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The Wiener Disorder Problem with Finite Horizon

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The Wiener disorder problem seeks to determine a stopping time which is as close as possible to the (unknown) time of 'disorder' when the drift of an observed Wiener process changes from one value to another. In this paper we present a solution of the Wiener disorder problem when the horizon is finite. The method of proof is based on reducing the initial problem to a parabolic free-boundary problem where the continuation region is determined by a continuous curved boundary. By means of the change-of-variable formula containing the local time of a diffusion process on curves we show that the optimal boundary can be characterized as a unique solution of the nonlinear integral equation.

1. Introduction

The Wiener disorder problem seeks to determine a stopping time which is as close as possible to the (unknown) time of 'disorder' when the drift of an observed Wiener process changes from one value to another. At least two Bayesian formulations of the problem have been studied so far (for more details on the history of these formulations and their interplay see [25; Chapter IV]). In the 'free' formulation (below referred to as the 'Bayesian problem') one minimizes a linear combination of the probability of a 'false alarm' and the expectation of a 'delay' in detecting the time of disorder correctly with no constraint on the former. In the 'fixed false-alarm' formulation (below referred to as the 'variational problem') one minimizes the same linear combination under the constraint that the probability of a 'false alarm' cannot exceed a given value. In these formulations it is customary assumed that the time of disorder is exponentially distributed and this methodology will be adopted in the present paper as well.

Disorder problems (as well as closely related 'change-point' problems and more general 'quickest detection' problems) have originally arisen and still play a prominent role in quality control where one observes the output of a production line and wishes to detect deviation from an acceptable level. After the introduction of the original control charts by Shewhart [21] various modifications of the problem have been recognized (see [13]) and implemented in a number

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of applied sciences (see [8]). These problems include: epidemiology (where one tests whether the incidence of a disease has remained constant over time and wishes to estimate the time of change in order to suggest possible causes); rhythm analysis in electrocardiograms (where the use of change detection methods constitutes a part of pattern recognition analysis); changes of the critical modes in electric-energy systems; the appearance of a target in radio/radar location; the appearance of 'breaks' in geological data; the beginning of earthquakes or tsunamis; seismic signal processing; the appearance of a shock wave front; the study of historical texts or manuscripts; the study of archeological sites, etc. Specific applications described in [1] include: statistical image processing and edge detection in noisy images; change-points in economic regression models (split or two-phase regression); detection of discontinuities in astrophysical time series with dependent data; changes in hazard rates as shown to occur after bone-marrow transplantation for leukemia patients; the comparison and matching of DNA sequences; the simultaneous estimation of smoothly varying parts and discontinuities of curves and surfaces. Applications in financial data analysis (detection of arbitrage) are recently discussed in [26].

In the situations described above one often wishes to decide whether a disorder appears before some fixed time in the future. Thus, from the standpoint of these particular applications, the *finite horizon* formulation of the disorder problem appears to be more desirable than the *infinite horizon* formulation of the same problem. It turns out, however, that the former problems are more difficult in continuous time and as such have not been studied so far. Clearly, among all processes that can be considered in the problem, the Wiener process and the Poisson process take a central place. Once these problems are understood sufficiently well, the study of problems including other processes may follow a similar line of arguments.

Shiryaev [22]–[24] derived an explicit solution of the Bayesian and variational problem for a Wiener process with infinite horizon by reducing the initial optimal stopping problem to a free-boundary problem for a differential operator (see also [27]). Some particular cases of the Bayesian problem for a Poisson process with infinite horizon were solved by Gal'chuk and Rozovskii [5] and Davis [2]. A complete solution of the latter problem was given in [18] by reducing the initial optimal stopping problem to a free-boundary problem for a differential-difference operator. The main aim of the present paper is to derive a solution of the Bayesian and variational problem for a Wiener process with *finite horizon*.

It is known that optimal stopping problems for Markov processes with finite horizon are inherently two-dimensional and thus analytically more difficult than those with infinite horizon. A standard approach for handling such a problem is to formulate a free-boundary problem for the (parabolic) operator associated with the (continuous) Markov process (see e.g. [11], [10], [6], [28], [7], [12]). Since solutions to such free-boundary problems are rarely known explicitly, the question often reduces to prove the existence and uniqueness of a solution to the free-boundary problem, which then leads to the optimal stopping boundary and the value function of the optimal stopping problem. In some cases the optimal stopping boundary has been characterized as a unique solution of the system of (at least) countably many nonlinear integral equations (see e.g. [7; Theorem 4.3]). A method of linearization was suggested in [14] with the aim of proving that only one equation from such a system may be sufficient to characterize the optimal stopping boundary uniquely. A complete proof of the latter fact in the case of a specific optimal stopping problem was given in [16] (see also [17]).

In Section 2 of the present paper we reduce the initial Bayesian problem to a finite-horizon optimal stopping problem for a diffusion process and the gain function containing an integral where the continuation region is determined by a continuous curved boundary. In order to find an analytic expression for the boundary we formulate an equivalent parabolic free-boundary problem for the infinitesimal operator of the strong Markov a posteriori probability process. By means of the method of proof proposed in [14] and [16], and using the change-of-variable formula from [15], we show that the optimal stopping boundary can be uniquely determined from a nonlinear Volterra integral equation of the second kind. This also leads to an explicit formula for the value (risk) function in terms of the optimal stopping boundary. In Section 3 we formulate the variational problem with finite horizon and construct an equivalent Bayesian problem. We then show that the optimality of the first hitting time of the a posteriori probability process to a continuous curved boundary can be deduced from the solution of the Bayesian problem. In Section 4 we present an explicit expression for the transition density function of the a posteriori probability process that is needed in the proof of Section 2.

The main results of the paper are stated in Theorem 2.1 and Theorem 3.1. The optimal sequential procedure in the initial Bayesian problem is displayed more explicitly in Remark 2.2. A simple numerical method for calculating the optimal boundary is presented in Remark 2.3.

2. Solution of the Bayesian problem

In the Bayesian problem with finite horizon (see [25; Chapter IV, Sections 3-4] for the infinite horizon case) it is assumed that we observe a trajectory of the Wiener process $X = (X_t)_{0 \le t \le T}$ with a drift changing from 0 to $\mu \neq 0$ at some random time θ taking the value 0 with probability π and being exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$.

2.1. For a precise probabilistic formulation of the Bayesian problem it is convenient to assume that all our considerations take place on a probability space $(\Omega, \mathcal{F}, P_\pi)$ where the probability measure P_π has the following structure:

$$P_\pi = \pi P^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda s} P^s \, ds \tag{2.1}$$

for $\pi \in [0,1]$ and P^s is a probability measure specified below for $s \ge 0$. Let θ be a nonnegative random variable satisfying $P_\pi[\theta = 0] = \pi$ and $P_\pi[\theta > t \,|\, \theta > 0] = e^{-\lambda t}$ for all $0 \le t \le T$ and some $\lambda > 0$, and let $W = (W_t)_{0 \le t \le T}$ be a standard Wiener process started at zero under P_π. It is assumed that θ and W are independent.

It is further assumed that we observe a process $X = (X_t)_{0 \le t \le T}$ satisfying the stochastic differential equation:

$$dX_t = \mu I(t \ge \theta) \, dt + \sigma \, dW_t \quad (X_0 = 0) \tag{2.2}$$

and thus being of the form:

$$X_t = \begin{cases} \sigma W_t & \text{if } \ t < \theta \\ \mu(t-\theta) + \sigma W_t & \text{if } \ t \ge \theta \end{cases} \tag{2.3}$$

where $\mu \neq 0$ and $\sigma > 0$ are given and fixed. Thus $P_\pi[X \in \cdot \,|\, \theta = s\,] = P^s[X \in \cdot\,]$ is the distribution law of a Wiener process with the diffusion coefficient $\sigma > 0$ and a drift changing from 0 to μ at time $s \geq 0$. It is assumed that the time θ of *'disorder'* is unknown (i.e. it cannot be observed directly).

Being based upon the continuous observation of X our task is to find a stopping time τ_* of X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \mid 0 \leq s \leq t)$ generated by X for $0 \leq t \leq T$) that is *'as close as possible'* to the unknown time θ. More precisely, the problem consists of computing the risk function:

$$V(\pi) = \inf_{0 \leq \tau \leq T} \Big(P_\pi[\tau < \theta] + c\, E_\pi[\tau - \theta]^+ \Big) \tag{2.4}$$

and finding the optimal stopping time τ_* at which the infimum in (2.4) is attained. Here $P_\pi[\tau < \theta]$ is the probability of a *'false alarm'*, $E_\pi[\tau - \theta]^+$ is the *'average delay'* in detecting the *'disorder'* correctly, and $c > 0$ is a given constant. Note that $\tau_* = T$ corresponds to the conclusion that $\theta \geq T$.

2.2. By means of standard arguments (see [25; pages 195-197]) one can reduce the Bayesian problem (2.4) to the optimal stopping problem:

$$V(\pi) = \inf_{0 \leq \tau \leq T} E_\pi \left[1 - \pi_\tau + c \int_0^\tau \pi_t \, dt \right] \tag{2.5}$$

for the a posteriori probability process $\pi_t = P_\pi[\theta \leq t | \mathcal{F}_t^X]$ for $0 \leq t \leq T$ with $P_\pi[\pi_0 = \pi] = 1$.

2.3. It can be shown (see [25; page 202]) that the likelihood ratio process $(\varphi_t)_{0 \leq t \leq T}$ defined by $\varphi_t = \pi_t/(1 - \pi_t)$ admits the representation:

$$\varphi_t = e^{Y_t} \left(\frac{\pi}{1 - \pi} + \lambda \int_0^t e^{-Y_s} \, ds \right) \tag{2.6}$$

where the process $(Y_t)_{0 \leq t \leq T}$ is given by:

$$Y_t = \lambda t + \frac{\mu}{\sigma^2} \left(X_t - \frac{\mu}{2} t \right). \tag{2.7}$$

It follows that the a posteriori probability process $(\pi_t)_{0 \leq t \leq T}$ can be expressed as:

$$\pi_t = \frac{\varphi_t}{1 + \varphi_t} \tag{2.8}$$

and hence solves the stochastic differential equation:

$$d\pi_t = \lambda (1 - \pi_t) \, dt + \frac{\mu}{\sigma} \pi_t (1 - \pi_t) \, d\overline{W}_t \quad (\pi_0 = \pi) \tag{2.9}$$

where the innovation process $(\overline{W}_t)_{0 \leq t \leq T}$ defined by:

$$\overline{W}_t = \frac{1}{\sigma} \Big(X_t - \mu \int_0^t \pi_s \, ds \Big) \tag{2.10}$$

is a standard Wiener process (see also [9; Chapter IX]). Using (2.6)-(2.8) it can be verified that $(\pi_t)_{0 \leq t \leq T}$ is a time-homogeneous (strong) Markov process under P_π for $\pi \in [0, 1]$ with respect to the natural filtration. As the latter clearly coincides with $(\mathcal{F}_t^X)_{0\le t\le T}$ it is also clear that the infimum in (2.5) can equivalently be taken over all stopping times of $(\pi_t)_{0\le t\le T}$.

2.4. In order to solve the problem (2.5) let us consider the extended optimal stopping problem for the Markov process $(t,\pi_t)_{0\le t\le T}$ given by:

$$V(t,\pi) = \inf_{0\le\tau\le T-t} E_{t,\pi}\left[G(\pi_{t+\tau}) + \int_0^\tau H(\pi_{t+s})\,ds\right] \tag{2.11}$$

where $P_{t,\pi}[\pi_t = \pi] = 1$, i.e. $P_{t,\pi}$ is a probability measure under which the diffusion process $(\pi_{t+s})_{0\le s\le T-t}$ solving (2.9) starts at π, the infimum in (2.11) is taken over all stopping times τ of $(\pi_{t+s})_{0\le s\le T-t}$, and we set $G(\pi) = 1-\pi$ and $H(\pi) = c\,\pi$ for all $\pi \in [0,1]$. Note that $(\pi_{t+s})_{0\le s\le T-t}$ under $P_{t,\pi}$ is equally distributed as $(\pi_s)_{0\le s\le T-t}$ under P_π. This fact will be frequently used in the sequel without further mentioning. Since G and H are bounded and continuous on $[0,1]$ it is possible to apply a version of Theorem 3 in [25; page 127] for a finite time horizon and by statement (2) of that theorem conclude that an optimal stopping time exists in (2.11).

2.5. Let us now determine the structure of the optimal stopping time in the problem (2.11). The facts derived in Subsections 2.5-2.8 will be summarized in Subsection 2.9 below.

(i) Note that by (2.9) we have:

$$G(\pi_{t+s}) = G(\pi) - \lambda \int_0^s (1-\pi_{t+u})\,du + M_s \tag{2.12}$$

where the process $(M_s)_{0\le s\le T-t}$ defined by $M_s = -\int_0^s (\mu/\sigma)\pi_{t+u}(1-\pi_{t+u})d\overline{W}_u$ is a continuous martingale under $P_{t,\pi}$. It follows from (2.12) using the optional sampling theorem (see e.g. [19; Chapter II, Theorem 3.2]) that:

$$E_{t,\pi}\left[G(\pi_{t+\sigma}) + \int_0^\sigma H(\pi_{t+u})\,du\right] = G(\pi) + E_{t,\pi}\left[\int_0^\sigma ((\lambda+c)\pi_{t+u} - \lambda)\,du\right] \tag{2.13}$$

for each stopping time σ of $(\pi_{t+s})_{0\le s\le T-t}$. Choosing σ to be the exit time from a small ball, we see from (2.13) that it is never optimal to stop when $\pi_{t+s} < \lambda/(\lambda+c)$ for $0\le s < T-t$. In other words, this shows that all points (t,π) for $0\le t<T$ with $0\le\pi<\lambda/(\lambda+c)$ belong to the continuation region:

$$C = \{(t,\pi)\in[0,T)\times[0,1] \mid V(t,\pi) < G(\pi)\}. \tag{2.14}$$

(ii) Recalling the solution to the problem (2.5) in the case of infinite horizon, where the stopping time $\tau_* = \inf\{\, t>0 \,|\, \pi_t \ge A_*\}$ is optimal and $0 < A_* < 1$ is uniquely determined from the equation (4.147) in [25; page 201], we see that all points (t,π) for $0\le t\le T$ with $A_* \le \pi \le 1$ belong to the stopping region. Moreover, since $\pi \mapsto V(t,\pi)$ with $0\le t\le T$ given and fixed is concave on $[0,1]$ (this is easily deduced using the same arguments as in [25; pages 197-198]), it follows directly from the previous two conclusions about the continuation and stopping region that there exists a function g satisfying $0 < \lambda/(\lambda+c) \le g(t) \le A_* < 1$ for all $0\le t\le T$ such that the continuation region is an open set of the form:

$$C = \{(t,\pi)\in[0,T)\times[0,1] \mid \pi < g(t)\} \tag{2.15}$$

and the stopping region is the closure of the set:

$$D = \{(t, \pi) \in [0, T) \times [0, 1] \mid \pi > g(t)\}. \tag{2.16}$$

(Below we will show that V is continuous so that C is open indeed. We will also see that $g(T) = \lambda/(\lambda + c)$.)

(iii) Since the problem (2.11) is time-homogeneous, in the sense that G and H are functions of space only (i.e. do not depend on time), it follows that the map $t \mapsto V(t, \pi)$ is increasing on $[0, T]$. Hence if (t, π) belongs to C for some $\pi \in [0, 1]$ and we take any other $0 \le t' < t \le T$, then $V(t', \pi) \le V(t, \pi) < G(\pi)$, showing that (t', π) belongs to C as well. From this we may conclude in (2.15)-(2.16) that the boundary $t \mapsto g(t)$ is decreasing on $[0, T]$.

(iv) Let us finally observe that the value function V from (2.11) and the boundary g from (2.15)-(2.16) also depend on T and let them denote here by V^T and g^T, respectively. Using the fact that $T \mapsto V^T(t, \pi)$ is a decreasing function on $[t, \infty)$ and $V^T(t, \pi) = G(\pi)$ for all $\pi \in [g^T(t), 1]$, we conclude that if $T < T'$, then $0 \le g^T(t) \le g^{T'}(t) \le 1$ for all $t \in [0, T]$. Letting T' in the previous expression go to ∞, we get that $0 < \lambda/(\lambda + c) \le g^T(t) \le A_* < 1$ and $A_* \equiv \lim_{T \to \infty} g^T(t)$ for all $t \ge 0$, where A_* is the optimal stopping point in the infinite horizon problem referred to above.

2.6. Let us now show that the value function $(t, \pi) \mapsto V(t, \pi)$ is continuous on $[0, T] \times [0, 1]$. For this it is enough to prove that:

$$\pi \mapsto V(t_0, \pi) \quad \text{is continuous at} \quad \pi_0 \tag{2.17}$$

$$t \mapsto V(t, \pi) \quad \text{is continuous at} \quad t_0 \quad \text{uniformly over} \quad \pi \in [\pi_0 - \delta, \pi_0 + \delta] \tag{2.18}$$

for each $(t_0, \pi_0) \in [0, T] \times [0, 1]$ with some $\delta > 0$ small enough (it may depend on π_0). Since (2.17) follows by the fact that $\pi \mapsto V(t, \pi)$ is concave on $[0, 1]$, it remains to establish (2.18).

For this, let us fix arbitrary $0 \le t_1 < t_2 \le T$ and $0 \le \pi \le 1$, and let $\tau_1 = \tau_*(t_1, \pi)$ denote the optimal stopping time for $V(t_1, \pi)$. Set $\tau_2 = \tau_1 \wedge (T - t_2)$ and note since $t \mapsto V(t, \pi)$ is increasing on $[0, T]$ and $\tau_2 \le \tau_1$ that we have:

$$\begin{aligned} 0 &\le V(t_2, \pi) - V(t_1, \pi) \\ &\le E_\pi\left[1 - \pi_{\tau_2} + c \int_0^{\tau_2} \pi_u \, du\right] - E_\pi\left[1 - \pi_{\tau_1} + c \int_0^{\tau_1} \pi_u \, du\right] \\ &\le E_\pi[\pi_{\tau_1} - \pi_{\tau_2}]. \end{aligned} \tag{2.19}$$

From (2.9) we find using the optional sampling theorem that:

$$E_\pi[\pi_\sigma] = \pi + \lambda \, E_\pi\left[\int_0^\sigma (1 - \pi_t) \, dt\right] \tag{2.20}$$

for each stopping time σ of $(\pi_t)_{0 \le t \le T}$. Hence by the fact that $\tau_1 - \tau_2 \le t_2 - t_1$ we get:

$$\begin{aligned} E_\pi[\pi_{\tau_1} - \pi_{\tau_2}] &= \lambda \, E_\pi\left[\int_0^{\tau_1} (1 - \pi_t) \, dt - \int_0^{\tau_2} (1 - \pi_t) \, dt\right] \\ &= \lambda \, E_\pi\left[\int_{\tau_2}^{\tau_1} (1 - \pi_t) \, dt\right] \le \lambda \, E_\pi[\tau_1 - \tau_2] \le \lambda \, (t_2 - t_1) \end{aligned} \tag{2.21}$$

for all $0 \le \pi \le 1$. Combining (2.19) with (2.21) we see that (2.18) follows. In particular, this shows that the instantaneous-stopping condition (2.43) is satisfied.

2.7. In order to prove that the smooth-fit condition (2.44) holds, i.e. that $\pi \mapsto V(t,\pi)$ is C^1 at $g(t)$, let us fix a point $(t,\pi) \in [0,T) \times (0,1)$ lying on the boundary g so that $\pi = g(t)$. Then for all $\varepsilon > 0$ such that $0 < \pi - \varepsilon < \pi$ we have:

$$\frac{V(t,\pi) - V(t,\pi-\varepsilon)}{\varepsilon} \ge \frac{G(\pi) - G(\pi-\varepsilon)}{\varepsilon} = -1 \tag{2.22}$$

and hence, taking the limit in (2.22) as $\varepsilon \downarrow 0$, we get:

$$\frac{\partial^- V}{\partial \pi}(t,\pi) \ge G'(\pi) = -1 \tag{2.23}$$

where the left-hand derivative in (2.23) exists (and is finite) by virtue of the concavity of $\pi \mapsto V(t,\pi)$ on $[0,1]$. Note that the latter will also be proved independently below.

Let us now fix some $\varepsilon > 0$ such that $0 < \pi - \varepsilon < \pi$ and consider the stopping time $\tau_\varepsilon = \tau_*(t, \pi-\varepsilon)$ being optimal for $V(t,\pi-\varepsilon)$. Note that τ_ε is the first exit time of the process $(\pi_{t+s})_{0\le s\le T-t}$ from the set C in (2.15). Then from (2.11) using equation (2.9) and the optional sampling theorem we obtain:

$$\begin{aligned} &V(t,\pi) - V(t,\pi-\varepsilon) \\ &\le E_\pi\left[1 - \pi_{\tau_\varepsilon} + c\int_0^{\tau_\varepsilon} \pi_u\,du\right] - E_{\pi-\varepsilon}\left[1 - \pi_{\tau_\varepsilon} + c\int_0^{\tau_\varepsilon} \pi_u\,du\right] \\ &= E_\pi\left[1 - \pi_{\tau_\varepsilon} + c\left(\tau_\varepsilon + \frac{\pi - \pi_{\tau_\varepsilon}}{\lambda}\right)\right] - E_{\pi-\varepsilon}\left[1 - \pi_{\tau_\varepsilon} + c\left(\tau_\varepsilon + \frac{\pi - \varepsilon - \pi_{\tau_\varepsilon}}{\lambda}\right)\right] \\ &= \left(\frac{c}{\lambda} + 1\right)\Big(E_{\pi-\varepsilon}[\pi_{\tau_\varepsilon}] - E_\pi[\pi_{\tau_\varepsilon}]\Big) + c\Big(E_\pi[\tau_\varepsilon] - E_{\pi-\varepsilon}[\tau_\varepsilon]\Big) + \varepsilon\frac{c}{\lambda} \end{aligned} \tag{2.24}$$

By (2.1) and (2.6)-(2.8) it follows that:

$$\begin{aligned} &E_{\pi-\varepsilon}[\pi_{\tau_\varepsilon}] - E_\pi[\pi_{\tau_\varepsilon}] \\ &= (\pi-\varepsilon)E^0[S(\pi-\varepsilon)] + (1-\pi+\varepsilon)\int_0^\infty \lambda e^{-\lambda s} E^s[S(\pi-\varepsilon)]\,ds \\ &\quad - \pi E^0[S(\pi)] - (1-\pi)\int_0^\infty \lambda e^{-\lambda s} E^s[S(\pi)]\,ds \\ &= \pi E^0[S(\pi-\varepsilon) - S(\pi)] + (1-\pi)\int_0^\infty \lambda e^{-\lambda s} E^s[S(\pi-\varepsilon) - S(\pi)]\,ds \\ &\quad - \varepsilon E^0[S(\pi-\varepsilon)] + \varepsilon\int_0^\infty \lambda e^{-\lambda s} E^s[S(\pi-\varepsilon)]\,ds \end{aligned} \tag{2.25}$$

where the function S is defined by:

$$S(\pi) = e^{Y_{\tau_\varepsilon}}\left(\frac{\pi}{1-\pi} + \lambda\int_0^{\tau_\varepsilon} e^{-Y_u}\,du\right) \Big/ \left(1 + e^{Y_{\tau_\varepsilon}}\left(\frac{\pi}{1-\pi} + \lambda\int_0^{\tau_\varepsilon} e^{-Y_u}\,du\right)\right). \tag{2.26}$$

By virtue of the mean value theorem there exists $\xi \in [\pi-\varepsilon, \pi]$ such that:

$$\begin{aligned} \text{(2.27)} \qquad & \pi E^0[S(\pi-\varepsilon)-S(\pi)] + (1-\pi)\int_0^\infty \lambda e^{-\lambda s} E^s[S(\pi-\varepsilon)-S(\pi)]\, ds \\ & = -\varepsilon \left(\pi E^0[S'(\xi)] + (1-\pi)\int_0^\infty \lambda e^{-\lambda s} E^s[S'(\xi)]\, ds \right) \end{aligned}$$

where S' is given by:

$$\text{(2.28)} \qquad S'(\xi) = e^{Y_{\tau_\varepsilon}} \Big/ \left((1-\xi)^2 \left(1 + e^{Y_{\tau_\varepsilon}} \left(\frac{\xi}{1-\xi} + \lambda \int_0^{\tau_\varepsilon} e^{-Y_u}\, du \right) \right)^2 \right).$$

Considering the second term on the right-hand side of (2.24) we find using (2.1) that:

$$\begin{aligned} \text{(2.29)} \qquad c\Big(E_\pi[\tau_\varepsilon] - E_{\pi-\varepsilon}[\tau_\varepsilon]\Big) &= c\varepsilon \Big(E^0[\tau_\varepsilon] + \int_0^\infty \lambda e^{-\lambda s} E^s[\tau_\varepsilon]\, ds \Big) \\ &= \frac{c\varepsilon}{1-\pi} \Big((1-2\pi) E^0[\tau_\varepsilon] + E_\pi[\tau_\varepsilon] \Big). \end{aligned}$$

Recalling that τ_ε is equally distributed as $\widetilde{\tau}_\varepsilon = \inf\{\, 0 \le s \le T-t \mid \pi_s^{\pi-\varepsilon} \ge g(t+s)\,\}$, where we write $\pi_s^{\pi-\varepsilon}$ to indicate dependance on the initial point $\pi-\varepsilon$ through (2.6) in (2.9) above, and considering the hitting time σ_ε to the constant level $\pi = g(t)$ given by $\sigma_\varepsilon = \inf\{\, s \ge 0 \mid \pi_s^{\pi-\varepsilon} \ge \pi \,\}$, it follows that $\widetilde{\tau}_\varepsilon \le \sigma_\varepsilon$ for every $\varepsilon > 0$ since g is decreasing, and $\sigma_\varepsilon \downarrow \sigma_0$ as $\varepsilon \downarrow 0$ where $\sigma_0 = \inf\{\, s > 0 \mid \pi_s^\pi \ge \pi \,\}$. On the other hand, since the diffusion process $(\pi_s^\pi)_{s \ge 0}$ solving (2.9) is regular (see e.g. [19; Chapter 7, Section 3]), it follows that $\sigma_0 = 0$ P_π-a.s. This in particular shows that $\tau_\varepsilon \to 0$ P_π-a.s. Hence we easily find that:

$$\text{(2.30)} \qquad S(\pi-\varepsilon) \to \pi, \quad S(\xi) \to \pi \quad \text{and} \quad S'(\xi) \to 1 \quad (P_\pi\text{-a.s.})$$

as $\varepsilon \downarrow 0$ for $s \ge 0$, and clearly $|S'(\xi)| \le K$ with some $K > 0$ large enough.

From (2.24) using (2.25)-(2.30) it follows that:

$$\text{(2.31)} \qquad \frac{V(t,\pi) - V(t,\pi-\varepsilon)}{\varepsilon} \le \left(\frac{c}{\lambda} + 1\right)\Big(-1 + o(1)\Big) + o(1) + \frac{c}{\lambda} = -1 + o(1)$$

as $\varepsilon \downarrow 0$ by the dominated convergence theorem and the fact that $P^0 \ll P_\pi$. This combined with (2.22) above proves that $V_\pi^-(t,\pi)$ exists and equals $G'(\pi) = -1$.

2.8. We proceed by proving that the boundary g is continuous on $[0,T]$ and that $g(T) = \lambda/(\lambda+c)$.

(i) Let us first show that the boundary g is right-continuous on $[0,T]$. For this, fix $t \in [0,T)$ and consider a sequence $t_n \downarrow t$ as $n \to \infty$. Since g is decreasing, the right-hand limit $g(t+)$ exists. Because $(t_n, g(t_n)) \in \overline{D}$ for all $n \ge 1$, and $\overline{D}$ is closed, we see that $(t, g(t+)) \in \overline{D}$. Hence by (2.16) we see that $g(t+) \ge g(t)$. The reverse inequality follows obviously from the fact that g is decreasing on $[0,T]$, thus proving the claim.

(ii) Suppose that at some point $t_* \in (0,T)$ the function g makes a jump, i.e. let $g(t_*-) > g(t_*) \ge \lambda/(\lambda+c)$. Let us fix a point $t' < t_*$ close to t_* and consider the half-open region

$R \subset C$ being a curved trapezoid formed by the vertices $(t', g(t'))$, $(t_*, g(t_*-))$, (t_*, π') and (t', π') with any π' fixed arbitrarily in the interval $(g(t_*), g(t_*-))$. Observe that the strong Markov property implies that the value function V from (2.11) is $C^{1,2}$ on C. Note also that the gain function G is C^2 in R so that by the Leibnitz-Newton formula using (2.43) and (2.44) it follows that:

$$V(t,\pi) - G(\pi) = \int_\pi^{g(t)} \int_u^{g(t)} \left(\frac{\partial^2 V}{\partial \pi^2}(t,v) - \frac{\partial^2 G}{\partial \pi^2}(v) \right) dv\, du \tag{2.32}$$

for all $(t,\pi) \in R$.

Since $t \mapsto V(t,\pi)$ is increasing, we have:

$$\frac{\partial V}{\partial t}(t,\pi) \geq 0 \tag{2.33}$$

for each $(t,\pi) \in C$. Moreover, since $\pi \mapsto V(t,\pi)$ is concave and (2.44) holds, we see that:

$$\frac{\partial V}{\partial \pi}(t,\pi) \geq -1 \tag{2.34}$$

for each $(t,\pi) \in C$. Finally, since the strong Markov property implies that the value function V from (2.11) solves the equation (2.42), using (2.33) and (2.34) we obtain:

$$\begin{aligned} \frac{\partial^2 V}{\partial \pi^2}(t,\pi) &= \frac{2\sigma^2}{\mu^2} \frac{1}{\pi^2(1-\pi)^2} \left(-c\pi - \lambda(1-\pi) \frac{\partial V}{\partial \pi}(t,\pi) - \frac{\partial V}{\partial t}(t,\pi) \right) \\ &\leq \frac{2\sigma^2}{\mu^2} \frac{1}{\pi^2(1-\pi)^2} \left(-c\pi + \lambda(1-\pi) \right) \leq -\varepsilon \frac{\sigma^2}{\mu^2} \end{aligned} \tag{2.35}$$

for all $t' \leq t < t_*$ and all $\pi' \leq \pi < g(t')$ with $\varepsilon > 0$ small enough. Note in (2.35) that $-c\pi + \lambda(1-\pi) < 0$ since all points (t,π) for $0 \leq t < T$ with $0 \leq \pi < \lambda/(\lambda+c)$ belong to C and consequently $g(t_*) \geq \lambda/(\lambda+c)$.

Hence by (2.32) using that $G_{\pi\pi} = 0$ we get:

$$V(t',\pi') - G(\pi') \leq -\varepsilon \frac{\sigma^2}{\mu^2} \frac{(g(t') - \pi')^2}{2} \to -\varepsilon \frac{\sigma^2}{\mu^2} \frac{(g(t_*-) - \pi')^2}{2} < 0 \tag{2.36}$$

as $t' \uparrow t_*$. This implies that $V(t_*, \pi') < G(\pi')$ which contradicts the fact that (t_*, π') belongs to the stopping region $\overline{D}$. Thus $g(t_*-) = g(t_*)$ showing that g is continuous at t_* and thus on $[0,T]$ as well.

(iii) We finally note that the method of proof from the previous part (ii) also implies that $g(T) = \lambda/(\lambda+c)$. To see this, we may let $t_* = T$ and likewise suppose that $g(T-) > \lambda/(\lambda+c)$. Then repeating the arguments presented above word by word we arrive to a contradiction with the fact that $V(T,\pi) = G(\pi)$ for all $\pi \in [\lambda/(\lambda+c), g(T-)]$ thus proving the claim.

2.9. Summarizing the facts proved in Subsections 2.5-2.8 above we may conclude that the following exit time is optimal in the extended problem (2.11):

$$\tau_* = \inf\{0 \leq s \leq T-t \mid \pi_{t+s} \geq g(t+s)\} \tag{2.37}$$

(the infimum of an empty set being equal $T-t$) where the boundary g satisfies the following properties:

$$g : [0,T] \to [0,1] \quad \text{is continuous and decreasing} \tag{2.38}$$

$$\lambda/(\lambda+c) \le g(t) \le A_* \quad \text{for all } \; 0 \le t \le T \tag{2.39}$$

$$g(T) = \lambda/(\lambda+c) \tag{2.40}$$

where A_* satisfying $0 < \lambda/(\lambda+c) < A_* < 1$ is the optimal stopping point for the infinite horizon problem uniquely determined from the transcendental equation (4.147) in [25; page 201].

Standard arguments imply that the infinitesimal operator $\mathbb{L}$ of the process $(t, \pi_t)_{0 \le t \le T}$ acts on a function $f \in C^{1,2}([0,T) \times [0,1])$ according to the rule:

$$(\mathbb{L}f)(t,\pi) = \left(\frac{\partial f}{\partial t} + \lambda(1-\pi)\frac{\partial f}{\partial \pi} + \frac{\mu^2}{2\sigma^2}\pi^2(1-\pi)^2 \frac{\partial^2 f}{\partial \pi^2} \right)(t,\pi) \tag{2.41}$$

for all $(t,\pi) \in [0,T) \times [0,1]$. In view of the facts proved above we are thus naturally led to formulate the following *free-boundary problem* for the unknown value function V from (2.11) and the unknown boundary g from (2.15)-(2.16):

$$(\mathbb{L}V)(t,\pi) = -c\pi \quad \text{for} \quad (t,\pi) \in C \tag{2.42}$$

$$V(t,\pi)\big|_{\pi=g(t)-} = 1 - g(t) \quad (\textit{instantaneous stopping}) \tag{2.43}$$

$$\frac{\partial V}{\partial \pi}(t,\pi)\big|_{\pi=g(t)-} = -1 \quad (\textit{smooth fit}) \tag{2.44}$$

$$V(t,\pi) < G(\pi) \quad \text{for} \quad (t,\pi) \in C \tag{2.45}$$

$$V(t,\pi) = G(\pi) \quad \text{for} \quad (t,\pi) \in D \tag{2.46}$$

where C and D are given by (2.15) and (2.16), and the condition (2.43) is satisfied for all $0 \le t \le T$ and the condition (2.44) is satisfied for all $0 \le t < T$.

Note that the superharmonic characterization of the value function (see [4] and [25]) implies that V from (2.11) is a largest function satisfying (2.42)-(2.43) and (2.45)-(2.46).

2.10. Making use of the facts proved above we are now ready to formulate the main result of this section.

Theorem 2.1. *In the free Bayesian formulation of the Wiener disorder problem (2.4)-(2.5) the optimal stopping time τ_* is explicitly given by:*

$$\tau_* = \inf\{ 0 \le t \le T \mid \pi_t \ge g(t) \} \tag{2.47}$$

where g can be characterized as a unique solution of the nonlinear integral equation:

$$\begin{aligned} E_{t,g(t)}[\pi_T] = g(t) &+ c \int_0^{T-t} E_{t,g(t)}[\pi_{t+u}\, I(\pi_{t+u} < g(t+u))]\, du \\ &+ \lambda \int_0^{T-t} E_{t,g(t)}[(1-\pi_{t+u})\, I(\pi_{t+u} > g(t+u))]\, du \end{aligned} \tag{2.48}$$

for $0 \le t \le T$ satisfying (2.38)-(2.40).

More explicitly, the three terms in the equation (2.48) are given as follows:

$$E_{t,g(t)}[\pi_T] = g(t) + (1 - g(t))\big(1 - e^{-\lambda(T-t)}\big) \tag{2.49}$$

$$E_{t,g(t)}[\pi_{t+u}\, I(\pi_{t+u} < g(t+u))] = \int_0^{g(t+u)} x \;\, p(g(t); u, x)\, dx \tag{2.50}$$

$$E_{t,g(t)}[(1 - \pi_{t+u})\, I(\pi_{t+u} > g(t+u))] = \int_{g(t+u)}^{1} (1 - x) \;\, p(g(t); u, x)\, dx \tag{2.51}$$

for $0 \le u \le T - t$ *with* $0 \le t \le T$, *where* p *is the transition density function of the process* $(\pi_t)_{0 \le t \le T}$ *given in (4.18) below.*

Proof. (i) The existence of a boundary g satisfying (2.38)-(2.40) such that τ_* from (2.47) is optimal in (2.4)-(2.5) was proved in Subsections 2.5-2.9 above. By the change-of-variable formula from [15] it follows that the boundary g solves the equation (2.48) (cf. (2.55)-(2.57) below). Thus it remains to show that the equation (2.48) has no other solution in the class of functions h satisfying (2.38)-(2.40).

Let us thus assume that a function h satisfying (2.38)-(2.40) solves the equation (2.48), and let us show that this function h must then coincide with the optimal boundary g. For this, let us introduce the function:

$$V^h(t,\pi) = \begin{cases} U^h(t,\pi) & \text{if } \ \pi < h(t) \\ G(\pi) & \text{if } \ \pi \ge h(t) \end{cases} \tag{2.52}$$

where the function U^h is defined by:

$$U^h(t,\pi) = E_{t,\pi}[G(\pi_T)] + c \int_0^{T-t} E_{t,\pi}[\pi_{t+u}\, I(\pi_{t+u} < h(t+u))]\, du \\ + \lambda \int_0^{T-t} E_{t,\pi}[(1 - \pi_{t+u})\, I(\pi_{t+u} > h(t+u))]\, du \tag{2.53}$$

for all $(t,\pi) \in [0,T) \times [0,1]$. Note that (2.53) with $G(\pi)$ instead of $U^h(t,\pi)$ on the left-hand side coincides with (2.48) when $\pi = g(t)$ and $h = g$. Since h solves (2.48) this shows that V^h is continuous on $[0,T) \times [0,1]$. We need to verify that V^h coincides with the value function V from (2.11) and that h equals g.

(ii) Using standard arguments based on the strong Markov property (or verifying directly) it follows that V^h i.e. U^h is $C^{1,2}$ on C_h and that:

$$(\mathbb{L}V^h)(t,\pi) = -c\pi \quad \text{for } \ (t,\pi) \in C_h \tag{2.54}$$

where C_h is defined as in (2.15) with h instead of g. Moreover, since $U^h_\pi := \partial U^h / \partial \pi$ is continuous on $[0,T) \times (0,1)$ (which is readily verified using the explicit expressions (2.49)-(2.51) above with π instead of $g(t)$ and h instead of g), we see that $V^h_\pi := \partial V^h / \partial \pi$ is continuous on $\overline{C}_h$. Finally, it is clear that V^h i.e. G is $C^{1,2}$ on $\overline{D}_h$, where D_h is defined as in (2.16) with h instead of g. Therefore, with $(t,\pi) \in [0,T) \times (0,1)$ given and fixed, the change-of-variable formula from [15] can be applied, and in this way we get:

$$
\begin{aligned}
(2.55) \qquad V^h(t+s, \pi_{t+s}) = V^h(t,\pi) &+ \int_0^s (\mathbb{L}V^h)(t+u, \pi_{t+u})\, I(\pi_{t+u} \neq h(t+u))\, du \\
&+ M_s^h + \frac{1}{2}\int_0^s \Delta_\pi V_\pi^h(t+u, \pi_{t+u})\, I(\pi_{t+u} = h(t+u))\, d\ell_u^h
\end{aligned}
$$

for $0 \leq s \leq T-t$ where $\Delta_\pi V_\pi^h(t+u, h(t+u)) = V_\pi^h(t+u, h(t+u)+) - V_\pi^h(t+u, h(t+u)-)$, the process $(\ell_s^h)_{0 \leq s \leq T-t}$ is the local time of $(\pi_{t+s})_{0 \leq s \leq T-t}$ at the boundary h given by:

$$
(2.56) \qquad \ell_s^h = P_{t,\pi} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(h(t+u) - \varepsilon < \pi_{t+u} < h(t+u) + \varepsilon)\, \frac{\mu^2}{\sigma^2} \pi_{t+u}^2 (1 - \pi_{t+u})^2\, du
$$

and $(M_s^h)_{0 \leq s \leq T-t}$ defined by $M_s^h = \int_0^s V_\pi^h(t+u, \pi_{t+u})\, I(\pi_{t+u} \neq h(t+u))(\mu/\sigma)\pi_{t+u}(1-\pi_{t+u})d\overline{W}_u$ is a martingale under $P_{t,\pi}$.

Setting $s = T-t$ in (2.55) and taking the $P_{t,\pi}$-expectation, using that V^h satisfies (2.54) in C_h and equals G in D_h, we get:

$$
\begin{aligned}
(2.57) \qquad E_{t,\pi}[G(\pi_T)] = V^h(t,\pi) &- c \int_0^{T-t} E_{t,\pi}[\pi_{t+u}\, I(\pi_{t+u} < h(t+u))]\, du \\
&- \lambda \int_0^{T-t} E_{t,\pi}[(1 - \pi_{t+u})\, I(\pi_{t+u} > h(t+u))]\, du + \frac{1}{2} F(t,\pi)
\end{aligned}
$$

where (by the continuity of the integrand) the function F is given by:

$$
(2.58) \qquad F(t,\pi) = \int_0^{T-t} \Delta_\pi V_\pi^h(t+u, h(t+u))\, d_u E_{t,\pi}[\ell_u^h]
$$

for all $(t,\pi) \in [0,T) \times [0,1]$. Thus from (2.57) and (2.52) we see that:

$$
(2.59) \qquad F(t,\pi) = \begin{cases} 0 & \text{if } \pi < h(t) \\ 2\,(U^h(t,\pi) - G(\pi)) & \text{if } \pi \geq h(t) \end{cases}
$$

where the function U^h is given by (2.53).

(iii) From (2.59) we see that if we are to prove that:

$$
(2.60) \qquad \pi \mapsto V^h(t,\pi) \quad \text{is} \quad C^1 \quad \text{at} \quad h(t)
$$

for each $0 \leq t < T$ given and fixed, then it will follow that:

$$
(2.61) \qquad U^h(t,\pi) = G(\pi) \quad \text{for all } \; h(t) \leq \pi \leq 1.
$$

On the other hand, if we know that (2.61) holds, then using the general fact obtained directly from the definition (2.52) above:

$$
(2.62) \qquad \frac{\partial}{\partial \pi}(U^h(t,\pi) - G(\pi))\Big|_{\pi = h(t)} = V_\pi^h(t, h(t)-) - V_\pi^h(t, h(t)+) = -\Delta_\pi V_\pi^h(t, h(t))
$$

for all $0 \le t < T$, we see that (2.60) holds too. The equivalence of (2.60) and (2.61) suggests that instead of dealing with the equation (2.59) in order to derive (2.60) above (which was the content of an earlier proof) we may rather concentrate on establishing (2.61) directly.

To derive (2.61) first note that using standard arguments based on the strong Markov property (or verifying directly) it follows that U^h is $C^{1,2}$ in D_h and that:

$$(\mathbb{L}U^h)(t,\pi) = -\lambda(1-\pi) \quad \text{for} \quad (t,\pi) \in D_h. \tag{2.63}$$

It follows that (2.55) can be applied with U^h instead of V^h, and this yields:

$$\begin{aligned} U^h(t+s, \pi_{t+s}) = U^h(t,\pi) - c \int_0^s \pi_{t+u}\, I(\pi_{t+u} < h(t+u))\, du \\ - \lambda \int_0^s (1-\pi_{t+u})\, I(\pi_{t+u} > h(t+u))\, du + N_s^h \end{aligned} \tag{2.64}$$

using (2.54) and (2.63) as well as that $\Delta_\pi U_\pi^h(t+u, h(t+u)) = 0$ for all $0 \le u \le s$ since U_π^h is continuous. In (2.64) we have $N_s^h = \int_0^s U_\pi^h(t+u, \pi_{t+u})\, I(\pi_{t+u} \ne h(t+u))\, (\mu/\sigma)\, \pi_{t+u}$ $(1-\pi_{t+u})\, d\overline{W}_u$ and $(N_s^h)_{0 \le s \le T-t}$ is a martingale under $P_{t,\pi}$.

For $h(t) \le \pi < 1$ consider the stopping time:

$$\sigma_h = \inf\{0 \le s \le T-t \mid \pi_{t+s} \le h(t+s)\}. \tag{2.65}$$

Then using that $U^h(t, h(t)) = G(h(t))$ for all $0 \le t < T$ since h solves (2.48), and that $U^h(T,\pi) = G(\pi)$ for all $0 \le \pi \le 1$, we see that $U^h(t+\sigma_h, \pi_{t+\sigma_h}) = G(\pi_{t+\sigma_h})$. Hence from (2.64) and (2.12) using the optional sampling theorem we find:

$$\begin{aligned} U^h(t,\pi) &= E_{t,\pi}[U^h(t+\sigma_h, \pi_{t+\sigma_h})] + c E_{t,\pi}\Big[\int_0^{\sigma_h} \pi_{t+u}\, I(\pi_{t+u} < h(t+u))\, du\Big] \\ &\quad + \lambda\, E_{t,\pi}\Big[\int_0^{\sigma_h} (1-\pi_{t+u})\, I(\pi_{t+u} > h(t+u))\, du\Big] \\ &= E_{t,\pi}[G(\pi_{t+\sigma_h})] + c E_{t,\pi}\Big[\int_0^{\sigma_h} \pi_{t+u}\, I(\pi_{t+u} < h(t+u))\, du\Big] \\ &\quad + \lambda\, E_{t,\pi}\Big[\int_0^{\sigma_h} (1-\pi_{t+u})\, I(\pi_{t+u} > h(t+u))\, du\Big] \\ &= G(\pi) - \lambda\, E_{t,\pi}\Big[\int_0^{\sigma_h} (1-\pi_{t+u})\, du\Big] + c E_{t,\pi}\Big[\int_0^{\sigma_h} \pi_{t+u}\, I(\pi_{t+u} < h(t+u))\, du\Big] \\ &\quad + \lambda\, E_{t,\pi}\Big[\int_0^{\sigma_h} (1-\pi_{t+u})\, I(\pi_{t+u} > h(t+u))\, du\Big] = G(\pi) \end{aligned} \tag{2.66}$$

since $\pi_{t+u} > h(t+u)$ for all $0 \le u < \sigma_h$. This establishes (2.61) and thus (2.60) holds as well.

It may be noted that a shorter but somewhat less revealing proof of (2.61) [and (2.60)] can be obtained by verifying directly (using the Markov property only) that the process:

$$\begin{aligned} U^h(t+s, \pi_{t+s}) + c \int_0^s \pi_{t+u}\, I(\pi_{t+u} < h(t+u))\, du \\ + \lambda \int_0^s (1-\pi_{t+u})\, I(\pi_{t+u} > h(t+u))\, du \end{aligned} \tag{2.67}$$

is a martingale under $P_{t,\pi}$ for $0 \le s \le T-t$. This verification moreover shows that the martingale property of (2.67) does not require that h is continuous and increasing (but only measurable). Taken together with the rest of the proof below this shows that the claim of uniqueness for the equation (2.48) holds in the class of continuous functions $h : [0,T] \to \mathbb{R}$ such that $0 \le h(t) \le 1$ for all $0 \le t \le T$.

(iv) Let us consider the stopping time:

$$\tau_h = \inf\{0 \le s \le T-t \mid \pi_{t+s} \ge h(t+s)\}. \tag{2.68}$$

Observe that, by virtue of (2.60), the identity (2.55) can be written as:

$$\begin{aligned} V^h(t+s, \pi_{t+s}) = V^h(t,\pi) &- c \int_0^s \pi_{t+u}\, I(\pi_{t+u} < h(t+u))\, du \\ &- \lambda \int_0^s (1-\pi_{t+u})\, I(\pi_{t+u} > h(t+u))\, du + M_s^h \end{aligned} \tag{2.69}$$

with $(M_s^h)_{0 \le s \le T-t}$ being a martingale under $P_{t,\pi}$. Thus, inserting τ_h into (2.69) in place of s and taking the $P_{t,\pi}$-expectation, by means of the optional sampling theorem we get:

$$V^h(t,\pi) = E_{t,\pi}\left[G(\pi_{t+\tau_h}) + c\int_0^{\tau_h} \pi_{t+u}\, du\right] \tag{2.70}$$

for all $(t,\pi) \in [0,T) \times [0,1]$. Then comparing (2.70) with (2.11) we see that:

$$V(t,\pi) \le V^h(t,\pi) \tag{2.71}$$

for all $(t,\pi) \in [0,T) \times [0,1]$.

(v) Let us now show that $h \le g$ on $[0,T]$. For this, recall that by the same arguments as for V^h we also have:

$$\begin{aligned} V(t+s, \pi_{t+s}) = V(t,\pi) &- c \int_0^s \pi_{t+u}\, I(\pi_{t+u} < g(t+u))\, du \\ &- \lambda \int_0^s (1-\pi_{t+u})\, I(\pi_{t+u} > g(t+u))\, du + M_s^g \end{aligned} \tag{2.72}$$

where $(M_s^g)_{0 \le s \le T-t}$ is a martingale under $P_{t,\pi}$. Fix some (t,π) such that $\pi > g(t) \vee h(t)$ and consider the stopping time:

$$\sigma_g = \inf\{0 \le s \le T-t \mid \pi_{t+s} \le g(t+s)\}. \tag{2.73}$$

Inserting σ_g into (2.69) and (2.72) in place of s and taking the $P_{t,\pi}$-expectation, by means of the optional sampling theorem we get:

$$\begin{aligned} E_{t,\pi}\left[V^h(t+\sigma_g, \pi_{t+\sigma_g}) + c\int_0^{\sigma_g} \pi_{t+u}\, du\right] &= G(\pi) \\ + E_{t,\pi}\left[\int_0^{\sigma_g} (c\pi_{t+u} - \lambda(1-\pi_{t+u}))\, I(\pi_{t+u} > h(t+u))\, du\right] & \end{aligned} \tag{2.74}$$

$$E_{t,\pi}\left[V(t+\sigma_g,\pi_{t+\sigma_g})+c\int_0^{\sigma_g}\pi_{t+u}\,du\right]=G(\pi)\tag{2.75}$$
$$+\,E_{t,\pi}\left[\int_0^{\sigma_g}(c\pi_{t+u}-\lambda(1-\pi_{t+u}))\,du\right].$$

Hence by means of (2.71) we see that:

$$E_{t,\pi}\left[\int_0^{\sigma_g}(c\pi_{t+u}-\lambda(1-\pi_{t+u}))\,I(\pi_{t+u}>h(t+u))\,du\right]\tag{2.76}$$
$$\geq E_{t,\pi}\left[\int_0^{\sigma_g}(c\pi_{t+u}-\lambda(1-\pi_{t+u}))\,du\right]$$

from where, by virtue of the continuity of h and g on $(0,T)$ and the first inequality in (2.39), it readily follows that $h(t)\leq g(t)$ for all $0\leq t\leq T$.

(vi) Finally, we show that h coincides with g. For this, let us assume that there exists some $t\in(0,T)$ such that $h(t)<g(t)$ and take an arbitrary π from $(h(t),g(t))$. Then inserting $\tau_*=\tau_*(t,\pi)$ from (2.37) into (2.69) and (2.72) in place of s and taking the $P_{t,\pi}$-expectation, by means of the optional sampling theorem we get:

$$E_{t,\pi}\left[G(\pi_{t+\tau_*})+c\int_0^{\tau_*}\pi_{t+u}\,du\right]=V^h(t,\pi)\tag{2.77}$$
$$+\,E_{t,\pi}\left[\int_0^{\tau_*}(c\pi_{t+u}-\lambda(1-\pi_{t+u}))\,I(\pi_{t+u}>h(t+u))\,du\right]$$

$$E_{t,\pi}\left[G(\pi_{t+\tau_*})+c\int_0^{\tau_*}\pi_{t+u}\,du\right]=V(t,\pi).\tag{2.78}$$

Hence by means of (2.71) we see that:

$$E_{t,\pi}\left[\int_0^{\tau_*}(c\pi_{t+u}-\lambda(1-\pi_{t+u}))\,I(\pi_{t+u}>h(t+u))\,du\right]\leq 0\tag{2.79}$$

which is clearly impossible by the continuity of h and g and the fact that $h\geq\lambda/(\lambda+c)$ on $[0,T]$. We may therefore conclude that V^h defined in (2.52) coincides with V from (2.11) and h is equal to g. This completes the proof of the theorem. □

Remark 2.2. Note that without loss of generality it can be assumed that $\mu>0$ in (2.2)-(2.3). In this case the optimal stopping time (2.47) can be equivalently written as follows:

$$\tau_*=\inf\{0\leq t\leq T\mid X_t\geq b^\pi(t,X_0^t))\}\tag{2.80}$$

where we set:

$$b^\pi(t,X_0^t)=\frac{\sigma^2}{\mu}\log\left(\frac{g(t)}{1-g(t)}\Big/\left(\frac{\pi}{1-\pi}+\lambda\int_0^t e^{-\lambda s}e^{-(\mu/\sigma^2)(X_s-\mu s/2)}ds\right)\right)+\left(\frac{\mu}{2}-\frac{\lambda\sigma^2}{\mu}\right)t\tag{2.81}$$

for $(t,\pi)\in[0,T]\times[0,1]$ and X_0^t denotes the sample path $s\mapsto X_s$ for $s\in[0,t]$.

The result proved above shows that the following sequential procedure is optimal: *Observe* X_t *for* $t \in [0,T]$ *and stop the observation as soon as* X_t *becomes greater than* $b^\pi(t, X_0^t)$ *for some* $t \in [0,T]$. *Then conclude that the drift has been changed from* 0 *to* μ.

Remark 2.3. In the preceding procedure we need to know the boundary b^π i.e. the boundary g. We proved above that g is a unique solution of the equation (2.48). This equation cannot be solved analytically but can be dealt with numerically. The following simple method can be used to illustrate the latter (better methods are needed to achieve higher precision around the singularity point $t = T$ and to increase the speed of calculation).

Set $t_k = kh$ for $k = 0, 1, \dots, n$ where $h = T/n$ and denote:

$$J(t, g(t)) = (1 - g(t))\left(1 - e^{-\lambda(T-t)}\right) \tag{2.82}$$

$$\begin{aligned} K(t, g(t); t+u, g(t+u)) = E_{t,g(t)}[c\pi_{t+u} I(\pi_{t+u} < g(t+u))] \\ + \lambda(1 - \pi_{t+u}) I(\pi_{t+u} > g(t+u))] \end{aligned} \tag{2.83}$$

upon recalling the explicit expressions (2.50) and (2.51) above.

Then the following discrete approximation of the integral equation (2.48) is valid:

$$J(t_k, g(t_k)) = \sum_{l=k}^{n-1} K(t_k, g(t_k); t_{l+1}, g(t_{l+1}))\, h \tag{2.84}$$

for $k = 0, 1, \dots, n-1$. Setting $k = n-1$ and $g(t_n) = \lambda/(\lambda + c)$ we can solve the equation (2.84) numerically and get a number $g(t_{n-1})$. Setting $k = n-2$ and using the values $g(t_{n-1})$, $g(t_n)$ we can solve (2.84) numerically and get a number $g(t_{n-2})$. Continuing the recursion we obtain $g(t_n), g(t_{n-1}), \dots, g(t_1), g(t_0)$ as an approximation of the optimal boundary g at the points $T, T-h, \dots, h, 0$ (cf. *Figure 1* above).

3. Solution of the variational problem

In the variational problem with finite horizon (see [25; Chapter IV, Sections 3-4] for the infinite horizon case) it is assumed that we observe a trajectory of the Wiener process $X = (X_t)_{0 \le t \le T}$ with a drift changing from 0 to $\mu \neq 0$ at some random time θ taking the value 0 with probability π and being exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$.

3.1. Adopting the setting and notation of Subsection 2.1 above, let $\mathcal{M}(\alpha, \pi)$ denote the class of stopping times τ of X satisfying $0 \le \tau \le T$ and:

$$P_\pi[\tau < \theta] \le \alpha \tag{3.1}$$

where $0 \le \alpha \le 1$ and $0 \le \pi \le 1$ are given and fixed. The variational problem seeks to determine a stopping time $\widehat{\tau}$ in the class $\mathcal{M}(\alpha, \pi)$ such that:

$$E_\pi[\widehat{\tau} - \theta]^+ \le E_\pi[\tau - \theta]^+ \tag{3.2}$$

for any other stopping time τ from $\mathcal{M}(\alpha, \pi)$. The stopping time $\widehat{\tau}$ is then said to be optimal in the variational problem (3.1)-(3.2).

3.2. To solve the variational problem (3.1)-(3.2) we will follow the train of thought from [25; Chapter IV, Section 3] which is based on exploiting the solution of the Bayesian problem found in Section 2 above. For this, let us first note that if $\alpha \geq 1-\pi$ then letting $\widehat{\tau} \equiv 0$ we see that $P_\pi[\widehat{\tau} < \theta] = P_\pi[0 < \theta] = 1-\pi \leq \alpha$ and clearly $E_\pi[\widehat{\tau}-\theta]^+ = E_\pi[-\theta]^+ = 0 \leq E[\tau-\theta]^+$ for every $\tau \in \mathcal{M}(\alpha,\pi)$ showing that $\widehat{\tau} \equiv 0$ is optimal in (3.1)-(3.2). Similarly, if $\alpha = e^{-\lambda T}(1-\pi)$ then letting $\widehat{\tau} \equiv T$ we see that $P_\pi[\widehat{\tau} < \theta] = P_\pi[T < \theta] = e^{-\lambda T}(1-\pi) = \alpha$ and clearly $E_\pi[\widehat{\tau}-\theta]^+ = E_\pi[T-\theta]^+ \leq E[\tau-\theta]^+$ for every $\tau \in \mathcal{M}(\alpha,\pi)$ showing that $\widehat{\tau} \equiv T$ is optimal in (3.1)-(3.2). The same argument also shows that $\mathcal{M}(\alpha,\pi)$ is empty if $\alpha < e^{-\lambda T}(1-\pi)$. We may thus conclude that the set of admissible α which lead to a nontrivial optimal stopping time $\widehat{\tau}$ in (3.1)-(3.2) equals $(e^{-\lambda T}(1-\pi), 1-\pi)$ where $\pi \in [0,1)$.

3.3. To describe the key technical points in the argument below leading to the solution of (3.1)-(3.2), let us consider the optimal stopping problem (2.11) with $c>0$ given and fixed. In this context set $V(t,\pi) = V(t,\pi;c)$ and $g(t) = g(t;c)$ to indicate the dependence on c and recall that $\tau_* = \tau_*(c)$ given in (2.47) is an optimal stopping time in (2.11). We then have:

$$g(t;c) \leq g(t;c') \ \text{ for all } \ t \in [0,T] \ \text{ if } \ c > c' \tag{3.3}$$

$$g(t;c) \uparrow 1 \ \text{ if } \ c \downarrow 0 \ \text{ for each } \ t \in [0,T] \tag{3.4}$$

$$g(t;c) \downarrow 0 \ \text{ if } \ c \uparrow \infty \ \text{ for each } \ t \in [0,T]. \tag{3.5}$$

To verify (3.3) let us assume that $g(t;c) > g(t;c')$ for some $t \in [0,T)$ and $c > c'$. Then for any $\pi \in (g(t;c'), g(t;c))$ given and fixed we have $V(t,\pi;c) < 1-\pi = V(t,\pi;c')$ contradicting the obvious fact that $V(t,\pi;c) \geq V(t,\pi;c')$ as it is clearly seen from (2.11). The relations (3.4) and (3.5) are verified in a similar manner.

3.4. Finally, to exhibit the optimal stopping time $\widehat{\tau}$ in (3.1)-(3.2) when $\alpha \in (e^{-\lambda T}(1-\pi), 1-\pi)$ and $\pi \in [0,1)$ are given and fixed, let us introduce the function:

$$u(c;\pi) = P_\pi[\tau_* < \theta] \tag{3.6}$$

for $c>0$ where $\tau_* = \tau_*(c)$ from (2.47) is an optimal stopping time in (2.5). Using that $P_\pi[\tau_* < \theta] = E_\pi[1-\pi_{\tau_*}]$ and (3.3) above it is readily verified that $c \mapsto u(c;\pi)$ is continuous and strictly increasing on $(0,\infty)$. [Note that a strict increase follows from the fact that $g(T;c) = \lambda/(\lambda+c)$.] From (3.4) and (3.5) we moreover see that $u(0+;\pi) = e^{-\lambda T}(1-\pi)$ due to $\tau_*(0+) \equiv T$ and $u(+\infty;\pi) = 1-\pi$ due to $\tau_*(+\infty) \equiv 0$. This implies that the equation:

$$u(c;\pi) = \alpha \tag{3.7}$$

has a unique root $c = c(\alpha)$ in $(0,\infty)$.

3.5. The preceding conclusions can now be used to formulate the main result of this section.

Theorem 3.1. *In the variational formulation of the Wiener disorder problem (3.1)-(3.2) there exists a non-trivial optimal stopping time* $\widehat{\tau}$ *if and only if:*

$$\alpha \in (e^{-\lambda T}(1-\pi), 1-\pi) \tag{3.8}$$

where $\pi \in [0,1)$. *In this case* $\widehat{\tau}$ *may be explicitly identified with* $\tau_* = \tau_*(c)$ *in (2.47) where* $g(t) = g(t;c)$ *is the unique solution of the integral equation (2.48) and* $c = c(\alpha)$ *is a unique root of the equation (3.7) on* $(0,\infty)$.

Proof. It remains us to show that $\widehat{\tau} = \tau_*(c)$ with $c = c(\alpha)$ and $\alpha \in (e^{-\lambda T}(1-\pi), 1-\pi)$ for $\pi \in [0,1)$ satisfies (3.2). For this note that since $P_\pi[\widehat{\tau} < \theta] = \alpha$ by construction, it follows by the optimality of $\tau_*(c)$ in (2.4) that:

$$\alpha + cE_\pi[\widehat{\tau} - \theta]^+ \le P_\pi[\tau < \theta] + cE_\pi[\tau - \theta]^+ \tag{3.9}$$

for any other stopping time τ with values in $[0,T]$. Moreover, if τ belongs to $\mathcal{M}(\alpha, \pi)$ then $P_\pi[\tau < \theta] \le \alpha$ and from (3.9) we see that $E_\pi[\widehat{\tau} - \theta]^+ \le E_\pi[\tau - \theta]^+$ establishing (3.2). The proof is complete. □

Remark 3.2. Recall from part (iv) of Subsection 2.5 above that $g(t; c) \le A_*(c)$ for all $0 \le t \le T$ where $0 < A_*(c) < 1$ is uniquely determined from the equation (4.147) in [25; page 201]. Since $A_*(c(\alpha)) = 1 - \alpha$ by Theorem 10 in [25; page 205] it follows that the optimal stopping boundary $t \mapsto g(t; c(\alpha))$ in (3.1)-(3.2) satisfies $g(t; c(\alpha)) \le 1 - \alpha$ for all $0 \le t \le T$.

4. Appendix

In this section we exhibit an explicit expression for the transition density function of the a posteriori probability process $(\pi_t)_{0 \le t \le T}$ given in (2.8) above.

4.1. Let $B = (B_t)_{t \ge 0}$ be a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$. With $t > 0$ and $\nu \in \mathbb{R}$ given and fixed recall from [29; page 527] that the random variable $A_t^{(\nu)} = \int_0^t e^{2(B_s + \nu s)} ds$ has the conditional distribution:

$$P\Big[A_t^{(\nu)} \in dz \,\Big|\, B_t + \nu t = y\Big] = a(t, y, z)\, dz \tag{4.1}$$

where the density function a for $z > 0$ is given by:

$$\begin{aligned} a(t, y, z) = {} & \frac{1}{\pi z^2} \exp\left(\frac{y^2 + \pi^2}{2t} + y - \frac{1}{2z}\Big(1 + e^{2y}\Big)\right) \\ & \times \int_0^\infty \exp\left(-\frac{w^2}{2t} - \frac{e^y}{z} \cosh(w)\right) \sinh(w) \sin\Big(\frac{\pi w}{t}\Big) dw. \end{aligned} \tag{4.2}$$

This implies that the random vector $\big(2(B_t + \nu t), A_t^{(\nu)}\big)$ has the distribution:

$$P\Big[2(B_t + \nu t) \in dy, A_t^{(\nu)} \in dz\Big] = b(t, y, z)\, dy\, dz \tag{4.3}$$

where the density function b for $z > 0$ is given by:

$$\begin{aligned} b(t, y, z) &= a\Big(t, \frac{y}{2}, z\Big) \frac{1}{2\sqrt{t}} \varphi\left(\frac{y - 2\nu t}{2\sqrt{t}}\right) \\ &= \frac{1}{(2\pi)^{3/2} z^2 \sqrt{t}} \exp\left(\frac{\pi^2}{2t} + \Big(\frac{\nu + 1}{2}\Big) y - \frac{\nu^2}{2} t - \frac{1}{2z}\Big(1 + e^y\Big)\right) \\ & \quad \times \int_0^\infty \exp\left(-\frac{w^2}{2t} - \frac{e^{y/2}}{z} \cosh(w)\right) \sinh(w) \sin\Big(\frac{\pi w}{t}\Big)\, dw \end{aligned} \tag{4.4}$$

and we set $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ for $x \in \mathbb{R}$ (for related expressions in terms of Hermite functions see [3] and [20]).

Denoting $I_t = \alpha B_t + \beta t$ and $J_t = \int_0^t e^{\alpha B_s + \beta s} ds$ with $\alpha \neq 0$ and $\beta \in \mathbb{R}$ given and fixed, and using that the scaling property of B implies:

$$(4.5) \quad P\left[\alpha B_t + \beta t \leq y, \int_0^t e^{\alpha B_s + \beta s}\, ds \leq z\right] = P\left[2(B_{t'} + \nu t') \leq y, \int_0^{t'} e^{2(B_s + \nu s)}\, ds \leq \frac{\alpha^2}{4}\, z\right]$$

with $t' = \alpha^2 t/4$ and $\nu = 2\beta/\alpha^2$, it follows by applying (4.3) and (4.4) that the random vector (I_t, J_t) has the distribution:

$$(4.6) \qquad P\Big[I_t \in dy, J_t \in dz\Big] = f(t, y, z)\, dy\, dz$$

where the density function f for $z > 0$ is given by:

$$\begin{aligned}(4.7) \qquad f(t, y, z) &= \frac{\alpha^2}{4}\, b\left(\frac{\alpha^2}{4}\, t,\, y,\, \frac{\alpha^2}{4}\, z\right) \\ &= \frac{2\sqrt{2}}{\pi^{3/2}\alpha^3}\, \frac{1}{z^2\sqrt{t}} \exp\left(\frac{2\pi^2}{\alpha^2 t} + \Big(\frac{\beta}{\alpha^2} + \frac{1}{2}\Big) y - \frac{\beta^2}{2\alpha^2}\, t - \frac{2}{\alpha^2 z}\Big(1 + e^y\Big)\right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{2w^2}{\alpha^2 t} - \frac{4e^{y/2}}{\alpha^2 z} \cosh(w)\right) \sinh(w)\, \sin\left(\frac{4\pi w}{\alpha^2 t}\right) dw.\end{aligned}$$

4.2. Letting $\alpha = -\mu/\sigma$ and $\beta = -\lambda - \mu^2/(2\sigma^2)$ it follows from the explicit expressions (2.6)-(2.7) and (2.3) that:

$$(4.8) \qquad P^0[\varphi_t \in dx] = P\Big[e^{-I_t}\Big(\frac{\pi}{1-\pi} + \lambda J_t\Big) \in dx\Big] = g(\pi; t, x)\, dx$$

where the density function g for $x > 0$ is given by:

$$\begin{aligned}(4.9) \qquad g(\pi; t, x) &= \frac{d}{dx} \int_{-\infty}^\infty \int_0^\infty I\Big(e^{-y}\Big(\frac{\pi}{1-\pi} + \lambda z\Big) \leq x\Big) f(t, y, z)\, dy\, dz \\ &= \int_{-\infty}^\infty f\Big(t, y, \frac{1}{\lambda}\Big(xe^y - \frac{\pi}{1-\pi}\Big)\Big) \frac{e^y}{\lambda}\, dy.\end{aligned}$$

Moreover, setting $\widetilde{I}_{t-s} = \alpha(B_t - B_s) + \beta(t-s)$ and $\widetilde{J}_{t-s} = \int_s^t e^{\alpha(B_u - B_s) + \beta(u-s)} du$ as well as $\widehat{I}_s = \alpha B_s + \widehat{\beta} s$ and $\widehat{J}_s = \int_0^s e^{\alpha B_u + \widehat{\beta} u} du$ with $\widehat{\beta} = -\lambda + \mu^2/(2\sigma^2)$, it follows from the explicit expressions (2.6)-(2.7) and (2.3) that:

$$\begin{aligned}(4.10) \qquad P^s[\varphi_t \in dx] &= P\Big[e^{-\gamma s} e^{-\widetilde{I}_{t-s}}\Big(e^{(\widehat{\beta} - \beta)s}\, e^{-\widehat{I}_s}\Big(\frac{\pi}{1-\pi} + \lambda \widehat{J}_s\Big) + \lambda e^{\gamma s} \widetilde{J}_{t-s}\Big) \in dx\Big] \\ &= h(s; \pi; t, x)\, dx\end{aligned}$$

for $0 < s < t$ where $\gamma = \mu^2/\sigma^2$. Since stationary independent increments of B imply that the random vector $(\widetilde{I}_{t-s}, \widetilde{J}_{t-s})$ is independent of $(\widehat{I}_s, \widehat{J}_s)$ and equally distributed as (I_{t-s}, J_{t-s}), we see upon recalling (4.8)-(4.9) that the density function h for $x>0$ is given by:

$$
\begin{aligned}
(4.11)\quad & h(s;\pi;t,x) \\
&= \frac{d}{dx}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} I\Big(e^{-\gamma s}e^{-y}\Big(e^{(\widehat{\beta}-\beta)s}w+\lambda e^{\gamma s}z\Big)\le x\Big) f(t-s,y,z)\,\widehat{g}(\pi;s,w)\,dy\,dz\,dw \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} f\Big(t-s,y,\frac{1}{\lambda}\big(xe^y-e^{(\widehat{\beta}-\beta-\gamma)s}w\big)\Big)\,\widehat{g}(\pi;s,w)\,\frac{e^y}{\lambda}\,dy\,dw
\end{aligned}
$$

where the density function $\widehat{g}$ for $w>0$ equals:

$$
\begin{aligned}
(4.12)\qquad \widehat{g}(\pi;s,w) &= \frac{d}{dx}\int_{-\infty}^{\infty}\int_0^{\infty} I\Big(e^{-y}\Big(\frac{\pi}{1-\pi}+\lambda z\Big)\le w\Big)\widehat{f}(s,y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\widehat{f}\Big(s,y,\frac{1}{\lambda}\Big(we^y-\frac{\pi}{1-\pi}\Big)\Big)\frac{e^y}{\lambda}\,dy
\end{aligned}
$$

and the density function $\widehat{f}$ for $z>0$ is defined as in (4.6)-(4.7) with $\widehat{\beta}$ instead of β.

Finally, by means of the same arguments as in (4.8)-(4.9) it follows from the explicit expressions (2.6)-(2.7) and (2.3) that:

$$
(4.13)\qquad P^t[\varphi_t\in dx]=P\Big[e^{-\widehat{I}_t}\Big(\frac{\pi}{1-\pi}+\lambda\widehat{J}_t\Big)\in dx\Big]=\widehat{g}(\pi;t,x)\,dx
$$

where the density function $\widehat{g}$ for $x>0$ is given by (4.12).

4.3. Noting by (2.1) that:

$$
(4.14)\quad P_\pi[\varphi_t\in dx]=\pi P^0[\varphi_t\in dx]+(1-\pi)\int_0^t\lambda e^{-\lambda s}P^s[\varphi_t\in dx]\,ds+(1-\pi)\,e^{-\lambda t}P^t[\varphi_t\in dx]
$$

we see by (4.8)+(4.10)+(4.13) that the process $(\varphi_t)_{0\le t\le T}$ has the marginal distribution:

$$
(4.15)\qquad P_\pi[\varphi_t\in dx]=q(\pi;t,x)\,dx
$$

where the transition density function q for $x>0$ is given by:

$$
(4.16)\qquad q(\pi;t,x)=\pi\,g(\pi;t,x)+(1-\pi)\int_0^t\lambda e^{-\lambda s}\,h(s;\pi;t,x)\,ds+(1-\pi)\,e^{-\lambda t}\,\widehat{g}(\pi;t,x)
$$

with g, h, $\widehat{g}$ from (4.9), (4.11), (4.12) respectively.

Hence by (2.8) we easily find that the process $(\pi_t)_{0\le t\le T}$ has the marginal distribution:

$$
(4.17)\qquad P_\pi[\pi_t\in dx]=p(\pi;t,x)\,dx
$$

where the transition density function p for $0<x<1$ is given by:

$$
(4.18)\qquad p(\pi;t,x)=\frac{1}{(1-x)^2}\,q\left(\pi;t,\frac{x}{1-x}\right).
$$

This completes the Appendix.

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