gains simultaneously, the investors playing on the *increase* may turn to a strategy consisting of buying a call option with a strike price L_2 and selling a call option with a higher strike price $K_2 > L_2$, while the investors playing on the *decrease* may turn to a strategy consisting of selling a call option with a strike price L_1 and buying a call option with a higher strike price $K_1 > L_1$. Such combinations are called spread options of "bull" and "bear", respectively, and their payoff functions are given by $G_2(x)$ and $-G_1(x)$ from (2.4), where x denotes the stock price (see Shiryaev [31; Chapter VI, Section 4e]). In the present paper we consider a contingent claim with arbitrary (random) times of exercise τ and cancellation σ, where according to the conditions of the claim the buyer can choose the exercise time τ and in case $\tau \le \sigma$ gets the value $G_2(X_\tau)$ from the seller, and the seller can choose the cancellation time σ and in case $\sigma < \tau$ gives the value $G_1(X_\sigma)$ to the buyer. Then by virtue of the fact that P_x is a martingale measure for the given market model (see e.g. Shiryaev et al [29; Section 1], Shiryaev [31; Chapter VII, Section 3g] and Kifer [20; Section 3]), the value (2.3) may be interpreted as a rational (fair) price of the mentioned contingent claim in the given model. We also observe that from the structure of the problem (2.3) it is intuitively clear that the buyer wants to stop when the process X comes close to L_1 (from above) while the seller wants to stop when the process X comes close to K_2 (from below) without waiting too long because of the punishment of discounting.

Taking into account the arguments stated above we will call the presented contingent claim a *spread game option*. Note that the structure of the given option differs from the structure of the game options considered in [20] and [24].

3. Solution of the free-boundary problem

By means of standard arguments it is shown that the infinitesimal operator $\mathbb{L}$ of the process X acts on an arbitrary function F from the class C^2 on $(0, \infty)$ according to the rule:

$$(\mathbb{L}F)(x) = rx\, F'(x) + (\theta^2 x^2/2)\, F''(x) \tag{3.1}$$

for all $x > 0$. In order to find explicit expressions for the unknown value function $V_*(x)$ from (2.3) and the boundaries A_* and B_* from (2.5)-(2.6), using the results of the general theory of optimal stopping problems for continuous time Markov processes as well as taking into account the results about the connection between optimal stopping games and free-boundary problems (see e.g. [12] and [30; Chapter III, Section 8] as well as [4]-[5]), we can formulate the following *free-boundary problem*:

$$(\mathbb{L}V)(x) = (\lambda + r)V(x) \quad \text{for} \quad A < x < B \tag{3.2}$$
$$V(A+) = A - L_1, \quad V(B-) = B - L_2 \quad (\textit{continuous fit}) \tag{3.3}$$
$$V(x) = G_1(x) \quad \text{for} \quad 0 < x < A, \quad V(x) = G_2(x) \quad \text{for} \quad x > B \tag{3.4}$$
$$G_2(x) < V(x) < G_1(x) \quad \text{for} \quad A < x < B \tag{3.5}$$

where $L_1 \le A \le D_1$ and $D_2 \le B \le K_2$ with $D_i = L_i(\lambda + r)/\lambda$, $i = 1, 2$. Moreover, we will further assume that the following conditions hold:

$$V'(A+) = V'(B-) = 1 \quad (\textit{smooth fit}). \tag{3.6}$$

By means of straightforward calculations it is shown (see e.g. [29; Section 8] or [31; Chapter VIII, Section 2a]) that the general solution of the equation (3.2) takes the form:

$$V(x) = C_1\, x^{\gamma_1} + C_2\, x^{\gamma_2} \tag{3.7}$$

where C_1 and C_2 are some arbitrary constants, and $\gamma_1 < 0 < 1 < \gamma_2$ are defined by:

$$\gamma_i = \left(\frac{1}{2} - \frac{r}{\theta^2}\right) + (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r}{\theta^2}\right)^2 + \frac{2(\lambda + r)}{\theta^2}} \tag{3.8}$$

for $i = 1, 2$. In this case, using the conditions (3.3), we get:

$$C_1\, A^{\gamma_1} + C_2\, A^{\gamma_2} = A - L_1, \quad C_1\, B^{\gamma_1} + C_2\, B^{\gamma_2} = B - L_2 \tag{3.9}$$

from where we find that in (3.7) we have:

$$C_1 = \frac{(A - L_1)(B/A)^{\gamma_2} - (B - L_2)}{A^{\gamma_1}[(B/A)^{\gamma_2} - (B/A)^{\gamma_1}]} \tag{3.10}$$
$$C_2 = \frac{B - L_2 - (A - L_1)(B/A)^{\gamma_1}}{A^{\gamma_2}[(B/A)^{\gamma_2} - (B/A)^{\gamma_1}]} \tag{3.11}$$

and hence, the solution of the system (3.2)-(3.4) takes the form:

$$\begin{aligned} V(x; A, B) = {} & \frac{(A - L_1)(B/A)^{\gamma_2} - (B - L_2)}{(B/A)^{\gamma_2} - (B/A)^{\gamma_1}} \left(\frac{x}{A}\right)^{\gamma_1} \\ & + \frac{B - L_2 - (A - L_1)(B/A)^{\gamma_1}}{(B/A)^{\gamma_2} - (B/A)^{\gamma_1}} \left(\frac{x}{A}\right)^{\gamma_2} \end{aligned} \tag{3.12}$$

for all $A < x < B$. Then, using the assumed smooth-fit conditions (3.6), we obtain:

$$\gamma_1\, C_1\, A^{\gamma_1 - 1} + \gamma_2\, C_2\, A^{\gamma_2 - 1} = 1, \quad \gamma_1\, C_1\, B^{\gamma_1 - 1} + \gamma_2\, C_2\, B^{\gamma_2 - 1} = 1 \tag{3.13}$$

from where, by virtue of the equalities (3.10)-(3.11), after some straightforward transformations we may conclude that the boundaries A and B should satisfy the following system of transcendental equations:

$$\left(\frac{B}{A}\right)^{\gamma_1} = \frac{(\gamma_2 - 1)B - \gamma_2 L_2}{(\gamma_2 - 1)A - \gamma_2 L_1} \tag{3.14}$$

$$\left(\frac{B}{A}\right)^{\gamma_2} = \frac{(1 - \gamma_1)B + \gamma_1 L_2}{(1 - \gamma_1)A + \gamma_1 L_1} \tag{3.15}$$

which is equivalent to the system:

$$\frac{(\gamma_2 - 1)A - \gamma_2 L_1}{A^{\gamma_1}} = \frac{(\gamma_2 - 1)B - \gamma_2 L_2}{B^{\gamma_1}} \tag{3.16}$$

$$\frac{(1 - \gamma_1)A + \gamma_1 L_1}{A^{\gamma_2}} = \frac{(1 - \gamma_1)B + \gamma_1 L_2}{B^{\gamma_2}} \tag{3.17}$$

where $L_1 \le A \le D_1$ and $D_2 \le B \le K_2$ with $D_i = L_i(\lambda + r)/\lambda$, $i = 1, 2$ (for an explanation of the latter inequalities see the text following (4.5) below).

In order to find sufficient conditions for the existence and uniqueness of a solution of the system of equations (3.16)-(3.17) for $L_1 \le A \le D_1$ and $D_2 \le B \le K_2$ let us use the idea of proof of the existence and uniqueness of solution of the system of transcendental equations (4.85) from [30; Chapter IV, Section 2]. For this let us define the functions $I_k(A)$ and $J_k(B)$, $k = 1, 2$, by:

$$I_1(A) = \frac{(\gamma_2 - 1)A - \gamma_2 L_1}{A^{\gamma_1}} \tag{3.18}$$

$$J_1(B) = \frac{(\gamma_2 - 1)B - \gamma_2 L_2}{B^{\gamma_1}} \tag{3.19}$$

$$I_2(A) = \frac{(1 - \gamma_1)A + \gamma_1 L_1}{A^{\gamma_2}} \tag{3.20}$$

$$J_2(B) = \frac{(1 - \gamma_1)B + \gamma_1 L_2}{B^{\gamma_2}} \tag{3.21}$$

for all A and B such that $L_1 \le A \le D_1$ and $D_2 \le B \le K_2$. By virtue of the fact that for the derivatives of the functions (3.18)-(3.21) the following expressions hold:

$$I_1'(A) = -\frac{(\gamma_1 - 1)(\gamma_2 - 1)(A - D_1)}{A^{\gamma_1 + 1}} < 0 \tag{3.22}$$

$$I_2'(A) = \frac{(\gamma_1 - 1)(\gamma_2 - 1)(A - D_1)}{A^{\gamma_2 + 1}} > 0 \tag{3.23}$$

for all $L_1 < A < D_1$ as well as:

$$J_1'(B) = -\frac{(\gamma_1 - 1)(\gamma_2 - 1)(B - D_2)}{B^{\gamma_1 + 1}} > 0 \tag{3.24}$$

$$J_2'(B) = \frac{(\gamma_1 - 1)(\gamma_2 - 1)(B - D_2)}{B^{\gamma_2 + 1}} < 0 \tag{3.25}$$

for all $D_2 < B < K_2$, we may therefore conclude that $I_1(A)$ decreases and $I_2(A)$ increases on the interval (L_1, D_1), while $J_1(B)$ increases and $J_2(B)$ decreases on the interval (D_2, K_2).

Let us further assume that the following conditions are satisfied:

$$\frac{(\gamma_2-1)L_1-\gamma_2L_1}{L_1^{\gamma_1}} \le \frac{(\gamma_2-1)K_2-\gamma_2L_2}{K_2^{\gamma_1}} \tag{3.26}$$

$$\frac{(1-\gamma_1)L_1+\gamma_1L_1}{L_1^{\gamma_2}} \le \frac{(1-\gamma_1)K_2+\gamma_1L_2}{K_2^{\gamma_2}} \tag{3.27}$$

and observe that by means of straightforward calculations it is verified that the following inequalities hold:

$$\frac{(\gamma_2-1)D_1-\gamma_2L_1}{D_1^{\gamma_1}} \ge \frac{(\gamma_2-1)D_2-\gamma_2L_2}{D_2^{\gamma_1}} \tag{3.28}$$

$$\frac{(1-\gamma_1)D_1+\gamma_1L_1}{D_1^{\gamma_2}} \ge \frac{(1-\gamma_1)D_2+\gamma_1L_2}{D_2^{\gamma_2}}. \tag{3.29}$$

Then it is easily seen that there exist A_1 and A_2 such that $L_1 \le A_1 \le A_2 \le D_1$ and being uniquely determined from the following equations:

$$\frac{(1-\gamma_1)A_1+\gamma_1L_1}{A_1^{\gamma_2}} = \frac{(1-\gamma_1)K_2+\gamma_1L_2}{K_2^{\gamma_2}} \tag{3.30}$$

$$\frac{(1-\gamma_1)A_2+\gamma_1L_1}{A_2^{\gamma_2}} = \frac{(1-\gamma_1)D_2+\gamma_1L_2}{D_2^{\gamma_2}}. \tag{3.31}$$

In this case from (3.16)-(3.17) it follows that for each A such that $A_1 \le A \le A_2$ there exist unique values $B_1(A)$ and $B_2(A)$, and according to the implicit function theorem, for the derivatives the following expressions hold:

$$B_1'(A) = \frac{I_1'(A)}{J_1'(B)} = \frac{A-D_1}{B-D_2}\left(\frac{B}{A}\right)^{\gamma_1+1} < 0 \tag{3.32}$$

$$B_2'(A) = \frac{I_2'(A)}{J_2'(B)} = \frac{A-D_1}{B-D_2}\left(\frac{B}{A}\right)^{\gamma_2+1} < 0 \tag{3.33}$$

from where it directly follows that:

$$\frac{B_2'(A)}{B_1'(A)} = \frac{A_1'(B)}{A_2'(B)} = \left(\frac{B}{A}\right)^{\gamma_2-\gamma_1} > 1 \tag{3.34}$$

for all $L_1 \le A_1 \le A \le A_2 \le D_1$. We also observe that by means of standard arguments it is shown that the inequalities $D_2 = B_2(A_2) \le B_1(A_2) \le B_1(A_1) \le B_2(A_1) = K_2$ hold. Taking into account the properties (3.32)-(3.34) we may therefore conclude that the system of equations (3.16)-(3.17) admits a unique solution A_* and B_* such that $L_1 \le A_* \le D_1$ and $D_2 \le B_* \le K_2$ with $D_i = L_i(\lambda+r)/\lambda$, $i = 1, 2$, so that, under the added conditions (3.26)-(3.27), the solution of the system (3.2)-(3.4)+(3.6) exists and is unique.

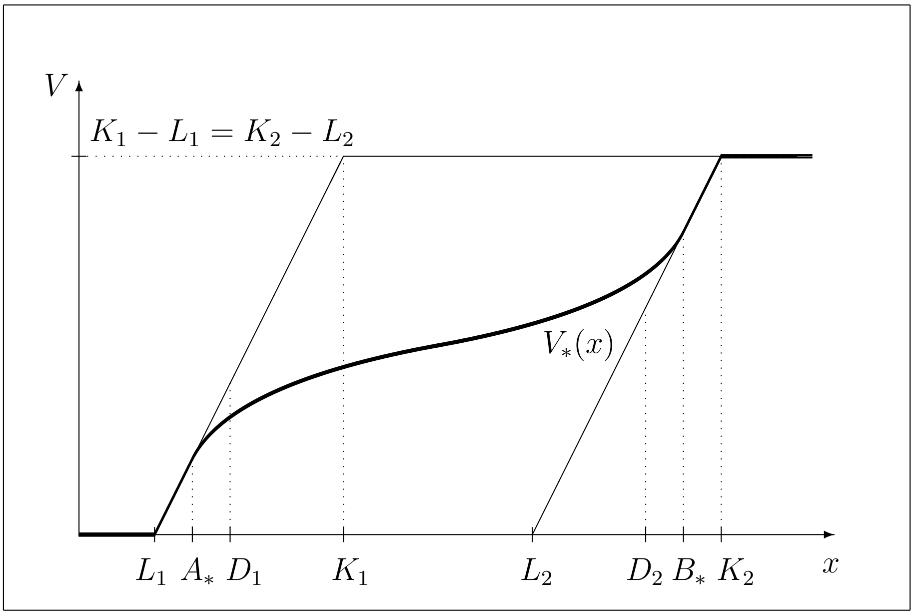

Figure 1. A computer drawing of the value function $V_*(x)$ and the optimal stopping boundaries A_* and B_*.

4. Main result and proof

Taking into account the facts proved above, let us now formulate the main assertion of the paper.

Theorem 4.1. *Let the process X be given by (2.1)-(2.2). Assume that the parameters r, θ, λ and L_i, K_i, $i = 1, 2$, are such that $0 < L_i < K_i$, $i = 1, 2$, as well as $L_1 < L_2$, $K_1 < K_2$, $K_1 - L_1 = K_2 - L_2$, $L_2(\lambda + r)/\lambda \le K_2$, and the conditions (3.26)-(3.27) are satisfied. Then the value function of the problem (2.3) takes the expression:*

$$V_*(x) = \begin{cases} G_1(x), & \text{if } 0 < x \le A_* \\ V(x; A_*, B_*), & \text{if } A_* < x < B_* \\ G_2(x), & \text{if } x \ge B_* \end{cases} \tag{4.1}$$

and the optimal stopping times σ_* and τ_* have the structure (2.5)-(2.6), where the function $V(x; A, B)$ is explicitly given by (3.12) and the optimal boundaries A_* and B_* satisfy the inequalities $L_1 \le A_* \le L_1(\lambda + r)/\lambda$ and $L_2(\lambda + r)/\lambda \le B_* \le K_2$ and are uniquely determined by the system of transcendental equations (3.16)-(3.17) [see Figure 1 above].

Proof. Let us show that the function (4.1) coincides with the value function (2.3) and the stopping times σ_* and τ_* from (2.5)-(2.6) with the boundaries A_* and B_* specified above are optimal. For this let us denote by $V(x)$ the right-hand side of the expression (4.1). In this case by means of straightforward calculations and the assumptions above it follows that the function $V(x)$ satisfies the system (3.2)-(3.4) and the conditions (3.6) as well as represents a difference of two convex functions where the latter is easily seen from (3.12). Then applying Itô-Tanaka-Meyer formula (see e.g. [13; Chapter V, Theorem 5.52] or [28; Chapter IV, Theorem 51]) to

$e^{-(\lambda+r)t}V(X_t)$, we obtain:

$$
\begin{aligned}
e^{-(\lambda+r)t}\, V(X_t) &= V(x) + M_t \\
&+ \int_0^t e^{-(\lambda+r)s}\, (\mathbb{L}V - (\lambda+r)V)(X_s) I(X_s \neq L_1, X_s \neq K_2)\, ds \\
&+ \frac{1}{2}\int_0^t e^{-(\lambda+r)s}\, I(X_s = L_1)\, d\ell_s^{L_1} - \frac{1}{2}\int_0^t e^{-(\lambda+r)s}\, I(X_s = K_2)\, d\ell_s^{K_2}
\end{aligned}
\tag{4.2}
$$

where the processes $(\ell_t^{L_1})_{t\ge 0}$ and $(\ell_t^{K_2})_{t\ge 0}$, the local time of X at the points L_1 and K_2, are defined by:

$$
\ell_t^{L_1} = P_x - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(L_1 - \varepsilon < X_s < L_1 + \varepsilon)\, \theta^2 X_s^2\, ds \tag{4.3}
$$

$$
\ell_t^{K_2} = P_x - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(K_2 - \varepsilon < X_s < K_2 + \varepsilon)\, \theta^2 X_s^2\, ds \tag{4.4}
$$

and the process $(M_t)_{t\ge 0}$ given by:

$$
M_t = \int_0^t e^{-(\lambda+r)s}\, V'(X_s) I(X_s \neq L_1, X_s \neq K_2)\, \theta X_s\, dB_s \tag{4.5}
$$

is a local martingale under the measure P_x with respect to $(\mathcal{F}_t^X)_{t\ge 0}$.

By virtue of the arguments from the previous section we may conclude that $(\mathbb{L}V - (\lambda + r)V)(x) \le 0$ for all $x > A_*$, $x \neq B_*$, $x \neq K_2$, and $(\mathbb{L}V - (\lambda+r)V)(x) \ge 0$ for all $0 < x < B_*$, $x \neq L_1$, $x \neq A_*$, where the boundaries A_* and B_* satisfy the inequalities $L_1 \le A_* \le L_1(\lambda+r)/\lambda$ and $L_2(\lambda+r)/\lambda \le B_* \le K_2 = K_1 - L_1 + L_2$. Moreover, by means of straightforward calculations it is shown that the function $V(x; A_*, B_*)$ is increasing (the derivative $V'(x; A_*, B_*)$ is positive) on the interval (A_*, B_*), and thus the property (3.5) also holds that together with (3.3)-(3.4) yields $V(x) \ge G_2(x)$ and $V(x) \le G_1(x)$ for all $x > 0$. By virtue of the fact that the time spent by the process X at the points L_1, A_*, B_* and K_2 is of Lebesgue measure zero, from the expression (4.2) it therefore follows that the inequalities:

$$
e^{-(\lambda+r)(\sigma_* \wedge \tau)}\, G_2(X_{\sigma_* \wedge \tau}) \le e^{-(\lambda+r)(\sigma_* \wedge \tau)}\, V(X_{\sigma_* \wedge \tau}) \le V(x) + M_{\sigma_* \wedge \tau} \tag{4.6}
$$

$$
e^{-(\lambda+r)(\sigma \wedge \tau_*)}\, G_1(X_{\sigma \wedge \tau_*}) \ge e^{-(\lambda+r)(\sigma \wedge \tau_*)}\, V(X_{\sigma \wedge \tau_*}) \ge V(x) + M_{\sigma \wedge \tau_*} \tag{4.7}
$$

are satisfied for any finite stopping times σ and τ of the process X.

Let $(\tau_n)_{n\in\mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process $(M_t)_{t\ge 0}$. Then using (4.6)-(4.7) and taking the expectations with respect to P_x, by means of the optional sampling theorem (see e.g. [14; Chapter I, Theorem 1.39]) we get:

$$
\begin{aligned}
&E_x\Big[e^{-(\lambda+r)(\sigma_* \wedge \tau \wedge \tau_n)} \Big(G_1(X_{\sigma_*}) I(\sigma_* < \tau \wedge \tau_n) + G_2(X_{\tau \wedge \tau_n}) I(\tau \wedge \tau_n \le \sigma_*)\Big)\Big] \\
&\le E_x\Big[e^{-(\lambda+r)(\sigma_* \wedge \tau \wedge \tau_n)}\, V(X_{\sigma_* \wedge \tau \wedge \tau_n})\Big] \le V(x) + E_x\big[M_{\sigma_* \wedge \tau \wedge \tau_n}\big] = V(x)
\end{aligned}
\tag{4.8}
$$

$$\begin{aligned}&E_x\Big[e^{-(\lambda+r)(\sigma\wedge\tau_*\wedge\tau_n)}\Big(G_1(X_{\sigma\wedge\tau_n})I(\sigma\wedge\tau_n<\tau_*)+G_2(X_{\tau_*})I(\tau_*\le\sigma\wedge\tau_n)\Big)\Big] \\ &\ge E_x\Big[e^{-(\lambda+r)(\sigma\wedge\tau_*\wedge\tau_n)}\,V(X_{\sigma\wedge\tau_*\wedge\tau_n})\Big]\ge V(x)+E_x\big[M_{\sigma\wedge\tau_*\wedge\tau_n}\big]=V(x)\end{aligned} \tag{4.9}$$

for all $x>0$. Hence, letting n go to infinity and using Fatou's lemma, we obtain that for any finite stopping times σ and τ the inequalities:

$$\begin{aligned}&E_x\Big[e^{-(\lambda+r)(\sigma_*\wedge\tau)}\Big(G_1(X_{\sigma_*})I(\sigma_*<\tau)+G_2(X_\tau)I(\tau\le\sigma_*)\Big)\Big] \\ &\le V(x)\le E_x\Big[e^{-(\lambda+r)(\sigma\wedge\tau_*)}\Big(G_1(X_\sigma)I(\sigma<\tau_*)+G_2(X_{\tau_*})I(\tau_*\le\sigma)\Big)\Big]\end{aligned} \tag{4.10}$$

hold for all $x>0$.

In order to show that the equalities in (4.10) are attained at σ_* and τ_* from (2.5)-(2.6), let us use the fact that the function $V(x)$ solves the equation (3.2) for all $A_*<x<B_*$. In this case by the expression (4.2) and the structure of the stopping times σ_* and τ_* it follows that the equality:

$$e^{-(\lambda+r)(\sigma_*\wedge\tau_*\wedge\tau_n)}\,V(X_{\sigma_*\wedge\tau_*\wedge\tau_n})=V(x)+M_{\sigma_*\wedge\tau_*\wedge\tau_n} \tag{4.11}$$

holds, from where, using the expressions (4.6)-(4.7), we may conclude that the inequalities:

$$-(K_1-L_1)\le M_{\sigma_*\wedge\tau_*\wedge\tau_n}\le K_2-L_2 \tag{4.12}$$

are satisfied for all $x>0$, where $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for $(M_t)_{t\ge0}$. Hence, letting n go to infinity in the expression (4.11) and using the conditions (3.3) as well as the obviously fulfilled property $P_x[\sigma_*\wedge\tau_*<\infty]=1$ (see e.g. [29; Section 8] or [31; Chapter VIII, Section 2a]), by means of the Lebesgue bounded convergence theorem we obtain the equality:

$$E_x\Big[e^{-(\lambda+r)(\sigma_*\wedge\tau_*)}\Big(G_1(X_{\sigma_*})\,I(\sigma_*<\tau_*)+G_2(X_{\tau_*})\,I(\tau_*\le\sigma_*)\Big)\Big]=V(x) \tag{4.13}$$

for all $x>0$, from where the desired assertion follows. □

5. Conclusions

Recall that throughout the paper and particularly in the proof of Theorem 4.1 we have used the assumption that $L_2(\lambda+r)/\lambda\le K_2$ among others. When the latter condition fails to hold but $L_1(\lambda+r)/\lambda\le K_1$ holds, let us set $B_*=K_2$ in (2.6) and consider the problem (2.3) as an optimal stopping problem for the seller. In this case we can also formulate the free-boundary problem (3.2)-(3.5), where $L_1\le A\le D_1$ and $B=K_2$ with $D_1=L_1(\lambda+r)/\lambda$, and assume that the following condition holds:

$$V'(A+)=1\quad(\textit{smooth fit}). \tag{5.1}$$

By means of the same arguments as in Section 3, using the assumed smooth-fit condition (5.1), it can be shown that the boundary A should satisfy the following transcendental equation:

$$\frac{\gamma_1}{A}\,\frac{(A-L_1)(K_2/A)^{\gamma_2}-(K_2-L_2)}{(K_2/A)^{\gamma_2}-(K_2/A)^{\gamma_1}}+\frac{\gamma_2}{A}\,\frac{(K_2-L_2)-(A-L_1)(K_2/A)^{\gamma_1}}{(K_2/A)^{\gamma_2}-(K_2/A)^{\gamma_1}}=1. \tag{5.2}$$

In order to find sufficient conditions for the existence and uniqueness of solution of the equation (5.2) let us define the function $H(A)$ by:

$$H(A) = [(\gamma_1 - 1)A - \gamma_1 L_1](K_2/A)^{\gamma_2} - [(\gamma_2 - 1)A - \gamma_2 L_1](K_2/A)^{\gamma_1} + (\gamma_2 - \gamma_1)(K_2 - L_2) \quad (5.3)$$

for all A such that $L_1 \le A \le D_1$. By virtue of the fact that for the derivative of the function (5.3) the following expression holds:

$$H'(A) = -\frac{(\gamma_1 - 1)(\gamma_2 - 1)(A - D_1)}{A} \left(\left(\frac{K_2}{A}\right)^{\gamma_2} - \left(\frac{K_2}{A}\right)^{\gamma_1} \right) < 0 \quad (5.4)$$

for all $L_1 < A < D_1$ we may therefore conclude that $H(A)$ increases on the interval (L_1, D_1). It thus follows that if the following conditions are satisfied:

$$[(\gamma_1 - 1)L_1 - \gamma_1 L_1](K_2/L_1)^{\gamma_2} - [(\gamma_2 - 1)L_1 - \gamma_2 L_1](K_2/L_1)^{\gamma_1} \ge (\gamma_1 - \gamma_2)(K_2 - L_2) \quad (5.5)$$
$$[(\gamma_1 - 1)D_1 - \gamma_1 L_1](K_2/D_1)^{\gamma_2} - [(\gamma_2 - 1)D_1 - \gamma_2 L_1](K_2/D_1)^{\gamma_1} \le (\gamma_1 - \gamma_2)(K_2 - L_2) \quad (5.6)$$

then the equation (5.2) admits a unique solution A_* such that $L_1 \le A_* \le D_1$, so that the solution of the system (3.2)-(3.4)+(5.1) with $B = K_2$ exists and is unique.

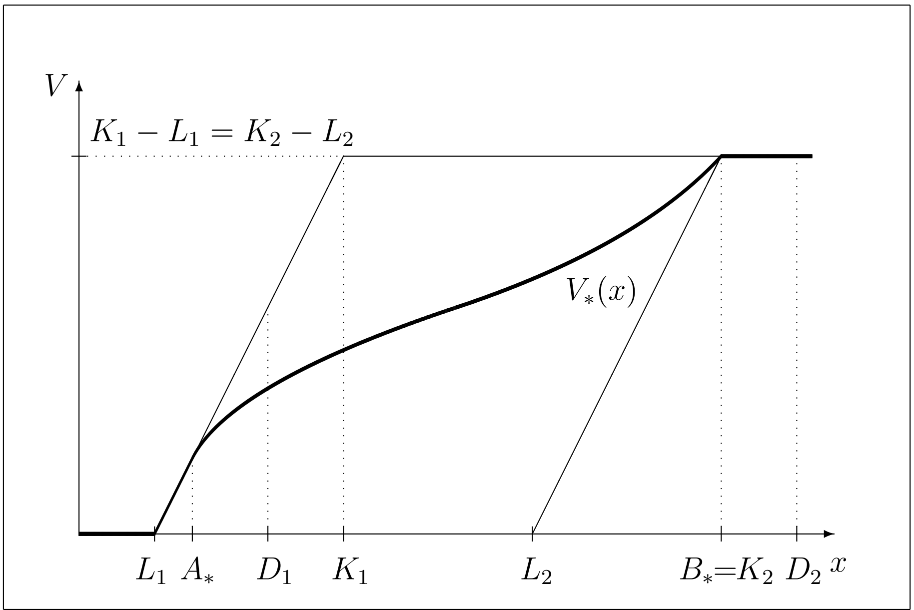

Figure 2. A computer drawing of the value function $V_*(x)$ and the optimal stopping boundaries A_* and K_2.

Taking into account the arguments above, let us formulate the following assertion.

Proposition 5.1. *Let the process X be given by (2.1)-(2.2). Assume that the parameters r, θ, λ and L_i, K_i, $i = 1, 2$, are such that $0 < L_i < K_i$, $i = 1, 2$, as well as $L_1 < L_2$, $K_1 < K_2$, $K_1 - L_1 = K_2 - L_2$, $L_1(\lambda + r)/\lambda \le K_1$, $L_2(\lambda + r)/\lambda > K_2$, and the conditions (5.5)-(5.6) are satisfied. Then the value function of the problem (2.3) takes the expression (4.1) and the optimal stopping times σ_* and τ_* have the structure (2.5)-(2.6) with $B_* = K_2$, where the function $V(x; A, B)$ is explicitly given by (3.12) and A_* satisfies the inequalities*

$L_1 \le A_* \le L_1(\lambda+r)/\lambda$ *and is uniquely determined by the transcendental equation (5.2) [see Figure 2 above].*

The verification of this assertion can be done by means of a slight modification of the arguments from the proof of Theorem 4.1 using also the facts that the condition (5.6) implies that $V'(K_2; D_1, K_2) < 1$ and the function $V'(K_2; A, K_2)$ is increasing in A on the interval (L_1, D_1). It is seen that the smooth-fit principle at the point B_* breaks down in this case. We also note that when the condition (5.5) fails to hold almost the same arguments show that (even when the condition $L_1(\lambda+r)/\lambda \le K_1$ fails to hold too) the assertion of Proposition 5.1 remains true with $A_* = L_1$ where the smooth-fit principle at A_* also breaks down [see Figure 3 below].

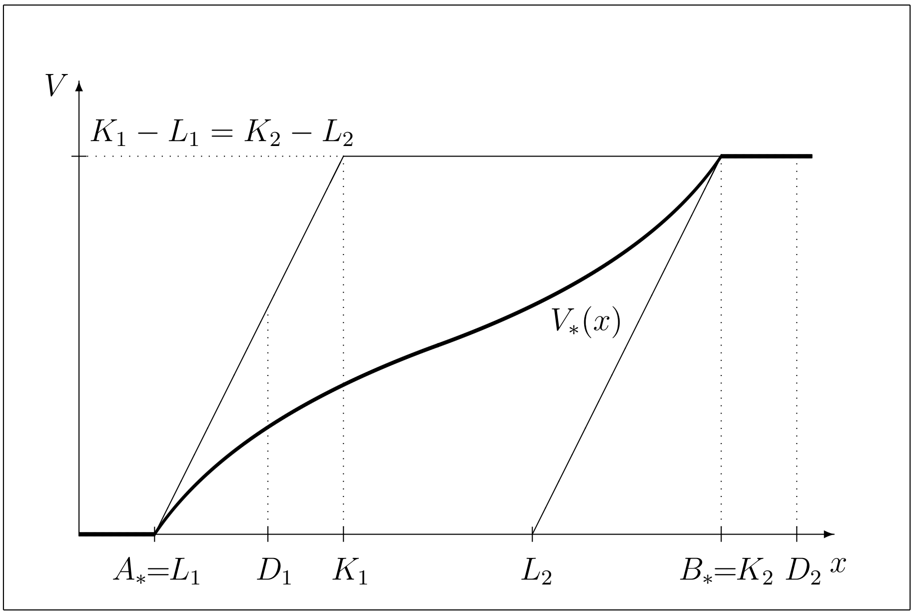

Figure 3. A computer drawing of the value function $V_*(x)$ and the optimal stopping boundaries L_1 and K_2.

Remark 5.2. We also mention that another interesting but difficult question is to present a complete description of the behavior of the optimal stopping boundaries A_* and B_* from (2.5)-(2.6) under the changing of the parameters of the model.

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