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# On Markovian short rates in term structure models driven by jump-diffusion processes

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In this paper a bond market model and the related term structure of interest rates are studied where prices of zero coupon bonds are driven by a jump-diffusion process. A criterion is derived on the deterministic forward rate volatilities under which the short rate process is Markovian. In the case that the volatilities depend on the short rate sufficient conditions are presented for the existence of a finite-dimensional Markovian realization of the term structure model.

## 1 Introduction

The question of the short rate having the Markov property has attracted a considerable attention in the literature on term structures of interest rates. The reason for that is the simplification of pricing formulas for bonds and derivatives in the underlying bond market in this case. When the bond market is driven by Wiener processes, the mentioned problem was studied by Carverhill [6]. He proved that the short rate process is Markovian within the Heath-Jarrow-Morton framework with a deterministic volatility function if and only if this volatility factorizes into a product of two functions depending only on the actual time and maturity time, respectively. A more general question of interest is the problem under which conditions there exists a multidimensional Markov process (a so-called state process) having the short rate process as one of its components such that the mentioned prices depend on this Markov process only. In this case the Markov process is said to be a *finite-dimensional Markovian realization* of the considered term structure model. Finite-dimensional realizations for bond markets driven by Wiener processes have been studied, for example, in Jeffrey [18], Ritchken and Sankarasubramanian [24], Bhar and Chiarella [1], Inui and Kijima [15], Chiarella and Kwon [7], Björk and Gombani [2], Björk and Svenson [5], Filipović [11], Filipović and Teichmann [12].

Eberlein and Raible [10] generalized Carverhill's result to that case under an additional assumption on the characteristic function of the marginal distribution (cf. (4.1) in [10]) which holds, in particular, for the class of hyperbolic processes (see also Eberlein [9]). Küchler and

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Naumann [20] extended that result by using other methods to all Lévy processes possible in the model of [10] and included new examples like the class of bilateral gamma processes and variance gamma processes (see e.g. Madan and Seneta [23], Madan [22] and Küchler and Tappe [21]) and for which the assumption (4.1) in [10] fails to hold. In the present paper we investigate a jump-diffusion bond market model and the corresponding term structure of interest rates. In the case of deterministic bond volatility structure we study the question under which conditions the short rate is Markovian. In generalization to the model of Eberlein and Raible [10] we allow the driving jump-diffusion process to have different volatility structure for its continuous martingale and pure jump part. Moreover, we are interested in the existence of a finite-dimensional Markovian realization for the term structure in the case where the bond price volatilities depend on the short rate.

The paper is organized as follows. In Section 2 we introduce a jump-diffusion bond market model and the corresponding term structure of interest rates which is a modification of models in [3] and [4]. In Section 3 we prove that under deterministic volatility structure the short rate is Markovian if and only if the forward rate volatilities of both continuous martingale and pure jump part decompose into products of two factors and have a common factor depending on the maturity time. The proof of this fact is based on the extensions of arguments in [6], [10], [19] and [20] to our jump-diffusion model. In Section 4 we consider the case where the bond price volatilities depend on the short rate. We prove that if the volatility of the continuous part satisfies the multiplicative-type condition (4.1) (cf. (2.11) in [24]) and the volatility of the jump part satisfies the additive-type condition (4.9), then the short rate is a component of a finite-dimensional Markov process and the bond price admits a finite-dimensional Markovian realization. We also show that if we replace the additive-type condition (4.9) by the multiplicative-type condition (4.28), then we obtain an infinite-dimensional state space for the term structure model.

# 2 The jump-diffusion bond market model

In this section, following [3], we define the basic objects of the bond market model driven by both a Wiener process and a Poisson random measure. For the terminology from stochastic analysis we refer to [17].

2.1. Suppose that on some stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T^*]}, Q)$ with an arbitrary but fixed finite time horizon $T^* > 0$ there exist a standard Wiener process $W = (W_t)_{t\in[0,T^*]}$ and a homogeneous Poisson random measure $\mu(dt, dx)$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ with the intensity measure (compensator) $\nu(dt, dx) = dt \otimes F(dx)$ where $F(dx)$ is a positive (nonzero) $\sigma$-finite measure on $\mathcal{B}(\mathbb{R})$ such that $\int (x^2 \wedge 1)F(dx) < \infty$. Let $W$ and $\mu$ be independent and let $(\mathcal{F}_t)_{t\in[0,T^*]}$ be their right-continuous completed natural filtration $\sigma\{W_s, \mu([0, s] \times A) \,|\, s \in [0, t], A \in \mathcal{B}(\mathbb{R})\}$, $t \in [0, T^*]$. The finiteness of $T^*$ is used in the proof of Theorem 3.3 below.

Let us consider a *term structure of bond prices* $\{P(t, T) \,|\, 0 \le t \le T \le T^*\}$ where the (positive) process $P = (P(t, T))_{t\in[0,T]}$ denotes the price of a zero coupon bond at time $t$ maturing at time $T$ and satisfies the normalization condition:

$$P(T, T) = 1 \tag{2.1}$$

for each $T \in [0, T^*]$. Let us suppose that for all but fixed $T \in [0, T^*]$ the logarithm of the bond price process $P = (P(t,T))_{t\in[0,T]}$ is given by the expression:

$$\log P(t,T) = \log P(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)\,dW_s + \int_0^t \int \delta(s,T,x)\,\mu(ds,dx) \tag{2.2}$$

where we assume:

$$\int_0^T \left( |\alpha(t,T)| + \sigma^2(t,T) + \int |\delta(t,T,x)|\,F(dx) \right) dt < \infty \quad (Q - \text{a.s.}) \tag{2.3}$$

for all $T \in [0,T^*]$, so that the integrals in (2.2) are well-defined. Here $\sigma(t,T)$ is measurable with respect to the predictable $\sigma$-algebra $\mathcal{P}$ on the initial stochastic base, and $\delta(t,T,x)$ is measurable with respect to $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ (we denote by $\mathcal{B}(\mathbb{R})$ the Borel $\sigma$-algebra on $\mathbb{R}$). It is assumed that for all $\omega \in \Omega$ and $x \in \mathbb{R}$ the functions $\sigma(t,T)$ and $\delta(t,T,x)$ defined on the triangle $\{(t,T)\,|\,0 \le t \le T \le T^*\}$ are twice continuously differentiable in the second variable and satisfy the condition:

$$\sigma(T,T) = \delta(T,T,x) = 0 \tag{2.4}$$

for all $x \in \mathbb{R}$ and $T \in [0,T^*]$. The function $\alpha(t,T)$ will be specified below. We will also suppose that we are allowed to differentiate under the integral sign, to interchange limits and integrals, as well as to interchange the order of integration and differentiation. (This is a common technical assumption, see e.g. Assumption 2.1 in [3].)

Assuming that for fixed $t \in [0,T]$ the bond price $P(t,T)$ is $(Q - \text{a.s.})$ continuously differentiable with respect to the variable $T$ on $[0,T^*]$, let us introduce the corresponding *term structure of interest rates* $\{f(t,T)\,|\,0 \le t \le T \le T^*\}$ where:

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T} \tag{2.5}$$

is the *instantaneous forward rate* contracted at time $t$ for maturity $T$. On the other hand, integrating equation (2.5) and using condition (2.1), we get:

$$P(t,T) = \exp\left( -\int_t^T f(t,u)\,du \right) \tag{2.6}$$

for all $0 \le t \le T \le T^*$, and hence, we see the one-to-one correspondence between the bond prices and the forward rates. Let us also define the *short rate* process $r = (r(t))_{t\in[0,T^*]}$ by:

$$r(t) = f(t,t) \tag{2.7}$$

being the forward rate at time $t$ for maturity $t$, and the associated with it *money account* process $B = (B(t))_{t\in[0,T^*]}$ by:

$$B(t) = \exp\left( \int_0^t r(s)\,ds \right) \tag{2.8}$$

playing the role of a *numéraire* in the model. Then setting:

$$\alpha(t,T) = r(t) - \frac{1}{2}\sigma^2(t,T) - \int \left( e^{\delta(t,T,x)} - 1 \right) F(dx) \tag{2.9}$$

for all $t \in [0, T]$ and assuming that the condition:

$$E\left[\exp\left(\int_0^T \left(\frac{1}{2}\sigma^2(t,T) + \int \left| e^{\delta(t,T,x)} - 1\right| F(dx)\right) dt\right)\right] < \infty \tag{2.10}$$

is satisfied for every $T \in [0, T^*]$, by means of the arguments in [17; Chapter II, Section 2], we conclude that the *discounted bond price* process $(P(t,T)/B(t))_{t\in[0,T]}$ forms an $(\mathcal{F}_t, Q)$ - martingale (see [4; Section 5]). Without going into details we note that the conditions (2.3) and (2.10) are imposed in order to avoid some technical difficulties.

Therefore, using the expression (2.9) for $\alpha(t,T)$, we get that under the measure $Q$ and for each $T \in [0, T^*]$ the logarithm of the bond price process (2.2) admits the representation:

$$\begin{aligned}\log P(t,T) = \log P(0,T)& \\ + \int_0^t r(s)\, ds - \frac{1}{2}\int_0^t \sigma^2(s,T)\, ds + \int_0^t \sigma(s,T)\, dW_s& \\ - \int_0^t \int \left(e^{\delta(s,T,x)} - 1\right) F(dx)\, ds + \int_0^t \int \delta(s,T,x)\, \mu(ds,dx)&,\end{aligned} \tag{2.11}$$

the forward rate process (2.5) takes the expression:

$$\begin{aligned}f(t,T) = f(0,T) + \int_0^t \sigma_T(s,T)\sigma(s,T)\, ds - \int_0^t \sigma_T(s,T)\, dW_s& \\ + \int_0^t \int e^{\delta(s,T,x)}\delta_T(s,T,x)\, F(dx)\, ds - \int_0^t \int \delta_T(s,T,x)\, \mu(ds,dx)&\end{aligned} \tag{2.12}$$

(the subscript $T$ at the functions $\sigma(t,T)$ and $\delta(t,T,x)$ denotes partial differentiation with respect to the second variable), as well as the short rate process (2.7) is given by:

$$\begin{aligned}r(t) = f(0,t) + \int_0^t \sigma_T(s,t)\sigma(s,t)\, ds - \int_0^t \sigma_T(s,t)\, dW_s& \\ + \int_0^t \int e^{\delta(s,t,x)}\delta_T(s,t,x)\, F(dx)\, ds - \int_0^t \int \delta_T(s,t,x)\, \mu(ds,dx)&.\end{aligned} \tag{2.13}$$

Hence, by means of (2.4), we get that the short rate process satisfies the equation:

$$dr(t) = f_T(t,t)\, dt - \sigma_T(t,t)\, dW_t - \int \delta_T(t,t,x)\, \mu(dt,dx) \tag{2.14}$$

where

$$\begin{aligned}f_T(t,t) = f_T(0,t)& \\ + \int_0^t [\sigma_{TT}(s,t)\sigma(s,t) + \sigma_T^2(s,t)]\, ds - \int_0^t \sigma_{TT}(s,t)\, dW_s& \\ + \int_0^t \int e^{\delta(s,t,x)}[\delta_{TT}(s,t,x) + \delta_T^2(s,t,x)]\, F(dx)\, ds - \int_0^t \int \delta_{TT}(s,t,x)\, \mu(ds,dx)&\end{aligned} \tag{2.15}$$

and all the integrals in (2.12) - (2.15) are well-defined by virtue of the assumption (2.10) and the allowance to differentiate under the integral sign.

If in the Heath-Jarrow-Morton approach (see [14]) one starts with the specification of the forward rates (2.12), then the discounted bond prices turn out to be $(\mathcal{F}_t, Q)$ - martingales, or in other words, $Q$ is a *martingale measure*. In this case, integrating expression (2.12), we easily get the following representation for the bond price process (2.6):

$$\begin{aligned} P(t,T) = &\frac{P(0,T)}{P(0,t)} \\ &\times \exp\Big( - \int_t^T \int_0^t \sigma_T(s,u)\sigma(s,u)\,ds\,du + \int_t^T \int_0^t \sigma_T(s,u)\,dW_s\,du \\ &- \int_t^T \int_0^t \int e^{\delta(s,u,x)} \delta_T(s,u,x)\,F(dx)\,ds\,du + \int_t^T \int_0^t \int \delta_T(s,u,x)\,\mu(ds,dx)\,du \Big) \end{aligned} \tag{2.16}$$

where by means of assumptions (2.4), we have:

$$\sigma(t,T) = \int_t^T \sigma_T(t,v)\,dv \quad \text{and} \quad \delta(t,T,x) = \int_t^T \delta_T(t,v,x)\,dv \tag{2.17}$$

for all $0 \le t \le T \le T^*$.

2.2. Let us now describe the place of our model defined in (2.11) - (2.17) in the wide spectrum of bond market models considered in the literature. If $\delta(t,T,x) = 0$ for all $0 \le t \le T \le T^*$ and $x \in \mathbb{R}$, then we receive the term structure model introduced by Heath, Jarrow and Morton [14]. In the case when $F(dx)$ is degenerated to the point mass $\lambda$ at 1 for some $\lambda > 0$, we get the model of Shirakawa [25]. If $F(\mathbb{R}) < \infty$ then we have a particular case of the model considered by Björk, Kabanov and Runggaldier [3] where a more general driving marked point process was considered. If $\sigma(t,T)$ and $\delta(t,T,x)$ admit the representations:

$$\sigma(t,T) = \int_t^T \beta(t,u)\,du \quad \text{and} \quad \delta(t,T,x) = x \int_t^T \gamma(t,u)\,du \tag{2.18}$$

with deterministic continuously differentiable functions $\beta(t,T)$ and $\gamma(t,T)$ such that $\beta(t,T)$ or $\gamma(t,T)$ is identical zero or $\beta(t,T) = c\,\gamma(t,T)$ for some constant $c \in \mathbb{R}$ and all $0 \le t \le T \le T^*$, then we obtain a special form of the model proposed by Eberlein and Raible [10] under the additional assumption on the driving Lévy process to have trajectories of finite variation. A more general formulation of jump-diffusion model than that defined in (2.11) - (2.17) was considered by Björk et al. [4].

2.3. Since under conditions (2.9) and (2.10) the process $(P(t,T)/B(t))_{t\in[0,T]}$ turns out to be an $(\mathcal{F}_t, Q)$ - martingale, by means of (2.8), it is easily shown that the bond price $P = (P(t,T))_{t\in[0,T]}$ can be computed as:

$$P(t,T) = E\left[ \exp\left( - \int_t^T r(s)\,ds \right) \Big| \mathcal{F}_t \right]. \tag{2.19}$$

Observe that if the short rate $r = (r(t))_{t\in[0,T^*]}$ is an $(\mathcal{F}_t, Q)$ - Markov process, then (2.19) takes the form:

$$P(t,T) = E\left[ \exp\left( - \int_t^T r(s)\,ds \right) \Big| r(t) \right], \tag{2.20}$$

hence $P(t,T) = H(t, r(t), T)$, $0 \le t \le T \le T^*$, for some measurable function $H$ defined on $[0,T] \times \mathbb{R} \times [0,T^*]$, and we see that the bond price can be evaluated by means of the Markovian short rate.

In general, the process $r = (r(t))_{t\in[0,T^*]}$ is possibly non-Markovian, but in some cases there exists a finite number of processes $(r_i(t))_{t\in[0,T^*]}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, such that the process $(r(t), r_1(t), \ldots, r_n(t))_{t\in[0,T^*]}$ is $(\mathcal{F}_t, Q)$ - Markovian. Then the bond price admits the following representation:

$$P(t,T) = E\left[\exp\left(-\int_t^T r(s)\,ds\right) \Big| \, r(t), r_1(t), \ldots, r_n(t)\right], \tag{2.21}$$

and hence $P(t,T) = H_n(t, r(t), r_1(t), \ldots, r_n(t), T)$, $0 \le t \le T \le T^*$, for some measurable function $H_n$ defined on $[0,T] \times \mathbb{R}^{n+1} \times [0,T^*]$, $n \in \mathbb{N}$. Therefore, the a priori infinite-dimensional bond price process can be realized by means of a finite-dimensional Markovian system. In this case the process $(r(t), r_1(t), \ldots, r_n(t))_{t\in[0,T^*]}$ is called a *finite-dimensional Markovian realization* and its components are called *state variables* of the term structure model.

# 3 Markovian short rates

In this section we consider an extension of the model proposed by Eberlein and Raible [10] by using jump-diffusion processes as driving terms. These processes are decomposed into a continuous part and a finite variation pure jump part, and both parts are allowed to have different volatility coefficients.

3.1. In addition to the previous assumptions and conditions (2.3) and (2.10), let us suppose that $\sigma(t,T)$ and $\delta(t,T,x)$ are given by (2.18) where $t \mapsto \beta(t,T)$ and $t \mapsto \gamma(t,T)$ are deterministic functions continuously differentiable on $[0,T]$ for $T \in [0,T^*]$. In this case, the short rate process (2.13) admits the representation:

$$\begin{aligned} r(t) &= f(0,t) \\ &\quad + \int_0^t \beta(s,t) \int_s^t \beta(s,u)\,du\,ds + \int_0^t \int e^{\int_s^t \gamma(s,u)du\,x} \gamma(s,t)\,x\,F(dx)\,ds - Z(t) \end{aligned} \tag{3.1}$$

where the process $Z = (Z(t))_{t\in[0,T^*]}$ is defined by:

$$Z(t) = \int_0^t \beta(s,t)\,dW_s + \int_0^t \gamma(s,t)\,dJ_s \tag{3.2}$$

and the process $J = (J_t)_{t\in[0,T^*]}$ is given by:

$$J_t = \int_0^t \int x\,\mu(ds,dx). \tag{3.3}$$

It follows by assumptions (2.3) and (2.18) that the value (3.3) is finite for all $t \in [0,T^*]$.

The main result of this section will be the criterion for the short rate $r = (r(t))_{t\in[0,T^*]}$ to be Markovian. We start with a slight extension of an assertion from [10] to the case of a jump-diffusion process.

**Lemma 3.1.** *Let $Z = (Z(t))_{t\in[0,T^*]}$ from (3.2) be an $(\mathcal{F}_t, Q)$ - Markov process. Then for all fixed $S$ and $T$ with $0 \le T \le S \le T^*$ there is a Borel function $G$ such that:*

$$\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t] = G\left(\int_0^T [\beta(t,T)\,dW_t + \gamma(t,T)\,dJ_t]\right) \quad (Q - \text{a.s.}). \tag{3.4}$$

**Proof.** If the process $Z = (Z(t))_{t\in[0,T^*]}$ is Markovian, then for all $0 \le T \le S \le T^*$ we have:

$$E[Z(S)\,|\,\mathcal{F}_T] = E[Z(S)\,|\,Z(T)] \quad (Q - \text{a.s.}). \tag{3.5}$$

Since the integrands $\beta(t,T)$ and $\gamma(t,T)$ for $0 \le t \le T \le T^*$ are deterministic, from the independence of increments of the processes $W = (W_t)_{t\in[0,T^*]}$ and $J = (J_t)_{t\in[0,T^*]}$ it follows that:

$$\begin{aligned} &E[Z(S)\,|\,\mathcal{F}_T] \\ &= E\left[\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\,\Big|\,\mathcal{F}_T\right] + E\left[\int_T^S [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\right] \end{aligned} \tag{3.6}$$

$(Q - \text{a.s.})$ and

$$\begin{aligned} &E[Z(S)\,|\,Z(T)] \\ &= E\left[\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\,\Big|\,Z(T)\right] + E\left[\int_T^S [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\right] \end{aligned} \tag{3.7}$$

$(Q - \text{a.s.})$. Hence, combining (3.5) - (3.7), we get:

$$\begin{aligned} &\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t] \\ &\quad = E\left[\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\,\Big|\,\int_0^T [\beta(t,T)\,dW_t + \gamma(t,T)\,dJ_t]\right] \end{aligned} \tag{3.8}$$

$(Q - \text{a.s.})$ that immediately implies the desired assertion. □

As a next step we shall prove a lemma being an extension of corresponding results from [10] and [20] to the case of a jump-diffusion process. The somewhat surprising fact here is that the constant $c$ is equal for both cases $j = 1$ and $j = 2$.

**Lemma 3.2.** *Let $f_j(t)$ and $g_j(t)$, $t \in [0,T]$, $j = 1,2$, be not identical zero continuously differentiable functions on $[0,T]$ for some $T > 0$. Assume that for a Borel function $G$ we have:*

$$\int_0^T f_1(t)\,dW_t + \int_0^T f_2(t)\,dJ_t = G\left(\int_0^T g_1(t)\,dW_t + \int_0^T g_2(t)\,dJ_t\right) \quad (Q - a.s.). \tag{3.9}$$

*Then $f_j(t) = c\,g_j(t)$ for $j = 1,2$ and all $t \in [0,T]$ as well as $G(x) = cx$ for some constant $c \in \mathbb{R}$ and Lebesgue almost all $x \in \mathbb{R}$.*

**Proof.** By virtue of independence of the processes $W$ and $J$, from (3.9) it follows that:

$$\int_0^T f_1(t)\,dW_t + u = G\left(\int_0^T g_1(t)\,dW_t + v\right) \quad (Q - \text{a.s.}) \tag{3.10}$$

for $Q(I_T^2(f_2), I_T^2(g_2))$ - almost all $(u,v) \in \mathbb{R}^2$, where we denote:

$$I_T^2(f_2) = \int_0^T f_2(t)\, dJ_t \quad \text{and} \quad I_T^2(g_2) = \int_0^T g_2(t)\, dJ_t. \tag{3.11}$$

Since $J$ is degenerated and $f_2(t)$ and $g_2(t)$ are not identical zero on $[0,T]$, there exists at least one pair $(u',v') \in \mathbb{R}^2$ such that the equality in (3.10) holds with $u = u'$ and $v = v'$. Then, using the fact that the only transformations mapping Gaussian random variables to Gaussian random variables are affine ones, we conclude that $G(x) = cx + d$ for Lebesgue almost all $x$ and some $c, d \in \mathbb{R}$, which do not depend on $u'$ and $v'$, and Lebesgue almost all $x \in \mathbb{R}$. It thus follows that the difference $I_T^1(f_1) - cI_T^1(g_1)$ with:

$$I_T^1(f_1) = \int_0^T f_1(t)\, dW_t \quad \text{and} \quad I_T^1(g_1) = \int_0^T g_1(t)\, dW_t \tag{3.12}$$

has zero variance, and hence $f_1(t) = cg_1(t)$ for all $t \in [0,T]$. Therefore, (3.9) implies:

$$\int_0^T (f_2(t) - cg_2(t) + 1)\, dJ_t - d = J_T \quad (Q - \text{a.s.}). \tag{3.13}$$

Then, by virtue of the result of [19], we conclude that $f_2(t) - cg_2(t) + 1 = c'$ for some constant $c' \in \mathbb{R}$ and all $t \in [0,T]$. Thus, from (3.13) it follows that $(c'-1)y = d$ for $Q(J_T)$ - almost all $y \in \mathbb{R}$.

Let us assume that $c' \neq 1$. Since $J$ is nondeterministic, we conclude that there exist two different points $y', y'' \in \mathbb{R}$ such that the last equality should hold with $y = y'$ and $y = y''$, respectively. This contradicts to the assumption $c' \neq 1$, which also implies that $d = 0$ and thus completes the proof of the lemma. □

3.2. We now formulate and prove the criterion for the short rate to be Markovian. Actually, we extend the results of [6], [10] and [20] to the case where the forward rate is driven by a jump-diffusion process.

**Theorem 3.3.** *Suppose that the functions $t \mapsto \beta(t,T)$ and $t \mapsto \gamma(t,T)$ defined in (2.18) are not identical zero on $[0,T]$ for each $T \in (0,T^*]$. Then the short rate process $r = (r(t))_{t\in[0,T^*]}$ is Markovian if and only if there are continuously differentiable functions $\eta(t)$ and $\kappa(t)$, $t \in [0,T^*]$, and a function $\zeta(T) > 0$, $T \in (0,T^*]$, such that:*

$$\beta(t,T) = \eta(t)\,\zeta(T) \quad \text{and} \quad \gamma(t,T) = \kappa(t)\,\zeta(T) \tag{3.14}$$

*for all $0 \le t \le T \le T^*$.*

**Proof.** (i) Let us first assume that $r = (r(t))_{t\in[0,T^*]}$ is a Markov process. Then by virtue of (3.1), the same holds for $Z = (Z(t))_{t\in[0,T^*]}$, and Lemma 3.1 shows that for each $0 < T \le S \le T^*$ there exists a Borel function $G$ such that (3.4) is valid.

Applying Lemma 3.2 to the functions $t \mapsto \beta(t,T^*)$ and $t \mapsto \beta(t,T)$ as well as $t \mapsto \gamma(t,T^*)$ and $t \mapsto \gamma(t,T)$, we get that there exists a function $\xi(T,T^*)$, not depending on $t$, such that:

$$\beta(t,T^*) = \xi(T,T^*)\,\beta(t,T) \quad \text{and} \quad \gamma(t,T^*) = \xi(T,T^*)\,\gamma(t,T) \tag{3.15}$$

for all $0 \le t \le T \le T^*$. Since the functions $\beta(t,T)$ and $\gamma(t,T)$ are assumed to be nonidentical zero on $[0,T]$, from (3.15) we conclude that $\xi(T,T^*) \neq 0$ for each $T \in (0,T^*]$. We have even $\xi(T,T^*) > 0$ because of $\xi(T^*,T^*) = 1$ and a continuity argument. (Note that the functions $T \mapsto \beta(t,T)$ and $T \mapsto \gamma(t,T)$ are continuous by the assumptions on the functions $\sigma(t,T)$ and $\delta(t,T,x)$.) Otherwise, it would follow that $\beta(t,T^*) = 0$ and $\gamma(t,T^*) = 0$ for all $t \in [0,T]$ and some $T \in (0,T^*]$, which is excluded by assumption. Therefore, defining $\eta(t) = \beta(t,T^*)$, $\kappa(t) = \gamma(t,T^*)$ and $\zeta(T) = 1/\xi(T,T^*) > 0$, we obtain the decompositions (3.14). The continuous differentiability of the functions $\eta(t)$ and $\kappa(t)$, $t \in [0,T^*]$, directly follows from the assumption of $t \mapsto \beta(t,T^*)$ and $t \mapsto \gamma(t,T^*)$ to be continuously differentiable on $[0,T^*]$, respectively.

(ii) Suppose now that the functions $\beta(t,T)$ and $\gamma(t,T)$, $0 \le t \le T \le T^*$, satisfy conditions (3.14). In this case the process $Z = (Z(t))_{t\in[0,T^*]}$ from (3.2) can be represented in the form:

$$Z(t) = \zeta(t) \left( \int_0^t \eta(s)\, dW_s + \int_0^t \kappa(s)\, dJ_s \right) \tag{3.16}$$

and hence, it is Markovian and so the process $r = (r(t))_{t\in[0,T^*]}$ is also. $\square$

**Remark 3.4.** Taking into account the expressions (2.18) we observe that in the case the conditions (3.14) are satisfied, the forward rate process (2.12) admits the representation:

$$\begin{aligned} f(t,T) = f(0,T) &+ \zeta(T) \int_0^t \eta^2(s) \int_s^T \zeta(u)\, du\, ds \\ &+ \zeta(T) \int_0^t \int e^{\kappa(s) \int_s^T \zeta(u)\, du\, x} \kappa(s)\, x\, F(dx)\, ds - \frac{\zeta(T)}{\zeta(t)} Z(t) \end{aligned} \tag{3.17}$$

and the short rate process (2.13) takes the expression:

$$\begin{aligned} r(t) = f(0,t) &+ \zeta(t) \int_0^t \eta^2(s) \int_s^t \zeta(u)\, du\, ds \\ &+ \zeta(t) \int_0^t \int e^{\kappa(s) \int_s^t \zeta(u)\, du\, x} \kappa(s)\, x\, F(dx)\, ds - Z(t) \end{aligned} \tag{3.18}$$

where the process $Z = (Z(t))_{t\in[0,T^*]}$ is given by (3.16).

**Example 3.5.** Suppose that $J = (J_t)_{t\in[0,T^*]}$ is a bilateral gamma process, that is a Lévy process with the triplet $(0,0,F(dx))$ where:

$$F(dx) = \left( \frac{\alpha_+}{x} e^{-\lambda_+ x} I\{x > 0\} + \frac{\alpha_-}{-x} e^{\lambda_- x} I\{x < 0\} \right) dx \tag{3.19}$$

and $\lambda_+, \lambda_-, \alpha_+, \alpha_-$ are some positive parameters (see e.g. [20; Section 5]). In this case, if $\left| \kappa(t) \int_t^T \zeta(u) du \right| < \min\{\lambda_+, \lambda_-\}$ for all $0 \le t \le T \le T^*$ and $x \in \mathbb{R}$, then conditions (2.3) and (2.10) are satisfied. Thus, the assertion of Theorem 3.4 holds and the expressions (3.17) and

(3.18) take the explicit form:

$$f(t,T) = f(0,T) + \zeta(T)\int_0^t \eta^2(s)\int_s^T \zeta(u)\,du\,ds \tag{3.20}$$
$$+\int_0^t \left(\frac{\alpha_+\kappa(s)\zeta(T)}{\lambda_+ - \kappa(s)\int_s^T\zeta(u)du} - \frac{\alpha_-\kappa(s)\zeta(T)}{\lambda_- + \kappa(s)\int_s^T\zeta(u)du}\right)ds - \frac{\zeta(T)}{\zeta(t)}Z(t)$$

and

$$r(t) = f(0,t) + \zeta(t)\int_0^t \eta^2(s)\int_s^t \zeta(u)\,du\,ds \tag{3.21}$$
$$+\int_0^t \left(\frac{\alpha_+\kappa(s)\zeta(t)}{\lambda_+ - \kappa(s)\int_s^t\zeta(u)du} - \frac{\alpha_-\kappa(s)\zeta(t)}{\lambda_- + \kappa(s)\int_s^t\zeta(u)du}\right)ds - Z(t)$$

where the process $Z = (Z(t))_{t\in[0,T^*]}$ is given by (3.16).

# 4 Finite-dimensional Markovian realizations

In this section we omit the assumption that $\sigma(t,T)$ and $\delta(t,T,x)$ are deterministic. We consider the case that $\sigma(t,T)$ and $\delta(t,T,x)$ depend on $r(t)$. For the model driven by a Wiener process this case was studied in [24]. We show that when $\delta(t,T,x)$ satisfies the additive-type condition (4.9), there exists a finite-dimensional Markovian realization of the term structure model. If (4.9) is substituted by the multiplicative-type condition (4.28), then the realization is, in general, of infinite dimension. In the sequel we assume that $F(\mathbb{R}) < \infty$. It means that the process (4.12) has finitely many jumps in finite intervals only and the proofs below rely heavily on this fact. The extension to the case $F(\mathbb{R}) = \infty$ seems to be a technical problem, but it remains open here.

4.1. In the sequel we will use the notation of Section 2 without assuming conditions (2.18). We immediately observe that by virtue of the conditions (2.3) and (2.10) and the assumptions of twice continuous differentiability of $\sigma(t,T)$ and $\delta(t,T,x)$ in the second variable for $\omega \in \Omega$ and $x \in \mathbb{R}$ fixed, as well as by the allowance to differentiate under the integral sign, all the integrals appearing in the sequel are well-defined.

Let us suppose that the condition:

$$\sigma_T(t,T) = \eta(t,r(t))\,\zeta(T) \tag{4.1}$$

holds for some deterministic continuous function $\eta(t,r)$ and twice continuously differentiable function $\zeta(T)$. (Note that (4.1) coincides with the condition (2.11) in [24].) In this case, applying (2.4), we get (here and in the sequel we assume $s \le t$):

$$\int_t^T \sigma_T(s,u)\,du = \theta(t,T)\sigma_T(s,t) \tag{4.2}$$

where

$$\theta(t,T) = \int_t^T \frac{\zeta(u)}{\zeta(t)}\,du. \tag{4.3}$$

Then, using the obvious fact that

$$\int_t^T \frac{\zeta(u)}{\zeta(t)} \int_t^u \frac{\zeta(v)}{\zeta(t)}\, dv\, du = \frac{\theta^2(t,T)}{2}, \tag{4.4}$$

we conclude

$$\int_t^T \sigma_T(s,u)\sigma(s,u)\, du = \theta(t,T)\sigma_T(s,t)\sigma(s,t) + \frac{\theta^2(t,T)}{2}\sigma_T^2(s,t). \tag{4.5}$$

Thus, changing the order of integration, we obtain:

$$\int_t^T \int_0^t \sigma_T(s,u)\, dW_s\, du - \int_t^T \int_0^t \sigma_T(s,u)\sigma(s,u)\, ds\, du = \theta(t,T)\chi(t) - \frac{\theta^2(t,T)}{2}\phi(t) \tag{4.6}$$

where the processes $(\phi(t))_{t\in[0,T^*]}$ and $(\chi(t))_{t\in[0,T^*]}$ are defined by:

$$\phi(t) = \int_0^t \sigma_T^2(s,t)\, ds \tag{4.7}$$

and

$$\chi(t) = \int_0^t \sigma_T(s,t)\, dW_s - \int_0^t \sigma_T(s,t)\sigma(s,t)\, ds. \tag{4.8}$$

Let us further assume that the function $\delta(t,T,x)$ has the structure

$$\delta(t,T,x) = \log[\kappa(t,r(t),x)\,\lambda(T)] \tag{4.9}$$

for some continuous function $\kappa(t,r,x)$ and twice continuously differentiable function $\lambda(T)$. In this case (under the condition $F(\mathbb{R}) < \infty$) it is easily seen that:

$$\int_t^T \int_0^t \int \delta_T(s,u,x)\,\mu(ds,dx)\, du = \log\left(\frac{\lambda(T)}{\lambda(t)}\right)\mu(t) \tag{4.10}$$

and

$$\int_t^T \int_0^t \int e^{\delta(s,u,x)}\delta_T(s,u,x)\, F(dx)\, ds\, du = (\lambda(T) - \lambda(t))\upsilon(t) \tag{4.11}$$

where the processes $(\mu(t))_{t\in[0,T^*]}$ and $(\upsilon(t))_{t\in[0,T^*]}$ are defined by:

$$\mu(t) = \int_0^t \int \mu(ds,dx) \tag{4.12}$$

and

$$\upsilon(t) = \int_0^t \int \kappa(s,r(s),x)\, F(dx)\, ds. \tag{4.13}$$

Summarizing the facts shown above let us formulate the main assertion of the section.

**Theorem 4.1.** *Assume that we have $F(\mathbb{R}) < \infty$ and conditions (4.1) and (4.9) are satisfied. Then the bond price (2.16) admits the representation:*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \tag{4.14}$$
$$\times \exp\left(\theta(t,T)\chi(t) - \frac{\theta^2(t,T)}{2}\phi(t) - (\lambda(T) - \lambda(t))\upsilon(t) + \log\frac{\lambda(T)}{\lambda(t)}\mu(t)\right)$$

*where $\theta(t,T)$, $\phi(t)$, $\chi(t)$, $\mu(t)$ and $\upsilon(t)$ are defined in (4.3), (4.7), (4.8), (4.12) and (4.13), respectively. If, in addition, the function $\eta(t,r)$ satisfies the Lipschitz condition:*

$$|\eta(t,r) - \eta(t,r')| \leq C|r - r'| \tag{4.15}$$

*for some $C > 0$ fixed, all $t \in [0,T^*]$ and $r, r' \in \mathbb{R}$, then $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{[0,T^*]}$ is a finite-dimensional Markovian realization.*

**Proof.** Since the representation (4.14) for the bond price (2.16) easily follows from (4.6), (4.10) and (4.11) above, it remains us to prove the Markov property of the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{[0,T^*]}$ under condition (4.15). For this, let us first observe that from (4.1) - (4.8) it follows that:

$$\int_0^t [\sigma_{TT}(s,t)\sigma(s,t) + \sigma_T^2(s,t)]\, ds - \int_0^t \sigma_{TT}(s,t)\, dW_s = -\frac{\zeta'(t)}{\zeta(t)}\chi(t) + \phi(t) \tag{4.16}$$

as well as from (4.9) - (4.13) it is seen that:

$$\int_0^t \int \delta_{TT}(s,t,x)\, \mu(ds,dx) = \frac{\lambda''(t)\lambda(t) - (\lambda'(t))^2}{\lambda^2(t)}\mu(t) \tag{4.17}$$

and

$$\int_0^t \int e^{\delta(s,t,x)}[\delta_{TT}(s,t,x) + \delta_T^2(s,t,x)]\, F(dx)\, ds = \lambda''(t)\upsilon(t). \tag{4.18}$$

Here, by virtue of (2.4), the processes (4.7) - (4.8) and (4.12) - (4.13) admit the representations:

$$d\phi(t) = \left[2\frac{\zeta'(t)}{\zeta(t)}\phi(t) + \eta^2(t,r(t))\zeta^2(t)\right] dt, \tag{4.19}$$

$$d\chi(t) = \left[\frac{\zeta'(t)}{\zeta(t)}\chi(t) - \phi(t)\right] dt + \eta(t,r(t))\zeta(t)\, dW_t, \tag{4.20}$$

$$d\mu(t) = \int \mu(dt,dx) \tag{4.21}$$

and

$$d\upsilon(t) = \int \kappa(t,r(t),x)\, F(dx)\, dt. \tag{4.22}$$

It follows from (4.16) - (4.18) that the short rate process (2.13) - (2.14) takes the expression:

$$dr(t) = f_T(t,t)\, dt - \eta(t,r(t))\zeta(t)\, dW_t - \frac{\lambda'(t)}{\lambda(t)}\int \mu(dt,dx) \tag{4.23}$$

where

$$f_T(t,t) = f_T(0,t) - \frac{\zeta'(t)}{\zeta(t)}\chi(t) + \phi(t) + \lambda''(t)\upsilon(t) - \frac{\lambda''(t)\lambda(t) - (\lambda'(t))^2}{\lambda^2(t)}\mu(t). \tag{4.24}$$

Observe that the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{t\in[0,T^*]}$ satisfies the multidimensional stochastic differential equation (4.19) - (4.23) with (4.24), which admits a solution for all possible initial conditions $(r_0, \phi_0, \chi_0, \mu_0, \upsilon_0)$ with nonnegative $\phi_0$ and $\mu_0$. We also note that, by virtue of the assumption $F(\mathbb{R}) < \infty$, the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{t\in[0,T^*]}$ has a finite number of jumps on each finite time interval. Hence, taking into account the fact that the function $\eta(t,r)$ is assumed to satisfy condition (4.15), by means of the strong uniqueness argument for solutions of stochastic differential equations (see e.g. [16; Chapter XIV, Theorem 14.95], [17; Section III, Theorem 2.32] and [13; Chapter IV, Theorem 1]) we conclude that the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{t\in[0,T^*]}$ is a unique strong solution of (4.19) - (4.23) with (4.24). Therefore, by virtue of the arguments in [13; Chapter VI, Section 1], it is Markovian. □

**Example 4.2.** Suppose that in the conditions of Theorem 4.1 we have $\sigma_T(t,T) = \eta(t,r(t))e^{-\beta T}$ and $\delta(t,T,x) = \rho(t,r(t),x) - \gamma T$ for some deterministic functions $\eta(t,r)$ and $\rho(t,r,x)$. Moreover, let $\beta$ and $\gamma$ be some nonzero constants. It follows that in this case $\zeta(T) = e^{-\beta T}$ and $\lambda(T) = e^{-\gamma T}$, and thus, the expression (4.14) takes the form:

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)} \\ &\times \exp\left(\frac{1-e^{-\beta(T-t)}}{\beta}\chi(t) - \frac{(1-e^{-\beta(T-t)})^2}{2\beta^2}\phi(t) - (e^{-\gamma T} - e^{-\gamma t})\upsilon(t) - \gamma(T-t)\mu(t)\right) \end{aligned} \tag{4.25}$$

where $\phi(t)$ and $\chi(t)$ defined in (4.7) and (4.8) admit the representations:

$$\phi(t) = e^{-2\beta t}\int_0^t \eta^2(s,r(s))\,ds \tag{4.26}$$

and

$$\chi(t) = e^{-\beta t}\int_0^t \eta(s,r(s))\,dW_s - e^{-\beta t}\int_0^t \eta^2(s,r(s))\frac{e^{-\beta s} - e^{-\beta t}}{\beta}\,ds \tag{4.27}$$

as well as $\mu(t)$ and $\upsilon(t)$ are given by (4.12) and (4.13) with $\kappa(t,r(t),x) = e^{\rho(t,r(t),x)}$.

**Remark 4.3.** Observe that if in the assumptions of Theorem 4.1 we have $\delta(t,T,x) = 0$ for all $0 \le t \le T \le T^*$ and $x \in \mathbb{R}$, then we get the result of [24].

4.2. Finally, we show that if instead of the additive-type condition (4.9) we consider one of multiplicative type, then the short rate process does not need to be a component of a finite-dimensional Markov process, but it can be a component of an infinite-dimensional one. For this, let us suppose that in the assumptions of Theorem 4.1 condition (4.9) is replaced by:

$$\delta_T(t,T,x) = \kappa(t,r(t),x)\,\lambda(T). \tag{4.28}$$

In this case it follows that:

$$\int_0^t \int \delta_{TT}(s,t,x)\,\mu(ds,dx) = \frac{\lambda'(t)}{\lambda(t)}\int_0^t \int \delta_T(s,t,x)\,\mu(ds,dx) \tag{4.29}$$

and

$$\int_0^t \int e^{\delta(s,t,x)} \delta_{TT}(s,t,x)\, F(dx)\, ds = \frac{\lambda'(t)}{\lambda(t)} \int_0^t \int e^{\delta(s,t,x)} \delta_T(s,t,x)\, F(dx)\, ds. \tag{4.30}$$

Thus, by means of the arguments of the proof of Theorem 4.1 (i), we obtain that the short rate process (2.13) - (2.14) satisfies the stochastic differential equation:

$$dr(t) = f_T(t,t)\, dt - \eta(t, r(t))\zeta(t)\, dW_t - \int \kappa(t, r(t), x)\lambda(t)\, \mu(dt, dx) \tag{4.31}$$

where

$$f_T(t,t) = f_T(0,t) - \frac{\zeta'(t)}{\zeta(t)}\chi(t) + \phi(t) + \frac{\lambda'(t)}{\lambda(t)}[\xi_1(t) - \psi(t)] + \xi_2(t), \tag{4.32}$$

the processes $(\phi(t))_{t\in[0,T^*]}$ and $(\chi(t))_{t\in[0,T^*]}$ are given by (4.7) - (4.8) as well as $(\psi(t))_{t\in[0,T^*]}$ and $(\xi_n(t))_{t\in[0,T^*]}$, $n \in \mathbb{N}$, are defined by:

$$\psi(t) = \int_0^t \int \kappa(s, r(s), x)\, \mu(ds, dx) \tag{4.33}$$

and

$$\xi_n(t) = \int_0^t \int e^{\delta(s,t,x)} \delta_T^n(s,t,x)\, F(dx)\, ds. \tag{4.34}$$

Differentiating the identities (4.33) and (4.34), we get:

$$d\psi(t) = \int \kappa(t, r(t), x)\, \mu(dt, dx) \tag{4.35}$$

and

$$d\xi_n(t) = \left(\xi_{n+1}(t) + n\frac{\lambda'(t)}{\lambda(t)}\xi_n(t)\right) dt + \int \kappa^n(t, r(t), x)\lambda^n(t)\, F(dx)\, dt. \tag{4.36}$$

**Remark 4.4.** From (4.19) - (4.20), (4.31) - (4.32), and (4.35) - (4.36) we see that in this case there is an infinite number of state variables $(\phi(t))_{t\in[0,T^*]}$, $(\chi(t))_{t\in[0,T^*]}$, $(\psi(t))_{t\in[0,T^*]}$, $(\xi_n(t))_{t\in[0,T^*]}$, $n \in \mathbb{N}$. In particular, the process $(r(t), \phi(t), \chi(t), \psi(t),\ \xi_1(t), \xi_2(t))_{t\in[0,T^*]}$ is not Markovian.

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