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On large deviations in testing Ornstein-Uhlenbeck-type models*

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We obtain exact large deviation rates for the log-likelihood ratio in testing models with observed Ornstein-Uhlenbeck processes and get explicit rates of decrease for the error probabilities of Neyman-Pearson, Bayes, and minimax tests. Moreover, we give expressions for the rates of decrease for the error probabilities of Neyman-Pearson tests in models with observed processes solving affine stochastic delay differential equations.

1 Introduction

Asymptotic properties of likelihood ratios play an important role in statistical testing problems. Sometimes they can be studied by using large deviation results, for example, in the case of binary statistical experiments. Chernoff [9] proved large deviation theorems for sums of i.i.d. observations. Bahadur [1]-[3] studied asymptotic efficiency of tests and estimates for observed sequences of random variables (see also Bahadur, Zabel and Gupta [4]). Birgé [8] applied the results of [9] to the investigation of the rate of decrease for error probabilities of Neyman-Pearson tests. Generalizations of the large deviation results to the case of semimartingale models and their applications are collected in the monograph [21]. Lin'kov [22] proved large deviation theorems for extended random variables and applied them to the investigation of general statistical experiments. Exact large deviation rates for the log-likelihood ratio in testing models with fractional Brownian motion were derived in [23]. In the present paper we derive an explicit form of large deviation theorems of Chernoff type for the log-likelihood ratio in testing models with Ornstein-Uhlenbeck processes by applying the large deviation results from the general continuous-time semimartingale framework of Lin'kov [21]. Note that the results

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in [4] were obtained by using the large deviation techniques for sequences of random variables. The problem of testing mean reversion for processes of Ornstein-Uhlenbeck type was earlier studied by Szimayer and Maller [27]. Note that the Ornstein-Uhlenbeck processes play a key role for modeling the behavior of interest rates in financial markets (see e.g. [29] or [5]).

In recent years, several statistical problems for models with time delay were studied. Dietz [11] considered an Ornstein-Uhlenbeck-type model with exponential decreasing memory and proved the local asymptotically mixed normality (in an extended sense) of the suitably normalized model. Gushchin and Küchler [12] - [14] derived local asymptotic properties of the likelihood process in (two-parameter) models connected with a special case of affine stochastic delay differential equations. Putschke [25] continued this investigation for a multi-parametric case of such affine delay equations. Küchler and Kutoyants [17] studied the asymptotic behavior of the maximum likelihood and Bayesian estimators of delay in a simple Ornstein-Uhlenbeck-type model. Küchler and Vasil'ev [18] investigated the almost sure consistency and asymptotic normality of sequential estimators for multi-parametric affine delay equations. Gushchin and Küchler [15] derived conditions under which a model with an affine stochastic delay differential equation satisfies the local asymptotic normality property and where the maximum likelihood and Bayesian estimators of a parameter are asymptotically normal and efficient. In this paper we consider the problem of testing hypotheses and study the asymptotic behavior of the error probabilities for Neyman-Pearson tests in Ornstein-Uhlenbeck-type models with delay. Asymptotic properties for tests of delay parameters in the cases of small noise and large sample size were recently studied by Kutoyants [19] - [20].

The paper is organized as follows. In Section 2, we cite large deviation results for the log-likelihood ratio process and their applications to the investigation of the rates of decrease for error probabilities of Neyman-Pearson, Bayes, and minimax tests (cf. [21] - [23]). In Section 3, by means of explicit expressions for Hellinger integrals, we obtain exact large deviation rates for the log-likelihood ratio in a model of testing hypotheses about the parameter of an Ornstein-Uhlenbeck process. We remark that there appears some kind of discontinuity in the solution when the basic hypothesis is altered. The results are applied to the investigation of the rates of decrease for error probabilities of the tests mentioned above. It seems possible to derive the analogues of some of these results in the models with discretely observed data (see e.g. [4]). For this, some essentially different techniques, which is applied by derivation of the large deviation results for testing models with sequences of random variables, should be used. Then the initial continuous-time assertions can be obtained as a limiting case of the corresponding results from the discrete-time case by applying the invariance principle. In Section 4, we get the rates of decrease for the error probabilities of Neyman-Pearson tests in models with processes that solve affine stochastic delay differential equations and give two illustrating examples.

2 Large deviation theorems and their applications

Suppose that $(\Omega, \mathcal{F}, P_0, P_1)$ is a binary statistical experiment and that $X = (X_t)_{t\geq 0}$ is a real-valued process. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by X, that is $\mathcal{F}_t = \sigma(X_s \,|\, 0 \leq s \leq t)$, $t \geq 0$. Let H_0 and H_1 be two statistical hypotheses under which the distribution of the observed process X is given by the different measures P_0 and P_1, respectively. We will consider the problem of testing the hypothesis H_0 against its alternative H_1. In this section we cite some results from Lin'kov [21] - [23] and use his notation.

2.1. Suppose that the measures P_0 and P_1 are locally equivalent on the filtration $(\mathcal{F}_t)_{t\geq 0}$ and introduce the *log-likelihood ratio* process $\Lambda = (\Lambda_t)_{t\geq 0}$ defined as the logarithm of the Radon-Nikodym derivative:

$$\Lambda_t = \log \frac{d(P_1|\mathcal{F}_t)}{d(P_0|\mathcal{F}_t)} \tag{2.1}$$

for all $t \geq 0$. Let the process $H(\varepsilon) = (H_t(\varepsilon))_{t\geq 0}$ be the *Hellinger integral* of the order $\varepsilon \in (-\infty, \infty)$ of the restrictions $P_1|\mathcal{F}_t$ and $P_0|\mathcal{F}_t$ given by:

$$H_t(\varepsilon) := H_t(\varepsilon; P_1, P_0) = E_0[\exp(\varepsilon \Lambda_t)] \tag{2.2}$$

for all $t \geq 0$ (see e.g. [16; Chapter IV, Section 1]). Note that the relationship $H_t(\varepsilon; P_0, P_1) = H_t(1-\varepsilon; P_1, P_0)$ holds for all $\varepsilon \in (-\infty, \infty)$ and $t \geq 0$.

We will say that the Hellinger integral (2.2) satisfies the *regularity condition* if for some function ψ_t, $t \geq 0$, such that $\psi_t \to \infty$ as $t \to \infty$, the (possibly infinite) limit

$$\varkappa(\varepsilon) := \lim_{t\to\infty} \psi_t^{-1} \log H_t(\varepsilon) \tag{2.3}$$

exists for all $\varepsilon \in (-\infty, \infty)$. It is known (see e.g. [10; Chapter III, Section 3]) that the function $\varkappa(\varepsilon)$ is a strictly convex and continuously differentiable function on $(\varepsilon_-, \varepsilon_+)$ with

$$-\infty \leq \varepsilon_- := \inf\{\varepsilon \,|\, \varkappa(\varepsilon) < \infty\} < \varepsilon_+ := \sup\{\varepsilon \,|\, \varkappa(\varepsilon) < \infty\} \leq \infty \tag{2.4}$$

where $\varepsilon_- \leq 0$ and $\varepsilon_+ \geq 1$. If $\varepsilon_- < 0$ or $\varepsilon_+ > 1$ then the derivatives $\varkappa'(0)$ and $\varkappa'(1)$ are well-defined, respectively.

For every $\gamma \in \mathbb{R}$ let us define the function $I(\gamma)$ as the *Legendre-Fenchel transform* of $\varkappa(\varepsilon)$ by:

$$I(\gamma) := \sup_{\varepsilon \in (\varepsilon_-, \varepsilon_+)} (\varepsilon\gamma - \varkappa(\varepsilon)) \tag{2.5}$$

(cf. e.g. [26]) with

$$\varkappa'(\varepsilon_-+) := \lim_{\varepsilon \downarrow \varepsilon_-} \varkappa'(\varepsilon) < \varkappa'(\varepsilon_+-) := \lim_{\varepsilon \uparrow \varepsilon_+} \varkappa'(\varepsilon) \tag{2.6}$$

and define the values:

$$\gamma_0 := \varkappa'(0) \quad \text{if} \quad \varepsilon_- < 0, \qquad \gamma_0 := \varkappa'(0+) = \lim_{\varepsilon \downarrow \varepsilon_-} \varkappa'(\varepsilon) \quad \text{if} \quad \varepsilon_- = 0 \tag{2.7}$$

$$\gamma_1 := \varkappa'(1) \quad \text{if} \quad \varepsilon_+ > 1, \qquad \gamma_1 := \varkappa'(1-) = \lim_{\varepsilon \uparrow \varepsilon_+} \varkappa'(\varepsilon) \quad \text{if} \quad \varepsilon_+ = 1 \tag{2.8}$$

where by virtue of the convexity of $\varkappa(\varepsilon)$ on $(\varepsilon_-, \varepsilon_+)$ we have $\gamma_0 < \gamma_1$.

The following assertion is a large deviation theorem of Chernoff type for the log-likelihood ratio process $\Lambda = (\Lambda_t)_{t\geq 0}$.

Proposition 2.1. *Let the regularity condition (2.3) be satisfied. Then the following conclusions are valid:*

(i) if $\gamma_0 < \varkappa'(\varepsilon_+-)$ *then for all* $\gamma \in (\gamma_0, \varkappa'(\varepsilon_+-))$ *we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t > \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t \geq \gamma] = -I(\gamma); \tag{2.9}$$

(ii) if $\varepsilon_- < 0$ and $\varkappa'(\varepsilon_-+) < \varkappa'(0)$ then for all $\gamma \in (\varkappa'(\varepsilon_-+), \varkappa'(0))$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t < \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t \le \gamma] = -I(\gamma); \tag{2.10}$$

(iii) if $\varkappa'(\varepsilon_-+) < \gamma_1$ then for all $\gamma \in (\varkappa'(\varepsilon_-+), \gamma_1)$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t < \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t \le \gamma] = \gamma - I(\gamma); \tag{2.11}$$

(iv) if $\varepsilon_+ > 1$ and $\varkappa'(1) < \varkappa'(\varepsilon_+-)$ then for all $\gamma \in (\varkappa'(1), \varkappa'(\varepsilon_+-))$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t > \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t \ge \gamma] = \gamma - I(\gamma). \tag{2.12}$$

This assertion is proved by using large deviation theorems for extended random variables in [22].

2.2. The results cited above give an opportunity to investigate the rate of decrease of the error probabilities for some statistical tests. In the rest of the section we refer some results about the asymptotic behavior of the error probabilities for Neyman-Pearson, Bayes, and minimax tests. The proofs of these results can be found in [22] (see also references in [23]).

Let α_t, $t \ge 0$, be an arbitrary function having values in $(0,1)$, and let $\delta_t(\alpha_t)$ be a *Neyman-Pearson* test of the level $\alpha_t \in (0,1)$ for testing the hypotheses H_0 and H_1 under the observations X_s, $0 \le s \le t$ (see e.g. [21; Chapter II, Section 2.1]). The following assertion describes the rate of decrease for the error probabilities of the first and second kind α_t and $\beta(\alpha_t)$, respectively, for the test $\delta_t(\alpha_t)$ under the regularity condition (2.3).

Proposition 2.2. *Let (2.3) be satisfied with $\gamma_0 < \gamma_1$. Then the following conclusions are valid:*

(i) for all $a \in (I(\gamma_0), I(\gamma_1))$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = -a \qquad \text{if and only if} \qquad \lim_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) = -b(a) \tag{2.13}$$

with

$$b(a) := a - \gamma(a) \in (I(\gamma_1) - \gamma_1, I(\gamma_0) - \gamma_0) \tag{2.14}$$

and $\gamma(a)$ is a unique solution of the equation $I(\gamma) = a$ with respect to $\gamma \in (\gamma_0, \gamma_1)$;

(ii) for all $a \in [0, I(\gamma_0)]$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = -a \qquad \text{implies} \qquad \limsup_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \le \gamma_0 - I(\gamma_0) \tag{2.15}$$

and for all $a \in [I(\gamma_1), \infty]$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = -a \qquad \text{implies} \qquad \liminf_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \ge \gamma_1 - I(\gamma_1); \tag{2.16}$$

(iii) for all $b \in [0, I(\gamma_1) - \gamma_1]$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) = -b \qquad \text{implies} \qquad \limsup_{t\to\infty} \psi_t^{-1} \log \alpha_t \le -I(\gamma_1) \tag{2.17}$$

and for all $b \in [I(\gamma_0) - \gamma_0, \infty]$ *we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) = -b \qquad \textit{implies} \qquad \liminf_{t\to\infty} \psi_t^{-1} \log \alpha_t \geq -I(\gamma_0). \tag{2.18}$$

These results under more restrictive conditions were proved in [21]. The *only if* part in (2.13) for the sequence of observed i.i.d. random variables was proved by Birgé [8].

Let δ_t^π be a *Bayes* test for testing the hypotheses H_0 and H_1 based on the observations X_s, $0 \leq s \leq t$, where π and $1-\pi$, $\pi \in [0,1]$, are the a priori probabilities of the hypotheses H_0 and H_1, respectively (see e.g. [21; Chapter II, Section 2.1]). The following assertion describes the rate of decrease for the error probabilities of the first and second kind $\alpha_t(\delta_t^\pi)$ and $\beta(\delta_t^\pi)$, and the risk $e(\delta_t^\pi)$ for the test δ_t^π under the regularity condition (2.3).

Proposition 2.3. *Let (2.3) be satisfied with* $\gamma_0 < 0 < \gamma_1$. *Then the following relationships hold:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \log e(\delta_t^\pi) = -I(0). \tag{2.19}$$

This assertion was proved by Chernoff [9] for the case of i.i.d. random variables. Under some other conditions the last equality in (2.19) was proved by Vajda [28].

Let δ_t^* be a *minimax* test for testing the hypotheses H_0 and H_1 under the observations X_s, $0 \leq s \leq t$ (see e.g. [7; Chapter III, Section 4]). The following assertion describes the rate of decrease for the error probabilities of the first and second kind $\alpha_t(\delta_t^*)$ and $\beta(\delta_t^*)$, and the risk $e(\delta_t^*)$ for the test δ_t^* under the regularity condition (2.3).

Proposition 2.4. *Suppose that (2.3) is satisfied with* $\gamma_0 < 0 < \gamma_1$. *Then we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t^*) = \lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t^*) = \lim_{t\to\infty} \psi_t^{-1} \log e(\delta_t^*) = -I(0). \tag{2.20}$$

3 Ornstein-Uhlenbeck models

In this section we consider a model where the observation process $X = (X_t)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$dX_t = -\theta X_t \, dt + dW_t \qquad (X_0 = x) \tag{3.1}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process and $\theta \geq 0$, $x \in \mathbb{R}$ are some given constants. We will study the problem of testing the simple hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1$.

Here we specify the results of the previous section for Ornstein-Uhlenbeck processes in both cases $\theta_1 > \theta_0 = 0$ and $\theta_1 > \theta_0 > 0$. It is remarkable that the first case cannot be obtained from the second one by letting $\theta_0 \downarrow 0$.

3.1. Since equation (3.1) has a pathwise unique continuous solution under both hypotheses H_0 and H_1, by means of the Girsanov formula for diffusion-type processes (see e.g. [24; Chapter VII, Theorem 7.19]) we may conclude that the measures P_0 and P_1 are locally equivalent on $(\mathcal{F}_t)_{t\geq 0}$, and under the hypothesis H_0 the log-likelihood ratio process (2.1) admits the representation:

$$\Lambda_t = (\theta_0 - \theta_1) \int_0^t X_s \, dW_s - \frac{(\theta_0 - \theta_1)^2}{2} \int_0^t X_s^2 \, ds. \tag{3.2}$$

By applying Itô's formula (see e.g. [24; Chapter IV, Theorem 4.4] or [16; Chapter I, Theorem 4.57]), from (3.1) it follows that under H_0 we have:

$$X_t^2 = x^2 + 2\int_0^t X_s\, dX_s + t = x^2 - 2\theta_0 \int_0^t X_s^2\, ds + 2\int_0^t X_s\, dW_s + t \tag{3.3}$$

and hence:

$$\int_0^t X_s\, dW_s = \frac{1}{2}\left(X_t^2 - x^2 + 2\theta_0 \int_0^t X_s^2\, ds - t \right). \tag{3.4}$$

Thus, by substituting the expression (3.4) into (3.2), we obtain that the Hellinger integral (2.2) takes the expression:

$$\begin{aligned} H_t(\varepsilon) &= E_0 \left[\exp\left(\frac{\varepsilon(\theta_0 - \theta_1)}{2} \left(X_t^2 - x^2 + 2\theta_0 \int_0^t X_s^2\, ds - t \right) - \frac{\varepsilon(\theta_0 - \theta_1)^2}{2} \int_0^t X_s^2\, ds \right) \right] \\ &= \exp\left(\frac{\varepsilon(\theta_1 - \theta_0)}{2} (x^2 + t) \right) E_0 \left[\exp\left(\frac{\varepsilon(\theta_0 - \theta_1)}{2} X_t^2 - \frac{\varepsilon(\theta_1^2 - \theta_0^2)}{2} \int_0^t X_s^2\, ds \right) \right]. \end{aligned} \tag{3.5}$$

In order to derive the large deviation results from the previous section for the model defined in (3.1) we should find a function ψ_t, $t \geq 0$, for which the regularity condition (2.3) is satisfied. For this, we will investigate the asymptotic behavior of the Hellinger integral (3.5) under $t \to \infty$.

3.2. First, let us suppose that $\theta_1 > \theta_0 = 0$ in (3.1). In this case the Hellinger integral (3.5) takes the form:

$$H_t(\varepsilon) = \exp\left(\frac{\varepsilon\theta_1}{2} (x^2 + t) \right) E_0 \left[\exp\left(-\frac{\varepsilon\theta_1}{2} X_t^2 - \frac{\varepsilon\theta_1^2}{2} \int_0^t X_s^2\, ds \right) \right]. \tag{3.6}$$

Assume that $\varepsilon > 0$ and define $\varphi := \varepsilon\theta_1/2$ and $\xi := \sqrt{\varepsilon}\theta_1$ (or $\xi := -\sqrt{\varepsilon}\theta_1$). Then, by using the Feynman-Kac formula, we obtain that the logarithm of the Hellinger integral (3.6) admits the representation:

$$\begin{aligned} \log H_t(\varepsilon) &= \varphi(x^2 + t) \\ &\quad - \frac{x^2[\xi \sinh(\xi t) + 2\varphi \cosh(\xi t)]}{2[\cosh(\xi t) + 2\varphi\xi^{-1} \sinh(\xi t)]} - \frac{1}{2} \log[\cosh(\xi t) + 2\varphi\xi^{-1} \sinh(\xi t)] \end{aligned} \tag{3.7}$$

(cf. the formula (1.9.3) in [6; Chapter II, Section 1]). It can be also shown that for $\varepsilon < 0$ and sufficiently large $t > 0$ we have $H_t(\varepsilon) = \infty$ in (3.6). Hence, by substituting the expression (3.7) into (2.3), taking $\psi_t = \theta_1 t$ and letting t go to ∞, we get:

$$\varkappa(\varepsilon) = -\frac{\sqrt{\varepsilon}(1 - \sqrt{\varepsilon})}{2} \quad \text{and} \quad \varkappa'(\varepsilon) = -\frac{1}{4\sqrt{\varepsilon}} + \frac{1}{2} \tag{3.8}$$

for $\varepsilon \in (\varepsilon_-, \varepsilon_+) = (0, \infty)$, so that $\varkappa'(\varepsilon_- +) = -\infty$, $\varkappa'(1) = 1/4$ and $\varkappa'(\varepsilon_+ -) = 1/2$.

It is easily seen that the function $I(\gamma)$ from (2.5) takes the expression:

$$I(\gamma) = \sup_{\varepsilon > 0} (\varepsilon\gamma - \varkappa(\varepsilon)) = \frac{1}{8(1 - 2\gamma)} \tag{3.9}$$

and the values in (2.7) - (2.8) can be calculated as $\gamma_0 = \varkappa'(\varepsilon_-+) = -\infty$ and $\gamma_1 = \varkappa'(1) = 1/4$ with $I(\gamma_0) = 0$ and $I(\gamma_1) = 1/4$.

Because in this case we have $\gamma_0 < 0 < \gamma_1$, from Propositions 2.1 - 2.4 and the formulas (3.8) - (3.9) we get that the following assertion holds.

Theorem 3.1. *In the model (3.1) of testing hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$ the following conclusions are valid with the functions $\psi_t = \theta_1 t$, $t \geq 0$, and $I(\gamma)$ from (3.9):*

(i) if $\gamma \in (\gamma_0, \varkappa'(\varepsilon_+-)) = (-\infty, 1/2)$ then (2.9) holds,
if $\gamma \in (\varkappa'(\varepsilon_-+), \varkappa'(0)) = (-\infty, 1/4)$ then (2.11) holds,
and if $\gamma \in (\varkappa'(1), \varkappa'(\varepsilon_+-)) = (1/4, 1/2)$ then (2.12) holds;

(ii) if $a \in (I(\gamma_0), I(\gamma_1)) = (0, 1/4)$ then (2.13) - (2.14) hold with $b(a) = a - 1/2 + 1/(16a)$;

(iii) if $a = I(\gamma_0) = 0$ then (2.15) holds,
if $a \in [I(\gamma_1), \infty] = [1/4, \infty]$ then (2.16) holds,
if $b = I(\gamma_1) - \gamma_1 = 0$ then (2.17) holds,
and if $b = I(\gamma_0) - \gamma_0 = \infty$ then (2.18) holds;

(iv) in the Bayes test we have (2.19), and for the minimax test (2.20) holds with $I(0) = 1/8$.

3.3. Now let us suppose that $\theta_1 > \theta_0 > 0$ in (3.1). In this case we assume that $\varepsilon > -\theta_0/(2(\theta_1^2 - \theta_0^2))$ and define $\varphi := \varepsilon(\theta_1 - \theta_0)/2$ and $\xi := \sqrt{2\varepsilon(\theta_1^2 - \theta_0^2)/\theta_0 + 1}$ (or $\xi := -\sqrt{2\varepsilon(\theta_1^2 - \theta_0^2)/\theta_0 + 1}$). This implies $(\xi^2 - 1)\theta_0/4 = \varepsilon(\theta_1^2 - \theta_0^2)/2$. Then, by using the Feynman-Kac formula, we obtain that the logarithm of the Hellinger integral (3.5) admits the representation:

$$\log H_t(\varepsilon) = \varphi(x^2 + t) + \frac{\theta_0 t}{2} + \frac{x^2}{4} - \frac{1}{2}\log[(1 + 4\varphi)\xi^{-1}\sinh(\theta_0\xi t) + \cosh(\theta_0\xi t)] \qquad (3.10)$$
$$+ \frac{x^2}{4\xi^{-1}\sinh(\theta_0\xi t)}\left(\frac{1}{(1 + 4\varphi)\xi^{-1}\sinh(\theta_0\xi t) + \cosh(\theta_0\xi t)} - \cosh(\theta_0\xi t)\right)$$

(cf. the formula (1.9.7) in [6; Chapter II, Section 7]). It can be also shown that for $\varepsilon < -\theta_0/(2(\theta_1^2 - \theta_0^2))$ and sufficiently large $t > 0$ we have $H_t(\varepsilon) = \infty$ in (3.5). Hence, by substituting the expression (3.10) into (2.3), taking $\psi_t = (\theta_1 - \theta_0)t$ and letting t go to ∞, we get:

$$\varkappa(\varepsilon) = \frac{\varepsilon}{2} - \frac{\sqrt{2\varepsilon\theta_0(\theta_1^2 - \theta_0^2) + \theta_0^2}}{2(\theta_1 - \theta_0)} + \frac{\theta_0}{2(\theta_1 - \theta_0)} \quad \text{and} \quad \varkappa'(\varepsilon) = \frac{1}{2} - \frac{\theta_0(\theta_0 + \theta_1)}{2\sqrt{2\varepsilon\theta_0(\theta_1^2 - \theta_0^2) + \theta_0^2}} \qquad (3.11)$$

for $\varepsilon \in (\varepsilon_-, \varepsilon_+) = (-\theta_0/(2(\theta_1^2 - \theta_0^2)), \infty)$ with $\varkappa'(\varepsilon_-+) = -\infty$, $\varkappa'(0) = (1 - \theta_0 - \theta_1)/2$, $\varkappa'(1) = 1/2 - \theta_0(\theta_0 + \theta_1)/(2\sqrt{2\theta_0(\theta_1^2 - \theta_0^2) + \theta_0^2})$, and $\varkappa'(\varepsilon_+-) = 1/2$.

It is easily seen that the function $I(\gamma)$ from (2.5) takes the expression:

$$I(\gamma) = \sup_{\varepsilon > \varepsilon_-}(\varepsilon\gamma - \varkappa(\varepsilon)) = \frac{\theta_0(1 - 2\gamma - \theta_0 - \theta_1)^2}{4(\theta_1^2 - \theta_0^2)(1 - 2\gamma)} \qquad (3.12)$$

and the values in (2.7) - (2.8) can be calculated as $\gamma_0 = \varkappa'(0) = (1 - \theta_0 - \theta_1)/2$, $\gamma_1 = \varkappa'(1) = 1/2 - \theta_0(\theta_0 + \theta_1)/(2\sqrt{2\theta_0(\theta_1^2 - \theta_0^2) + \theta_0^2})$ with $I(\gamma_0) = 0$ and $I(\gamma_1) = (\theta_0 - \sqrt{2\theta_0(\theta_1^2 - \theta_0^2) + \theta_0^2})^2/(4(\theta_1 - \theta_0)\sqrt{2\theta_0(\theta_1^2 - \theta_0^2) + \theta_0^2})$.

Because in this case we have $\gamma_0 < \gamma_1$, from Propositions 2.1 - 2.4 and the formulas (3.11) - (3.12) we get that the following assertion holds.

Theorem 3.2. *In the model (3.1) of testing hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$ with $\theta_1 > \theta_0 > 0$ the following conclusions are valid with the functions $\psi_t = (\theta_1 - \theta_0)t$, $t \geq 0$, and $I(\gamma)$ from (3.12):*

(i) if $\gamma \in (\gamma_0, \varkappa'(\varepsilon_+-)) = ((1-\theta_0-\theta_1)/2, 1/2)$ then (2.9) holds,
if $\gamma \in (\varkappa'(\varepsilon_-+), \varkappa'(0)) = (-\infty, (1-\theta_0-\theta_1)/2)$ then (2.10) holds,
if $\gamma \in (\varkappa'(\varepsilon_-+), \gamma_1) = (-\infty, 1/2 - \theta_0(\theta_0+\theta_1)/(2\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}))$ then (2.11) holds,
and if $\gamma \in (\varkappa'(1), \varkappa'(\varepsilon_+-)) = (1/2 - \theta_0(\theta_0+\theta_1)/(2\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2})), 1/2)$ then (2.12) holds;

(ii) if $a \in (0, I(\gamma_1)) = (0, (\theta_0 - \sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2})^2/(4(\theta_1-\theta_0)\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}))$ then (2.13) - (2.14) holds with $b(a) = (1-\theta_0-\theta_1)/2 - (\theta_0+\theta_1)(a(\theta_1-\theta_0) - \sqrt{a\theta_0(\theta_1-\theta_0)+a^2(\theta_1-\theta_0)^2})/\theta_0$;

(iii) if $a = I(\gamma_0) = 0$ then (2.15) holds,
if $a \in [I(\gamma_1), \infty] = [(\theta_0 - \sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2})^2/(4(\theta_1-\theta_0)\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}), \infty]$ then (2.16) holds,
if $b \in [0, I(\gamma_1)-\gamma_1] = [0, (\theta_0 - \sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2})^2/(4(\theta_1-\theta_0)\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}) - 1/2 + \theta_0(\theta_0+\theta_1)/(2\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2})]$ then (2.17) holds,
and if $b \in [I(\gamma_0)-\gamma_0, \infty] = [-(1-\theta_0-\theta_1)/2, \infty]$ then (2.18) holds;

(iv) if $\gamma_0 < 0 < \gamma_1$ holds, then in the Bayes test we have (2.19), and for the minimax test (2.20) holds with $I(0) = \theta_0(1-\theta_0-\theta_1)^2/(4(\theta_1^2-\theta_0^2))$.

Remark 3.3. The cases $\theta_0 > \theta_1 = 0$ and $\theta_0 > \theta_1 > 0$ can be dealt with similarly as above by virtue of the property $H_t(\varepsilon; P_0, P_1) = H_t(1-\varepsilon; P_1, P_0)$ for all $\varepsilon \in (-\infty, \infty)$ and $t \geq 0$.

Remark 3.4. The question if the derived rate bounds are optimal as well as the second order expansions for $\log \beta_t(\alpha_t)$ remain as open problems here.

4 Ornstein-Uhlenbeck-type models with delay

In this section we consider a model where the observed process $X = (X_t)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$dX_t = \int_{-r}^{0} X_{t+s}\, a(ds)\, dt + dW_t \qquad (X_t = Z_t \quad \text{for} \quad t \in [-r, 0]) \tag{4.1}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process independent of the initial (continuous) process $Z = (Z_t)_{-r\leq t\leq 0}$, and $a(ds)$ is a finite signed measure on $[-r, 0]$ for some $r > 0$ fixed. From the arguments in [15; Section 3] it follows that for given W, Z and $a(ds)$ there exists a pathwise unique continuous process $X = (X_t)_{t\geq -r}$ satisfying (4.1). Let us denote by $\mathbb{M}_s$ the set of all signed measures $a(ds)$ on $[-r, 0]$ such that a stationary solution of (4.1) exists (for necessary and sufficient conditions for the existence of a stationary solution of (4.1) see [13] and [15; Section 3]). We will study the problem of testing the simple hypothesis $H_0 : a(ds) \equiv a_0(ds)$ against the simple alternative $H_1 : a(ds) \equiv a_1(ds)$, where $a_i(ds) \in \mathbb{M}_s$ for $i = 0, 1$, and $a_0(ds) \not\equiv a_1(ds)$.

4.1. By using the arguments in [15; Section 3] we may conclude that equation (4.1) has a unique continuous stationary solution under both hypotheses $H_0 : a(ds) \equiv a_0(ds)$ and $H_1 : a(ds) \equiv a_1(ds)$, and the measures P_0 and P_1 are locally equivalent on $(\mathcal{F}_t)_{t\geq -r}$ with $\mathcal{F}_t = \sigma(X_s \,|\, -r \leq s \leq t)$ for $t \geq -r$ (here we set $\mathcal{F}_t = \sigma(Z_s \,|\, -r \leq s \leq t)$ for $-r \leq t \leq 0$). Then, by means of the Girsanov-type formula (5.1) in [15], we get that under the hypothesis H_0 the log-likelihood ratio process (2.1) admits the representation:

$$\Lambda_t = \log \frac{d(P_1|\mathcal{F}_0)}{d(P_0|\mathcal{F}_0)} + \int_0^t Y_s \, dW_s - \frac{1}{2} \int_0^t Y_s^2 \, ds \tag{4.2}$$

where the process $Y = (Y_t)_{t\geq 0}$ is defined by:

$$Y_t = \int_{-r}^0 X_{t+s} \, [a_1(ds) - a_0(ds)] \tag{4.3}$$

so that the Hellinger integral (2.2) takes the form:

$$H_t(\varepsilon) = E_0 \left[\exp \left(\varepsilon \log \frac{d(P_1|\mathcal{F}_0)}{d(P_0|\mathcal{F}_0)} + \varepsilon \int_0^t Y_s \, dW_s - \frac{\varepsilon}{2} \int_0^t Y_s^2 \, ds \right) \right]. \tag{4.4}$$

We should note that in the most cases of the model defined in (4.1), it seems to be impossible to calculate the Hellinger integral (4.4) in an explicit way, unless when $a_i(ds)$, $i = 0, 1$, are some Dirac measures at the point zero. Using the arguments in [21; Theorems 3.1.4, 3.2.2] we now describe the asymptotic behavior of the error probabilities for Neyman-Pearson tests.

Theorem 4.1. *Let α_t, $t \geq 0$, be the error probability of the first kind of the Neyman-Pearson test in the model (4.1) of testing hypothesis $H_0 : a(ds) \equiv a_0(ds)$ against the alternative $H_1 : a(ds) \equiv a_1(ds)$ with $a_i(ds) \in \mathbb{M}_s$, $i = 0, 1$, such that $a_0(ds) \not\equiv a_1(ds)$. Then for the function ψ_t, $t \geq 0$, given by:*

$$\psi_t = E_0 \left[\frac{1}{2} \int_0^t Y_s^2 \, ds \right] \tag{4.5}$$

we have

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = 0 \qquad \textit{implies} \qquad \limsup_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \leq -1, \tag{4.6}$$

and if the condition

$$H_t(\varepsilon'; P_1, P_0) < \infty \quad \textit{for some} \quad \varepsilon' < 0 \quad \textit{and all} \quad t \geq 0 \tag{4.7}$$

is satisfied, then

$$\lim_{t\to\infty} \psi_t^{-1} \log(1 - \alpha_t) = 0 \qquad \textit{implies} \qquad \liminf_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \geq -1. \tag{4.8}$$

Proof. Since in the assumptions above $a_i(ds) \in \mathbb{M}_s$ for $i = 0, 1$, by means of the arguments in [15; Sections 3 and 5], we may conclude that there exists a positive constant B_* depending on $a_0(ds)$ (see the formula (3.13) in [15]) and a constant $C_r > 0$ depending only on r from the formula (5.2) in [15] such that:

$$E_0[Y_t^2] \geq C_r B_* \|a_1 - a_0\|_D^2 \tag{4.9}$$

for all $t \geq 0$ (see the formula (5.7) in [15]). Here $\|a_1 - a_0\|_D$ is the dual Lipschitz norm from the formula (3.16) in [15] being strictly positive when $a_0(ds) \not\equiv a_1(ds)$. Thus, changing the order of integration and expectation in (4.5), from (4.9) we obtain that $\psi_t \to \infty$ under $t \to \infty$.

Let us choose ε and δ such that $0 < \varepsilon < \delta/2 < \delta < 1$ (when (4.7) holds, also $\varepsilon' \leq \delta < \delta/2 < \varepsilon < 0$) and $p = \delta/\varepsilon$, $q = \delta/(\delta - \varepsilon)$ such that $1/p + 1/q = 1$. Then standard tricks with Hölder's inequality (see e.g. [21; Theorem 3.1.4]) imply that for the Hellinger integral (4.4) we have:

$$H_t(\varepsilon) = H_0(\delta)^{\varepsilon/\delta} \left(E_0 \left[\exp \left(-\frac{\varepsilon}{(\delta - \varepsilon)} \frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \right)^{(\delta-\varepsilon)/\delta} \tag{4.10}$$

and applying Jensen's inequality to the right-hand side of (4.10) we get:

$$H_t(\varepsilon) = H_0(\delta)^{\varepsilon/\delta} \left(E_0 \left[\exp \left(-\mathrm{sgn}(\delta) \frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \right)^{\varepsilon/\delta} . \tag{4.11}$$

Observe that from Jensen's and Lyapunov's inequalities as well as by the monotonicity of the logarithmic function it follows that for given δ we have:

$$\log E_0 \left[\exp \left(-\frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \leq -\delta(1-\delta) E_0 \left[\frac{1}{2} \int_0^t Y_s^2 \, ds \right] . \tag{4.12}$$

Thus, letting t go to ∞ in (4.11), by using the property $\psi_t \to \infty$ as $t \to \infty$ and the fact that $H_0(\varepsilon)$ in (4.4) is finite (since the restrictions $P_0 | \mathcal{F}_0$ and $P_1 | \mathcal{F}_0$ are equivalent), by means of (4.12), we obtain:

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \limsup_{t \to \infty} \varepsilon^{-1} \psi_t^{-1} \log H_t(\varepsilon) \\ &\leq \limsup_{\delta \downarrow 0} \limsup_{t \to \infty} \delta^{-1} \psi_t^{-1} \log E_0 \left[\exp \left(-\frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \leq -1 \end{aligned} \tag{4.13}$$

and (when (4.7) holds) also:

$$\begin{aligned} &\liminf_{\varepsilon \uparrow 0} \liminf_{t \to \infty} \varepsilon^{-1} \psi_t^{-1} \log H_t(\varepsilon) \\ &\geq \liminf_{\delta \uparrow 0} \liminf_{t \to \infty} \delta^{-1} \psi_t^{-1} \log E_0 \left[\exp \left(-\frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \geq -1. \end{aligned} \tag{4.14}$$

Therefore, by virtue of [21; Theorems 2.3.1 and 2.3.3] we may conclude that (4.6) and (when (4.7) holds, also (4.8)) are satisfied. □

Corollary 4.2. From the arguments above it is easily seen that if condition (4.7) is satisfied, then we have the following more exact result:

$$\lim_{t \to \infty} \psi_t^{-1} \log \alpha_t = \lim_{t \to \infty} \psi_t^{-1} \log(1 - \alpha_t) = 0 \qquad \text{implies} \qquad \lim_{t \to \infty} \psi_t^{-1} \log \beta(\alpha_t) = -1. \tag{4.15}$$

4.2. In the rest of the section we give some examples of models of the type (4.1) in which condition (4.7) holds.

Example 4.3. Suppose that in (4.1) we have $Z_t = 0$ for $t \in [-r, 0]$ and $a_i(ds) \equiv -\theta_i \delta_0$ for $i = 0, 1$ with $\theta_1 > \theta_0 > 0$, where δ_0 denotes the Dirac measure having the mass one at the point zero. In this case there exists a stationary solution of equation (4.1), so that we have $a_i(ds) \in \mathbb{M}_s$ for $i = 0, 1$. Then from the results of the previous section it follows that condition (4.7) is satisfied, for example, with $\varepsilon' = -\theta_0/(4(\theta_1^2 - \theta_0^2))$, so that we have the exact result (4.15) with $\psi_t = (\theta_0 - \theta_1)^2 t/(2\theta_0) - (\theta_0 - \theta_1)^2 (1 - e^{-2\theta_0 t})/(4\theta_0^2)$, $t \geq 0$, in (4.5).

Example 4.4. Suppose that in (4.1) we have $Z_t = 0$ for $t \in [-r, 0]$, $a_0(ds) \equiv -\theta_0 \delta_0$ and $a_1(ds) \equiv -\theta_1 \delta_{-r}$ for some $\theta_0 > 0$, $0 < \theta_1 < \pi r/2$ and $r > 0$. This means that we consider a problem of testing hypothesis *'there is no delay'* against the alternative *'there is a delay'*. In this case there also exists a stationary solution of equation (4.1), so that we have $a_i(ds) \in \mathbb{M}_s$ for $i = 0, 1$. Some estimation problems for this type of models were considered in [12] and [17]. Let us introduce the process $M = (M_t)_{t \geq 0}$ given by:

$$M_t = \int_0^t (\theta_0 X_s - \theta_1 X_{s-r}) \, dW_s \quad \text{with} \quad \langle M \rangle_t = \int_0^t (\theta_0 X_s - \theta_1 X_{s-r})^2 \, ds. \tag{4.16}$$

Then it follows that the Hellinger integral (4.4) takes the form:

$$H_t(\varepsilon) = E_0 \left[\exp \left(\varepsilon M_t - \varepsilon \langle M \rangle_t / 2 \right) \right] \tag{4.17}$$

(with $H_0(\varepsilon) = 1$ since $Z \equiv 0$). If the following conditions hold:

$$E_0 \left[\exp \left(2\varepsilon^2 \langle M \rangle_t \right) \right] < \infty \qquad \text{and} \qquad E_0 \left[\exp \left(\varepsilon(2\varepsilon - 1) \langle M \rangle_t \right) \right] < \infty \tag{4.18}$$

then, by means of Cauchy-Schwarz inequality, we have for (4.17):

$$H_t(\varepsilon) \leq \left\{ E_0 \left[\exp \left(2\varepsilon M_t - (2\varepsilon)^2 \langle M \rangle_t / 2 \right) \right] \right\}^{1/2} \left\{ E_0 \left[\exp \left(\varepsilon(2\varepsilon - 1) \langle M \rangle_t \right) \right] \right\}^{1/2}. \tag{4.19}$$

From the formula (1.9.3) in [6; Chapter II, Section 7] it is easily seen that:

$$E_0 \left[\exp \left(\frac{\theta_0}{8} \int_0^t X_s^2 \, ds \right) \right] < \infty \tag{4.20}$$

and since under hypothesis H_0 we have:

$$\int_0^t (\theta_0 X_s - \theta_1 X_{s-r})^2 \, ds \leq 2\theta_0^2 \int_0^t X_s^2 \, ds + 2\theta_1^2 \int_0^t X_{s-r}^2 \, ds \leq 2(\theta_0^2 + \theta_1^2) \int_0^t X_s^2 \, ds \tag{4.21}$$

we may conclude that the conditions $4\varepsilon^2(\theta_0^2 + \theta_1^2) \leq \theta_0/8$ and $2\varepsilon(2\varepsilon - 1)(\theta_0^2 + \theta_1^2) \leq \theta_0/8$ guarantee that (4.18) - (4.19) holds, and hence (4.17) is finite. Thus, condition (4.7) is satisfied, for example, with $\varepsilon' = -\theta_0/(128(\theta_0^2 + \theta_1^2))$, so that we have the exact result (4.15) with $\psi_t = \theta_0 t/4 - \theta_1 e^{\theta_0 r}(t - r)/2 + \theta_1^2 (t - r)/(4\theta_0) - (1 - e^{-2\theta_0 t})/8 + \theta_1 e^{-\theta_0 r}/(4\theta_0) - \theta_1 e^{-2\theta_0 (t - r/2)}/(4\theta_0) - \theta_1^2 (1 - e^{-2\theta_0 (t-r)})/(8\theta_0^2)$, $t \geq r$, in (4.5).

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