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Pricing and filtering in a two-dimensional dividend switching model*

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We study a model of a financial market in which the dividend rates of two risky assets change their initial values to other constant ones at the times at which certain unobservable external events occur. The asset price dynamics are described by geometric Brownian motions with random drift rates switching at exponential random times, that are independent of each other and the constantly correlated driving Brownian motions. We obtain closed form expressions for the rational values of European contingent claims through the filtering estimates of occurrence of the switching times and their conditional probability density derived given the filtration generated by the underlying asset price processes.

1 Introduction

In the present paper, we introduce a model for two assets paying dividends with rates changing their initial values to other constant ones at the times at which certain unobservable events occur. Such a model is related to a financial market in which the occurrence of some external

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events leads to changes of the dividend rates of the underlying assets. For instance, such a situation may happen when the failure of a large industrial company or some important political decision taken by the parliament can affect the dividend policy of the issuing firms. Note that other models with random dividends have been earlier considered in the literature (see, e.g. Geske [6]), where the possibility of the significance of stochastic dividend effects on the rational values of contingent claims was emphasized. We propose a dividend switching model for several asset prices, which reflects an influence of certain unobservable external events, that appears to be new for the related literature, to the best of our knowledge.

The purpose of the present paper is to derive closed form expressions for rational values of European contingent claims in the model described above. Suppose that the dynamics of the underlying asset prices are described by geometric Brownian motions with random drift rates having the following structure with respect to the risk-neutral probability measure. We assume that the drift rates change their initial values to other constant ones at exponentially distributed random times, that are independent of each other and the constantly correlated driving Brownian motions. The rational values of the contingent claims are thus expressed through the transition density of a two-dimensional geometric Brownian motion. The results of the paper can naturally be extended to the case of a model with several underlying assets, the price processes of which are driven by constantly correlated Brownian motions with switching drift rates.

This paper continues the investigation of information-based approach for derivative pricing, initiated by Brody, Hughston and Macrina [2], for the case of a multi-dimensional diffusion model with switching random drifts. The hidden Markov model with a two-dimensional observation process represents an extension of the model with one-dimensional observations introduced by Shiryaev [9] (see also [11; Chapter IV, Section 4]) with the aim to provide a sequential procedure of detecting an unobservable switching (disorder) time. Models with more complicated hidden continuous-time Markov chains as unobservable signals have been studied in the literature and the corresponding finite-dimensional systems of Markovian filtering estimates have been derived (see, e.g. Liptser and Shiryaev [8; Chapter IX] or Elliott, Aggoun and Moore [4] for further developments). The analysis of such models represents an important part of general stochastic filtering theory (see, e.g. Kallianpur [7] for an extensive overview).

The present paper can also be considered as a companion one to [5], where the key argument of solving the problem of pricing of contingent claims was based on the Markov property of the underlying two-dimensional structural diffusion model. In the present paper, we propose a simple derivation of the three-dimensional Markovian system of stochastic differential equations for the posterior probabilities, being filtering estimates of the occurrence of the switching times, driven by a two-dimensional innovation Brownian motion. Since the transition density of the multi-dimensional Markov process formed by the asset prices together with the resulting posterior probabilities certainly has a complicated structure, the main tool of deriving the pricing formulas is based on the application of the so-called key lemma of credit risk theory.

The paper is organized as follows. In Section 2, we introduce a model with two underlying risky assets having the structure of dividend rates described above. In Section 3, we derive stochastic differential equations for the posterior probabilities of occurrence of the external events and get explicit expressions for their conditional probability density given the accessible filtration generated by the market prices of risky assets. In Section 4, we obtain closed form expressions for the rational prices of European contingent claims under the partial information generated by the price dynamics of the underlying assets. The main results of the paper are stated in Propositions 3.1 and 4.1.

2 The model

In this section, we introduce a model for two assets with switching dividend rates.

2.1 The dynamics of asset prices

Let us suppose that on a probability space $(\Omega, \mathcal{G}, P)$ there exist two random times τ_i, $i = 1, 2$, and two standard Brownian motions $W^i = (W^i_t)_{t \geq 0}$, $i = 1, 2$. Assume that $P(\tau_i > t) = e^{-\lambda_i t}$ and $\langle W^1, W^2 \rangle_t = \rho t$, for all $t \geq 0$ and some $\lambda_i > 0$ and $\rho \in (-1, 1)$ fixed. Let the processes $S^i = (S^i_t)_{t \geq 0}$, $i = 1, 2$, be given by:

$$S^i_t = \exp\left(\left(r - \frac{\sigma_i^2}{2} - \delta_{i,0}\right) t - (\delta_{i,1} - \delta_{i,0})\,(t - \tau_i)^+ + \sigma_i\, W^i_t\right) \tag{2.1}$$

where $(t - \tau_i)^+ = \max\{t - \tau_i, 0\}$, and σ_i, $\delta_{i,j}$ are some strictly positive constants, for every $i = 1, 2$ and $j = 0, 1$. Assume that τ_i, $i = 1, 2$, are independent of each other and of the Brownian motions W^i, $i = 1, 2$. The processes S^i, $i = 1, 2$, describe the risk-neutral dynamics of the prices of dividend paying assets issued by the two firms, and τ_i, $i = 1, 2$, are random times at which some unobservable events occur, leading to the changes of the dividend rates. In more details, for every $i = 1, 2$ fixed, the i-th asset pays dividends at the rate $\delta_{i,0}$ until the time τ_i at which the i-th event occurs and the dividend rate is changed to $\delta_{i,1}$. Here, $r \geq 0$ is the interest rate of a riskless banking account, and σ_i is the volatility coefficient. It follows from an application of Itô's formula that the process S^i given by (2.1) admits the representation:

$$dS_t^i = (r - \delta_{i,0} - (\delta_{i,1} - \delta_{i,0})\, I(\tau_i \leq t))\, S_t^i\, dt + \sigma_i\, S_t^i\, dW_t^i \tag{2.2}$$

where $I(\cdot)$ denotes the indicator function.

2.2 The payoffs of European contingent claims

In what follows, we determine the rational (no-arbitrage) prices of European contingent claims with payoffs of the form $C(S_T^1, S_T^2)$, for some non-negative measurable functions $C(s_1, s_2)$, $s_i > 0$, $i = 1, 2$, and a fixed time horizon $T > 0$. We assume that the information available from the market is generated by the asset prices only. This is related to a situation where small investors trading in the market cannot observe the times at which the external events τ_i, $i = 1, 2$, occur. The rational (no-arbitrage) price process $V = (V_t)_{0 \leq t \leq T}$ of such a claim is given by:

$$V_t = E[e^{-r(T-t)}\, C(S_T^1, S_T^2) \,|\, \mathcal{F}_t^S] \tag{2.3}$$

for any $0 \leq t \leq T$, where the expectation is taken with respect to the equivalent martingale measure under which the dynamics of S^i, $i = 1, 2$, are given by (2.2). Here, we denote by $(\mathcal{F}_t^S)_{t \geq 0}$ the natural filtration of the couple of processes (S^1, S^2), that is, $\mathcal{F}_t^S = \sigma(S_u^1, S_u^2 \,|\, 0 \leq u \leq t)$ for all $t \geq 0$. This filtration reflects the information flow which is accessible for investors trading in the market. Observe that the value in (2.3) can be decomposed as:

$$\begin{aligned} V_t &= E[e^{-r(T-t)}\, C(S_T^1, S_T^2)\, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{F}_t^S] \\ &+ \sum_{i=1}^{2} E[e^{-r(T-t)}\, C(S_T^1, S_T^2)\, I(\tau_{3-i} \leq T < \tau_i) \,|\, \mathcal{F}_t^S] \\ &+ \sum_{i=1}^{2} E[e^{-r(T-t)}\, C(S_T^1, S_T^2)\, I(\tau_{3-i} < \tau_i \leq T) \,|\, \mathcal{F}_t^S] \end{aligned} \tag{2.4}$$

for all $0 \leq t \leq T$.

Remark 2.1. Observe that the assumption that every external event affects the price of one of the assets only does not restrict the generality, since the driving Brownian motions are supposed to be correlated. More precisely, let us assume that the asset price processes

$S^{*i} = (S^{*i}_t)_{t \geq 0}$, $i = 1, 2$, are defined by:

$$S^{*i}_t = \exp\left(\left(r - \frac{(\sigma^*_i)^2}{2} - \delta^*_{i,0}\right) t - (\delta^*_{i,1} - \delta^*_{i,0})\,(t - \tau_1)^+ - (\delta^*_{i,2} - \delta^*_{i,0})\,(t - \tau_2)^+ + \sigma^*_i\, W^{*i}_t\right) \tag{2.5}$$

where $W^{*i} = (W^{*i}_t)_{t \geq 0}$, $i = 1, 2$, are standard Brownian motions on the initial probability space and $\langle W^{*1}, W^{*2} \rangle_t = \rho^* t$ for all $t \geq 0$. It is shown by means of standard arguments that, under the assumption that $\delta_{i,0} \neq \delta_{i,1}$, $i = 1, 2$, the equalities:

$$\ln S^{*i}_t = \sum_{j=1}^{2} \left(\frac{\delta^*_{i,j} - \delta^*_{i,0}}{\delta_{j,1} - \delta_{j,0}}\right) \ln S^j_t \quad \text{and thus} \quad W^{*i}_t = \frac{1}{\sigma^*_i} \sum_{j=1}^{2} \left(\frac{\delta^*_{i,j} - \delta^*_{i,0}}{\delta_{j,1} - \delta_{j,0}}\right) \sigma_j W^j_t \tag{2.6}$$

hold, where the parameters $\delta^*_{i,0}$ and $\sigma^*_i > 0$ can be explicitly identified by:

$$\delta^*_{i,0} = r - \frac{(\sigma^*_i)^2}{2} - \sum_{j=1}^{2} \left(r - \frac{\sigma_j^2}{2} - \delta_{j,0}\right) \left(\frac{\delta^*_{i,j} - \delta^*_{i,0}}{\delta_{j,1} - \delta_{j,0}}\right) \tag{2.7}$$

and

$$(\sigma^*_i)^2 = \sum_{j=1}^{2} \left(\frac{\delta^*_{i,j} - \delta^*_{i,0}}{\delta_{j,1} - \delta_{j,0}}\right)^2 \sigma_j^2 + 2 \left(\frac{\delta^*_{i,1} - \delta^*_{i,0}}{\delta_{1,1} - \delta_{1,0}}\right) \left(\frac{\delta^*_{i,2} - \delta^*_{i,0}}{\delta_{2,1} - \delta_{2,0}}\right) \sigma_1 \sigma_2 \rho \tag{2.8}$$

as well as $\rho^* \in (-1, 1)$ is given by:

$$\rho^* = \frac{1}{\sigma^*_1 \sigma^*_2} \left(\sum_{j=1}^{2} \sigma_j^2 \prod_{i=1}^{2} \frac{\delta^*_{i,j} - \delta^*_{i,0}}{\delta_{j,1} - \delta_{j,0}} + \left(\prod_{i=1}^{2} \frac{\delta^*_{i,i} - \delta^*_{i,0}}{\delta_{i,1} - \delta_{i,0}} + \prod_{i=1}^{2} \frac{\delta^*_{i,3-i} - \delta^*_{i,0}}{\delta_{3-i,1} - \delta_{3-i,0}} \right) \sigma_1 \sigma_2 \rho \right) \tag{2.9}$$

for every $i = 1, 2$. Note that the assets S^i, $i = 1, 2$, can be expressed through S^{*i}, $i = 1, 2$, and the parameters are specified by means of the inverse linear transformation associated with that of (2.6). It also follows from the expressions of (2.6)-(2.9) that any contingent claim depending on the assets S^{*i}, $i = 1, 2$, with a given positive measurable payoff function $C(s_1, s_2)$ can be represented as $C^*(S^1, S^2) = C(S^{*1}, S^{*2})$, for some appropriately constructed function $C^*(s_1, s_2)$. We therefore conclude that the consideration of a model with two changes in the dividend rates of both assets such as in (2.5) is equivalent to the use of the model with only one change in every underlying asset such as in (2.1), so that, the choice of the latter model does not yield any sensible loss of generality.

2.3 Some filtrations and distribution laws

Let us introduce the process $X^{i,j} = (X^{i,j}_t)_{t \geq 0}$ by:

$$X^{i,j}_t = \exp\left(\left(r - \frac{\sigma_i^2}{2} - \delta_{i,j}\right) t + \sigma_i\, W^i_t\right) \tag{2.10}$$

for all $t \geq 0$, and every $i = 1, 2$ and $j = 0, 1$. Note that we have $S_t^i = X_t^{i,0}$ for $0 \leq t < \tau_i$, and $S_t^i = X_t^{i,0} e^{\gamma_i (t - \tau_i)}$ for $t \geq \tau_i$, where we set $\gamma_i = \delta_{i,0} - \delta_{i,1}$ for $i = 1, 2$.

Let us denote by $(\mathcal{F}_t^X)_{t \geq 0}$ the natural filtration of the couple of processes $(X^{1,0}, X^{2,0})$, that is, $\mathcal{F}_t^X = \sigma(X_u^{1,0}, X_u^{2,0} \,|\, 0 \leq u \leq t)$ for all $t \geq 0$. For every $i = 1, 2$, let $H^i = (H_t^i)_{t \geq 0}$ be the indicator process associated with the random time τ_i and defined by $H_t^i = I(\tau_i \leq t)$, and let $(\mathcal{H}_t^i)_{t \geq 0}$ be its natural filtration, so that, $\mathcal{H}_t^i = \sigma(H_u^i \,|\, 0 \leq u \leq t)$ for all $t \geq 0$. Let us also define the filtrations $(\mathcal{G}_t^i)_{t \geq 0}$ and $(\mathcal{G}_t)_{t \geq 0}$ by $\mathcal{G}_t^i = \mathcal{F}_t^X \vee \mathcal{H}_t^i$ and $\mathcal{G}_t = \mathcal{F}_t^X \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2$, respectively, so that, the equality $\mathcal{G}_t \equiv \mathcal{F}_t^X \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2 = \mathcal{F}_t^S \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2$ holds for all $t \geq 0$. It is further assumed that all the considered filtrations are right-continuous and completed by all the sets of P-measure zero.

As it follows from the results below, the two-dimensional process (S^1, S^2) has a complicated Markovian structure on its natural filtration $(\mathcal{F}_t^S)_{t \geq 0}$, so that, the direct computation of the conditional expectations in (2.4) should be avoided. Therefore, taking into account the tower property for conditional expectations, we obtain from the expression in (2.1) that the value process of (2.4) admits the representation:

$$\begin{aligned} V_t = \; & E[E[e^{-r(T-t)} \, C(X_T^{1,0}, X_T^{2,0}) \, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{G}_t] \,|\, \mathcal{F}_t^S] \\ & + \sum_{i=1}^{2} E[E[e^{-r(T-t)} \, C(X_T^{i,0}, X_T^{3-i,0} e^{\gamma_{3-i}(T - \tau_{3-i})}) \, I(\tau_{3-i} \leq T < \tau_i) \,|\, \mathcal{G}_t] \,|\, \mathcal{F}_t^S] \\ & + \sum_{i=1}^{2} E[E[e^{-r(T-t)} \, C(X_T^{i,0} e^{\gamma_i (T - \tau_i)}, X_T^{3-i,0} e^{\gamma_{3-i}(T - \tau_{3-i})}) \, I(\tau_{3-i} < \tau_i \leq T) \,|\, \mathcal{G}_t] \,|\, \mathcal{F}_t^S] \end{aligned} \tag{2.11}$$

for all $0 \leq t \leq T$. In order to compute the expectations in (2.11), we will use the fact that $(X_T^{i,j} / X_t^{i,j}, X_T^{3-i,\ell} / X_t^{3-i,\ell})$ is a couple of log-normal random variables and its density function $g_{j,\ell}$ defined by:

$$P(X_T^{i,j} / X_t^{i,j} \in dy, X_T^{3-i,\ell} / X_t^{3-i,\ell} \in dz) = g_{j,\ell}(T - t; y, z) \, dy dz \tag{2.12}$$

admits the representation:

$$\begin{aligned} g_{j,\ell}(u; y, z) = \; & \frac{1}{2\pi u y z \sigma_i \sigma_{3-i} \sqrt{1 - \rho^2}} \\ & \times \exp\bigg(-\frac{1}{(1 - \rho^2)} \bigg(\frac{(\ln y - \mu_{i,j} u)^2}{2\sigma_i^2 u} + \frac{(\ln z - \mu_{3-i,\ell} u)^2}{2\sigma_{3-i}^2 u} - \frac{(\ln y - \mu_{i,j} u)(\ln z - \mu_{3-i,\ell} u)\rho}{\sigma_i \sigma_{3-i} u} \bigg) \bigg) \end{aligned} \tag{2.13}$$

for all $u, y, z > 0$ and every $j, \ell = 0, 1$ (see, e.g. [8; Chapter XIII, Section 1]). Here and after, we set $\mu_{i,j} = r - \sigma_i^2/2 - \delta_{i,j}$ for every $i = 1, 2$ and $j = 0, 1$, in order to simplify further notations.

3 Filtering equations and conditional densities

In this section, we derive stochastic differential equations for the posterior probabilities of occurrence of the external events and their conditional probability density with respect to the accessible filtration $(\mathcal{F}_t^S)_{t \geq 0}$.

3.1 Posterior probabilities

Let us introduce the processes $\Phi^i = (\Phi_t^i)_{t \geq 0}$ and $\Psi^i = (\Psi_t^i)_{t \geq 0}$ defined by:

$$\Phi_t^i = \lambda_i \int_0^t \frac{Z_t^{i,0}}{Z_v^{i,0}} \, dv \quad \text{and} \quad \Psi_t^i = \lambda_{3-i} \int_0^t \Phi_u^i \, \frac{Z_t^{i,0}}{Z_u^{i,0}} \frac{Z_t^{3-i,1}}{Z_u^{3-i,1}} \, du \tag{3.1}$$

where the process $Z^{i,j} = (Z_t^{i,j})_{t \geq 0}$ is given by:

$$Z_t^{i,j} = \exp\left(\lambda_i \, t + \frac{\mu_{i,1} - \mu_{i,0}}{\sigma_i^2 (1 - \rho^2)} \Big(\ln S_t^i - \frac{\mu_{i,1} + \mu_{i,0}}{2} \, t - \frac{\sigma_i \rho}{\sigma_{3-i}} \big(\ln S_t^{3-i} - \mu_{3-i,j} \, t \big) \Big) \right) \tag{3.2}$$

in terms of the logarithm of the asset price process S^i having the form:

$$\ln S_t^i = \mu_{i,0} \, t + (\mu_{i,1} - \mu_{i,0}) \, (t - \tau_i)^+ + \sigma_i \, W_t^i \tag{3.3}$$

for all $t \geq 0$, where $\mu_{i,j} = r - \sigma_i^2/2 - \delta_{i,j}$ for every $i = 1, 2$ and $j = 0, 1$. Here, the process Φ^i is the likelihood-ratio process corresponding to the case of $\tau_i \leq t < \tau_{3-i}$ (see [11; Chapter IV, Section 4]), and the process Ψ^i is the likelihood-ratio process corresponding to the case of $\tau_i < \tau_{3-i} \leq t$, for all $t \geq 0$ and every $i = 1, 2$.

By means of standard arguments similar to those in [10; Chapter IV, Section 4] (which are compressed in [11; Chapter IV, Section 4]), resulting from the application of the generalized Bayes' formula (see, e.g. [8; Theorem 7.23]), it is shown that the (conditional) posterior probability processes $\Pi = (\Pi_t)_{t \geq 0}$ and $\Pi^i = (\Pi_t^i)_{t \geq 0}$ defined by $\Pi_t = P(\tau_1 \leq t, \tau_2 \leq t \, | \, \mathcal{F}_t^S)$ and $\Pi_t^i = P(\tau_i \leq t \, | \, \mathcal{F}_t^S)$, respectively, take the form:

$$\Pi_t = \frac{\Psi_t}{1 + \Xi_t} \quad \text{and} \quad \Pi_t^i = \frac{\Upsilon_t^i}{1 + \Xi_t} \tag{3.4}$$

where the processes $\Psi = (\Psi_t)_{t \geq 0}$, $\Upsilon^i = (\Upsilon_t^i)_{t \geq 0}$ and $\Xi = (\Xi_t)_{t \geq 0}$ are given by:

$$\Psi_t = \Psi_t^i + \Psi_t^{3-i}, \quad \Upsilon_t^i = \Phi_t^i + \Psi_t \quad \text{and} \quad \Xi_t = \Phi_t^i + \Phi_t^{3-i} + \Psi_t \tag{3.5}$$

for all $t \geq 0$ and every $i = 1, 2$.

3.2 Filtering equations

Applying Itô's formula, we deduce that the process $Z^{i,j}$ from (3.2) admits the representation:

$$dZ_t^{i,j} = Z_t^{i,j} \left(\lambda_i \, dt + \frac{\mu_{i,1} - \mu_{i,0}}{\sigma_i^2 (1-\rho^2)} \Big(d \ln S_t^i - \mu_{i,0} \, dt - \frac{\sigma_i \rho}{\sigma_{3-i}} \big(d \ln S_t^{3-i} - \mu_{3-i,j} \, dt \big) \Big) \right) \tag{3.6}$$

with $Z_0^{i,j} = 1$. Then, defining the process $U^i = (U_t^i)_{t \geq 0}$ by $U_t^i = Z_t^{i,0} Z_t^{3-i,1}$ and the process $Y^i = (Y_t^i)_{t \geq 0}$ by:

$$Y_t^i = \frac{\mu_{i,1} - \mu_{i,0}}{\sigma_i^2 (1-\rho^2)} \Big(\ln S_t^i - \mu_{i,0} \, t - \frac{\sigma_i \rho}{\sigma_{3-i}} \big(\ln S_t^{3-i} - \mu_{3-i,0} \, t \big) \Big) \tag{3.7}$$

we see that the following expression holds:

$$dU_t^i = U_t^i \left((\lambda_i + \lambda_{3-i}) \, dt + dY_t^i + dY_t^{3-i} \right) \tag{3.8}$$

with $U_0^i = 1$, for all $t \geq 0$, and every $i = 1, 2$ and $j = 0, 1$.

Hence, using Itô's formula again, we obtain that the processes Φ^i and Ψ^i from (3.1) solve the stochastic differential equations:

$$d\Phi_t^i = \lambda_i \big(1 + \Phi_t^i \big) \, dt + \Phi_t^i \, dY_t^i \tag{3.9}$$

with $\Phi_0^i = 0$, and

$$d\Psi_t^i = \big(\lambda_{3-i} \, \Phi_t^i + (\lambda_i + \lambda_{3-i}) \, \Psi_t^i \big) \, dt + \Psi_t^i \, (dY_t^i + dY_t^{3-i}) \tag{3.10}$$

with $\Psi_0^i = 0$. Thus, the processes defined in (3.5) admit the representations:

$$d\Psi_t = \big(\lambda_{3-i} \Phi_t^i + \lambda_i \Phi_t^{3-i} + (\lambda_i + \lambda_{3-i}) \, \Psi_t \big) \, dt + \Psi_t \, (dY_t^i + dY_t^{3-i}) \tag{3.11}$$

with $\Psi_0^i = 0$,

$$d\Upsilon_t^i = \big(\lambda_i (1 + \Xi_t) + \lambda_{3-i} \Upsilon_t^i \big) \, dt + \Upsilon_t^i \, dY_t^i + \Psi_t \, dY_t^{3-i} \tag{3.12}$$

with $\Upsilon_0^i = 0$, and

$$d\Xi_t = (\lambda_i + \lambda_{3-i})(1 + \Xi_t) \, dt + \Upsilon_t^i \, dY_t^i + \Upsilon_t^{3-i} \, dY_t^{3-i} \tag{3.13}$$

with $\Xi_0 = 0$, where the process Y^i is given by (3.7), for every $i = 1, 2$.

By means of straightforward computations, we therefore conclude that the processes defined in (3.4) solve the stochastic differential equations:

$$
\begin{aligned}
d\Pi_t = {} & \big(\lambda_i(\Pi_t^{3-i} - \Pi_t) + \lambda_{3-i}(\Pi_t^i - \Pi_t)\big)\, dt \\
& + \Pi_t(1 - \Pi_t^i)\bigg(\frac{\mu_{i,1} - \mu_{i,0}}{\sigma_i^2(1-\rho^2)}\Big(d\ln S_t^i - \big(\mu_{i,0} + (\mu_{i,1} - \mu_{i,0})\Pi_t^i\big)\, dt \\
& \qquad - \frac{\sigma_i \rho}{\sigma_{3-i}}\big(d\ln S_t^{3-i} - \big(\mu_{3-i,0} + (\mu_{3-i,1} - \mu_{3-i,0})\Pi_t^{3-i}\big)\, dt\big)\Big)\bigg) \\
& + \Pi_t(1 - \Pi_t^{3-i})\bigg(\frac{\mu_{3-i,1} - \mu_{3-i,0}}{\sigma_{3-i}^2(1-\rho^2)}\Big(d\ln S_t^{3-i} - \big(\mu_{3-i,0} + (\mu_{3-i,1} - \mu_{3-i,0})\Pi_t^{3-i}\big)\, dt \\
& \qquad - \frac{\sigma_{3-i} \rho}{\sigma_i}\big(d\ln S_t^i - \big(\mu_{i,0} + (\mu_{i,1} - \mu_{i,0})\Pi_t^i\big)\, dt\big)\Big)\bigg)
\end{aligned} \tag{3.14}
$$

with $\Pi_0 = 0$, and

$$
\begin{aligned}
d\Pi_t^i = {} & \lambda_i(1 - \Pi_t^i)\, dt \\
& + \Pi_t^i(1 - \Pi_t^i)\bigg(\frac{\mu_{i,1} - \mu_{i,0}}{\sigma_i^2(1-\rho^2)}\Big(d\ln S_t^i - \big(\mu_{i,0} + (\mu_{i,1} - \mu_{i,0})\Pi_t^i\big)\, dt \\
& \qquad - \frac{\sigma_i \rho}{\sigma_{3-i}}\big(d\ln S_t^{3-i} - \big(\mu_{3-i,0} + (\mu_{3-i,1} - \mu_{3-i,0})\Pi_t^{3-i}\big)\, dt\big)\Big)\bigg) \\
& + (\Pi_t - \Pi_t^i \Pi_t^{3-i})\bigg(\frac{\mu_{3-i,1} - \mu_{3-i,0}}{\sigma_{3-i}^2(1-\rho^2)}\Big(d\ln S_t^{3-i} - \big(\mu_{3-i,0} + (\mu_{3-i,1} - \mu_{3-i,0})\Pi_t^{3-i}\big)\, dt \\
& \qquad - \frac{\sigma_{3-i} \rho}{\sigma_i}\big(d\ln S_t^i - \big(\mu_{i,0} + (\mu_{i,1} - \mu_{i,0})\Pi_t^i\big)\, dt\big)\Big)\bigg)
\end{aligned} \tag{3.15}
$$

with $\Pi_0^i = 0$, where we recall that $\mu_{i,j} = r - \sigma_i^2/2 - \delta_{i,j}$ for every $i = 1, 2$ and $j = 0, 1$.

3.3 Innovation processes

Using the filtering arguments from [8; Chapter IX, Section 1], for every $i = 1, 2$, we obtain that the process S^i from (2.1) and (2.2) admits the representation:

$$
dS_t^i = \big(r - \delta_{i,0} - (\delta_{i,1} - \delta_{i,0})\, \Pi_t^i\big)\, S_t^i\, dt + \sigma_i\, S_t^i\, d\overline{W}_t^i \tag{3.16}
$$

in the filtration $(\mathcal{F}_t^S)_{t \ge 0}$. Here, the innovation process $\overline{W}^i = (\overline{W}_t^i)_{t \ge 0}$ defined by:

$$
\overline{W}_t^i = \int_0^t \frac{dS_u^i}{\sigma_i S_u^i} - \frac{1}{\sigma_i} \int_0^t \big(r - \delta_{i,0} - (\delta_{i,1} - \delta_{i,0})\, \Pi_u^i\big)\, du \tag{3.17}
$$

is a standard Brownian motion in the filtration $(\mathcal{F}_t^S)_{t \ge 0}$, according to P. Lévy's characterization theorem. Moreover, it can be shown by means of standard arguments that $\langle \overline{W}^1, \overline{W}^2 \rangle_t = \rho t$

holds for all $t \geq 0$, and the natural filtration of $\overline{W}^i$, $i = 1, 2$, coincides with $(\mathcal{F}_t^S)_{t \geq 0}$. It thus follows that the processes Π and Π^i from (3.4) and (3.14)-(3.15) solve the stochastic differential equations:

$$\begin{aligned} d\Pi_t = & \big(\lambda_i (\Pi_t^{3-i} - \Pi_t) + \lambda_{3-i} (\Pi_t^i - \Pi_t) \big)\, dt \\ & + \frac{\delta_{i,0} - \delta_{i,1}}{\sigma_i \sqrt{1-\rho^2}}\, \Pi_t (1 - \Pi_t^i)\, d\widehat{W}_t^i + \frac{\delta_{3-i,0} - \delta_{3-i,1}}{\sigma_{3-i} \sqrt{1-\rho^2}}\, \Pi_t (1 - \Pi_t^{3-i})\, d\widehat{W}_t^{3-i} \end{aligned} \tag{3.18}$$

with $\Pi_0 = 0$, and

$$\begin{aligned} d\Pi_t^i = & \lambda_i (1 - \Pi_t^i)\, dt \\ & + \frac{\delta_{i,0} - \delta_{i,1}}{\sigma_i \sqrt{1-\rho^2}}\, \Pi_t^i (1 - \Pi_t^i)\, d\widehat{W}_t^i + \frac{\delta_{3-i,0} - \delta_{3-i,1}}{\sigma_{3-i} \sqrt{1-\rho^2}}\, (\Pi_t - \Pi_t^i \Pi_t^{3-i})\, d\widehat{W}_t^{3-i} \end{aligned} \tag{3.19}$$

with $\Pi_0^i = 0$, where the process $\widehat{W}^i = (\widehat{W}_t^i)_{t \geq 0}$ defined by:

$$\widehat{W}_t^i = \frac{\overline{W}_t^i - \rho \overline{W}_t^{3-i}}{\sqrt{1-\rho^2}} \tag{3.20}$$

is also a standard Brownian motion, for every $i = 1, 2$. It can be also shown by means of standard arguments that $\langle \widehat{W}^1, \widehat{W}^2 \rangle_t = \rho t$ holds for all $t \geq 0$, and the natural filtration of $\widehat{W}^i$, $i = 1, 2$, coincides with $(\mathcal{F}_t^S)_{t \geq 0}$.

3.4 Conditional densities

Let us now find an expression for the family of conditional probability density processes $(\alpha_t(u,v))_{t \geq 0}$ defined from the representation:

$$P(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t^S) = \int_u^\infty \int_v^\infty \alpha_t(a,b)\, \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b}\, da db \tag{3.21}$$

for all $t, u, v \geq 0$. Applying the generalized Bayes' formula (see, e.g. [8; Theorem 7.23]), we obtain that the conditional probability in (3.21) can be expressed as:

$$\begin{aligned} P(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t^S) = & \int_{t \vee u}^\infty \int_{t \vee v}^\infty \frac{e^{(\lambda_1+\lambda_2)t}}{1 + \Xi_t}\, \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b}\, da db \\ + \int_u^{t \vee u} \int_{t \vee v}^\infty & \frac{e^{\lambda_1 a + \lambda_2 t}}{1 + \Xi_t} \frac{Z_t^{1,0}}{Z_a^{1,0}}\, \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b}\, da db + \int_{t \vee u}^\infty \int_v^{t \vee v} \frac{e^{\lambda_1 t + \lambda_2 b}}{1 + \Xi_t} \frac{Z_t^{2,0}}{Z_b^{2,0}}\, \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b}\, da db \\ + \int_u^{t \vee u} \int_v^{t \vee v} & \frac{e^{\lambda_1 a + \lambda_2 b}}{1 + \Xi_t} \left(\frac{Z_t^{1,0}}{Z_a^{1,0}} \frac{Z_t^{2,1}}{Z_b^{2,1}} I(a < b) + \frac{Z_t^{2,0}}{Z_b^{2,0}} \frac{Z_t^{1,1}}{Z_a^{1,1}} I(b < a) \right) \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b}\, da db \end{aligned} \tag{3.22}$$

for $t, u, v \geq 0$. Here, the processes $Z^{i,j}$ and Ξ are defined in (3.2) and (3.5) above, for every $i = 1, 2$ and $j = 0, 1$. Therefore, the conditional probability density in (3.21) takes the form:

$$\alpha_t(u,v) = \frac{e^{\lambda_1(u\wedge t)+\lambda_2(v\wedge t)}}{1+\Xi_t} \left(\frac{Z_t^{1,0}}{Z_{u\wedge t}^{1,0}} \frac{Z_t^{2,1}}{Z_{v\wedge t}^{2,1}} I(u<v) + \frac{Z_t^{2,0}}{Z_{v\wedge t}^{2,0}} \frac{Z_t^{1,1}}{Z_{u\wedge t}^{1,1}} I(v<u) \right) \tag{3.23}$$

for all $t, u, v \geq 0$. Furthermore, by virtue of the definition of the processes in (3.1)-(3.2) and (3.4)-(3.5), applying standard arguments, we verify that:

$$\int_0^\infty \int_0^\infty \alpha_t(u,v)\, \lambda_1 \lambda_2 e^{-\lambda_1 u - \lambda_2 v}\, du dv = 1 \tag{3.24}$$

as expected. This shows the regularity of the family of conditional probability density processes $(\alpha_t(u,v))_{t\geq 0}$.

Summarizing the facts proved above, let us formulate the following assertion.

Proposition 3.1. *In the two-dimensional model for S^i, $i = 1, 2$, of (2.1)-(2.2) and (3.16) with partial information contained in $(\mathcal{F}_t^S)_{t\geq 0}$, the posterior probability (Π, Π^1, Π^2) from (3.4) and (3.18)-(3.19) is a three-dimensional time-homogeneous Markov process, while $(S^1, S^2, \Pi, \Pi^1, \Pi^2)$ forms a five-dimensional time-homogeneous Markov process. Moreover, the conditional probability density $\alpha_t(u,v)$ defined in (3.21) admits the representation (3.23), where the processes $Z^{i,j}$, $i = 1, 2$, $j = 0, 1$, and Ξ are given by (3.2) and (3.5), respectively.*

Let us now make a short note that links the two-dimensional model of (2.1)-(2.2) and the initial one-dimensional one considered in [11; Chapter IV, Section 4].

Corollary 3.2. *Observe that, in the case of $\rho = 0$, it follows from the structure of the processes in (3.1)-(3.2) and (3.4)-(3.5) that $P(\tau_1 \leq t, \tau_2 \leq t \,|\, \mathcal{F}_t^S) = P(\tau_1 \leq t \,|\, \mathcal{F}_t^S) P(\tau_2 \leq t \,|\, \mathcal{F}_t^S)$, so that the property $\Pi_t = \Pi_t^1 \Pi_t^2$ holds for all $t \geq 0$. In that case, the filtering equations in (3.15) and (3.19) take the form of the stochastic differential equations in (4.149) and (4.150) from [11; Chapter IV, Section 4].*

4 Computation of rational prices

In this section, we compute the three conditional expectations of the expressions in (2.4) and (2.11). To simplify the notation, without loss of generality we further assume that the payoffs are already discounted by the dynamics of the bank account, which is equivalent to letting the interest rate r be equal to zero.

4.1 The first term

Let us begin by computing the first term in (2.11). For this, we first observe that:

$$\begin{aligned} & E[C(X_T^{1,0}, X_T^{2,0})\, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{G}_t] \\ & = I(t < \tau_1 \wedge \tau_2)\, E[C(X_T^{1,0}, X_T^{2,0})\, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{G}_t] \\ & = I(t < \tau_1 \wedge \tau_2)\, E[C(X_t^{1,0}(X_T^{1,0}/X_t^{1,0}), X_t^{2,0}(X_T^{2,0}/X_t^{2,0}))\, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{G}_t] \end{aligned} \tag{4.1}$$

holds for all $0 \le t \le T$. Then, applying the key lemma (see, e.g. [3; page 122] or [1; Section 5.1]) for the filtrations $(\mathcal{G}_t)_{t \ge 0}$ and $(\mathcal{F}_t^X)_{t \ge 0}$ and taking into account the independence of τ_i, $i = 1, 2$, and $X^{i,0}$, $i = 1, 2$, we get:

$$\begin{aligned} & I(t < \tau_1 \wedge \tau_2)\, E[C(X_t^{1,0}(X_T^{1,0}/X_t^{1,0}), X_t^{2,0}(X_T^{2,0}/X_t^{2,0}))\, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{G}_t] \\ & = I(t < \tau_1 \wedge \tau_2)\, \frac{E[C(X_t^{1,0}(X_T^{1,0}/X_t^{1,0}), X_t^{2,0}(X_T^{2,0}/X_t^{2,0}))\, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{F}_t^X]}{P(t < \tau_1 \wedge \tau_2 \,|\, \mathcal{F}_t^X)} \\ & = I(t < \tau_1 \wedge \tau_2)\, \frac{C_0(T, T-t, X_t^{1,0}, X_t^{2,0})}{P(t < \tau_1 \wedge \tau_2 \,|\, \mathcal{F}_t^X)} = \frac{I(t < \tau_1 \wedge \tau_2)}{P(t < \tau_1 \wedge \tau_2)}\, C_0(T, T-t, S_t^1, S_t^2) \end{aligned} \tag{4.2}$$

where, by virtue of the independence of increments of $\ln X^{i,0}$, we have:

$$\begin{aligned} & C_0(T, T-t, s_1, s_2) = E[C(s_1(X_T^{1,0}/X_t^{1,0}), s_2(X_T^{2,0}/X_t^{2,0}))]\, P(T < \tau_1 \wedge \tau_2) \\ & = e^{-(\lambda_1+\lambda_2)T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(s_1 y, s_2 z)\, g_{0,0}(T-t; y, z)\, dy dz \end{aligned} \tag{4.3}$$

for each $0 \le t \le T$, and the function $g_{0,0}$ is given in (2.13) above. Hence, by means of the tower property for conditional expectations, using the fact that the arguments from the previous section yield $P(t < \tau_1 \wedge \tau_2 \,|\, \mathcal{F}_t^S) = \Pi_t - \Pi_t^1 - \Pi_t^2 + 1$, we obtain from (4.1) and (4.2) that:

$$\begin{aligned} & E[C(S_T^1, S_T^2)\, I(T < \tau_1 \wedge \tau_2) \,|\, \mathcal{F}_t^S] \\ & = \frac{P(t < \tau_1 \wedge \tau_2 \,|\, \mathcal{F}_t^S)}{P(t < \tau_1 \wedge \tau_2)}\, C_0(T, T-t, S_t^1, S_t^2) = \frac{\Pi_t - \Pi_t^1 - \Pi_t^2 + 1}{e^{-(\lambda_1+\lambda_2)t}}\, C_0(T, T-t, S_t^1, S_t^2) \end{aligned} \tag{4.4}$$

holds for all $0 \le t \le T$, where the function $C_0(T, T-t, s_1, s_2)$ is given by (4.3) above.

4.2 The second term

Let us continue with computing the second term in (2.11). For this, we observe that:

$$
\begin{aligned}
&E[C(X_T^{i,0}, X_T^{3-i,0} e^{\gamma_{3-i}(T-\tau_{3-i})})\, I(\tau_{3-i} \le T < \tau_i) \,|\, \mathcal{G}_t] \qquad (4.5)\\
&= E[C(X_T^{i,0}, X_T^{3-i,0} e^{\gamma_{3-i}(T-\tau_{3-i})})\, I(\tau_{3-i} \le t < T < \tau_i) \,|\, \mathcal{G}_t]\\
&\quad + E[C(X_T^{i,0}, X_T^{3-i,0} e^{\gamma_{3-i}(T-\tau_{3-i})})\, I(t < \tau_{3-i} \le T < \tau_i) \,|\, \mathcal{G}_t]\\
&= I(\tau_{3-i} \le t < \tau_i)\, E[C(X_t^{i,0}(X_T^{i,0}/X_t^{i,0}), S_t^{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\, I(T < \tau_i) \,|\, \mathcal{G}_t]\\
&\quad + I(t < \tau_1 \wedge \tau_2)\, E[C(X_t^{i,0}(X_T^{i,0}/X_t^{i,0}), X_t^{3-i,0} e^{\gamma_{3-i}(T-\tau_{3-i})}(X_T^{3-i,0}/X_t^{3-i,0}))\, I(t < \tau_{3-i} \le T < \tau_i) \,|\, \mathcal{G}_t]
\end{aligned}
$$

holds for all $0 \le t \le T$. Then, applying the key lemma for the filtrations $(\mathcal{G}_t)_{t\ge0}$ and $(\mathcal{G}_t^{3-i})_{t\ge0}$ and taking into account the independence of τ_i and τ_{3-i}, $X^{i,j}$, $i = 1, 2$, $j = 0, 1$, we get:

$$
\begin{aligned}
&I(\tau_{3-i} \le t < \tau_i)\, E[C(X_t^{i,0}(X_T^{i,0}/X_t^{i,0}), S_t^{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\, I(T < \tau_i) \,|\, \mathcal{G}_t] \qquad (4.6)\\
&= I(\tau_{3-i} \le t < \tau_i)\, \frac{E[C(X_t^{i,0}(X_T^{i,0}/X_t^{i,0}), S_t^{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\, I(T < \tau_i) \,|\, \mathcal{G}_t^{3-i}]}{P(\tau_{3-i} \le t < \tau_i \,|\, \mathcal{G}_t^{3-i})}\\
&= I(\tau_{3-i} \le t < \tau_i)\, \frac{C_{1,i}^0(T, T-t, X_t^{i,0}, S_t^{3-i})}{P(\tau_{3-i} \le t < \tau_i \,|\, \mathcal{G}_t^{3-i})} = \frac{I(\tau_{3-i} \le t < \tau_i)}{P(t < \tau_i)} C_{1,i}^0(T, T-t, S_t^i, S_t^{3-i})
\end{aligned}
$$

where, by virtue of the independence of increments of $\ln X^{i,j}$, we have:

$$
\begin{aligned}
&C_{1,i}^0(T, T-t, s_i, s_{3-i}) = E[C(s_1(X_T^{i,0}/X_t^{i,0}), s_2(X_T^{3-i,1}/X_t^{3-i,1}))]\, P(T < \tau_i) \qquad (4.7)\\
&= e^{-\lambda_i T} \int_0^\infty \int_0^\infty C(s_i y, s_{3-i} z)\, g_{0,1}(T-t; y, z)\, dy dz
\end{aligned}
$$

for each $0 \le t \le T$, and the function $g_{0,1}$ is given in (2.13) above. Hence, by means of the tower property for conditional expectations and the fact that the arguments from the previous section yield $P(\tau_{3-i} \le t < \tau_i \,|\, \mathcal{F}_t^S) = \Pi_t^{3-i} - \Pi_t$, we obtain from (4.5) and (4.6) that:

$$
\begin{aligned}
&E[C(S_T^i, S_T^{3-i})\, I(\tau_{3-i} \le t < T < \tau_i) \,|\, \mathcal{F}_t^S] \qquad (4.8)\\
&= \frac{P(\tau_{3-i} \le t < \tau_i \,|\, \mathcal{F}_t^S)}{P(t < \tau_i)} C_{1,i}^0(T, T-t, S_t^i, S_t^{3-i}) = \frac{\Pi_t^{3-i} - \Pi_t}{e^{-\lambda_i t}} C_{1,i}^0(T, T-t, S_t^i, S_t^{3-i})
\end{aligned}
$$

holds for all $0 \le t \le T$, where the function $C_{1,i}^0(T, T-t, s_i, s_{3-i})$ is given in (4.7) above.

Now, applying the key lemma for the filtrations $(\mathcal{G}_t)_{t\ge0}$ and $(\mathcal{F}_t^X)_{t\ge0}$ and taking into account the independence of τ_i, $i = 1, 2$, and $X^{i,0}$, $i = 1, 2$, we get:

$$
\begin{aligned}
& I(t < \tau_1 \wedge \tau_2) \qquad (4.9) \\
& \quad \times E[C(X^{i,0}_t (X^{i,0}_T / X^{i,0}_t), X^{3-i,0}_t e^{\gamma_{3-i}(T-\tau_{3-i})} (X^{3-i,0}_T / X^{3-i,0}_t)) \, I(t < \tau_{3-i} \le T < \tau_i) \, | \, \mathcal{G}_t] \\
& = I(t < \tau_1 \wedge \tau_2) \\
& \quad \times \frac{E[C(X^{i,0}_t (X^{i,0}_T / X^{i,0}_t), X^{3-i,0}_t e^{\gamma_{3-i}(T-\tau_{3-i})} (X^{3-i,0}_T / X^{3-i,0}_t)) \, I(t < \tau_{3-i} \le T < \tau_i) \, | \, \mathcal{F}^X_t]}{P(t < \tau_1 \wedge \tau_2 \, | \, \mathcal{F}^X_t)} \\
& = I(t < \tau_1 \wedge \tau_2) \, \frac{C^1_{1,i}(T, T-t, X^{i,0}_t, X^{3-i,0}_t)}{P(t < \tau_1 \wedge \tau_2 \, | \, \mathcal{F}^X_t)} = \frac{I(t < \tau_1 \wedge \tau_2)}{P(t < \tau_1 \wedge \tau_2)} \, C^1_{1,i}(T, T-t, S^i_t, S^{3-i}_t)
\end{aligned}
$$

where, by virtue of the independence of increments of $\ln X^{i,0}$, we have:

$$
\begin{aligned}
& C^1_{1,i}(T, T-t, s_i, s_{3-i}) \qquad (4.10) \\
& = E[C(s_i (X^{i,0}_T / X^{i,0}_t), s_{3-i} e^{\gamma_{3-i}(T-\tau_{3-i})} (X^{3-i,0}_T / X^{3-i,0}_t)) \, I(t < \tau_{3-i} \le T)] \, P(T < \tau_i) \\
& = e^{-\lambda_i T} \int_t^T \int_0^\infty \int_0^\infty C(s_i y, s_{3-i} e^{\gamma_{3-i}(T-v)} z) \, \lambda_{3-i} e^{-\lambda_{3-i} v} \, g_{0,0}(T-t; y, z) \, dv dy dz
\end{aligned}
$$

for each $0 \le t \le T$, and the function $g_{0,0}$ is given in (2.13) above. Hence, by means of the tower property, we obtain from (4.5) and (4.9) that:

$$
\begin{aligned}
& E[C(S^i_T, S^{3-i}_T) \, I(t < \tau_{3-i} \le T < \tau_i) \, | \, \mathcal{F}^S_t] \qquad (4.11) \\
& = \frac{P(t < \tau_1 \wedge \tau_2 \, | \, \mathcal{F}^S_t)}{P(t < \tau_1 \wedge \tau_2)} C^1_{1,i}(T, T-t, S^i_t, S^{3-i}_t) = \frac{\Pi_t - \Pi^1_t - \Pi^2_t + 1}{e^{-(\lambda_1 + \lambda_2)t}} \, C^1_{1,i}(T, T-t, S^i_t, S^{3-i}_t)
\end{aligned}
$$

holds for all $0 \le t \le T$, where the function $C^1_{1,i}(T, T-t, s_i, s_{3-i})$ is given in (4.10) above.

4.3 The third term

Let us complete with computing the third term in (2.11). For this, we observe that:

$$
\begin{aligned}
& E[C(X^{i,0}_T e^{\gamma_i(T-\tau_i)}, X^{3-i,0}_T e^{\gamma_{3-i}(T-\tau_{3-i})}) \, I(\tau_{3-i} < \tau_i \le T) \, | \, \mathcal{G}_t] \qquad (4.12) \\
& = E[C(X^{i,0}_T e^{\gamma_i(T-\tau_i)}, X^{3-i,0}_T e^{\gamma_{3-i}(T-\tau_{3-i})}) \, I(\tau_{3-i} < \tau_i \le t) \, | \, \mathcal{G}_t] \\
& \quad + E[C(X^{i,0}_T e^{\gamma_i(T-\tau_i)}, X^{3-i,0}_T e^{\gamma_{3-i}(T-\tau_{3-i})}) \, I(\tau_{3-i} \le t < \tau_i \le T) \, | \, \mathcal{G}_t] \\
& \quad + E[C(X^{i,0}_T e^{\gamma_i(T-\tau_i)}, X^{3-i,0}_T e^{\gamma_{3-i}(T-\tau_{3-i})}) \, I(t < \tau_{3-i} < \tau_i \le T) \, | \, \mathcal{G}_t] \\
& = I(\tau_{3-i} < \tau_i \le t) \, E[C(S^i_t (X^{i,1}_T / X^{i,1}_t), S^{3-i}_t (X^{3-i,1}_T / X^{3-i,1}_t)) \, | \, \mathcal{G}_t] \\
& \quad + I(\tau_{3-i} \le t < \tau_i) \, E[C(X^{i,0}_t e^{\gamma_i(T-\tau_i)} (X^{i,0}_T / X^{i,0}_t), S^{3-i}_t (X^{3-i,1}_T / X^{3-i,1}_t)) \, I(t < \tau_i \le T) \, | \, \mathcal{G}_t] \\
& \quad + I(t < \tau_1 \wedge \tau_2) \\
& \qquad \times E[C(X^{i,0}_t e^{\gamma_i(T-\tau_i)} (X^{i,0}_T / X^{i,0}_t), X^{3-i,0}_t e^{\gamma_{3-i}(T-\tau_{3-i})} (X^{3-i,0}_T / X^{3-i,0}_t)) \, I(t < \tau_{3-i} < \tau_i \le T) \, | \, \mathcal{G}_t]
\end{aligned}
$$

holds for all $0 \le t \le T$. Firstly, using the independence of increments of $\ln X^{i,1}$, $i = 1, 2$, we get:

$$
\begin{aligned}
&I(\tau_{3-i} < \tau_i \le t)\, E[C(S_t^i(X_T^{i,1}/X_t^{i,1}), S_t^{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\,|\,\mathcal{G}_t] \\
&= I(\tau_{3-i} < \tau_i \le t)\, C_{2,i}^0(T, T-t, S_t^i, S_t^{3-i})
\end{aligned}
\tag{4.13}
$$

where we have:

$$
\begin{aligned}
&C_{2,i}^0(T, T-t, s_i, s_{3-i}) = E[C(s_i(X_T^{i,1}/X_t^{i,1}), s_{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\,|\,\mathcal{G}_t] \\
&= \int_0^\infty \int_0^\infty C(s_i y, s_{3-i} z)\, g_{1,1}(T-t; y, z)\, dydz
\end{aligned}
\tag{4.14}
$$

for each $0 \le t \le T$, and the function $g_{1,1}$ is given in (2.13) above. Hence, using the tower property and the explicit form of the conditional density of τ_i, $i = 1, 2$, derived in the previous section, we obtain from (4.12) and (4.13) that:

$$
\begin{aligned}
&E[C(S_T^i, S_T^{3-i})\, I(\tau_{3-i} < \tau_i \le t)\,|\,\mathcal{F}_t^S] \\
&= P(\tau_{3-i} < \tau_i \le t\,|\,\mathcal{F}_t^S)\, C_{2,i}^0(T, T-t, S_t^i, S_t^{3-i}) \\
&= \int_0^t \int_0^t \alpha_t(u, v)\, I(v < u)\, dudv\, C_{2,i}^0(T, T-t, S_t^i, S_t^{3-i})
\end{aligned}
\tag{4.15}
$$

holds for all $0 \le t \le T$, where the function $C_{2,i}^0(T, T-t, s_i, s_{3-i})$ is given in (4.14) and the density $\alpha_t(u, v)$ is given in (3.23) above.

Secondly, applying the key lemma for the filtrations $(\mathcal{G}_t)_{t \ge 0}$ and $(\mathcal{G}_t^{3-i})_{t \ge 0}$ and taking into account the independence of τ_i and τ_{3-i}, $X^{i,j}$, $i = 1, 2$, $j = 0, 1$, we get:

$$
\begin{aligned}
&I(\tau_{3-i} \le t < \tau_i)\, E[C(X_t^{i,0} e^{\gamma_i(T-\tau_i)}(X_T^{i,0}/X_t^{i,0}), S_t^{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\, I(t < \tau_i \le T)\,|\,\mathcal{G}_t] \\
&= I(\tau_{3-i} \le t < \tau_i) \frac{E[C(X_t^{i,0} e^{\gamma_i(T-\tau_i)}(X_T^{i,0}/X_t^{i,0}), S_t^{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\, I(t < \tau_i \le T)\,|\,\mathcal{G}_t^{3-i}]}{P(\tau_{3-i} \le t < \tau_i\,|\,\mathcal{G}_t^{3-i})} \\
&= I(\tau_{3-i} \le t < \tau_i)\, \frac{C_{2,i}^1(T, T-t, X_t^{i,0}, S_t^{3-i})}{P(\tau_{3-i} \le t < \tau_i\,|\,\mathcal{G}_t^{3-i})} = \frac{I(\tau_{3-i} \le t < \tau_i)}{P(t < \tau_i)} C_{2,i}^1(T, T-t, S_t^1, S_t^2)
\end{aligned}
\tag{4.16}
$$

where, by virtue of the independence of increments of $\ln X^{i,j}$, we have:

$$
\begin{aligned}
&C_{2,i}^1(T, T-t, s_i, s_{3-i}) \\
&= E[C(s_i e^{\gamma_i(T-\tau_i)}(X_T^{i,0}/X_t^{i,0}), s_{3-i}(X_T^{3-i,1}/X_t^{3-i,1}))\, I(t < \tau_i \le T)\,|\,\mathcal{G}_t^{3-i}] \\
&= \int_t^T \int_0^\infty \int_0^\infty C(s_i e^{\gamma_i(T-u)} y, s_{3-i} z)\, \lambda_i e^{-\lambda_i u}\, g_{0,1}(T-t; y, z)\, dudydz
\end{aligned}
\tag{4.17}
$$

for each $0 \le t \le T$, and the function $g_{0,1}$ is given in (2.13) above. Hence, by means of the tower property, we obtain from (4.12) and (4.16) that:

$$E[C(S_T^i, S_T^{3-i})\, I(\tau_{3-i} \le t < \tau_i \le T) \,|\, \mathcal{F}_t^S] \tag{4.18}$$
$$= \frac{P(\tau_{3-i} \le t < \tau_i \,|\, \mathcal{F}_t^S)}{P(t < \tau_i)}\, C_{2,i}^1(T, T-t, S_t^i, S_t^{3-i}) = \frac{\Pi_t^{3-i} - \Pi_t}{e^{-\lambda_i t}}\, C_{2,i}^1(T, T-t, S_t^i, S_t^{3-i})$$

holds for all $0 \le t \le T$, where the function $C_{2,i}^1(T, T-t, s_i, s_{3-i})$ is given in (4.17) above.

Finally, applying the key lemma for the filtrations $(\mathcal{G}_t)_{t\ge0}$ and $(\mathcal{F}_t^X)_{t\ge0}$ and taking into account the independence of τ_i, $i = 1, 2$, and $X^{i,0}$, $i = 1, 2$, we get:

$$I(t < \tau_1 \wedge \tau_2) \tag{4.19}$$
$$\times\, E[C(X_t^{i,0} e^{\gamma_i(T-\tau_i)}(X_T^{i,0}/X_t^{i,0}), X_t^{3-i,0} e^{\gamma_{3-i}(T-\tau_{3-i})}(X_T^{3-i,0}/X_t^{3-i,0}))\, I(t < \tau_{3-i} < \tau_i \le T) \,|\, \mathcal{G}_t]$$
$$= I(t < \tau_1 \wedge \tau_2)$$
$$\times\, \frac{E[C(X_t^{i,0} e^{\gamma_i(T-\tau_i)}(X_T^{i,0}/X_t^{i,0}), X_t^{3-i,0} e^{\gamma_{3-i}(T-\tau_{3-i})}(X_T^{3-i,0}/X_t^{3-i,0}))\, I(t < \tau_{3-i} < \tau_i \le T) \,|\, \mathcal{F}_t^X]}{P(t < \tau_1 \wedge \tau_2 \,|\, \mathcal{F}_t^X)}$$
$$= I(t < \tau_1 \wedge \tau_2)\, \frac{C_{2,i}^2(T, T-t, X_t^{i,0}, X_t^{3-i,0})}{P(t < \tau_1 \wedge \tau_2 \,|\, \mathcal{F}_t^X)} = \frac{I(t < \tau_1 \wedge \tau_2)}{P(t < \tau_1 \wedge \tau_2)}\, C_{2,i}^2(T, T-t, S_t^i, S_t^{3-i})$$

where, by virtue of the independence of increments of $\ln X^{i,0}$, we have:

$$C_{2,i}^2(T, T-t, s_i, s_{3-i}) \tag{4.20}$$
$$= E[C(s_i e^{\gamma_i(T-\tau_i)}(X_T^{i,0}/X_t^{i,0}), s_{3-i} e^{\gamma_{3-i}(T-\tau_{3-i})}(X_T^{3-i,0}/X_t^{3-i,0}))\, I(t < \tau_{3-i} < \tau_i \le T) \,|\, \mathcal{F}_t^X]$$
$$= \int_t^T \int_t^T \int_0^\infty \int_0^\infty C(s_i e^{\gamma_i(T-u)} y, s_{3-i} e^{\gamma_{3-i}(T-v)} z)\, I(v < u)$$
$$\times\, \lambda_1 \lambda_2 e^{-\lambda_1 u - \lambda_2 v}\, g_{0,0}(T-t; y, z)\, du dv dy dz$$

for each $0 \le t \le T$, and the function $g_{0,0}$ is given in (2.13) above. Hence, by means of the tower property, we obtain from (4.5) and (4.19) that:

$$E[C(S_T^i, S_T^{3-i})\, I(t < \tau_{3-i} < \tau_i \le T) \,|\, \mathcal{F}_t^S] \tag{4.21}$$
$$= \frac{P(t < \tau_1 \wedge \tau_2 \,|\, \mathcal{F}_t^S)}{P(t < \tau_1 \wedge \tau_2)}\, C_{2,i}^2(T, T-t, S_t^i, S_t^{3-i}) = \frac{\Pi_t - \Pi_t^1 - \Pi_t^2 + 1}{e^{-(\lambda_1+\lambda_2)t}}\, C_{2,i}^2(T, T-t, S_t^i, S_t^{3-i})$$

holds for all $0 \le t \le T$, where the function $C_{2,i}^2(T, T-t, s_i, s_{3-i})$ is given in (4.20) above.

Therefore, summarizing the facts proved above, we are now ready to formulate the following assertion.

Proposition 4.1. *Let the interest rate r of the bank account be equal to zero. The rational price of the European contingent claim in (2.4) and (2.11) under the partial information contained in $(\mathcal{F}_t^S)_{t \geq 0}$ is given by the sum of the terms in (4.4), (4.8), (4.11), (4.15), (4.18) and (4.21).*

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