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# The Wiener Sequential Testing Problem with Finite Horizon

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We present a solution of the Bayesian problem of sequential testing of two simple hypotheses about the mean value of an observed Wiener process on the time interval with finite horizon. The method of proof is based on reducing the initial optimal stopping problem to a parabolic free-boundary problem where the continuation region is determined by two continuous curved boundaries. By means of the change-of-variable formula containing the local time of a diffusion process on curves we show that the optimal boundaries can be characterized as a unique solution of the coupled system of two nonlinear integral equations.

## 1. Introduction

The problem of sequential testing of two simple hypotheses about the mean value of an observed Wiener process seeks to determine as soon as possible and with minimal probability error which of the given two values is a true mean. The problem admits two different formulations (cf. Wald [20]). In the Bayesian formulation it is assumed that the unknown mean has a given distribution, and in the variational formulation no probabilistic assumption about the unknown mean is made a priori. In this paper we only study the Bayesian formulation.

The history of the problem is long and we only mention a few points starting with Wald and Wolfowitz [21]-[22] who used the Bayesian approach to prove the optimality of the sequential probability ratio test (SPRT) in the variational problem for i.i.d. sequences of observations. Dvoretzky, Kiefer and Wolfowitz [1] stated without proof that if the continuous-time log-likelihood ratio process has stationary independent increments, then the SPRT remains optimal in the variational problem. Mikhalevich [9] and Shiryaev [17] (see also [18; Chapter IV]) derived an explicit solution of the Bayesian and variational problem for a Wiener process with infinite horizon by reducing the initial optimal stopping problem to a free-boundary problem for a differential operator. A complete proof of the statement from [1] (under some mild assumptions) was given by Irle and Schmitz [4]. An explicit solution of the Bayesian and variational problem for a Poisson process with infinite horizon was derived in [15] by reducing the initial

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optimal stopping problem to a free-boundary problem for a differential-difference operator. The main aim of the present paper is to derive a solution of the Bayesian problem for a Wiener process with *finite horizon*.

It is known that optimal stopping problems for Markov processes with finite horizon are inherently two-dimensional and thus analytically more difficult than those with infinite horizon. A standard approach for handling such a problem is to formulate a free-boundary problem for the (parabolic) operator associated with the (continuous) Markov process (see e.g. [8], [3], [19], [5], [10]). Since solutions to such free-boundary problems are rarely known explicitly, the question often reduces to prove the existence and uniqueness of a solution to the free-boundary problem, which then leads to the optimal stopping boundary and the value function of the optimal stopping problem. In some cases the optimal stopping boundary has been characterized as a unique solution of the system of (at least) countably many nonlinear integral equations (see e.g. [5; Theorem 4.3]). A method of linearization was suggested in [11] with the aim of proving that only one equation from such a system may be sufficient to characterize the optimal stopping boundary uniquely. A complete proof of the latter fact in the case of a specific optimal stopping problem was given in [13] (see also [14]).

In the present paper we reduce the initial Bayesian problem to a finite-horizon optimal stopping problem for a diffusion process and a non-smooth gain function where the continuation region is determined by two continuous curved boundaries. In order to find an analytic expression for the boundaries we formulate an equivalent parabolic free-boundary problem for the infinitesimal operator of the strong Markov a posteriori probability process. By means of the method of proof proposed in [11] and [13], and using the change-of-variable formula from [12], we show that the optimal stopping boundaries can be uniquely determined from a coupled system of nonlinear Volterra integral equations of the second kind. This also leads to the explicit formula for the value (risk) function in terms of the optimal stopping boundaries.

The main result of the paper is stated in Theorem 2.1. The optimal sequential procedure in the initial Bayesian problem is displayed more explicitly in Remark 2.2. A simple numerical method for calculating the optimal boundaries is presented in Remark 2.3.

## 2. Solution of the Bayesian problem

In the Bayesian formulation of the problem with finite horizon (see [18; Chapter IV, Sections 1-2] for the infinite horizon case) it is assumed that we observe a trajectory of the Wiener process $X = (X_t)_{0 \le t \le T}$ with drift $\theta\mu$ where the random variable $\theta$ may be 1 or 0 with probability $\pi$ or $1-\pi$, respectively.

2.1. For a precise probabilistic formulation of the Bayesian problem it is convenient to assume that all our considerations take place on a probability space $(\Omega, \mathcal{F}, P_\pi)$ where the probability measure $P_\pi$ has the following structure:

$$P_\pi = \pi P_1 + (1-\pi) P_0 \tag{2.1}$$

for $\pi \in [0,1]$. Let $\theta$ be a random variable taking two values 1 and 0 with probabilities $P_\pi[\theta = 1] = \pi$ and $P_\pi[\theta = 0] = 1-\pi$, and let $W = (W_t)_{0 \le t \le T}$ be a standard Wiener process started at zero under $P_\pi$. It is assumed that $\theta$ and $W$ are independent.

It is further assumed that we observe a process $X = (X_t)_{0 \le t \le T}$ of the form:

$$X_t = \theta \mu t + \sigma W_t \tag{2.2}$$

where $\mu \neq 0$ and $\sigma^2 > 0$ are given and fixed. Thus $P_\pi[X \in \cdot \mid \theta = i] = P_i[X \in \cdot\,]$ is the distribution law of a Wiener process with drift $i\mu$ and diffusion coefficient $\sigma^2 > 0$ for $i = 0, 1$, so that $\pi$ and $1 - \pi$ play the role of a priori probabilities of the statistical hypotheses:

$$H_1 : \theta = 1 \quad \text{and} \quad H_0 : \theta = 0 \tag{2.3}$$

respectively.

Being based upon the continuous observation of $X$ our task is to test sequentially the hypotheses $H_1$ and $H_0$ with a minimal loss. For this, we consider a sequential decision rule $(\tau, d)$, where $\tau$ is a stopping time of the observed process $X$ (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \mid 0 \le s \le t)$ generated by $X$ for $0 \le t \le T$), and $d$ is an $\mathcal{F}_\tau^X$-measurable random variable taking values 0 and 1. After stopping the observation at time $\tau$, the terminal decision function $d$ indicates which hypothesis should be accepted according to the following rule: if $d = 1$ we accept $H_1$, and if $d = 0$ we accept $H_0$. The problem then consists of computing the risk function:

$$V(\pi) = \inf_{(\tau,d)} E_\pi[\tau + aI(d = 0, \theta = 1) + bI(d = 1, \theta = 0)] \tag{2.4}$$

and finding the optimal decision rule $(\tau_*, d_*)$ at which the infimum in (2.4) is attained. Here $E_\pi[\tau]$ is the average loss due to a cost of the observations, and $aP_\pi[d = 0, \theta = 1] + bP_\pi[d = 1, \theta = 0]$ is the average loss due to a wrong terminal decision, where $a > 0$ and $b > 0$ are given constants.

2.2. By means of standard arguments (see [18; pages 166-167]) one can reduce the Bayesian problem (2.4) to the optimal stopping problem:

$$V(\pi) = \inf_{0 \le \tau \le T} E_\pi[\tau + a\pi_\tau \wedge b(1 - \pi_\tau)] \tag{2.5}$$

for the a posteriori probability process $\pi_t = P_\pi[\theta = 1 | \mathcal{F}_t^X]$ for $0 \le t \le T$ with $P_\pi[\pi_0 = \pi] = 1$. Setting $c = b/(a+b)$ the optimal decision function is then given by $d_* = 1$ if $\pi_{\tau_*} \ge c$ and $d_* = 0$ if $\pi_{\tau_*} < c$.

2.3. It can be shown (see [18; pages 180-181]) that the likelihood ratio process $(\varphi_t)_{0 \le t \le T}$ defined as the Radon-Nikodym derivative:

$$\varphi_t = \frac{dP_1|_{\mathcal{F}_t^X}}{dP_0|_{\mathcal{F}_t^X}} \tag{2.6}$$

admits the following representation:

$$\varphi_t = \exp\left(\frac{\mu}{\sigma^2}\left(X_t - \frac{\mu}{2}t\right)\right) \tag{2.7}$$

while the a posteriori probability process $(\pi_t)_{0 \le t \le T}$ can be expressed as:

$$\pi_t = \left(\frac{\pi}{1-\pi}\varphi_t\right) \Big/ \left(1 + \frac{\pi}{1-\pi}\varphi_t\right) \tag{2.8}$$

and hence solves the stochastic differential equation:

$$d\pi_t = \frac{\mu}{\sigma}\pi_t(1-\pi_t)\, d\overline{W}_t \quad (\pi_0 = \pi) \tag{2.9}$$

where the innovation process $(\overline{W}_t)_{0\le t\le T}$ defined by:

$$\overline{W}_t = \frac{1}{\sigma}\Big(X_t - \mu \int_0^t \pi_s\, ds\Big) \tag{2.10}$$

is a standard Wiener process (see also [7; Chapter IX]). Using (2.7) and (2.8) it can be verified that $(\pi_t)_{0\le t\le T}$ is a time-homogeneous (strong) Markov process under $P_\pi$ with respect to the natural filtration. As the latter clearly coincides with $(\mathcal{F}_t^X)_{0\le t\le T}$ it is also clear that the infimum in (2.5) can equivalently be taken over all stopping times of $(\pi_t)_{0\le t\le T}$.

2.4. In order to solve the problem (2.5) let us consider the extended optimal stopping problem for the Markov process $(t,\pi_t)_{0\le t\le T}$ given by:

$$V(t,\pi) = \inf_{0\le\tau\le T-t} E_{t,\pi}[G(t+\tau, \pi_{t+\tau})] \tag{2.11}$$

where $P_{t,\pi}[\pi_t = \pi] = 1$, i.e. $P_{t,\pi}$ is a probability measure under which the diffusion process $(\pi_{t+s})_{0\le s\le T-t}$ solving (2.9) starts at $\pi$, the infimum in (2.11) is taken over all stopping times $\tau$ of $(\pi_{t+s})_{0\le s\le T-t}$, and we set $G(t,\pi) = t + a\pi \wedge b(1-\pi)$ for $(t,\pi) \in [0,T]\times[0,1]$. Since $G$ is bounded and continuous on $[0,T]\times[0,1]$ it is possible to apply a version of Theorem 3 in [18; page 127] for a finite time horizon and by statement (2) of that theorem conclude that an optimal stopping time exists in (2.11).

2.5. Let us now determine the structure of the optimal stopping time in the problem (2.11).

(i) It follows from (2.9) that the scale function of $(\pi_t)_{t\ge0}$ is given by $S(x) = x$ for $x\in[0,1]$ and the speed measure of $(\pi_t)_{t\ge0}$ is given by $m(dx) = (2\sigma/\mu)\, dx/(x(1-x))$ for $x \in \langle 0,1\rangle$. Hence the Green function of $(\pi_t)_{t\ge0}$ on $[\pi_0,\pi_1]\subset\langle 0,1\rangle$ is given by $G_{\pi_0,\pi_1}(x,y) = (\pi_1 - x)(y - \pi_0)/(\pi_1-\pi_0)$ for $\pi_0\le y\le x$ and $G_{\pi_0,\pi_1}(x,y) = (\pi_1-y)(x-\pi_0)/(\pi_1-\pi_0)$ for $x\le y\le \pi_1$.

Set $H(\pi) = a\pi\wedge b(1-\pi)$ for $\pi\in[0,1]$ and let $d = H(c)$. Take $\varepsilon\in\langle 0,d\rangle$ and denote by $\pi_0 = \pi_0(\varepsilon)$ and $\pi_1 = \pi_1(\varepsilon)$ the unique points $0<\pi_0<c<\pi_1<1$ satisfying $H(\pi_0) = H(\pi_1) = d-\varepsilon$. Let $\sigma_\varepsilon = \inf\{\, t>0 \mid \pi_t \notin \langle \pi_0,\pi_1\rangle \,\}$ and set $\sigma_\varepsilon^T = \sigma_\varepsilon \wedge T$. Then $\sigma_\varepsilon$ and $\sigma_\varepsilon^T$ are stopping times and it is easily verified that:

$$E_c[\sigma_\varepsilon^T] \le E_c[\sigma_\varepsilon] = \int_{\pi_0}^{\pi_1} G_{\pi_0,\pi_1}(x,y)\, m(dy) \le K\varepsilon^2 \tag{2.12}$$

for some $K>0$ large enough (not depending on $\varepsilon$). Similarly, we find that:

$$\begin{aligned} E_c[H(\pi_{\sigma_\varepsilon^T})] &= E_c[H(\pi_{\sigma_\varepsilon}) I(\sigma_\varepsilon < T)] + E_c[H(\pi_T) I(\sigma_\varepsilon \ge T)] \\ &\le d-\varepsilon + d\, P_c[\sigma_\varepsilon > T] \le d - \varepsilon + (d/T)\, E_c[\sigma_\varepsilon] \le d-\varepsilon + L\,\varepsilon^2 \end{aligned} \tag{2.13}$$

where $L = dK/T$.

Combining (2.12) and (2.13) we see that:

$$E_c[G(\sigma_\varepsilon^T, \pi_{\sigma_\varepsilon^T})] = E_c[\sigma_\varepsilon^T + H(\pi_{\sigma_\varepsilon^T})] \le d - \varepsilon + (K{+}L)\,\varepsilon^2 \tag{2.14}$$

for all $\varepsilon \in \langle 0, d\rangle$. Choosing $\varepsilon > 0$ in (2.14) small enough we see that $E_c[G(\sigma_\varepsilon^T, \pi_{\sigma_\varepsilon^T})] < d$. Using the fact that $G(t,\pi) = t + H(\pi)$ is linear in $t$, and $T > 0$ above is arbitrary, this shows that it is never optimal to stop in (2.11) when $\pi_{t+s} = c$ for $0 \le s < T-t$. In other words, this shows that all points $(t,c)$ for $0 \le t < T$ belong to the continuation region:

$$C = \{ (t,\pi) \in [0,T\rangle \times [0,1] \mid V(t,\pi) < G(t,\pi) \}. \tag{2.15}$$

(ii) Recalling the solution to the problem (2.5) in the case of infinite horizon, where the stopping time $\tau_* = \inf\{\, t > 0 \mid \pi_t \notin \langle A_*, B_*\rangle \,\}$ is optimal and $0 < A_* < c < B_* < 1$ are uniquely determined from the system (4.85) in [18; page 185], we see that all points $(t,\pi)$ for $0 \le t \le T$ with either $0 \le \pi \le A_*$ or $B_* \le \pi \le 1$ belong to the stopping region. Moreover, since $\pi \mapsto V(t,\pi)$ with $0 \le t \le T$ given and fixed is concave on $[0,1]$ (this is easily deduced using the same arguments as in [6; page 105] or [18; page 168]), it follows directly from the previous two conclusions about the continuation and stopping region that there exist functions $g_0$ and $g_1$ satisfying $0 < A_* \le g_0(t) < c < g_1(t) \le B_* < 1$ for all $0 \le t < T$ such that the continuation region is an open set of the form:

$$C = \{ (t,\pi) \in [0,T\rangle \times [0,1] \mid \pi \in \langle g_0(t), g_1(t)\rangle \} \tag{2.16}$$

and the stopping region is the closure of the set:

$$D = \{ (t,\pi) \in [0,T\rangle \times [0,1] \mid \pi \in [0, g_0(t)\rangle \cup \langle g_1(t), 1] \}. \tag{2.17}$$

(Below we will show that $V$ is continuous so that $C$ is open indeed. We will also see that $g_0(T) = g_1(T) = c$.)

(iii) Since the problem (2.11) is time-homogeneous, in the sense that $G(t,\pi) = t + H(\pi)$ is linear in $t$ and $H$ depends on $\pi$ only, it follows that the map $t \mapsto V(t,\pi) - t$ is increasing on $[0,T]$. Hence if $(t,\pi)$ belongs to $C$ for some $\pi \in \langle 0,1\rangle$ and we take any other $0 \le t' < t \le T$, then $V(t',\pi) - G(t',\pi) = V(t',\pi) - t' - H(\pi) \le V(t,\pi) - t - H(\pi) = V(t,\pi) - G(t,\pi) < 0$, showing that $(t',\pi)$ belongs to $C$ as well. From this we may conclude in (2.16)-(2.17) that the boundary $t \mapsto g_0(t)$ is increasing and the boundary $t \mapsto g_1(t)$ is decreasing on $[0,T]$.

(iv) Let us finally observe that the value function $V$ from (2.11) and the boundaries $g_0$ and $g_1$ from (2.16)-(2.17) also depend on $T$ and let them denote here by $V^T$, $g_0^T$ and $g_1^T$, respectively. Using the fact that $T \mapsto V^T(t,\pi)$ is a decreasing function on $[t,\infty\rangle$ and $V^T(t,\pi) = G(t,\pi)$ for all $\pi \in [0, g_0^T(t)] \cup [g_1^T(t), 1]$, we conclude that if $T < T'$, then $0 \le g_0^{T'}(t) \le g_0^T(t) < c < g_1^T(t) \le g_1^{T'}(t) \le 1$ for all $t \in [0,T\rangle$. Letting $T'$ in the previous expression go to $\infty$, we get that $0 < A_* \le g_0^T(t) < c < g_1^T(t) \le B_* < 1$ with $A_* \equiv \lim_{T\to\infty} g_0^T(t)$ and $B_* \equiv \lim_{T\to\infty} g_1^T(t)$ for all $t \ge 0$, where $A_*$ and $B_*$ are the optimal stopping points in the infinite horizon problem referred to above.

2.6. Let us now show that the value function $(t,\pi) \mapsto V(t,\pi)$ is continuous on $[0,T] \times [0,1]$. For this it is enough to prove that:

$$\pi \mapsto V(t_0,\pi) \quad \text{is continuous at} \quad \pi_0 \tag{2.18}$$

$$t \mapsto V(t,\pi) \quad \text{is continuous at} \quad t_0 \quad \text{uniformly over} \quad \pi \in [\pi_0 - \delta, \pi_0 + \delta] \tag{2.19}$$

for each $(t_0,\pi_0) \in [0,T] \times [0,1]$ with some $\delta > 0$ small enough (it may depend on $\pi_0$). Since (2.18) follows by the fact that $\pi \mapsto V(t,\pi)$ is concave on $[0,1]$, it remains to establish (2.19).

For this, let us fix arbitrary $0 \le t_1 < t_2 \le T$ and $0 < \pi < 1$, and let $\tau_1 = \tau_*(t_1,\pi)$ denote the optimal stopping time for $V(t_1,\pi)$. Set $\tau_2 = \tau_1 \wedge (T-t_2)$ and note since $t \mapsto V(t,\pi)$ is increasing on $[0,T]$ and $\tau_2 \le \tau_1$ that we have:

$$\begin{aligned}(2.20)\qquad 0 \le V(t_2,\pi) - V(t_1,\pi) &\le E_\pi[(t_2+\tau_2) + H(\pi_{t_2+\tau_2})] - E_\pi[(t_1+\tau_1) + H(\pi_{t_1+\tau_1})] \\ &\le (t_2 - t_1) + E_\pi[H(\pi_{t_2+\tau_2}) - H(\pi_{t_1+\tau_1})]\end{aligned}$$

where we recall that $H(\pi) = a\pi \wedge b(1-\pi)$ for $\pi \in [0,1]$. Observe further that:

$$(2.21)\qquad E_\pi[H(\pi_{t_2+\tau_2}) - H(\pi_{t_1+\tau_1})] = \sum_{i=0}^{1} \frac{1+(-1)^i(1-2\pi)}{2} E_i[h(\varphi_{\tau_2}) - h(\varphi_{\tau_1})]$$

where for each $\pi \in \langle 0,1\rangle$ given and fixed the function $h$ is defined by:

$$(2.22)\qquad h(x) = H\left(\left(\frac{\pi}{1-\pi}x\right) \Big/ \left(1+\frac{\pi}{1-\pi}x\right)\right)$$

for all $x>0$. Then for any $0 < x_1 < x_2$ given and fixed it follows by the mean value theorem (note that $h$ is $C^1$ on $\langle 0,\infty\rangle$ except one point) that there exists $\xi \in [x_1,x_2]$ such that:

$$(2.23)\qquad |h(x_2) - h(x_1)| \le |h'(\xi)|\,(x_2 - x_1)$$

where the derivative $h'$ at $\xi$ satisfies:

$$(2.24)\quad |h'(\xi)| = \left|H'\left(\left(\frac{\pi}{1-\pi}\xi\right) \Big/ \left(1+\frac{\pi}{1-\pi}\xi\right)\right)\right| \frac{\pi(1-\pi)}{(1-\pi+\pi\xi)^2} \le K\,\frac{\pi(1-\pi)}{(1-\pi)^2} = K\,\frac{\pi}{1-\pi}$$

with some $K>0$ large enough.

On the other hand, the explicit expression (2.7) yields:

$$(2.25)\qquad \varphi_{\tau_2} - \varphi_{\tau_1} = \varphi_{\tau_2}\left(1 - \frac{\varphi_{\tau_1}}{\varphi_{\tau_2}}\right) = \varphi_{\tau_2}\left(1 - \exp\left(\frac{\mu}{\sigma^2}(X_{\tau_1} - X_{\tau_2}) - \frac{\mu^2}{2\sigma^2}(\tau_1 - \tau_2)\right)\right)$$

and thus the strong Markov property (stationary independent increments) together with the representation (2.2) and the fact that $\tau_1 - \tau_2 \le t_2 - t_1$ implies:

$$\begin{aligned}(2.26)\qquad & E_i(|\varphi_{\tau_2} - \varphi_{\tau_1}|) \\ &= E_i\left[\left|\varphi_{\tau_2}\left(1 - \exp\left(\frac{\mu}{\sigma}(W_{\tau_1} - W_{\tau_2}) - (-1)^i\frac{\mu^2}{2\sigma^2}(\tau_1 - \tau_2)\right)\right)\right|\right] \\ &= E_i\left[\varphi_{\tau_2}\, E_i\left[\left|1 - \exp\left(\frac{\mu}{\sigma}(W_{\tau_1} - W_{\tau_2}) - (-1)^i\frac{\mu^2}{2\sigma^2}(\tau_1 - \tau_2)\right)\right| \,\Big|\, \mathcal{F}^X_{\tau_2}\right]\right] \\ &\le E_i[\varphi_{\tau_2}]\, E_i\left[\sup_{0\le t\le t_2-t_1} \exp\left(\frac{\mu}{\sigma}W_t + \frac{\mu^2}{2\sigma^2}t\right) - 1\right]\end{aligned}$$

for $i=0,1$. Since it easily follows that:

$$E_i[\varphi_{\tau_2}] = E_i\left[\exp\left(\frac{\mu}{\sigma}W_{\tau_2} - (-1)^i\frac{\mu^2}{2\sigma^2}\tau_2\right)\right] \le \exp\left(\frac{\mu^2}{\sigma^2}(T-t_2)\right) \le \exp\left(\frac{\mu^2}{\sigma^2}T\right) \tag{2.27}$$

from (2.22)-(2.27) we get:

$$E_i(|h(\varphi_{\tau_2}) - h(\varphi_{\tau_1})|) \le K\,\frac{\pi}{1-\pi}\,E_i(|\varphi_{\tau_2}-\varphi_{\tau_1}|) \le K\,\frac{\pi}{1-\pi}\,L(t_2-t_1) \tag{2.28}$$

where the function $L$ is defined by:

$$L(t_2-t_1) = \exp\left(\frac{\mu^2}{\sigma^2}T\right)E_i\left[\sup_{0\le t\le t_2-t_1}\exp\left(\frac{\mu}{\sigma}W_t + \frac{\mu^2}{2\sigma^2}t\right) - 1\right]. \tag{2.29}$$

Therefore, combining (2.28) with (2.20)-(2.21) above, we obtain:

$$V(t_2,\pi) - V(t_1,\pi) \le (t_2-t_1) + K\,\frac{\pi}{1-\pi}\,L(t_2-t_1) \tag{2.30}$$

from where, by virtue of the fact that $L(t_2-t_1)\to 0$ in (2.29) as $t_2-t_1\downarrow 0$, we easily conclude that (2.19) holds. In particular, this shows that the instantaneous-stopping conditions (2.50) are satisfied.

2.7. In order to prove that the smooth-fit conditions (2.51) hold, i.e. that $\pi\mapsto V(t,\pi)$ is $C^1$ at $g_0(t)$ and $g_1(t)$, let us fix a point $(t,\pi)\in[0,T\rangle\times\langle 0,1\rangle$ lying on the boundary $g_0$ so that $\pi=g_0(t)$. Then for all $\varepsilon>0$ such that $\pi<\pi+\varepsilon<c$ we have:

$$\frac{V(t,\pi+\varepsilon)-V(t,\pi)}{\varepsilon} \le \frac{G(t,\pi+\varepsilon)-G(t,\pi)}{\varepsilon} \tag{2.31}$$

and hence, taking the limit in (2.31) as $\varepsilon\downarrow 0$, we get:

$$\frac{\partial^+ V}{\partial\pi}(t,\pi) \le \frac{\partial G}{\partial\pi}(t,\pi) \tag{2.32}$$

where the right-hand derivative in (2.32) exists (and is finite) by virtue of the concavity of $\pi\mapsto V(t,\pi)$ on $[0,1]$. Note that the latter will also be proved independently below.

Let us now fix some $\varepsilon>0$ such that $\pi<\pi+\varepsilon<c$ and consider the stopping time $\tau_\varepsilon=\tau_*(t,\pi+\varepsilon)$ being optimal for $V(t,\pi+\varepsilon)$. Note that $\tau_\varepsilon$ is the first exit time of the process $(\pi_{t+s})_{0\le s\le T-t}$ from the set $C$ in (2.16). Then by (2.1) and (2.8) it follows using the mean value theorem that there exists $\xi_i\in[\pi,\pi+\varepsilon]$ such that:

$$\begin{aligned} V(t,\pi+\varepsilon)-V(t,\pi) &\ge E_{\pi+\varepsilon}[G(t+\tau_\varepsilon,\pi_{t+\tau_\varepsilon})] - E_\pi[G(t+\tau_\varepsilon,\pi_{t+\tau_\varepsilon})] \\ &= \sum_{i=0}^{1} E_i[S_i(\pi+\varepsilon)-S_i(\pi)] = \varepsilon\sum_{i=0}^{1}E_i[S_i'(\xi_i)] \end{aligned} \tag{2.33}$$

where the function $S_i$ is defined by:

$$S_i(\pi) = \frac{1+(-1)^i(1-2\pi)}{2}\,G\left(t+\tau_\varepsilon,\frac{\pi}{1-\pi}\varphi_{\tau_\varepsilon}\Big/\left(1+\frac{\pi}{1-\pi}\varphi_{\tau_\varepsilon}\right)\right) \tag{2.34}$$

and its derivative $S_i'$ at $\xi_i$ is given by:

$$S_i'(\xi_i) = (-1)^{i+1}\, G\left(t+\tau_\varepsilon, \frac{\xi_i}{1-\xi_i}\varphi_{\tau_\varepsilon}\Big/\left(1+\frac{\xi_i}{1-\xi_i}\varphi_{\tau_\varepsilon}\right)\right) + \frac{1+(-1)^i(1-2\xi_i)}{2}\,\frac{\partial G}{\partial \pi}\left(t+\tau_\varepsilon, \frac{\xi_i}{1-\xi_i}\varphi_{\tau_\varepsilon}\Big/\left(1+\frac{\xi_i}{1-\xi_i}\varphi_{\tau_\varepsilon}\right)\right)\frac{\varphi_{\tau_\varepsilon}}{(1-\xi_i+\xi_i\varphi_{\tau_\varepsilon})^2} \tag{2.35}$$

for $i=0,1$. Since $g_0$ is increasing it is easily verified using (2.7)-(2.8) and the fact that $t\mapsto(\pm\mu/(2\sigma))\,t$ is a lower function for the standard Wiener process $W$ that $\tau_\varepsilon\to 0$ ($P_i$-a.s.) and thus $\varphi_{\tau_\varepsilon}\to 1$ ($P_i$-a.s.) as $\varepsilon\downarrow 0$ for $i=0,1$. Hence we easily find:

$$S_i'(\xi_i)\to(-1)^{i+1}G(t,\pi)+\frac{1+(-1)^i(1-2\pi)}{2}\,\frac{\partial G}{\partial\pi}(t,\pi)\quad(P_i\text{-a.s.}) \tag{2.36}$$

as $\varepsilon\downarrow 0$, and clearly $|S_i'(\xi_i)|\le K_i$ with some $K_i>0$ large enough for $i=0,1$.

It thus follows from (2.33) using (2.36) that:

$$\frac{V(t,\pi+\varepsilon)-V(t,\pi)}{\varepsilon}\ge\sum_{i=0}^{1}E_i[S_i'(\xi_i)]\to\frac{\partial G}{\partial\pi}(t,\pi) \tag{2.37}$$

as $\varepsilon\downarrow 0$ by the dominated convergence theorem. This combined with (2.31) above proves that $V_\pi^+(t,\pi)$ exists and equals $G_\pi(t,\pi)$. The smooth-fit at the boundary $g_1$ is proved analogously.

2.8. We proceed by proving that the boundaries $g_0$ and $g_1$ are continuous on $[0,T]$ and that $g_0(T)=g_1(T)=c$.

(i) Let us first show that the boundaries $g_0$ and $g_1$ are right-continuous on $[0,T]$. For this, fix $t\in[0,T\rangle$ and consider a sequence $t_n\downarrow t$ as $n\to\infty$. Since $g_i$ is monotone, the right-hand limit $g_i(t+)$ exists for $i=0,1$. Because $(t_n,g_i(t_n))\in\overline{D}$ for all $n\ge 1$, and $\overline{D}$ is closed, we see that $(t,g_i(t+))\in\overline{D}$ for $i=0,1$. Hence by (2.17) we see that $g_0(t+)\le g_0(t)$ and $g_1(t+)\ge g_1(t)$. The reverse inequalities follow obviously from the fact that $g_0$ is increasing and $g_1$ is decreasing on $[0,T]$, thus proving the claim.

(ii) Suppose that at some point $t_*\in\langle 0,T\rangle$ the function $g_1$ makes a jump, i.e. let $g_1(t_*-)>g_1(t_*)\ge c$. Let us fix a point $t'<t_*$ close to $t_*$ and consider the half-open region $R\subset C$ being a curved trapezoid formed by the vertexes $(t',g_1(t'))$, $(t_*,g_1(t_*-))$, $(t_*,\pi')$ and $(t',\pi')$ with any $\pi'$ fixed arbitrarily in the interval $\langle g_1(t_*),g_1(t_*-)\rangle$. Observe that the strong Markov property implies that the value function $V$ from (2.11) is $C^{1,2}$ on $C$. Note also that the gain function $G$ is $C^{1,2}$ in $R$ so that by the Leibnitz-Newton formula using (2.50) and (2.51) it follows that:

$$V(t,\pi)-G(t,\pi)=\int_\pi^{g_1(t)}\int_u^{g_1(t)}\left(\frac{\partial^2 V}{\partial\pi^2}-\frac{\partial^2 G}{\partial\pi^2}\right)(t,v)\,dv\,du \tag{2.38}$$

for all $(t,\pi)\in R$.

Let us fix some $(t,\pi)\in C$ and take an arbitrary $\varepsilon>0$ such that $(t+\varepsilon,\pi)\in C$. Then denoting by $\tau_\varepsilon=\tau_*(t+\varepsilon,\pi)$ the optimal stopping time for $V(t+\varepsilon,\pi)$, we have:

$$\begin{aligned}\frac{V(t+\varepsilon,\pi)-V(t,\pi)}{\varepsilon}&\ge\frac{E_{t+\varepsilon,\pi}[G(t+\varepsilon+\tau_\varepsilon,\pi_{t+\varepsilon+\tau_\varepsilon})]-E_{t,\pi}[G(t+\tau_\varepsilon,\pi_{t+\tau_\varepsilon})]}{\varepsilon}\\&=\frac{E_\pi[G(t+\varepsilon+\tau_\varepsilon,\pi_{\tau_\varepsilon})-G(t+\tau_\varepsilon,\pi_{\tau_\varepsilon})]}{\varepsilon}=1\end{aligned} \tag{2.39}$$

and thus, taking the limit in (2.39) as $\varepsilon \downarrow 0$, we get:

$$\frac{\partial V}{\partial t}(t,\pi) \geq \frac{\partial G}{\partial t}(t,\pi) = 1 \tag{2.40}$$

at each $(t,\pi) \in C$.

Since the strong Markov property implies that the value function $V$ from (2.11) solves the equation (2.49), using (2.40) we obtain:

$$\frac{\partial^2 V}{\partial \pi^2}(t,\pi) = -\frac{2\sigma^2}{\mu^2}\,\frac{1}{\pi^2(1-\pi)^2}\,\frac{\partial V}{\partial t}(t,\pi) \leq -\varepsilon\,\frac{\sigma^2}{\mu^2} \tag{2.41}$$

for all $t' \leq t < t_*$ and all $\pi' \leq \pi < g_1(t')$ with $\varepsilon > 0$ small enough.

Hence by (2.38) using that $G_{\pi\pi} = 0$ we get:

$$V(t',\pi') - G(t',\pi') \leq -\varepsilon\,\frac{\sigma^2}{\mu^2}\,\frac{(g_1(t')-\pi')^2}{2} \longrightarrow -\varepsilon\,\frac{\sigma^2}{\mu^2}\,\frac{(g_1(t_*-)-\pi')^2}{2} < 0 \tag{2.42}$$

as $t' \uparrow t_*$. This implies that $V(t_*,\pi') < G(t_*,\pi')$ which contradicts the fact that $(t_*,\pi')$ belongs to the stopping region $\overline{D}$. Thus $g_1(t_*-) = g_1(t_*)$ showing that $g_1$ is continuous at $t_*$ and thus on $[0,T]$ as well. A similar argument shows that the function $g_0$ is continuous on $[0,T]$.

(iii) We finally note that the method of proof from the previous part (ii) also implies that $g_0(T) = g_1(T) = c$. To see this, we may let $t_* = T$ and likewise suppose that $g_1(T-) > c$. Then repeating the arguments presented above word by word we arrive to a contradiction with the fact that $V(T,\pi) = G(T,\pi)$ for all $\pi \in [c, g_1(T-)]$ thus proving the claim.

2.9. Summarizing the facts proved in Subsections 2.5-2.8 above we may conclude that the following exit time is optimal in the extended problem (2.11):

$$\tau_* = \inf\{\, 0 \leq s \leq T-t \mid \pi_{t+s} \notin \langle g_0(t+s), g_1(t+s)\rangle \,\} \tag{2.43}$$

(the infimum of an empty set being equal $T-t$) where the two boundaries $(g_0, g_1)$ satisfy the following properties (see *Figure 1* below):

$$g_0 : [0,T] \rightarrow [0,1] \quad \text{is continuous and increasing} \tag{2.44}$$

$$g_1 : [0,T] \rightarrow [0,1] \quad \text{is continuous and decreasing} \tag{2.45}$$

$$A_* \leq g_0(t) < c < g_1(t) \leq B_* \quad \text{for all} \quad 0 \leq t < T \tag{2.46}$$

$$g_i(T) = c \quad \text{for} \quad i = 0,1 \tag{2.47}$$

where $A_*$ and $B_*$ satisfying $0 < A_* < c < B_* < 1$ are the optimal stopping points for the infinite horizon problem uniquely determined from the system of transcendental equations (4.85) in [18; page 185].

Standard arguments imply that the infinitesimal operator $\mathbb{L}$ of the process $(t,\pi_t)_{0 \leq t \leq T}$ acts on a function $f \in C^{1,2}([0,T\rangle \times [0,1])$ according to the rule:

$$(\mathbb{L}f)(t,\pi) = \left( \frac{\partial f}{\partial t} + \frac{\mu^2}{2\sigma^2}\pi^2(1-\pi)^2 \frac{\partial^2 f}{\partial \pi^2} \right)(t,\pi) \tag{2.48}$$

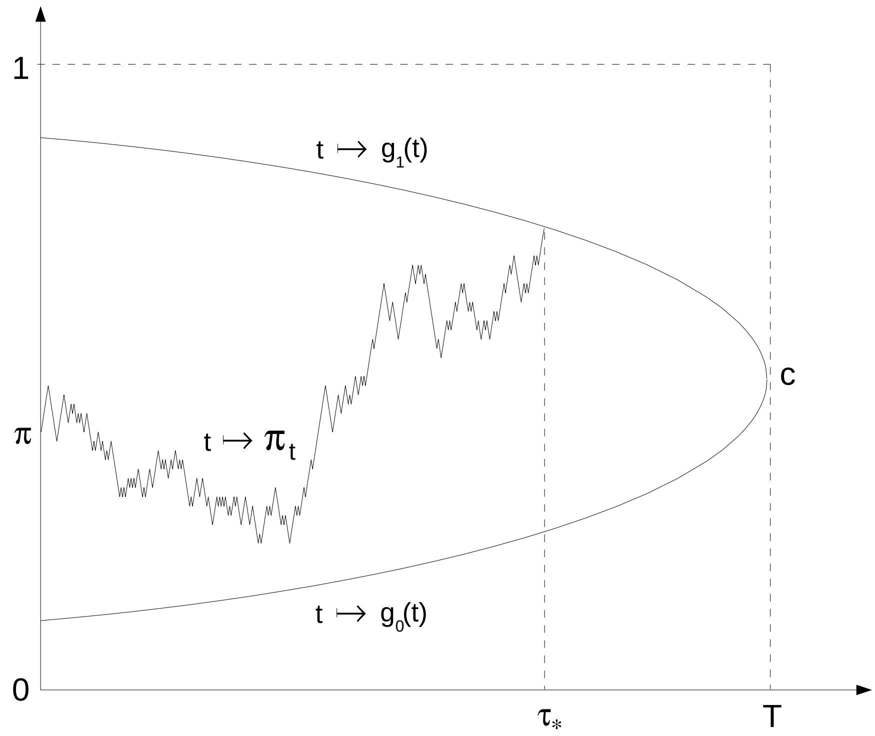


**Figure 1.** A computer drawing of the optimal stopping boundaries $g_0$ and $g_1$ from Theorem 2.1. In the case above it is optimal to accept the hypothesis $H_1$.

for all $(t,\pi)\in[0,T\rangle\times[0,1]$. In view of the facts proved above we are thus naturally led to formulate the following *free-boundary problem* for the unknown value function $V$ from (2.11) and the unknown boundaries $(g_0,g_1)$ from (2.16)-(2.17):

$$(\mathbb{L}V)(t,\pi)=0 \quad \text{for} \quad (t,\pi)\in C \tag{2.49}$$

$$V(t,\pi)\Big|_{\pi=g_0(t)+}=t+ag_0(t), \quad V(t,\pi)\Big|_{\pi=g_1(t)-}=t+b(1-g_1(t)) \tag{2.50}$$

$$\frac{\partial V}{\partial\pi}(t,\pi)\Big|_{\pi=g_0(t)+}=a, \quad \frac{\partial V}{\partial\pi}(t,\pi)\Big|_{\pi=g_1(t)-}=-b \tag{2.51}$$

$$V(t,\pi)<G(t,\pi) \quad \text{for} \quad (t,\pi)\in C \tag{2.52}$$

$$V(t,\pi)=G(t,\pi) \quad \text{for} \quad (t,\pi)\in D \tag{2.53}$$

where $C$ and $D$ are given by (2.16) and (2.17), and the *instantaneous-stopping conditions* (2.50) are satisfied for all $0\le t\le T$ and the *smooth-fit conditions* (2.51) are satisfied for all $0\le t<T$.

Note that the superharmonic characterization of the value function (see [2] and [18]) implies that $V$ from (2.11) is a largest function satisfying (2.49)-(2.50) and (2.52)-(2.53).

2.10. Making use of the facts proved above we are now ready to formulate the main result of the paper. Below we set $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy$ for $x \in \mathbb{R}$.

**Theorem 2.1.** *In the Bayesian problem (2.4)-(2.5) of testing two simple hypotheses (2.3) the optimal decision rule* $(\tau_*, d_*)$ *is explicitly given by:*

$$\tau_* = \inf\{ 0 \le t \le T \mid \pi_t \notin \langle g_0(t), g_1(t) \rangle \} \tag{2.54}$$

$$d_* = \begin{cases} 1 \text{ (accept } H_1) & \textit{if } \pi_{\tau_*} = g_1(\tau_*) \\ 0 \text{ (accept } H_0) & \textit{if } \pi_{\tau_*} = g_0(\tau_*) \end{cases} \tag{2.55}$$

*where the two boundaries* $(g_0, g_1)$ *can be characterized as a unique solution of the coupled system of nonlinear integral equations:*

$$\begin{aligned} E_{t,g_i(t)}[a\pi_T \wedge b(1-\pi_T)] &= a g_i(t) \wedge b(1-g_i(t)) \\ &+ \sum_{j=0}^{1} \int_0^{T-t} (-1)^j P_{t,g_i(t)}[\pi_{t+u} \le g_j(t+u)]\, du \quad (i = 0, 1) \end{aligned} \tag{2.56}$$

*for* $0 \le t \le T$ *satisfying (2.44)-(2.47) [see Figure 1 above].*

*More explicitly, the six terms in the system (2.56) read as follows:*

$$\begin{aligned} &E_{t,g_i(t)}[a\pi_T \wedge b(1-\pi_T)] = \\ &= g_i(t) \int_{-\infty}^{\infty} \frac{a g_i(t) \exp\big(\mu z\sqrt{T-t}/\sigma + \mu^2(T-t)/(2\sigma^2)\big) \wedge b(1-g_i(t))}{1 - g_i(t) + g_i(t)\exp\big(\mu z\sqrt{T-t}/\sigma + \mu^2(T-t)/(2\sigma^2)\big)}\, \varphi(z)\, dz \\ &+ (1-g_i(t)) \int_{-\infty}^{\infty} \frac{a g_i(t) \exp\big(\mu z\sqrt{T-t}/\sigma - \mu^2(T-t)/(2\sigma^2)\big) \wedge b(1-g_i(t))}{1 - g_i(t) + g_i(t)\exp\big(\mu z\sqrt{T-t}/\sigma - \mu^2(T-t)/(2\sigma^2)\big)}\, \varphi(z)\, dz \end{aligned} \tag{2.57}$$

$$\begin{aligned} &P_{t,g_i(t)}[\pi_{t+u} \le g_j(t+u)] \\ &= g_i(t)\Phi\left( \frac{\sigma}{\mu\sqrt{u}} \log\left( \frac{g_j(t+u)}{1-g_j(t+u)} \frac{1-g_i(t)}{g_i(t)} \right) - \frac{\mu\sqrt{u}}{2\sigma} \right) \\ &+ (1-g_i(t))\Phi\left( \frac{\sigma}{\mu\sqrt{u}} \log\left( \frac{g_j(t+u)}{1-g_j(t+u)} \frac{1-g_i(t)}{g_i(t)} \right) + \frac{\mu\sqrt{u}}{2\sigma} \right) \end{aligned} \tag{2.58}$$

*for* $0 \le u \le T-t$ *with* $0 \le t \le T$ *and* $i, j = 0, 1$.

*[Note that in the case when* $a = b$ *we have* $c = 1/2$ *and the system (2.56) reduces to one equation only since* $g_1 = 1 - g_0$ *by symmetry.]*

**Proof.** (i) The existence of boundaries $(g_0, g_1)$ satisfying (2.44)-(2.47) such that $\tau_*$ from (2.54) is optimal in (2.4)-(2.5) was proved in Subsections 2.5-2.9 above. By the change-of-variable formula from [12] it follows that the boundaries $(g_0, g_1)$ solve the system (2.56) (cf. (2.62)-(2.64) below). Thus it remains to show that the system (2.56) has no other solution in the class of functions $(h_0, h_1)$ satisfying (2.44)-(2.47).

Let us thus assume that two functions $(h_0, h_1)$ satisfying (2.44)-(2.47) solve the system (2.56), and let us show that these two functions $(h_0, h_1)$ must then coincide with the optimal boundaries $(g_0, g_1)$. For this, let us introduce the function:

$$V^h(t,\pi) = \begin{cases} U^h(t,\pi) & \text{if } (t,\pi) \in C_h \\ G(t,\pi) & \text{if } (t,\pi) \in \overline{D}_h \end{cases} \tag{2.59}$$

where the function $U^h$ is defined by:

$$U^h(t,\pi) = E_{t,\pi}[G(T,\pi_T)] - \int_0^{T-t} P_{t,\pi}[(t+u,\pi_{t+u}) \in D_h]\, du \tag{2.60}$$

for all $(t,\pi) \in [0,T\rangle \times [0,1]$ and the sets $C_h$ and $D_h$ are defined as in (2.16) and (2.17) with $h_i$ instead of $g_i$ for $i=0,1$. Note that (2.60) with $G(t,\pi)$ instead of $U^h(t,\pi)$ on the left-hand side coincides with (2.56) when $\pi = g_i(t)$ and $h_j = g_j$ for $i,j=0,1$. Since $(h_0,h_1)$ solve (2.56) this shows that $V^h$ is continuous on $[0,T\rangle \times [0,1]$. We need to verify that $V^h$ coincides with the value function $V$ from (2.11) and that $h_i$ equals $g_i$ for $i=0,1$.

(ii) Using standard arguments based on the strong Markov property (or verifying directly) it follows that $V^h$ i.e. $U^h$ is $C^{1,2}$ on $C_h$ and that:

$$(\mathbb{L}V^h)(t,\pi) = 0 \quad \text{for } (t,\pi) \in C_h. \tag{2.61}$$

Moreover, since $U^h_\pi$ is continuous on $[0,T\rangle \times \langle 0,1\rangle$ (which is readily verified using the explicit expressions (2.57) and (2.58) above with $\pi$ instead of $g_i(t)$ and $h_j$ instead of $g_j$ for $i,j=0,1$), we see that $V^h_\pi$ is continuous on $\overline{C}_h$. Finally, since $h_0(t) \in \langle 0,c\rangle$ and $h_1(t) \in \langle c,1\rangle$ we see that $V^h$ i.e. $G$ is $C^{1,2}$ on $\overline{D}_h$. Therefore, with $(t,\pi) \in [0,T\rangle \times \langle 0,1\rangle$ given and fixed, the change-of-variable formula from [12] can be applied, and in this way we get:

$$\begin{aligned} V^h(t+s,\pi_{t+s}) &= V^h(t,\pi) \\ &+ \int_0^s (\mathbb{L}V^h)(t+u,\pi_{t+u}) I(\pi_{t+u} \neq h_0(t+u), \pi_{t+u} \neq h_1(t+u))\, du \\ &+ M^h_s + \frac{1}{2}\sum_{i=0}^{1} \int_0^s \Delta_\pi V^h_\pi(t+u,\pi_{t+u}) I(\pi_{t+u} = h_i(t+u))\, d\ell^{h_i}_u \end{aligned} \tag{2.62}$$

for $0 \le s \le T-t$ where $\Delta_\pi V^h_\pi(t+u,h_i(t+u)) = V^h_\pi(t+u,h_i(t+u)+) - V^h_\pi(t+u,h_i(t+u)-)$, the process $(\ell^{h_i}_s)_{0\le s\le T-t}$ is the local time of $(\pi_{t+s})_{0\le s\le T-t}$ at the boundary $h_i$ given by:

$$\ell^{h_i}_s = P_{t,\pi}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(h_i(t+u)-\varepsilon < \pi_{t+u} < h_i(t+u)+\varepsilon)\, \frac{\mu^2}{\sigma^2}\pi^2_{t+u}(1-\pi_{t+u})^2\, du \tag{2.63}$$

for $i=0,1$, and $(M^h_s)_{0\le s\le T-t}$ defined by $M^h_s = \int_0^s V^h_\pi(t+u,\pi_{t+u})\, I(\pi_{t+u} \neq h_0(t+u), \pi_{t+u} \neq h_1(t+u))\,(\mu/\sigma)\,\pi_{t+u}\,(1-\pi_{t+u})\, d\overline{W}_u$ is a martingale under $P_{t,\pi}$.

Setting $s = T-t$ in (2.62) and taking the $P_{t,\pi}$-expectation, using that $V^h$ satisfies (2.61) in $C_h$ and equals $G$ in $D_h$, we get:

$$E_{t,\pi}[G(T,\pi_T)] = V^h(t,\pi) + \int_0^{T-t} P_{t,\pi}[(t+u,\pi_{t+u}) \in D_h]\, du + \frac{1}{2}F(t,\pi) \tag{2.64}$$

where (by the continuity of the integrand) the function $F$ is given by:

$$F(t,\pi) = \sum_{i=0}^{1} \int_0^{T-t} \Delta_\pi V^h_\pi(t+u, h_i(t+u))\, d_u E_{t,\pi}[\ell^{h_i}_u] \tag{2.65}$$

for all $(t,\pi) \in [0,T\rangle \times [0,1]$ and $i=0,1$. Thus from (2.64) and (2.59) we see that:

$$F(t,\pi) = \begin{cases} 0 & \text{if } (t,\pi) \in C_h \\ 2\,(U^h(t,\pi) - G(t,\pi)) & \text{if } (t,\pi) \in \overline{D}_h \end{cases} \tag{2.66}$$

where the function $U^h$ is given by (2.60).

(iii) From (2.66) we see that if we are to prove that:

$$\pi \mapsto V^h(t,\pi) \quad \text{is} \quad C^1 \quad \text{at} \quad h_i(t) \tag{2.67}$$

for each $0 \le t < T$ given and fixed and $i = 0,1$, then it will follow that:

$$U^h(t,\pi) = G(t,\pi) \quad \text{for all } (t,\pi) \in \overline{D}_h. \tag{2.68}$$

On the other hand, if we know that (2.68) holds, then using the general facts obtained directly from the definition (2.59) above:

$$\frac{\partial}{\partial \pi}(U^h(t,\pi) - G(t,\pi))\Big|_{\pi=h_0(t)} = V^h_\pi(t,h_0(t)+) - V^h_\pi(t,h_0(t)-) = \Delta_\pi V^h_\pi(t,h_0(t)) \tag{2.69}$$

$$\frac{\partial}{\partial \pi}(U^h(t,\pi) - G(t,\pi))\Big|_{\pi=h_1(t)} = V^h_\pi(t,h_1(t)-) - V^h_\pi(t,h_1(t)+) = -\Delta_\pi V^h_\pi(t,h_1(t)) \tag{2.70}$$

for all $0 \le t < T$, we see that (2.67) holds too. The equivalence of (2.67) and (2.68) suggests that instead of dealing with the equation (2.66) in order to derive (2.67) above we may rather concentrate on establishing (2.68) directly.

To derive (2.68) first note that using standard arguments based on the strong Markov property (or verifying directly) it follows that $U^h$ is $C^{1,2}$ in $D_h$ and that:

$$(\mathbb{L}U^h)(t,\pi) = 1 \quad \text{for } (t,\pi) \in D_h. \tag{2.71}$$

It follows that (2.62) can be applied with $U^h$ instead of $V^h$, and this yields:

$$U^h(t+s, \pi_{t+s}) = U^h(t,\pi) + \int_0^s I((t+u, \pi_{t+u}) \in D_h)\, du + N^h_s \tag{2.72}$$

using (2.61) and (2.71) as well as that $\Delta_\pi U^h_\pi(t+u, h_i(t+u)) = 0$ for all $0 \le u \le s$ and $i = 0,1$ since $U^h_\pi$ is continuous. In (2.72) we have $N^h_s = \int_0^s U^h_\pi(t+u, \pi_{t+u})\, I(\pi_{t+u} \ne h_0(t+u), \pi_{t+u} \ne h_1(t+u))\, (\mu/\sigma)\, \pi_{t+u}(1-\pi_{t+u})\, d\overline{W}_u$ and $(N^h_s)_{0 \le s \le T-t}$ is a martingale under $P_{t,\pi}$.

Next note that (2.62) applied to $G$ instead of $V^h$ yields:

$$G(t+s, \pi_{t+s}) = G(t,\pi) + \int_0^s I(\pi_{t+u} \ne c)\, du - \frac{a+b}{2}\, \ell^c_s + M_s \tag{2.73}$$

using that $\mathbb{L}G = 1$ off $[0,T]\times\{c\}$ as well as that $\Delta_\pi G_\pi(t+u,c) = -b-a$ for $0 \le u \le s$. In (2.73) we have $M_s = \int_0^s G_\pi(t+u,\pi_{t+u})\, I(\pi_{t+u} \ne c)\, (\mu/\sigma)\, \pi_{t+u}(1-\pi_{t+u})\, d\overline{W}_u = \int_0^s [a\, I(\pi_{t+u} < c) - b\, I(\pi_{t+u} > c)]\, (\mu/\sigma)\, \pi_{t+u}(1-\pi_{t+u})\, d\overline{W}_u$ and $(M_s)_{0 \le s \le T-t}$ is a martingale under $P_{t,\pi}$.

For $0 < \pi \le h_0(t)$ or $h_1(t) \le \pi < 1$ consider the stopping time:

$$\sigma_h = \inf\{0 \le s \le T-t \mid \pi_{t+s} \in [h_0(t+s), h_1(t+s)]\}. \tag{2.74}$$

Then using that $U^h(t,h_i(t)) = G(t,h_i(t))$ for all $0 \le t < T$ and $i = 0,1$ since $(h_0,h_1)$ solve (2.56), and that $U^h(T,\pi) = G(T,\pi)$ for all $0 \le \pi \le 1$, we see that $U^h(t+\sigma_h, \pi_{t+\sigma_h}) = G(t+\sigma_h, \pi_{t+\sigma_h})$. Hence from (2.72) and (2.73) using the optional sampling theorem (see e.g. [16; Chapter II, Theorem 3.2]) we find:

$$\begin{aligned} U^h(t,\pi) &= E_{t,\pi}[U^h(t+\sigma_h, \pi_{t+\sigma_h})] - E_{t,\pi}\Big[\int_0^{\sigma_h} I((t+u,\pi_{t+u}) \in D_h)\, du\Big] \\ &= E_{t,\pi}[G(t+\sigma_h, \pi_{t+\sigma_h})] - E_{t,\pi}\Big[\int_0^{\sigma_h} I((t+u,\pi_{t+u}) \in D_h)\, du\Big] \\ &= G(t,\pi) + E_{t,\pi}\Big[\int_0^{\sigma_h} I(\pi_{t+u} \ne c)\, du\Big] \\ &\quad - E_{t,\pi}\Big[\int_0^{\sigma_h} I((t+u,\pi_{t+u}) \in D_h)\, du\Big] = G(t,\pi) \end{aligned} \tag{2.75}$$

since $\pi_{t+u} \ne c$ and $(t+u, \pi_{t+u}) \in D_h$ for all $0 \le u < \sigma_h$. This establishes (2.68) and thus (2.67) holds as well.

It may be noted that a shorter but somewhat less revealing proof of (2.68) [and (2.67)] can be obtained by verifying directly (using the Markov property only) that the process:

$$U^h(t+s, \pi_{t+s}) - \int_0^s I((t+u,\pi_{t+u}) \in D_h)\, du \tag{2.76}$$

is a martingale under $P_{t,\pi}$ for $0 \le s \le T-t$. This verification moreover shows that the martingale property of (2.76) does not require that $h_0$ and $h_1$ are continuous and monotone (but only measurable). Taken together with the rest of the proof below this shows that the claim of uniqueness for the equation (2.56) holds in the class of continuous functions $h_0$ and $h_1$ from $[0,T]$ to $\mathbb{R}$ such that $0 < h_0(t) < c$ and $c < h_1(t) < 1$ for all $0 < t < T$.

(iv) Let us consider the stopping time:

$$\tau_h = \inf\{0 \le s \le T-t \mid \pi_{t+s} \notin \langle h_0(t+s), h_1(t+s)\rangle\}. \tag{2.77}$$

Observe that, by virtue of (2.67), the identity (2.62) can be written as:

$$V^h(t+s, \pi_{t+s}) = V^h(t,\pi) + \int_0^s I((t+u,\pi_{t+u}) \in D_h)\, du + M_s^h \tag{2.78}$$

with $(M_s^h)_{0 \le s \le T-t}$ being a martingale under $P_{t,\pi}$. Thus, inserting $\tau_h$ into (2.78) in place of $s$ and taking the $P_{t,\pi}$-expectation, by means of the optional sampling theorem we get:

$$V^h(t,\pi) = E_{t,\pi}[G(t+\tau_h, \pi_{t+\tau_h})] \tag{2.79}$$

for all $(t,\pi)\in[0,T\rangle\times[0,1]$. Then comparing (2.79) with (2.11) we see that:

$$V(t,\pi)\le V^h(t,\pi) \tag{2.80}$$

for all $(t,\pi)\in[0,T\rangle\times[0,1]$.

(v) Let us now show that $g_0\le h_0$ and $h_1\le g_1$ on $[0,T]$. For this, recall that by the same arguments as for $V^h$ we also have:

$$V(t+s,\pi_{t+s})=V(t,\pi)+\int_0^s I((t+u,\pi_{t+u})\in D)\,du+M_s^g \tag{2.81}$$

where $(M_s^g)_{0\le s\le T-t}$ is a martingale under $P_{t,\pi}$. Fix some $(t,\pi)$ belonging to both $D$ and $D_h$ (firstly below $g_0$ and $h_0$ and then above $g_1$ and $h_1$) and consider the stopping time:

$$\sigma_g=\inf\{0\le s\le T-t\mid \pi_{t+s}\in[g_0(t+s),g_1(t+s)]\}. \tag{2.82}$$

Inserting $\sigma_g$ into (2.78) and (2.81) in place of $s$ and taking the $P_{t,\pi}$-expectation, by means of the optional sampling theorem we get:

$$E_{t,\pi}[V^h(t+\sigma_g,\pi_{t+\sigma_g})]=G(t,\pi)+E_{t,\pi}\left[\int_0^{\sigma_g} I((t+u,\pi_{t+u})\in D_h)\,du\right] \tag{2.83}$$

$$E_{t,\pi}[V(t+\sigma_g,\pi_{t+\sigma_g})]=G(t,\pi)+E_{t,\pi}[\sigma_g]. \tag{2.84}$$

Hence by means of (2.80) we see that:

$$E_{t,\pi}\left[\int_0^{\sigma_g} I((t+u,\pi_{t+u})\in D_h)\,du\right]\ge E_{t,\pi}[\sigma_g] \tag{2.85}$$

from where, by virtue of the continuity of $h_i$ and $g_i$ on $\langle 0,T\rangle$ for $i=0,1$, it readily follows that $D\subseteq D_h$, i.e. $g_0(t)\le h_0(t)$ and $h_1(t)\le g_1(t)$ for all $0\le t\le T$.

(vi) Finally, we show that $h_i$ coincides with $g_i$ for $i=0,1$. For this, let us assume that there exists some $t\in\langle 0,T\rangle$ such that $g_0(t)<h_0(t)$ or $h_1(t)<g_1(t)$ and take an arbitrary $\pi$ from $\langle g_0(t),h_0(t)\rangle$ or $\langle h_1(t),g_1(t)\rangle$, respectively. Then inserting $\tau_*=\tau_*(t,\pi)$ from (2.43) into (2.78) and (2.81) in place of $s$ and taking the $P_{t,\pi}$-expectation, by means of the optional sampling theorem we get:

$$E_{t,\pi}[G(t+\tau_*,\pi_{t+\tau_*})]=V^h(t,\pi)+E_{t,\pi}\left[\int_0^{\tau_*} I((t+u,\pi_{t+u})\in D_h)\,du\right] \tag{2.86}$$

$$E_{t,\pi}[G(t+\tau_*,\pi_{t+\tau_*})]=V(t,\pi). \tag{2.87}$$

Hence by means of (2.80) we see that:

$$E_{t,\pi}\left[\int_0^{\tau_*} I((t+u,\pi_{t+u})\in D_h)\,du\right]\le 0 \tag{2.88}$$

which is clearly impossible by the continuity of $h_i$ and $g_i$ for $i=0,1$. We may therefore conclude that $V^h$ defined in (2.59) coincides with $V$ from (2.11) and $h_i$ is equal to $g_i$ for $i=0,1$. This completes the proof of the theorem. $\square$

**Remark 2.2.** Note that without loss of generality it can be assumed that $\mu > 0$ in (2.2). In this case the optimal decision rule (2.54)-(2.55) can be equivalently written as follows:

$$\tau_* = \inf\{0 \le t \le T \mid X_t \notin \langle b_0^\pi(t), b_1^\pi(t)\rangle\} \tag{2.89}$$

$$d_* = \begin{cases} 1 \text{ (accept } H_1) & \text{if } X_{\tau_*} = b_1^\pi(\tau_*) \\ 0 \text{ (accept } H_0) & \text{if } X_{\tau_*} = b_0^\pi(\tau_*) \end{cases} \tag{2.90}$$

where we set:

$$b_i^\pi(t) = \frac{\sigma^2}{\mu} \log\left(\frac{1-\pi}{\pi}\frac{g_i(t)}{1-g_i(t)}\right) + \frac{\mu}{2}t \tag{2.91}$$

for $t \in [0,T]$, $\pi \in [0,1]$ and $i = 0,1$.

The result proved above shows that the following sequential procedure is optimal. *Observe $X_t$ for $t \in [0,T]$ and stop the observation as soon as $X_t$ becomes either greater than $b_1^\pi(t)$ or smaller than $b_0^\pi(t)$ for some $t \in [0,T]$. In the first case conclude that the drift equals $\mu$, and in the second case conclude that the drift equals $0$.*

**Remark 2.3.** In the preceding procedure we need to know the boundaries $(b_0^\pi, b_1^\pi)$ i.e. the boundaries $(g_0, g_1)$. We proved above that $(g_0, g_1)$ is a unique solution of the system (2.56). This system cannot be solved analytically but can be dealt with numerically. The following simple method can be used to illustrate the latter (better methods are needed to achieve higher precision around the singularity point $t = T$ and to increase the speed of calculation).

Set $t_k = kh$ for $k = 0,1,\dots,n$ where $h = T/n$ and denote:

$$J(t, g_i(t)) = E_{t,g_i(t)}[a\pi_T \wedge b(1-\pi_T)] - ag_i(t) \wedge b(1-g_i(t)) \tag{2.92}$$

$$K(t, g_i(t); t+u, g_0(t+u), g_1(t+u)) = \sum_{j=0}^{1} (-1)^j P_{t,g_i(t)}[\pi_{t+u} \le g_j(t+u)] \tag{2.93}$$

for $i = 0,1$ upon recalling the explicit expressions (2.57) and (2.58) above. Note that $K$ always depends on both $g_0$ and $g_1$.

Then the following discrete approximation of the integral equations (2.56) is valid:

$$J(t_k, g_i(t_k)) = \sum_{l=k}^{n-1} K(t_k, g_i(t_k); t_{l+1}, g_0(t_{l+1}), g_1(t_{l+1}))\, h \quad (i = 0,1) \tag{2.94}$$

for $k = 0,1,\dots,n-1$. Setting $k = n-1$ and $g_0(t_n) = g_1(t_n) = c$ we can solve the system of two equations (2.94) numerically and get numbers $g_0(t_{n-1})$ and $g_1(t_{n-1})$. Setting $k = n-2$ and using the values $g_0(t_{n-1})$, $g_0(t_n)$, $g_1(t_{n-1})$, $g_1(t_n)$ we can solve (2.94) numerically and get numbers $g_0(t_{n-2})$ and $g_1(t_{n-2})$. Continuing the recursion we obtain $g_i(t_n), g_i(t_{n-1}), \dots, g_i(t_1), g_i(t_0)$ as an approximation of the optimal boundary $g_i$ at the points $T, T-h, \dots, h, 0$ for $i = 0,1$ (cf. *Figure 1* above).

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