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On the structure of discounted optimal stopping problems for one-dimensional diffusions

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We connect two approaches for solving discounted optimal stopping problems for one-dimensional time-homogeneous regular diffusion processes on infinite time intervals. The optimal stopping rule is assumed to be the first exit time of the underlying process from a region restricted by two constant boundaries. We provide an explicit decomposition of the reward process into a product of a gain function of the boundaries and a uniformly integrable martingale inside the continuation region. This martingale plays a key role for stating sufficient conditions for the optimality of the first exit time. We also consider several illustrating examples of rational valuation of perpetual American strangle options.

1 Introduction

Optimal stopping problems have as an objective to search for random times at which the underlying stochastic processes should be stopped, with the aim to optimize the expected values of

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given reward functionals. The majority of explicitly solvable stopping problems with exponential discounting are essentially those for one-dimensional diffusion processes with infinite time horizon. The optimal stopping times are then the first times at which the underlying processes exit certain regions restricted by constant boundaries. We study the connection between the following two approaches existing in the literature, which are applied for computing explicit expressions for the value functions and the boundaries in such optimal stopping problems.

In the *free-boundary* approach, described profoundly in the monograph of Peskir and Shiryaev [26], the associated free-boundary problem for a differential operator is formulated for the unknown value function and the stopping boundaries. Although such a free-boundary problem usually has multiple solutions, the appropriate one can be specified as the solution which satisfies certain additional conditions such as smooth fit, natural boundary, normal entrance, normal reflection, etc. (see [26; Chapter IV, Section 9.1] for an extensive overview of proofs of these principles). By means of the verification arguments from stochastic analysis, including an extended Itô-Tanaka formula and the optional sampling theorem, it is then shown that the resulting solution of the free-boundary problem provides the solution of the initial optimal stopping problem.

In the *martingale* approach due to Beibel and Lerche [5] and [6], applied to optimal stopping problems with infinite time horizon, the discounted reward process is decomposed into a product of a positive uniformly integrable martingale and a gain function of the current state of the underlying one-dimensional diffusion process. It is then shown that the optimization of the gain function over all admissible stopping points gives the value of the initial optimal stopping problem, whenever the optimal stopping time is finite almost surely with respect to the probability measure constructed by means of the positive martingale (see also Lerche and Urusov [19]). This approach is closely connected to the fundamental principle of Snell [33] of the least superharmonic characterization of the value function and provides explicit solutions of optimal stopping problems for continuous Markov processes. Furthermore, it can easily be verified that the solution of the optimal stopping problem, obtained by means of the approach of Beibel and Lerche, satisfies the associated free-boundary problem. It can also be shown directly that the optimization of the resulting gain function yields that the smooth-fit condition holds for the value function at the optimal stopping boundaries.

In the present paper, we assume that the optimal stopping rule is the first time at which the underlying one-dimensional diffusion process exits a region restricted by two constant boundaries. We characterize the value function in Theorem 3.1, as a solution of the associated free-boundary problem satisfying the smooth-fit conditions at the optimal boundaries. We show how the gain function and the positive uniformly integrable martingale inside the continuation region can be specified explicitly from the solution of the free-boundary problem. This martingale appears further in Theorem 3.2, where sufficient conditions are given for the optimality of the first exit time. We illustrate our results by several examples of exponentially discounted optimal stopping problems in one-dimensional diffusion models.

The paper is organized as follows. In Section 2, we introduce the setting of an exponentially discounted optimal stopping problem for a one-dimensional time-homogeneous regular diffusion process and formulate the associated free-boundary problem. We derive a closed form solution of the latter problem and decompose it into a form which is amenable for the comparison of the two approaches. In Section 3, we verify that the solution of the free-boundary problem provides the solution of the initial optimal stopping problem and show that its value coincides with the resulting gain from the decomposition obtained. In Section 4, we provide several examples of discounted optimal stopping problems for one-dimensional linear diffusions, which arise mainly from the rational valuation of perpetual American strangle options.

2 Preliminaries

In this section, we give a formulation of an exponentially discounted optimal stopping problem for a one-dimensional diffusion process and solve the associated ordinary differential free-boundary problem. We also provide a decomposition of the solution, which is related to the martingale approach of Beibel and Lerche.

2.1. Formulation of the problem. For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $W = (W_t)_{t \geq 0}$. Let $X = (X_t)_{t \geq 0}$ be a process solving the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t \quad (X_0 = x) \tag{2.1}$$

where $\mu(x)$ and $\sigma(x) > 0$ are some continuously differentiable functions on $(0, \infty)$. The latter assumption guarantees the existence of a pathwise unique solution of the equation in (2.1), for a given starting point $x > 0$ (see, e.g. Karatzas and Shreve [15; Chapter V, Theorem 2.5] and Rogers and Williams [28; Chapter V, Section 44]). It follows that X is a regular diffusion process, in the sense of Karlin and Taylor [16; Chapter XV], on its state space which is assumed to be the positive half line $(0, \infty)$. Let us consider an optimal stopping problem with the value function

$$V_*(x) = \sup_{\tau} E_x \left[e^{-r\tau} \, H(X_\tau) \right] \tag{2.2}$$

where the supremum is taken over all stopping times τ, with respect to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of the process X. Here, $H(x)$ is a payoff function, $r > 0$ is a discounting rate, and E_x denotes the expectation under the assumption that $X_0 = x$, for some $x > 0$. Such optimal stopping problems have been considered in Dynkin [11], Fakeev [12], Mucci [21], Salminen [29], Øksendal and Reikvam [23], and Beibel and Lerche [6] among others (see also Bensoussan and Lions [7; Theorem 3.19] and Øksendal [22; Chapter X]), for regular diffusion processes with general payoffs and infinite time horizon. More recently, such optimal stopping problems were studied in Dayanik and Karatzas [10], Alvarez [2] and [3], Peskir and Shiryaev [26], and Lamberton and Zervos [18] (see the latter references for an extensive discussion).

2.2. Structure of the optimal stopping time. It follows from the general theory of optimal stopping for Markov processes (see, e.g. [26; Chapter I, Section 2.1]) that the optimal stopping time in the problem of (2.2) is given by

$$\tau_* = \inf\{t \geq 0 \,|\, V_*(X_t) = H(X_t)\} \tag{2.3}$$

whenever it exists, under appropriate conditions on the payoff function $H(x)$. For simplicity of exposition, we further assume that the payoff $H(x)$ is a positive continuous function which is continuously differentiable on $(0, \infty)$ except at a finite number of points $c_i > 0$, $i = 1, \dots, n$, for some $n \in \mathbb{N} \cup \{0\}$, and its first derivative $H'(x)$ is continuously differentiable on (c_{i-1}, c_i), for every $i = 1, \dots, n+1$, with $c_0 = 0$ and $c_{n+1} = \infty$. Note that almost all payoff functions, which are considered in the examples below, belong to this class. In that case, the stopping time τ_* turns out to be the first time at which the process X exits a region that consists of a countable union of disjoint (possibly unbounded) open (in the relative topology) intervals (a_i, b_i), such that $0 \le a_i < b_i \le \infty$, $i \in \mathbb{N}$ (see, e.g. [2; Section 3] and [26; Chapter I, Section 2.2]). In order to simplify the exposition, we further search for an optimal stopping time of the form

$$\tau_* = \inf\{t \ge 0 \,|\, X_t \notin (a_*, b_*)\} \tag{2.4}$$

for some numbers $0 \le a_* < b_* \le \infty$ to be determined (see [26; Chapter IV, Section 9] for a similar setting).

2.3. The free-boundary problem. By means of standard arguments (see, e.g. [15; Chapter V, Section 5.1]), it can be shown that the infinitesimal operator $\mathbb{L}$ of the process X acts on an arbitrary twice continuously differentiable locally bounded function $F(x)$ according to the rule

$$(\mathbb{L}F)(x) = \mu(x)\, F'(x) + \frac{1}{2}\, \sigma^2(x)\, F''(x) \tag{2.5}$$

for all $x > 0$. In order to find explicit expressions for the unknown value function $V_*(x)$ from (2.2) and the unknown boundaries a_* and b_* from (2.4), we may use the results of the general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [31; Chapter III, Section 8] and [26; Chapter IV, Section 8]). We formulate the associated free-boundary problem

$$(\mathbb{L}V)(x) = rV(x) \quad \text{for} \quad a < x < b \tag{2.6}$$

$$V(a+) = H(a) \quad \text{and} \quad V(b-) = H(b) \quad (\textit{instantaneous stopping}) \tag{2.7}$$

$$V'(a+) = H'(a) \quad \text{and} \quad V'(b-) = H'(b) \quad (\textit{smooth fit}) \tag{2.8}$$

$$V(x) = H(x) \quad \text{for} \quad x < a \quad \text{and} \quad x > b \tag{2.9}$$

$$V(x) > H(x) \quad \text{for} \quad a < x < b \tag{2.10}$$

for some $0 < a < b < \infty$ fixed. Note that the conditions in (2.8) are naturally used for the value functions at the optimal stopping boundaries for the underlying regular diffusions X, whenever the coefficients $\mu(x)$ and $\sigma(x)$ as well as the payoff functions $H(x)$ are continuously differentiable at the boundaries a and b (see [26; Chapter IV, Section 9] for an extensive overview).

2.4. Solution of the free-boundary problem. We now look for functions which solve the free-boundary problem stated in (2.6)–(2.8). Let $U_+(x)$ and $U_-(x)$ be two fundamental positive solutions (i.e. nontrivial linearly independent particular solutions) of the second order ordinary differential equation in (2.6). Without loss of generality, we may assume that $U_+(x)$

and $U_-(x)$ are (strictly) increasing and decreasing functions on $(0, \infty)$, respectively. Note that the (strictly) convex functions $U_+(x)$ and $U_-(x)$ can be represented as the Laplace transforms of the first passage times of the process X on constant boundaries (see, e.g. [14; Chapter IV, Section 4.6] or [28; Chapter V, Section 50]). It was proved in [29; Theorem 2.7] that the functions $U_+(x)$ and $U_-(x)$ turn out to be *minimal* r-excessive functions of the regular diffusion X, in the sense of the fact that the canonical diffusion processes, which correspond to the associated normal transition functions, converge to single points almost surely. The general solution of the second order ordinary differential equation in (2.6) is thus given by

$$V(x) = C_+ \, U_+(x) + C_- \, U_-(x) \tag{2.11}$$

where C_+ and C_- are some arbitrary constants. Hence, applying the instantaneous-stopping conditions from (2.7) to the function in (2.11), we get that the equalities

$$C_+ \, U_+(a) + C_- \, U_-(a) = H(a) \quad \text{and} \quad C_+ \, U_+(b) + C_- \, U_-(b) = H(b) \tag{2.12}$$

hold for some $0 < a < b < \infty$. Solving the system of equations in (2.12), we obtain the function

$$V(x; a, b) = C_+(a, b) \, U_+(x) + C_-(a, b) \, U_-(x) \tag{2.13}$$

which satisfies the system in (2.6)–(2.7) when we put

$$C_+(a, b) = \frac{H(a)U_-(b) - H(b)U_-(a)}{U_+(a)U_-(b) - U_+(b)U_-(a)} \tag{2.14}$$

and

$$C_-(a, b) = \frac{H(b)U_+(a) - H(a)U_+(b)}{U_+(a)U_-(b) - U_+(b)U_-(a)} \tag{2.15}$$

for $0 < a < b < \infty$. Applying the smooth-fit conditions from (2.8) to the function in (2.13), we therefore obtain that the equalities

$$C_+(a, b) \, U'_+(a) + C_-(a, b) \, U'_-(a) = H'(a) \tag{2.16}$$

$$C_+(a, b) \, U'_+(b) + C_-(a, b) \, U'_-(b) = H'(b) \tag{2.17}$$

hold with $C_+(a, b)$ and $C_-(a, b)$ given by (2.14) and (2.15), under the assumption that the function $H(x)$ is continuously differentiable at the boundaries a and b. It is shown by means of standard arguments that the system in (2.16)–(2.17) is equivalent to

$$\frac{H(a)U_+'(a) - H'(a)U_+(a)}{U_+'(a)U_-(a) - U_+(a)U_-'(a)} = \frac{H(b)U_+'(b) - H'(b)U_+(b)}{U_+'(b)U_-(b) - U_+(b)U_-'(b)} \tag{2.18}$$

$$\frac{H(a)U_-'(a) - H'(a)U_-(a)}{U_+'(a)U_-(a) - U_+(a)U_-'(a)} = \frac{H(b)U_-'(b) - H'(b)U_-(b)}{U_+'(b)U_-(b) - U_+(b)U_-'(b)} \tag{2.19}$$

for some $0 < a < b < \infty$ (compare with [29; Theorem 4.7], [2; Theorem 2] and [10; Corollary 7.2]). Note that, in the case of a general payoff function $H(x)$, the solution a_* and b_* of the system in (2.18)–(2.19) may not be unique. However, we will prove the uniqueness of such a couple for a particular payoff function in Examples 4.2 and 4.6 below. Sufficient conditions for uniqueness of solutions of more general optimal stopping game problems were derived in [3; Theorem 4.3] in terms of the infinitesimal characteristics and the payoff functions in a slightly less general (from the point of view of differentiability) framework.

2.5. Some decompositions of the value function. It follows from (2.13) that the function $V(x; a, b)$ admits the representation

$$V(x; a, b) = G(a, b)\Big(p(a, b)\, U_+(x) + (1 - p(a, b))\, U_-(x)\Big) \tag{2.20}$$

for any $x \in (a, b)$ fixed. Here, the gain function $G(a, b)$ is defined by

$$G(a, b) = C_+(a, b) + C_-(a, b) \tag{2.21}$$

so that the function $p(a, b)$ is given by

$$p(a, b) = \frac{C_+(a, b)}{C_+(a, b) + C_-(a, b)} \tag{2.22}$$

for $0 < a < b < \infty$. Hence, the expressions for $C_+(a, b)$ and $C_-(a, b)$ in (2.14) and (2.15) imply that the formulas in (2.21) and (2.22) take the form

$$G(a, b) = \frac{H(b)(U_+(a) - U_-(a)) - H(a)(U_+(b) - U_-(b))}{U_+(a)U_-(b) - U_+(b)U_-(a)} \tag{2.23}$$

and

$$p(a, b) = \frac{H(a)U_-(b) - H(b)U_-(a)}{H(b)(U_+(a) - U_-(a)) - H(a)(U_+(b) - U_-(b))} \tag{2.24}$$

for $0 < a < b < \infty$. We also observe that, by means of straightforward computations, it is shown that the system of equations in (2.18)–(2.19) is equivalent to the system

$$\frac{\partial G}{\partial a}(a, b) = 0 \quad \text{and} \quad \frac{\partial G}{\partial b}(a, b) = 0. \tag{2.25}$$

This means that solutions of the former system are critical points of the gain function $G(a, b)$.

3 Main results

In this section, we formulate and prove the main assertions of the paper, which build a connection between the free-boundary approach developed in Peskir and Shiryaev [26] and the martingale approach of Beibel and Lerche [5] and [6].

3.1. Verification lemma. Let us first prove the corresponding verification assertion and show how the uniformly integrable martingale can be specified from the solution of the free-boundary problem in (2.6)–(2.8). Such assertions were proved in Peskir and Shiryaev [26] among other references in different settings, and in Lamberton and Zervos [18] under less restrictive assumptions on the payoff function. For the case of optimal stopping games, a similar assertion was proved in Alvarez [3].

Theorem 3.1 *Let the process X be a pathwise unique solution of the stochastic differential equation in (2.1). Suppose that the payoff $H(x)$ is a positive continuous function such that $H'(x)$ is continuously differentiable on (c_{i-1}, c_i), $i = 1, \ldots, n+1$, $n \in \mathbb{N} \cup \{0\}$, for some $0 = c_0 < c_1 < \cdots < c_n < c_{n+1} = \infty$. Assume that the stopping time τ_* has the structure of the first exit time of the process X from the interval (a_*, b_*) as in (2.4), where the couple a_* and b_*, such that $0 < a_* < b_* < \infty$, is a unique solution of the system of equations in (2.18)–(2.19). Then, the value function of the optimal stopping problem in (2.2) has the form*

$$V_*(x) = \begin{cases} V(x; a_*, b_*), & \text{if } a_* < x < b_*, \\ H(x), & \text{if } x \le a_* \text{ or } x \ge b_*, \end{cases} \tag{3.1}$$

where $V(x; a, b)$ is given by (2.20), and τ_* from (2.4) is an optimal stopping time.

Moreover, the function $V(x; a_*, b_*)$ admits the representation

$$V(x; a_*, b_*) = G(a_*, b_*)\, E_x \left[M^*_{\tau_*} \right] \tag{3.2}$$

for $x \in (a_*, b_*)$, where the process $M^* = (M^*_t)_{t \ge 0}$ defined by

$$M^*_t = e^{-r(\tau_* \wedge t)} \Big(p(a_*, b_*)\, U_+(X_{\tau_* \wedge t}) + (1 - p(a_*, b_*))\, U_-(X_{\tau_* \wedge t}) \Big) \tag{3.3}$$

is a uniformly integrable martingale, and the functions $G(a, b)$ and $p(a, b)$ are given by (2.21) and (2.22), respectively.

Proof: In order to verify the assertions stated above, let us show that the function defined in (3.1) coincides with the value function in (2.2), and that the stopping time τ_* in (2.4) is optimal with the boundaries a_* and b_* specified above. For this, let us denote by $V(x)$ the right-hand side of the expression in (3.1). Taking into account the assumptions of continuous differentiability of the coefficients $\mu(x)$ and $\sigma(x)$ in (2.1), we may conclude from the equation in (2.6) that the derivative $V'(x)$ is continuously differentiable on (a_*, b_*). Hence, according to the conditions of (2.7)–(2.9), applying the local time-space formula from [24] (see also [25] and [26; Chapter II, Section 3.5] for a summary of the related results as well as further references), we get

$$\begin{aligned} e^{-rt} V(X_t) = V(x) + M'_t & \\ + \int_0^t e^{-rs} \, (\mathbb{L}V - rV)(X_s) \, I(X_s \neq a_*, X_s \neq b_*, X_s \neq c_i, i = 1, \ldots, n) \, ds & \\ + \sum_{i=1}^n \frac{1}{2} \int_0^t e^{-rs} \Big(V'(c_i+) - V'(c_i-) \Big) \, I(X_s = c_i) \, d\ell^i_s & \end{aligned} \tag{3.4}$$

for all $t \geq 0$, where the process $M' = (M'_t)_{t \geq 0}$ defined by

$$M'_t = \int_0^t e^{-rs} \, V'(X_s) \, I(X_s \neq c_i, i = 1, \ldots, n) \, \sigma(X_s) \, dW_s \tag{3.5}$$

is a local martingale with respect to P_x, which is a probability measure under which the process X starts at $x > 0$. Here, the process $\ell^i = (\ell^i_t)_{t \geq 0}$ defined by

$$\ell^i_t = P_x - \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^t I(c_i - \delta < X_s < c_i + \delta) \, \sigma^2(X_s) \, ds \tag{3.6}$$

is the local time of the process X at the point $c_i > 0$ at which the derivative $H'(x)$ does not exist, and $I(\cdot)$ denotes the indicator function.

Suppose that at some $y > 0$, such that either $y < a_*$ or $y > b_*$ holds with $y \neq c_i$, $i = 1, \ldots, n$, we have $(\mathbb{L}H - rH)(y) > 0$. Then, there exists some $\varepsilon > 0$ and $\delta > 0$ such that either $y + \delta < a_*$ or $y - \delta > b_*$, while none of the points c_i, $i = 1, \ldots, n$, belong to the interval $(y - \delta, y + \delta)$, and $(\mathbb{L}H - rH)(x) \geq \varepsilon$ holds for all $x \in (y - \delta, y + \delta)$. We thus see that it would be worthy to continue the observation at least until the first time at which the process X exits the neighborhood $(y - \delta, y + \delta)$, for $y > 0$ fixed. However, this fact contradicts the assumption about the structure of the optimal stopping time in (2.4). We may therefore conclude that $(\mathbb{L}H - rH)(x) \leq 0$ holds for any $x < a_*$ and $x > b_*$ such that $x \neq c_i$, $i = 1, \ldots, n$. These arguments also show that if $(\mathbb{L}H - rH)(y) > 0$ holds then the expression in (3.4) yields that y belongs to the interval (a_*, b_*). The formula in (3.4) also implies that if $H'(y+) - H'(y-) > 0$ holds for some $y > 0$ then y belongs to the interval (a_*, b_*) as well. The latter fact follows from the property that the term in the third line of (3.4) is increasing more rapidly than the one in the second line there, on small time intervals under the assumptions above.

Suppose now that $V(z) \leq H(z)$ for some $z \in (a_*, b_*)$. Then, there exist points z' and z'' such that $a_* < z' \leq z'' < b_*$ and the equality $V(z) = H(z)$ holds for $z = z'$ and $z = z''$, as well as the equalities $V'(z'-) = H'(z')$ and $V'(z''+) = H'(z'')$ are satisfied, whenever the payoff function $H(x)$ is continuously differentiable at z' and z''. Hence, according to the arguments of the previous section, this would lead to the fact that the free-boundary problem in (2.6)–(2.8) would have the function $V(x)$ with the corresponding couple of boundaries a_* and z' or z'' and b_* as its solution, respectively. However, this fact contradicts the assumption of uniqueness of solution of the system in (2.18)–(2.19) and thus shows that the inequality $V(x) > H(x)$ should hold for all $x \in (a_*, b_*)$.

Getting all these arguments together with the fact proved above that the function $V(x)$ and the boundaries a_* and b_* satisfy the system in (2.6)–(2.10), we conclude that $(\mathbb{L}V - rV)(x) \leq 0$ holds for any $x > 0$ such that $x \neq a_*$, $x \neq b_*$, $x \neq c_i$, $i = 1, \ldots, n$, as well as $V(x) \geq H(x)$ holds for all $x > 0$. It follows from the continuous differentiability of the functions $\mu(x)$ and $\sigma(x)$ in (2.1) that the time spent by X at the points a_*, b_* and c_i, $i = 1, \ldots, n$, is of Lebesgue measure zero, and thus, the indicators which appear in the integrals in the second line of (3.4) and in (3.5) can be ignored (see, e.g. [8; Chapter II, Section 1]). Hence, the expression in (3.4) yields that the inequalities

$$e^{-r\tau}\, H(X_\tau) \leq e^{-r\tau}\, V(X_\tau) \leq V(x) + M'_\tau \tag{3.7}$$

hold for any stopping time τ of the process X started at $x > 0$.

Let $(\tau_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process M'. Taking the expectations with respect to the probability measure P_x in (3.7), by means of the optional sampling theorem (see, e.g. [15; Chapter I, Theorem 3.22]), we get that the inequalities

$$E_x\left[e^{-r(\tau\wedge\tau_n)}\,H(X_{\tau\wedge\tau_n})\right]\le E_x\left[e^{-r(\tau\wedge\tau_n)}\,V(X_{\tau\wedge\tau_n})\right] \tag{3.8}$$
$$\le V(x)+E_x\left[M'_{\tau\wedge\tau_n}\right]=V(x)$$

hold for all $x>0$. Hence, letting n go to infinity and using Fatou's lemma, we obtain

$$E_x\left[e^{-r\tau}\,H(X_\tau)\right]\le E_x\left[e^{-r\tau}\,V(X_\tau)\right]\le V(x) \tag{3.9}$$

for any stopping time τ and all $x>0$. By virtue of the structure of the stopping time in (2.4), it is readily seen that the equalities in (3.9) hold with τ_* instead of τ when either $x\le a_*$ or $x\ge b_*$.

It remains to show that the equalities are attained in (3.9) when τ_* replaces τ for $a_*<x<b_*$. By virtue of the fact that the function $V(x;a_*,b_*)$ and the boundaries a_* and b_* satisfy the conditions in (2.6) and (2.7), it follows from the expression in (3.4) and the structure of the stopping time in (2.4) that the equalities

$$e^{-r(\tau_*\wedge\tau_n)}\,V(X_{\tau_*\wedge\tau_n};a_*,b_*)=G(a_*,b_*)\,M^*_{\tau_n}=V(x)+M'_{\tau_*\wedge\tau_n} \tag{3.10}$$

are satisfied for all $x\in(a_*,b_*)$ and any localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ of M' and thus of M^* too, where the process M^* is defined in (3.3). Observe that the assumptions on $H(x)$, together with the explicit form of the function in (2.13)–(2.15) as well as the properties of the functions $U_+(x)$ and $U_-(x)$, yield that the condition

$$E_x\Big[\sup_{t\ge0}e^{-r(\tau_*\wedge t)}\,V(X_{\tau_*\wedge t};a_*,b_*)\Big]<\infty \tag{3.11}$$

holds for all $x\in(a_*,b_*)$, as well as the variable $e^{-r\tau_*}V(X_{\tau_*};a_*,b_*)$ is equal to zero on the event $\{\tau_*=\infty\}$ (P_x-a.s.). Hence, taking into account the property in (3.11), we conclude from the expression in (3.10) that the process $(M'_{\tau_*\wedge t})_{t\ge0}$ is a uniformly integrable martingale. Note that such properties remain also true for certain payoff functions $H(x)$ and underlying processes X in the one stopping boundary cases $0=a_*<b_*<\infty$ and $0<a_*<b_*=\infty$. Therefore, taking the expectations in (3.10) and letting n go to infinity, we apply Lebesgue's dominated convergence theorem to obtain the equalities

$$E_x\left[e^{-r\tau_*}\,H(X_{\tau_*})\right]=E_x\left[e^{-r\tau_*}\,V(X_{\tau_*};a_*,b_*)\right]=G(a_*,b_*)\,E_x\left[M^*_{\tau_*}\right]=V(x) \tag{3.12}$$

for all $x \in (a_*, b_*)$. The latter, together with the inequalities in (3.9), implies the fact that $V(x)$ coincides with the value function $V_*(x)$ from (2.2), and τ_* from (2.4) is an optimal stopping time. Note that these arguments also yield the expression in (3.2) as well as the fact that M^* is a uniformly integrable martingale. □

3.2. Optimality of the solution. Observe again that, since a_* and b_* is a couple solving the smooth-fit equations in (2.18)–(2.19), it is then automatically a critical point of the function $G(a, b)$. Moreover, the assertion of Theorem 3.1 implies the fact that the function $G(a, b)$ defined in (2.21) attains a *local* maximum at the couple a_* and b_*. In order to prove this claim, we observe that, for any $x_* \in (a_*, b_*)$ fixed, we may put $U_+(x_*) = U_-(x_*) = 1$. This can be done without loss of generality, since the functions $U_+(x)$ and $U_-(x)$ were chosen up to multiplicative constants, as fundamental solutions of the second order ordinary differential equation in (2.6). Suppose now that there exist $a' \in (a_* - \delta, a_* + \delta)$ and $b' \in (b_* - \delta, b_* + \delta)$ such that $G(a', b') > G(a_*, b_*)$ holds for some $\delta > 0$ with $a_* + \delta < x_* < b_* - \delta$. In this case, it would follow from the expression in (2.20) that $V(x_*; a', b') > V(x_*; a_*, b_*)$ holds too. However, this fact contradicts the assertion of Theorem 3.1 stating that τ_*, being of the type of (2.4), is optimal. Therefore, we have that $V(x_*; a, b) \leq V(x_*; a_*, b_*)$ holds for all $a \in (a_* - \delta, a_* + \delta)$ and $b \in (b_* - \delta, b_* + \delta)$, thus proving the claim.

3.3. The martingale approach of Beibel and Lerche. We now describe shortly the martingale approach of Beibel and Lerche [5] and [6]. To stay in the framework of those papers, we assume that the starting value x is chosen as x_* such that $U_+(x_*) = U_-(x_*) = 1$. In this case, the value of the expected reward functional from (2.2) admits the representation

$$V_*(x_*) = \sup_{\tau} E_{x_*} \left[M_\tau \left(\frac{H(X_\tau)}{pU_+(X_\tau) + (1-p)U_-(X_\tau)} \right) \right] \tag{3.13}$$

for any $0 < p < 1$ fixed, where the process $M = (M_t)_{t \geq 0}$ defined by

$$M_t = e^{-rt} \Big(p\, U_+(X_t) + (1-p)\, U_-(X_t) \Big) \tag{3.14}$$

forms a martingale. It thus follows that the solution of the initial optimal stopping problem is reduced to the maximization of the expression inside the inner brackets of (3.13) over $0 < p < 1$. More precisely, the following assertions hold (see [5; Subsection 2.3] and [6; Section 3]).

Theorem 3.2 *If $0 < p^* < 1$ is such that the expression*

$$0 < \sup_{a \le x_*} \frac{H(a)}{p^* U_+(a) + (1-p^*) U_-(a)} = \sup_{x_* \le b} \frac{H(b)}{p^* U_+(b) + (1-p^*) U_-(b)} = C^* < \infty \tag{3.15}$$

holds, then C^* coincides with the value $V_*(x_*)$ in (3.13)*.*

Moreover, if there exist numbers a_* and b_* such that $0 < a_* < x_* < b_* < \infty$ and the equalities

$$C^* = \frac{H(a_*)}{p^* U_+(a_*) + (1-p^*) U_-(a_*)} = \frac{H(b_*)}{p^* U_+(b_*) + (1-p^*) U_-(b_*)} \tag{3.16}$$

are satisfied, then the stopping time τ_* from (2.4) *is optimal in the problem of* (3.13)*.*

Observe that, for instance, such a number p^* exists when the situation

$$0 < \sup_{a \le x_*} \frac{H(a)}{U_-(a)} < \sup_{x_* \le b} \frac{H(b)}{U_-(b)} < \infty \quad \text{and} \quad 0 < \sup_{x_* \le b} \frac{H(b)}{U_+(b)} < \sup_{a \le x_*} \frac{H(a)}{U_+(a)} < \infty \tag{3.17}$$

is realized (see [5; Subsection 2.3] and [6; Section 3] for further details). Note that the condition of (3.17) is satisfied in Examples 4.2 and 4.6 below within the setting of two stopping boundaries. Furthermore, in all other examples, both approaches lead to the same results. We also observe that when the process X starts at some point $x \neq x_*$ such that $a_* < x < b_*$, it follows from (3.13) and (3.14) that the equalities

$$V_*(x) = C^* \, E_x\big[M_{\tau_*}\big] = C^* \Big(p^* \, U_+(x) + (1-p^*) \, U_-(x)\Big) \tag{3.18}$$

hold. Here, the mixing constant p^* defines the corresponding *representing* measure on the minimal r-excessive functions $U_+(x)$ and $U_-(x)$ in the sense of Salminen [29; Section 3].

3.4. Connection of the two approaches. We shall now compare the outcomes of both approaches. For this, let us assume that the process X starts at $x_* > 0$, so that we have $M_0^* = 1$ in (3.3). It follows from [27; Chapter VIII, Proposition 1.13] that there exists a probability measure P^* being locally equivalent to P on the filtration $(\mathcal{F}_t)_{t \ge 0}$ and such that its density process is given by

$$\frac{dP^*}{dP}\Big|_{\mathcal{F}_t} = M_t^* \tag{3.19}$$

for all $t \ge 0$, so that the restrictions $P^* \,|\, \mathcal{F}_{\tau_*}$ and $P \,|\, \mathcal{F}_{\tau_*}$ are equivalent on the event $\{\tau_* < \infty\}$. By virtue of the fact that $e^{-r\tau_*} \, H(X_{\tau_*})$ is equal to zero on the event $\{\tau_* = \infty\}$ (P-a.s.), we see from (3.10) that M^*_∞ is zero on $\{\tau_* = \infty\}$ (P-a.s.) too. Hence, the assertion of Theorem 3.1 implies the fact that the expressions

$$V(x_*; a_*, b_*) = G(a_*, b_*)\, E_{x_*}\left[M^*_{\tau_*}\, I(\tau_* < \infty) + M^*_{\tau_*}\, I(\tau_* = \infty)\right] \tag{3.20}$$
$$= G(a_*, b_*)\, P^*_{x_*}(\tau_* < \infty)$$

hold for $a_* < x_* < b_*$, where $P^*_{x_*}$ denotes the appropriate probability measure under which the process X starts at $x_* > 0$. Note that if $P^*_{x_*}(\tau_* < \infty) = 1$ holds, then we see from (3.20) that $V(x_*; a_*, b_*)$ is equal to $G(a_*, b_*)$. On the other hand, it follows from the result of Theorem 3.2 that C^* coincides with $V(x_*; a_*, b_*)$, and thus, $G(a_*, b_*)$ is equal to C^* in this case. Moreover, the explicit expression in (2.24) and the second equality in (3.16) directly imply the fact that $p(a_*, b_*)$ is equal to p^*. Thus, we can replace p^* by $p(a, b)$ and $p(a_*, b_*)$ in the expressions of (3.15) and (3.16), respectively. We may therefore conclude that the process M^* from (3.3) coincides with the martingale M from [5; Subsection 2.3] and [6; Section 3], which also appears in (3.13) and takes the form of (3.14) with $p = p^*$, when it is considered up to the time τ_*.

4 Some examples

In this section, we illustrate the results obtained above on several examples of discounted optimal stopping problems for one-dimensional linear diffusions. We demonstrate in Examples 4.2 and 4.6 that the connection between the two approaches established in Theorems 3.1 and 3.2 holds, under the assumption that the optimal stopping time is the first exit time of the underlying process from an open interval. We include Examples 4.3 and 4.4 to show that the smooth-fit conditions for the value functions at the optimal stopping boundaries may break down or even the optimal stopping time fails to exist, when the payoff function does not satisfy the appropriate assumptions. We also consider the case of symmetric payoffs in Example 4.1 and the case in which one of the boundaries is reflecting in Example 4.5 for completeness.

4.1. Some problems for a geometric Brownian motion. Let us first assume that $\mu(x) = (r - \delta)x$ and $\sigma(x) = \theta x$ in (2.1), for all $x > 0$ and some $0 < \delta < r$ and $\theta > 0$ fixed. In that case, the process X turns out to be a geometric Brownian motion with the state space $(0, \infty)$, and the fundamental solutions of the second order ordinary differential equation in (2.6) are given by $U_+(x) = x^{\gamma_+}$ and $U_-(x) = x^{\gamma_-}$ with

$$\gamma_\pm = \frac{1}{2} - \frac{r-\delta}{\theta^2} \pm \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\theta^2}\right)^2 + \frac{2r}{\theta^2}} \tag{4.1}$$

so that $\gamma_- < 0 < 1 < \gamma_+$ holds.

Example 4.1 (Symmetric smooth payoffs.) Let us first consider a continuously differentiable payoff function $H(x)$ such that $H(x) = H(1/x)$ holds for $x > 0$, and assume that $(r-\delta)/\theta^2 = 1/2$ (see [5; Subsection 2.3]). In this case, we have $a_* = 1/b_*$, so that $p(a,b) \equiv 1/2$ and the gain function takes the form $G(a,b) \equiv 2C_+(a,b) \equiv 2C_-(a,b) = 2H(a)/(a^\eta + a^{-\eta})$ with $b = 1/a$ and $\eta = \sqrt{2r/\theta^2}$. The latter function attains its maximum at a_* being a solution of

$$\eta \, \frac{H(a)}{a} \, \frac{a^\eta - a^{-\eta}}{a^\eta + a^{-\eta}} = H'(a) \tag{4.2}$$

whenever it exists, for $0 < a = 1/b < \infty$. These facts follow either from the equations in (2.18)–(2.19) or from the equation in (3.16) (see also [6; Subsection 4.1]). It then follows from Theorem 3.1 that a_* and $b_* = 1/a_*$ are optimal stopping boundaries in the corresponding optimal stopping problem, provided that the function $H(x)$ is chosen such that the optimal stopping time is of the structure (2.4).

Example 4.2 (Perpetual American strangle option.) Let us now consider the function $H(x) = (L-x)^+ \vee (x-K)^+$, for $x > 0$, where $0 < L < K$ are given constants (see [5; Subsection 2.4]). In this case, the gain function $G(a,b)$ admits the representation

$$G(a,b) = \frac{(b-K)(a^{\gamma_+} - a^{\gamma_-}) - (L-a)(b^{\gamma_+} - b^{\gamma_-})}{a^{\gamma_+} b^{\gamma_-} - b^{\gamma_+} a^{\gamma_-}} \tag{4.3}$$

so that the function $p(a,b)$ takes the form

$$p(a,b) = \frac{(L-a)b^{\gamma_-} - (b-K)a^{\gamma_-}}{(b-K)(a^{\gamma_+} - a^{\gamma_-}) - (L-a)(b^{\gamma_+} - b^{\gamma_-})} \tag{4.4}$$

for some $0 < a < L < K < b < \infty$. The function in (4.3) attains its maximum value at the couple a_* and b_* being the unique solution of the system in (2.18)–(2.19) which takes the form

$$I_+(a) = J_+(b) \quad \text{and} \quad I_-(a) = J_-(b) \tag{4.5}$$

with

$$I_+(a) = \frac{(\gamma_- - 1)a - \gamma_- L}{a^{\gamma_+}} \quad \text{and} \quad J_+(b) = \frac{(1-\gamma_-)b + \gamma_- K}{b^{\gamma_+}} \tag{4.6}$$

$$I_-(a) = \frac{(1-\gamma_+)a + \gamma_+ L}{a^{\gamma_-}} \quad \text{and} \quad J_-(b) = \frac{(\gamma_+ - 1)b - \gamma_+ K}{b^{\gamma_-}} \tag{4.7}$$

for all $0 < a < L < K < b < \infty$. It then follows from Theorem 3.1 that a_* and b_* are optimal stopping boundaries in the corresponding optimal stopping problem. We also note that the assumptions of Theorem 3.2 are satisfied, so that the value of $G(a_*, b_*)$ coincides with the value of C^* from [5; Subsection 2.4] when $0 < L < 1 < K$, under $x_* = 1$ (see Figure 4.1 below).

In order to show the existence and uniqueness of a solution of the system of equations in (4.5), we use the idea of proof of the existence and uniqueness of solutions applied for the systems of equations in (4.73)–(4.74) from [31; Chapter IV, Section 2] and (3.16)–(3.17) from [13; Section 3]. For this, we observe that, for the derivatives of the functions in (4.6)–(4.7), the expressions

$$I'_+(a) = -\frac{(\gamma_+ - 1)(\gamma_- - 1)a - \gamma_+\gamma_- L}{a^{\gamma_+ + 1}} \equiv -\frac{(\gamma_+ - 1)(\gamma_- - 1)(a - \overline{L})}{a^{\gamma_+ + 1}} < 0 \tag{4.8}$$

$$J'_+(b) = \frac{(\gamma_+ - 1)(\gamma_- - 1)b - \gamma_+\gamma_- K}{b^{\gamma_+ + 1}} \equiv \frac{(\gamma_+ - 1)(\gamma_- - 1)(b - \overline{K})}{b^{\gamma_+ + 1}} < 0 \tag{4.9}$$

$$I'_-(a) = \frac{(\gamma_+ - 1)(\gamma_- - 1)a - \gamma_+\gamma_- L}{a^{\gamma_- + 1}} \equiv \frac{(\gamma_+ - 1)(\gamma_- - 1)(a - \overline{L})}{a^{\gamma_- + 1}} > 0 \tag{4.10}$$

$$J'_-(b) = -\frac{(\gamma_+ - 1)(\gamma_- - 1)b - \gamma_+\gamma_- K}{b^{\gamma_- + 1}} \equiv -\frac{(\gamma_+ - 1)(\gamma_- - 1)(b - \overline{K})}{b^{\gamma_- + 1}} > 0 \tag{4.11}$$

hold for all $0 < a < \overline{L} < \overline{K} < b < \infty$ with

$$\overline{L} = \frac{\gamma_+\gamma_- L}{(\gamma_+ - 1)(\gamma_- - 1)} \equiv \frac{rL}{\delta} \quad \text{and} \quad \overline{K} = \frac{\gamma_+\gamma_- K}{(\gamma_+ - 1)(\gamma_- - 1)} \equiv \frac{rK}{\delta}. \tag{4.12}$$

Hence, the function $I_+(a)$ decreases on the interval $(0, \overline{L})$ with $I_+(0+) = \infty$ and $I_+(\overline{L}) = \gamma_- L/[(\gamma_+ - 1)\overline{L}^{\gamma_+}] < 0$, so that the range of its values is given by the interval $(I_+(\overline{L}), \infty)$. The function $J_+(b)$ decreases on the interval $(\overline{K}, \infty)$ with $J_+(\overline{K}) = -\gamma_- K/[(\gamma_+ - 1)\overline{K}^{\gamma_+}] > 0$ and $J_+(\infty) = 0$, so that the range of its values is given by the interval $(0, J_+(\overline{K}))$. The function $I_-(a)$ increases on the interval $(0, \overline{L})$ with $I_-(0+) = 0$ and $I_-(\overline{L}) = -\gamma_+ L/[(\gamma_- - 1)\overline{L}^{\gamma_-}] > 0$, so that the range of its values is given by the interval $(0, I_-(\overline{L}))$. The function $J_-(b)$ increases on the interval $(\overline{K}, \infty)$ with $J_-(\overline{K}) = \gamma_+ K/[(\gamma_- - 1)\overline{K}^{\gamma_-}] < 0$ and $J_-(\infty) = \infty$, so that the range of its values is given by the interval $(J_-(\overline{K}), \infty)$.

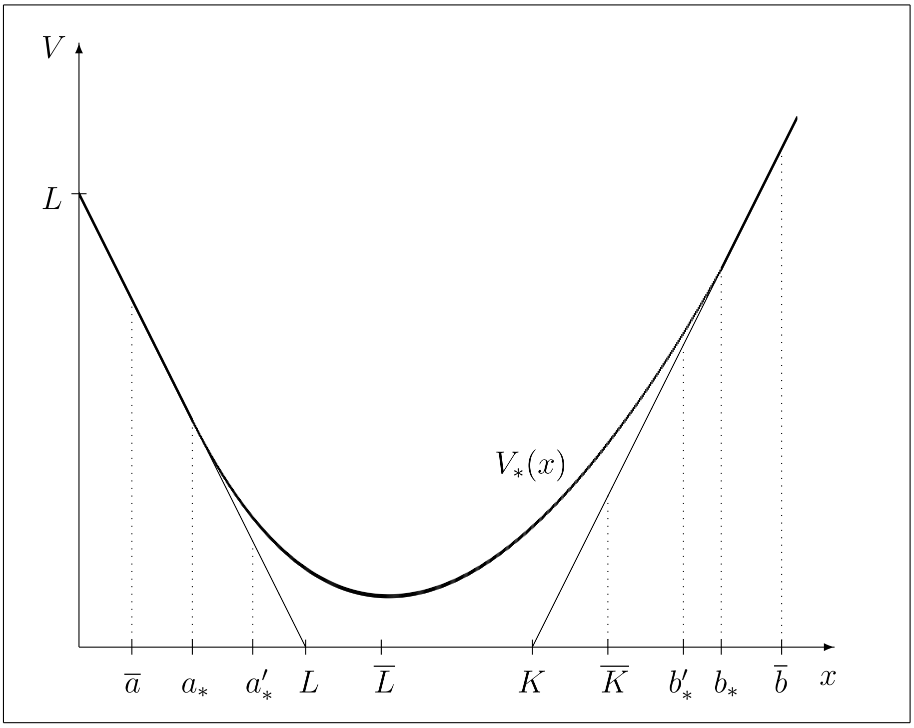

Figure 4.1. A computer drawing of the value function $V_*(x)$ and the optimal stopping boundaries a_* and b_* in Example 4.2.

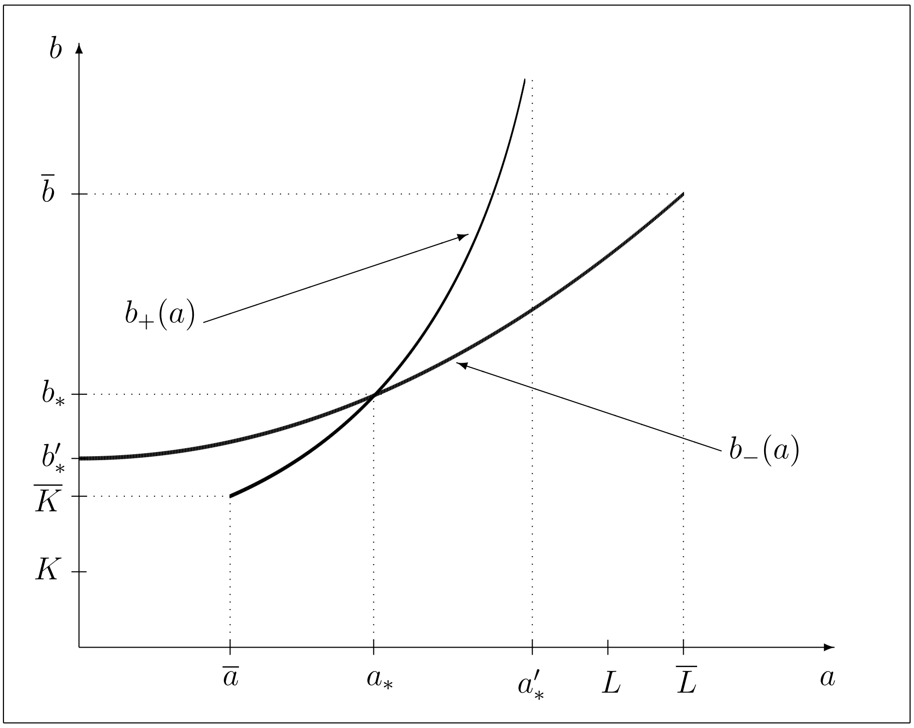

Figure 4.2. A computer drawing of the functions $b_+(a)$ and $b_-(a)$.

It thus follows from the left-hand equation in (4.5) that, for each $b \in (\overline{K}, \infty)$, there exists a unique number $a \in (\overline{a}, a'_*)$, where $\overline{a}$ is uniquely determined by the equation $I_+(\overline{a}) = J_+(\overline{K})$. It also follows from the right-hand equation in (4.5) that, for each $a \in (0, \overline{L})$, there exists a unique number $b \in (b'_*, \overline{b})$, where $\overline{b}$ is uniquely determined by the equation $I_-(\overline{L}) = J_-(\overline{b})$ (see Figure 4.1 above). Here, the numbers a'_* and b'_* defined by

$$a'_* = \frac{\gamma_- L}{\gamma_- - 1} \quad \text{and} \quad b'_* = \frac{\gamma_+ K}{\gamma_+ - 1} \tag{4.13}$$

are the optimal stopping boundaries for the cases of perpetual American put ($K = \infty$) and call ($L = 0$) options, respectively (see [20]). We may therefore conclude that the equations in (4.5) uniquely define the function $b_+(a)$ on $(\overline{a}, a'_*)$ with the range $(\overline{K}, \infty)$ and the function $b_-(a)$ on $(0, \overline{L})$ with the range $(b'_*, \overline{b})$, respectively. This fact directly yields that, for each point $a \in (\overline{a}, a'_*)$, there exist unique values $b_+(a)$ and $b_-(a)$ belonging to $(\overline{K}, \infty)$, that together with the inequalities $\overline{K} < b_-(0+) \equiv b'_* < b_-(\overline{L}) < \infty \equiv b_+(a'_*)$ guarantees the existence of exactly one intersection point with the coordinates a_* and b_* of the curves associated with the functions $b_+(a)$ and $b_-(a)$ on the interval $(\overline{a}, a'_*)$ such that $b'_* < b_+(a_*) \equiv b_* \equiv b_-(a_*) < \overline{b}$ holds (see Figure 4.2 above). This completes the proof of the claim.

Example 4.3 (Perpetual American capped strangle option.) In the assumptions of Example 4.2, let us change the payoff function $H(x)$ by $\widetilde{H}(x) = [(L-x)^+ \wedge (L-p)] \vee [(x-K)^+ \wedge (q-K)]$, for $x > 0$, where $0 < p < L < K < q$ are some given constants (see [9; Section 1] for the one-sided case). In this case, the gain function $\widetilde{G}(a,b)$ defined as $G(a,b)$ in (4.3) with $(L-a) \wedge (L-p)$ instead of $(L-a)$ and $(b-K) \wedge (q-K)$ instead of $(b-K)$ attains its unique maximum value at $\widetilde{a}_* = a_* \vee p$ and $\widetilde{b}_* = b_* \wedge q$, where the couple a_* and b_* is a unique solution of the system in (4.5). It follows from Theorem 3.1 that $\widetilde{a}_*$ and $\widetilde{b}_*$ are the optimal stopping boundaries in the corresponding optimal stopping problem. However, we see that the smooth-fit conditions in (2.8) do not necessarily hold when the payoff function $\widetilde{H}(x)$ is not smooth at $\widetilde{a}_*$ and $\widetilde{b}_*$.

Example 4.4 (Perpetual American barrier strangle option.) In the assumptions of Example 4.2, let us now change the payoff function $H(x)$ by $\widehat{H}(x) = [(L-x)^+ I(x > p)] \vee [(x-K)^+ I(x < q)]$, for $x > 0$, where $0 < p < L < K < q$ are some given constants. In this case, the gain function $\widehat{G}(a,b)$ defined as $G(a,b)$ in (4.3) with $(L-a)I(a>p)$ instead of $(L-a)$ and $(b-K)I(b<q)$ instead of $(b-K)$ attains its unique maximum value at $\widehat{a}_* = a_*$ and $\widehat{b}_* = b_*$ whenever $a_* > p$ and $b_* < q$, where the couple a_* and b_* is a unique solution of the system in (4.5). It follows from Theorem 3.1 that $\widehat{a}_*$ and $\widehat{b}_*$ are the optimal stopping boundaries in the corresponding optimal stopping problem whenever $a_* > p$ and $b_* < q$, and there exists no optimal stopping time otherwise.

4.2. A problem for a geometric Brownian motion with reflection. Let us now consider the process X solving the stochastic differential equation

$$dX_t = -\rho X_t \, dt + \theta X_t \, dW_t + I(X_t = 1) \, dN_t \quad (X_0 = x) \tag{4.14}$$

where $N = (N_t)_{t \geq 0}$ is an increasing process changing its value only when X started at $x > 1$ arrives at the point 1 being the instantaneously reflecting boundary, and $0 < r < \rho$ are some given constants.

Example 4.5 (Perpetual American lookback call option with floating strike.) Let us consider the function $H(x) = (x-K)^+$, for $x > 1$, with some $K > 1$ (see [30] and [32; Chapter VIII, Section 2d]). In this case, we have $a_* = 1$ as well as $p(1,b) \equiv \gamma_-/(\gamma_- - \gamma_+)$, and the gain function takes the form $G(1,b) = (b-K)^+/[p(1,b)b^{\gamma_+} + (1-p(1,b))b^{\gamma_-}]$ (see [6; Section 4.3]). The latter attains its maximum at some point b_* on the interval (K,∞). Note that, in this setting, the maximization of the function $G(1,b)$ is equivalent to the application of the condition

$$V'(1+) = 0 \quad (\textit{normal reflection}) \tag{4.15}$$

to the value function of the associated optimal stopping problem, instead of the corresponding smooth-fit condition in (2.8). Adapting the arguments of the proof of Theorem 3.1 to this case, we obtain that $a_* = 1$ is a reflecting boundary for the process X and $b_* > K$ is an optimal stopping boundary in (2.4).

4.3. A problem for a linear diffusion. Let us finally assume that $\mu(x) = 1 - \rho x$ and $\sigma(x) = \theta x$ in (2.1), for all $x > 0$ and some $0 < r < \rho$ and $\theta > 0$ fixed. In that case, the process X turns out to be a linear diffusion with the state space $(0,\infty)$ and the fundamental solutions of the second order ordinary differential equation in (2.6) are given by

$$U_+(x) = x^{\lambda_+} \frac{\Psi(-\lambda_+, 1-\lambda_+ + \lambda_-; 2/(\theta^2 x))}{\Psi(-\lambda_+, 1-\lambda_+ + \lambda_-; 2/\theta^2)} \tag{4.16}$$

and

$$U_-(x) = x^{\lambda_+} \frac{\Phi(-\lambda_+, 1-\lambda_+ + \lambda_-; 2/(\theta^2 x))}{\Phi(-\lambda_+, 1-\lambda_+ + \lambda_-; 2/\theta^2)} \tag{4.17}$$

with

$$\lambda_\pm = \frac{1}{2} + \frac{\rho}{\theta^2} \pm \sqrt{\left(\frac{1}{2} + \frac{\rho}{\theta^2}\right)^2 + \frac{2r}{\theta^2}} \tag{4.18}$$

so that $\lambda_- < 0 < 1 + 2\rho/\theta^2 < \lambda_+$ holds. Here, we denote by

$$\Phi(\alpha, \beta; z) = 1 + \sum_{k=1}^\infty \frac{(\alpha)_k}{(\beta)_k} \frac{z^k}{k!} \tag{4.19}$$

and

$$\Psi(\alpha, \beta; z) = \frac{\pi}{\sin(\pi\beta)} \left(\frac{\Phi(\alpha, \beta; z)}{\Gamma(1+\alpha-\beta)\Gamma(\beta)} - z^{1-\beta} \frac{\Phi(1+\alpha-\beta, 2-\beta; z)}{\Gamma(\alpha)\Gamma(2-\beta)} \right) \tag{4.20}$$

Kummer's confluent hypergeometric functions of the first and second kind, respectively, for $\beta \neq 0, -1, -2, \dots$ and $(\beta)_k = \beta(\beta+1)\cdots(\beta+k-1)$, $k \in \mathbb{N}$, where the series in (4.19) converges under all $z > 0$ (see, e.g. [1; Chapter XIII] and [4; Chapter VI]), and Γ denotes Euler's Gamma function.

Example 4.6 (Perpetual American integral strangle option with floating strikes.) Let us consider the function $H(x) = (L-x)^+ \vee (x-K)^+$, for $x > 0$, where $0 < L < K$ are given constants (see [17] for the one-sided case). In this case, the gain function $G(a,b)$ admits the representation of (2.23), so that the function $p(a,b)$ takes the form of (2.24) with $H(a) = L - a$ and $H(b) = b - K$, as well as $U_+(x)$ and $U_-(x)$ given by (4.16) and (4.17), for some $0 < a < L < K < b < \infty$. The resulting function $G(a,b)$ attains its maximum value at the couple a_* and b_* being the unique solution of the system in (2.18)–(2.19) which takes the form of (4.5) with

$$I_+(a) = e^{-2/(\theta^2 a)}\, a^{-\lambda_-} \bigg([\lambda_+ L - (\lambda_+ - 1)a]\, \Psi\left(-\lambda_+, 1 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 a}\right) - \frac{2\lambda_+(L-a)}{\theta^2 a}\, \Psi\left(1 - \lambda_+, 2 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 a}\right)\bigg) \tag{4.21}$$

$$J_+(b) = e^{-2/(\theta^2 b)}\, b^{-\lambda_-} \bigg([(\lambda_+ - 1)b - \lambda_+ K]\, \Psi\left(-\lambda_+, 1 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 b}\right) - \frac{2\lambda_+(b-K)}{\theta^2 b}\, \Psi\left(1 - \lambda_+, 2 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 b}\right)\bigg) \tag{4.22}$$

$$I_-(a) = e^{-2/(\theta^2 a)}\, a^{-\lambda_-} \bigg([\lambda_+ L - (\lambda_+ - 1)a]\, \Phi\left(-\lambda_+, 1 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 a}\right) + \frac{2\lambda_+(L-a)}{\theta^2(1-\lambda_+ + \lambda_-)a}\, \Phi\left(1 - \lambda_+, 2 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 a}\right)\bigg) \tag{4.23}$$

$$J_-(b) = e^{-2/(\theta^2 b)}\, b^{-\lambda_-} \bigg([(\lambda_+ - 1)b - \lambda_+ K]\, \Phi\left(-\lambda_+, 1 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 b}\right) + \frac{2\lambda_+(b-K)}{\theta^2(1-\lambda_+ + \lambda_-)b}\, \Phi\left(1 - \lambda_+, 2 - \lambda_+ + \lambda_-; \frac{2}{\theta^2 b}\right)\bigg) \tag{4.24}$$

for all $0 < a < L < K < b < \infty$. Here, we have used the fact that the function $D(x) = U'_+(x)U_-(x) - U_+(x)U'_-(x)$ admits the representation

$$D(x) = B\, \exp\left(-\int^x \frac{2(1-\rho y)}{\theta^2 y^2}\, dy\right) = B\, x^{\lambda_+ + \lambda_- - 1}\, e^{2/(\theta^2 x)} \tag{4.25}$$

for all $x > 0$ and some constant $B > 0$ representing the constant Wronskian determinant of the fundamental solutions $U_+(x)$ and $U_-(x)$ (see, e.g. [8; Chapter II, Section 1]). It then follows from Theorem 3.1 that a_* and b_* are optimal stopping boundaries in the corresponding optimal stopping problem. The uniqueness of solution of the system in (4.5) with the functions in (4.21)–(4.24) can be proved using the same schema of arguments as in Example 4.2 above, so that we omit the proof. Moreover, it can be shown directly using the arguments in [5; Subsection 2.3] and [6; Section 3] that the assumptions of Theorem 3.2 hold when $0 < L < 1 < K$, under $x_* = 1$.

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References

[1] Abramovitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* National Bureau of Standards. Wiley, New York.

[2] Alvarez, L. H. R. (2001). Reward functionals, salvage values, and optimal stopping. *Mathematical Methods of Operations Research* **54** (315–337).

[3] Alvarez, L. H. R. (2008). A class of solvable stopping games. *Applied Mathematics and Optimization* **49** (265–295).

[4] Bateman, H. and Erdélyi, A. (1953). *Higher Transcendental Functions.* Mc Graw-Hill, New York.

[5] Beibel, M. and Lerche, H. R. (1997). A new look at optimal stopping problems related to mathematical finance. *Statistica Sinica* **7** (93–108).

[6] Beibel, M. and Lerche, H. R. (2000). A note on optimal stopping of regular diffusions under random discounting. *Theory of Probability and Its Applications* **45** (657–669).

[7] Bensoussan, A. and Lions, J. L. (1982). *Applications of Variational Inequalities in Stochastic Control.* North Holland, Amsterdam.

[8] Borodin, A. N. and Salminen, P. (2002). *Handbook of Brownian Motion.* (Second Edition) Birkhäuser, Basel.

[9] Broadie, M. and Detemple, J. (1995). American capped call options on dividend-paying assets. *Review of Financial Studies* **8** (161–191).

[10] Dayanik, S. and Karatzas, I. (2003). On the optimal stopping problem for one-dimensional diffusions. *Stochastic Processes and Their Applications* **107** (173–212).

[11] Dynkin, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Soviet Mathematical Doklady* **4** (627–629).

[12] Fakeev, A. G. (1970). Optimal stopping rules for stochastic processes with continuous parameter. *Theory of Probability and Its Applications* **15** (324–331).

[13] Gapeev, P. V. (2005). The spread option optimal stopping game. *Exotic Option Pricing and Advanced Levy Models.* Kyprianou, A., Schoutens, W. and Wilmott, P. (eds.) Wiley, Chichester (293–305).

[14] Itô, K. and McKean, H. P. (1965). *Diffusion Processes and Their Sample Paths.* Springer, Berlin.

[15] Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus.* (Second Edition) Springer, New York.

[16] Karlin, S. and Taylor, H. M. (1981). *A Second Course in Stochastic Processes.* Academic Press, London – New York.

[17] Kramkov, D. O. and Mordecki, E. (1994). Integral opton. *Theory of Probability and Its Applications* **39**(1) (201–211).

[18] Lamberton, D. and Zervos, M. (2006). On the problem of optimally stopping a one-dimensional Itô diffusion. *Preprint.*

[19] Lerche, H. R. and Urusov, M. (2007). Optimal stopping via measure transformation: The Beibel–Lerche approach. *Stochastics* **3–4** (275–291).

[20] McKean, H. P. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Industrial Management Review* **6** (32–39).

[21] Mucci, A. G. (1978). Existence and explicit determination of optimal stopping times. *Stochastic Processes and Their Applications* **8** (33–58).

[22] Øksendal, B. (1998). *Stochastic Differential Equations. An Introduction with Applications.* (Fifth Edition) Springer, Berlin.

[23] Øksendal, B. and Reikvam, K. (1998). Viscosity solutions of optimal stopping problems. *Stochastics and Stochastics Reports* **62** (285–301).

[24] Peskir, G. (2005). A change-of-variable formula with local time on curves. *Journal of Theoretical Probability* **18** (499–535).

[25] Peskir, G. (2007). A change-of-variable formula with local time on surfaces. *Séminaire de Probababilité XL.* Lecture Notes in Mathematics **1899**, Springer (69–96).

[26] Peskir, G. and Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Birkhäuser, Basel.

[27] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion.* (Third Edition) Springer, Berlin.

[28] Rogers, L. C. G. and Williams, D. (1987). *Diffusions, Markov Processes and Martingales II. Itô Calculus.* Wiley, New York.

[29] Salminen, P. (1985). Optimal stopping of one-dimensional diffusions. *Mathematische Nachrichten* **124** (85–101).

[30] Shepp, L. A. and Shiryaev, A. N. (1994). A new look at the pricing of Russian options. *Theory of Probability and Its Applications* **39**(1) (103–119).

[31] Shiryaev, A. N. (1978). *Optimal Stopping Rules.* Springer, Berlin.

[32] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance.* World Scientific, Singapore.

[33] Snell, J. L. (1952). Applications of martingale system theorems. *Transactions of American Mathematical Society* **73** (293–312).