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# An optimal stopping problem in a diffusion-type model with delay

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We present an explicit solution to an optimal stopping problem in a model described by a stochastic delay differential equation with an exponential delay measure. The method of proof is based on reducing the initial problem to a free-boundary problem and solving the latter by means of the smooth-fit condition. The problem can be interpreted as pricing special perpetual average American put options in a diffusion-type model with delay.

## 1 Introduction

The main aim of this paper is to present a solution to the optimal stopping problem (3) for the process $X$ that solves the stochastic differential equation (1) with an exponential delay measure on an infinite time interval. This problem is related to the option pricing theory in mathematical finance, where the process $X$ can describe the logarithm of the price of a risky asset (e.g. a stock) on a financial market. In that case, the value (3) can be formally interpreted as a *fair price* of a *special perpetual average American put option* in a diffusion-type market model with delay. In this model, the dynamics of the price depends on its deviation from the running average over past values.

In recent years, several control problems for models described by stochastic delay differential equations were studied. Øksendal and Sulem [14] proved maximum principles for certain classes of such models and applied them to solving some problems related to finance. Elsanosi, Øksendal and Sulem [4] proved a verification theorem of variational inequality type and applied it to finding explicit solutions for some classes of optimal harvesting delay problems. Larssen [9] established the dynamic programming principle for stochastic delay differential equations.

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Larssen and Risebro [10] exhibited certain classes of delayed control problems that can be reduced to ordinary control problems. In this paper, we show how an explicit solution to an optimal stopping problem in a model described by a stochastic delay differential equation can be derived.

The paper is organized as follows. In Section 2, using change-of-measure arguments, for the initial problem (3) we construct an equivalent optimal stopping problem for the one-dimensional Markov deviation process. In order to find explicit expressions for the value function and the optimal boundary, we formulate an associated free-boundary problem. In Section 3, we derive a solution to the free-boundary problem, which can be expressed by the confluent hypergeometric function and thus admits a representation in closed form. In Section 4, we verify that the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping problem. The main result of the paper is stated in Theorem 4.1.

# 2 Formulation of the problem

First, let us give a precise description of the diffusion-type model with delay.

2.1. Suppose that on some probability space $(\Omega, \mathcal{F}, P)$ there exists a standard Wiener process $W = (W_t)_{t \geq 0}$ and a continuous process $X = (X_t)_{t \in \mathbb{R}}$ solving the stochastic differential equation:

$$dX_t = -(\theta^2/2)(X_t - \lambda Y_t)^2 \, dt + \theta(X_t - \lambda Y_t) \, dW_t \quad \text{for} \quad t \geq 0, \quad X_0 = x, \tag{1}$$

where the process $Y = (Y_t)_{t \geq 0}$ is defined by:

$$Y_t = \int_{-\infty}^0 e^{\lambda s} X_{t+s} \, ds, \quad X_t = X_t^0 \quad \text{for} \quad t \leq 0, \tag{2}$$

for some $\theta > 0$, $\lambda > 0$, and $x \in \mathbb{R}$ given and fixed. Here $X_t^0$, $t \leq 0$, is a (deterministic) bounded measurable function. Since the exponential of $X$ is a local martingale, the process $X$ can be thought of describing the logarithm of a (discounted) stock price on a financial market. The goal of this paper is to compute the value:

$$V_* = \sup_\tau E\Big[e^{-\delta\tau}\left(Ke^{\lambda Y_\tau} - e^{X_\tau}\right)^+\Big], \tag{3}$$

where the supremum is taken over all finite stopping times $\tau$ of the process $X$ (i.e. stopping times with respect to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of $X$), and to determine an optimal stopping time at which the supremum in (3) is attained. The value (3) can be interpreted as an *arbitrage-free price* of a special average American put option, where $K > 0$ and $\delta > 0$ are some given constants. Some other optimal stopping problems for geometric Brownian motion with gain functions containing integrals were solved in [8] and [15]. Note that a different class of optimal stopping problems can be obtained when the underlying process is Markovian, but at the same time, there is a delay in the available information as in [13].

By differentiation it can be shown that the process $Y$ admits the representation:

$$dY_t = Z_t \, dt, \quad Y_0 = y, \tag{4}$$

where the process $Z = (Z_t)_{t \geq 0}$ is defined by:

$$Z_t = X_t - \lambda Y_t \tag{5}$$

for all $t \geq 0$. The process $Z$ defined in (5) expresses the deviation of the logarithm of the present value of the process $X$ from its exponentially weighted average $\lambda Y$. By means of Itô's formula (see e.g. [11; Theorem 4.4] or [7; Theorem I.4.57]) it can be shown that the *deviation process* $Z$ solves the stochastic differential equation:

$$dZ_t = -(\theta^2/2)\, Z_t\, (Z_t + 2\lambda/\theta^2)\, dt + \theta Z_t\, dW_t, \quad Z_0 = z, \tag{6}$$

which admits the explicit solution:

$$Z_t = \frac{\exp\big(\theta\, W_t - (\lambda + \theta^2/2)\, t\big)}{1/z + (\theta^2/2) \int_0^t \exp\big(\theta\, W_s - (\lambda + \theta^2/2)\, s\big)\, ds} \tag{7}$$

for $z \neq 0$ (cf. e.g. [12; Example 5.15] or [5; Chapter IV]). From the structure of the solution (7) it follows that the process $Z$ started at some $z < 0$ remains negative and explodes in finite time with positive probability. On the other hand, started at some $z > 0$ the solution $Z$ exists globally and remains positive, while started at $z = 0$ it is trapped at the same point. To avoid degeneracy we thus further assume that $z > 0$.

Observe that from the one-to-one correspondence (5) between the processes $(X_t, Y_t)_{t \geq 0}$ and $(Z_t, Y_t)_{t \geq 0}$, by virtue of (4) it follows that the natural filtration of the process $Z$ coincides with $(\mathcal{F}_t)_{t \geq 0}$, and from (6)-(7) it is seen that the latter coincides with the natural filtration of the process $W$. Note that $Y_0 = y$ in (4) and $Z_0 = z$ in (6) can be straightforwardly expressed by means of the initial function $X_t^0$, $t \leq 0$. It therefore follows that the value (3) takes the form:

$$V_* = \sup_\tau E\Big[e^{-\delta\tau + X_\tau}\, \big(Ke^{-Z_\tau} - 1\big)^+\Big], \tag{8}$$

where the supremum can equivalently be taken over all finite stopping times of the process $Z$. Taking into account the multiplicative structure of the gain function in (8), without loss of generality we can further assume that $x = 0$.

2.2. Let us define the process $\widetilde{W} = (\widetilde{W}_t)_{t \geq 0}$ by:

$$\widetilde{W}_t = W_t - \int_0^t \theta Z_s\, ds, \tag{9}$$

where $Z = (Z_t)_{t \geq 0}$ is given by (5)-(7). Hence, from (6) it follows that the process $Z$ solves the stochastic differential equation:

$$dZ_t = -(3\theta^2/2)\, Z_t\, (Z_t + 2\lambda/(3\theta^2))\, dt + \theta Z_t\, d\widetilde{W}_t, \quad Z_0 = z, \tag{10}$$

which admits the explicit solution:

$$Z_t = \frac{\exp\big(\theta\, \widetilde{W}_t - (\lambda + \theta^2/2)\, t\big)}{1/z + (3\theta^2/2) \int_0^t \exp\big(\theta\, \widetilde{W}_s - (\lambda + \theta^2/2)\, s\big)\, ds} \tag{11}$$

for $z > 0$. Substituting the expression (11) into (9), we see that $\widetilde{W}$ is a diffusion-type process with respect to the Wiener process $W$ (cf. e.g. [11; Section IV.2]), and its natural filtration clearly coincides with $(\mathcal{F}_t)_{t \ge 0}$.

Let us denote by $A_t(\widetilde{W})$ the right-hand side of the expression (11). Taking into account the assumption $z > 0$ and using the continuity of $A_t(\widetilde{W})$ and $A_t(W)$, we get:

$$P\left[\int_0^t A_s^2(\widetilde{W})\, ds < \infty\right] = P\left[\int_0^t A_s^2(W)\, ds < \infty\right] = 1 \tag{12}$$

for all $t \ge 0$. Then, by means of the result of [11; Theorem 7.6], we obtain:

$$E\left[\exp\left(\int_0^t \theta Z_s\, dW_s - \frac{1}{2}\int_0^t \theta^2 Z_s^2\, ds\right)\right] = 1 \tag{13}$$

for all $t \ge 0$. Hence, following the arguments in [20; Section 7] and [17; Section 2] (see also [19; Chapter VIII, Section 2d]), we apply the results of [11; Theorem 7.1] and [16; Theorem A.6.1] and conclude that there exists a probability measure $\widetilde{P}$ being locally equivalent to $P$ with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$ and such that its density process is given by:

$$\frac{d\widetilde{P}|\mathcal{F}_t}{dP|\mathcal{F}_t} = \exp\left(\int_0^t \theta Z_s\, dW_s - \frac{1}{2}\int_0^t \theta^2 Z_s^2\, ds\right) \tag{14}$$

for all $t \ge 0$. Thus, by Girsanov's theorem (see e.g. [11; Theorem 6.3] or [12; Theorem 8.6.4]) it follows that the process $\widetilde{W} = (\widetilde{W}_t)_{t \ge 0}$ defined in (9) is a standard Wiener process under the measure $\widetilde{P}$. By using (11) it can be verified that $Z = (Z_t)_{t \ge 0}$ is a time-homogeneous (strong) Markov process under $\widetilde{P}$ with respect to its natural filtration, which coincides with $(\mathcal{F}_t)_{t \ge 0}$.

Observe that (14) also implies that for any finite stopping time $\tau$ with respect to $(\mathcal{F}_t)_{t \ge 0}$ the restriction $\widetilde{P} \,|\, \mathcal{F}_\tau$ is equivalent to $P \,|\, \mathcal{F}_\tau$. Then, using the explicit expressions (1) and (5) as well as the assumption $x = 0$, we obtain the following representation:

$$\frac{d\widetilde{P}|\mathcal{F}_\tau}{dP|\mathcal{F}_\tau} = e^{X_\tau} \tag{15}$$

for all finite stopping times $\tau$. It therefore follows that for computing the value (8) we can consider the following optimal stopping problem for the Markov process $Z$ given by:

$$V_*(z) = \sup_\tau \widetilde{E}_z\left[e^{-\delta\tau}\big(Ke^{-Z_\tau} - 1\big)^+\right], \tag{16}$$

where $\widetilde{P}_z$ denotes the law of the diffusion process started at the point $z > 0$ and solving equation (10), and the supremum is taken over all finite stopping times of $Z$. Thus, we may say that the deviation process $Z$ plays the role of a *sufficient statistic* in the optimal stopping problem (16). We will search for an optimal stopping time in (16) of the following form:

$$\tau_* = \inf\{t \ge 0 \,|\, Z_t \le B_*\}, \tag{17}$$

where $B_*$ is the largest number from $0 < z \le \log K$ such that $V_*(z) = Ke^{-z} - 1$. The point $B_*$ is called an *optimal stopping boundary.* Note that if $K \le 1$ and $z > 0$ then the problem (16) becomes trivial, so that we further assume that $K > 1$.

2.3. Standard arguments based on the application of Itô's formula (see e.g. [12; Theorem 7.3.3]) imply that in this case the infinitesimal generator $\mathbb{L}$ of the process $Z$ acts on a function $F \in C^2(0,\infty)$ like:

$$(\mathbb{L}F)(z) = -(3\theta^2/2)\, z\,(z + 2\lambda/(3\theta^2))\, F'(z) + (\theta^2/2)\, z^2\, F''(z) \tag{18}$$

for all $z > 0$. In order to find the unknown value function $V_*(z)$ from (16) and the unknown boundary $B_*$ from (17), we refer to the general theory of optimal stopping problems for continuous time Markov processes (see e.g. [6] and [18; Section III.8]) and formulate the following *free-boundary problem*:

$$(\mathbb{L}V)(z) = \delta V(z) \quad \text{for} \quad z > B \tag{19}$$

$$V(B+) = Ke^{-B} - 1 \quad (\textit{continuous fit}) \tag{20}$$

$$V(z) = (Ke^{-z} - 1)^+ \quad \text{for} \quad z < B \tag{21}$$

$$V(z) > (Ke^{-z} - 1)^+ \quad \text{for} \quad z > B \tag{22}$$

where $0 < B \le \log K$ and (20) plays the role of an instantaneous-stopping condition. Observe that the superharmonic characterization of the value function (see [3] and [18]) implies that $V_*(z)$ is the smallest function satisfying (19)-(22) with the boundary $B_*$. Because of the continuity of the process $Z$ we also assume that the following condition holds:

$$V'(B+) = -Ke^{-B} \quad (\textit{smooth fit}). \tag{23}$$

# 3 Solution of the free-boundary problem

Let us now derive a solution to the free-boundary problem formulated above.

3.1. By means of straightforward calculations it can be checked that equation (19) has the general solution:

$$V(z) = C_1\, z^{\gamma} U\left(\gamma, 2\gamma - \frac{2\lambda}{\theta^2}; 3z\right) + C_2\, z^{\gamma} L\left(-\gamma, 2\gamma - \frac{2\lambda}{\theta^2} - 1; 3z\right), \tag{24}$$

where $C_1$ and $C_2$ are some arbitrary constants and $\gamma$ is given by:

$$\gamma = \frac{1}{2} + \frac{\lambda}{\theta^2} + \sqrt{\left(\frac{1}{2} + \frac{\lambda}{\theta^2}\right)^2 + \frac{2\delta}{\theta^2}}. \tag{25}$$

Here $U(a,b;z)$ is the confluent hypergeometric function, which admits the integral representation:

$$U(a,b;z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}\, t^{a-1} (1+t)^{b-a-1}\, dt \tag{26}$$

for $a > 0$ and $b > 1$ (see e.g. [1; Chapter XIII] or [2; Chapter VI] with a different parametrization), and $L(a,b;z)$ is the generalized Laguerre polynomial function defined by:

$$L(a,b;z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+b+1)}{\Gamma(b+k+1)\Gamma(a-k+1)} \frac{(-1)^k z^k}{k!} = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}\, {}_1F_1(a, -b-1; z) \tag{27}$$

(see e.g. [1; Chapter XXII] or [2; Chapter X]), where ${}_1F_1$ is the Kummer confluent hypergeometric function and $\Gamma$ denotes the Euler Gamma function.

It thus follows that in (24) we have $C_2 = 0$, since otherwise $V(z) \to \pm\infty$ as $z \to \infty$, which should be excluded due to the obvious fact that the value function (16) is decreasing and bounded for all $z > 0$. Hence, imposing conditions (20) and (23) on the function (24), we obtain the following equalities:

$$C_1 \, B^\gamma U\left(\gamma, 2\gamma - \frac{2\lambda}{\theta^2}; 3B\right) = Ke^{-B} - 1 \tag{28}$$

$$\gamma C_1 \, B^{\gamma-1} U\left(\gamma, 2\gamma - \frac{2\lambda}{\theta^2}; 3B\right) - 3\gamma C_1 \, B^\gamma U\left(\gamma + 1, 2\gamma - \frac{2\lambda}{\theta^2} + 1; 3B\right) = -Ke^{-B}. \tag{29}$$

By solving equations (28)-(29) it therefore follows that the solution of system (19)-(20)+(23) is given by:

$$V(z; B_*) = \big(Ke^{-B_*} - 1\big)\Big(\frac{z}{B_*}\Big)^\gamma \frac{U(\gamma, 2\gamma - 2\lambda/\theta^2; 3z)}{U(\gamma, 2\gamma - 2\lambda/\theta^2; 3B_*)} \tag{30}$$

for all $z > B_*$, where $B_*$ satisfies the transcendental equation:

$$\gamma \, \frac{3B \, U(\gamma + 1, 2\gamma - 2\lambda/\theta^2 + 1; 3B)}{U(\gamma, 2\gamma - 2\lambda/\theta^2; 3B)} - \gamma = \frac{KBe^{-B}}{Ke^{-B} - 1}. \tag{31}$$

3.2. In order to prove the existence and uniqueness of the solution of equation (31) on the interval $(0, \log K)$, let us denote by $G(B)$ the left-hand side and by $H(B)$ the right-hand side of equation (31). Then $H(B)$ is a strictly increasing function on $(0, \log K)$ with $H(0+) = 0$ and $H(\log K-) = \infty$. Thus, if we deduce that $G(B)$ is a decreasing function on $(0, \log K)$ such that $G(0+) > 0$, then we will be able to conclude that there exists a unique solution $B_*$ of equation (31) on the interval $(0, \log K)$.

To prove $G(0+) > 0$, let us note that by applying the change-of-variable formula to (26) it follows that:

$$U(a, b; z) = \frac{z^{1-b}}{\Gamma(a)} \int_0^\infty e^{-u} u^{a-1} (u + z)^{b-a-1} \, du, \tag{32}$$

which directly implies:

$$\lim_{z \downarrow 0} \frac{z \, U(a+1, b+1; z)}{U(a, b; z)} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+1)\Gamma(b-1)} = \frac{b-1}{a} \tag{33}$$

for any $b > a + 1 > 1$ fixed. Hence, inserting $a = \gamma$ and $b = 2\gamma - 2\lambda/\theta^2$ as well as $z = 3B$ into (33), for the left-hand side of (31) we get:

$$\lim_{B \downarrow 0} G(B) = \lim_{B \downarrow 0} \gamma \, \frac{3B \, U(\gamma + 1, 2\gamma - 2\lambda/\theta^2 + 1; 3B)}{U(\gamma, 2\gamma - 2\lambda/\theta^2; 3B)} - \gamma = \gamma - \frac{2\lambda}{\theta^2} - 1 > 0. \tag{34}$$

To derive the monotonicity of $G(B)$, let us observe that from representation (32) it follows that:

$$\frac{z \, U(a+1, b+1; z)}{U(a, b; z)} = \frac{\int_0^\infty e^{-u} u^a (u + z)^{b-a-1} \, du}{a \int_0^\infty e^{-u} u^{a-1} (u + z)^{b-a-1} \, du} = \frac{\int_0^\infty u (u + z) f_z(u) \, du}{a \int_0^\infty (u + z) f_z(u) \, du}, \tag{35}$$

where $f_z(u) = C(z) e^{-u} u^{a-1} (u+z)^{b-a-2}$, $u \geq 0$, is a probability density with some normalizing constant $C(z)$ for any $b > a+1 > 1$ fixed. Then, applying the Cauchy-Schwarz or Jensen inequality and taking into account the fact that $b-a-1>0$, we obtain:

$$
\begin{aligned}
&\frac{d}{dz}\left(\frac{z\, U(a+1,b+1;z)}{U(a,b;z)}\right) \qquad (36)\\
&= \frac{(b-a-1)\left(\int_0^\infty u f_z(u)\,du \int_0^\infty (u+z) f_z(u)\,du - \int_0^\infty u(u+z) f_z(u)\,du \int_0^\infty f_z(u)\,du\right)}{\left(a\int_0^\infty (u+z) f_z(u)\,du\right)^2}\\
&\leq \frac{(b-a-1)\left(\int_0^\infty u^2 f_z(u)\,du + z\int_0^\infty u f_z(u)\,du - \int_0^\infty u(u+z) f_z(u)\,du \int_0^\infty f_z(u)\,du\right)}{\left(a\int_0^\infty (u+z) f_z(u)\,du\right)^2} = 0.
\end{aligned}
$$

Thus, setting $a=\gamma$ and $b = 2\gamma - 2\lambda/\theta^2$ as well as $z = 3B$, we may conclude that $G(B)$, being the left-hand side of (31), is decreasing on $(0, \log K)$. This completes the proof of uniqueness.

3.3. So far, we have seen that $V(z) = V(z; B_*)$ satisfies equation (19), and conditions (20) and (23) hold with $B = B_*$. Let us now show that inequality (22) is also satisfied. For this, we take logarithms on both sides of (22) and observe that, in view of equality (20) and the fact that $V(z;B_*)$ is positive, it suffices to verify the inequality:

$$
\frac{d}{dz}\log V(z;B_*) > \frac{d}{dz}\log(Ke^{-z}-1) \tag{37}
$$

for $B_* < z < \log K$. By using the definition of $V(z;B_*)$ in (30) and (31), it is straightforward to see that inequality (37) is equivalent to:

$$
\gamma\,\frac{3z\, U(\gamma+1, 2\gamma-2\lambda/\theta^2+1; 3z)}{U(\gamma, 2\gamma-2\lambda/\theta^2; 3z)} - \gamma < \frac{Kze^{-z}}{Ke^{-z}-1} \tag{38}
$$

for $B_* < z < \log K$. Thus, following the arguments above and using the monotonicity properties of the functions $G(z)$ and $H(z)$ (coinciding with the left-hand and right-hand sides of (38)) on the interval $(0, \log K)$, we may conclude that inequality (38) holds for $B_* < z < \log K$. The latter fact directly implies that (22) is satisfied with $V(z) = V(z;B_*)$ and $B = B_*$.

# 4 Main result and proof

We are now in a position to formulate and prove the main assertion of the paper.

**Theorem 4.1.** *Let the process $Z$ be given by (10)-(11) with $z>0$ and assume that $K>1$. Then the value function of the problem (16) takes the form:*

$$
V_*(z) = \begin{cases} V(z;B_*), & \text{if} \quad z > B_* \\ Ke^{-z}-1, & \text{if} \quad z \leq B_* \end{cases} \tag{39}
$$

*and the optimal stopping time $\tau_*$ has the structure (17), where the function $V(z;B_*)$ is given by (30) and the boundary $B_*$ is the unique solution of the transcendental equation (31).*

**Proof.** It remains to show that the function (39) coincides with the value function (16) and that the stopping time $\tau_*$ from (17) with the boundary $B_*$ specified above is optimal. Let us denote by $V(z)$ the right-hand side of the expression (39). It follows by construction from the previous section that the function $V(z)$ solves the system (19)-(22), and condition (23) is satisfied. Thus, applying Itô's formula to $e^{-\delta t}V(Z_t)$, we obtain:

$$e^{-\delta t}\, V(Z_t) = V(z) + \int_0^t e^{-\delta s}\, (\mathbb{L}V - \delta V)(Z_s)\, ds + \widetilde{M}_t, \tag{40}$$

where the process $(\widetilde{M}_t)_{t\geq 0}$ defined by:

$$\widetilde{M}_t = \int_0^t e^{-\delta s}\, V'(Z_s)\, \theta Z_s\, d\widetilde{W}_s \tag{41}$$

is a continuous local martingale under $\widetilde{P}_z$. Observe that the time spent by the process $Z$ at the boundary $B_*$ is of Lebesgue measure zero, which allows to extend $(\mathbb{L}V - \delta V)(z)$ arbitrarily to $z = B_*$.

Due to the properties (20)-(23), a Taylor expansion shows that $V''(B_*-) \leq V''(B_*+)$ holds, which by the form of the generator in (18) directly implies that $(\mathbb{L}V - \delta V)(B_*-) \leq (\mathbb{L}V - \delta V)(B_*+) = 0$. Moreover, it can be checked that:

$$\frac{d}{dz}(\mathbb{L}V - \delta V)(z) = -(\mathbb{L}V - \delta V)(z) + (4\theta^2 + \lambda)Ke^{-z} + \delta \tag{42}$$

for all $0 < z < B_*$, from where we may conclude that $(\mathbb{L}V - \delta V)(z)$ is increasing and thus negative on $(0, B_*)$. This together with (19) yields $(\mathbb{L}V - \delta V)(z) \leq 0$ for all $z > 0$. From expression (40) it therefore follows that the inequalities:

$$e^{-\delta\tau}\left(Ke^{-Z_\tau} - 1\right)^+ \leq e^{-\delta\tau}\, V(Z_\tau) \leq V(z) + \widetilde{M}_\tau \tag{43}$$

hold for any finite stopping time $\tau$ of the process $Z$ started at $z > 0$. Let $(\sigma_n)_{n\in\mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process $(\widetilde{M}_t)_{t\geq 0}$. Taking in (43) the expectation with respect to the measure $\widetilde{P}_z$, by means of the optional sampling theorem (see e.g. [7; Theorem I.1.39] or [16; Theorem II.3.2]), we get:

$$\begin{aligned} &\widetilde{E}_z\left[e^{-\delta(\tau\wedge\sigma_n)}\left(Ke^{-Z_{\tau\wedge\sigma_n}} - 1\right)^+\right] \leq \widetilde{E}_z\left[e^{-\delta(\tau\wedge\sigma_n)}\, V(Z_{\tau\wedge\sigma_n})\right] \\ &\leq V(z) + \widetilde{E}_z\left[\widetilde{M}_{\tau\wedge\sigma_n}\right] = V(z) \end{aligned} \tag{44}$$

for all $z > 0$. Hence, letting $n \to \infty$ and using Fatou's lemma, we obtain that for any finite stopping time $\tau$ the inequalities:

$$\widetilde{E}_z\left[e^{-\delta\tau}\left(Ke^{-Z_\tau} - 1\right)^+\right] \leq \widetilde{E}_z\left[e^{-\delta\tau}\, V(Z_\tau)\right] \leq V(z) \tag{45}$$

are satisfied for all $z > 0$.

By virtue of the fact that the function $V(z)$ together with the boundary $B_*$ satisfy the system (19)-(22), by the structure of the stopping time $\tau_*$ in (17) and by expression (40) it follows that the equality:

$$e^{-\delta(\tau_*\wedge\sigma_n)}\, V(Z_{\tau_*\wedge\sigma_n}) = V(z) + \widetilde{M}_{\tau_*\wedge\sigma_n} \tag{46}$$

holds. Then, using the expression (43) and the fact that the function $V(z)$ is decreasing, we infer the inequalities:

$$-V(z) \le \widetilde{M}_{\tau_* \wedge \sigma_n} \le V(B_* \wedge z) - V(z) \tag{47}$$

for all $z > 0$. Note that from (11) it follows that $Z_t$ tends to zero as $t \to \infty$ ($\widetilde{P}_z$-a.s.), and the latter fact implies that for the stopping time (17) we have $\widetilde{P}_z[\tau_* < \infty] = 1$ for all $z > 0$. Hence, letting $n \to \infty$ in (46) and using conditions (20)-(21) as well as the property $V(B_* \wedge z) < \infty$, we can apply the Lebesgue dominated convergence theorem to obtain the equality:

$$\widetilde{E}_z \left[ e^{-\delta \tau_*} \left( K e^{-Z_{\tau_*}} - 1 \right)^+ \right] = V(z) \tag{48}$$

for all $z > 0$, which together with (45) directly implies the desired assertion. □

**Remark 4.2.** Let us briefly consider the dependence of the solution on the *deviation parameter* $\lambda$, which reflects the impact of the delay. For this, let us denote by $V_*(z;\lambda)$ the value function from (16) and by $B_*(\lambda)$ the optimal stopping boundary from (17), where we underline the dependence on $\lambda$. Then, by the comparison theorem for stochastic differential equations applied to (10) and by the structure of the value function (16) it follows that $V_*(z;\lambda)$ increases in $\lambda$. Hence, a simple comparison argument yields that $B_*(\lambda)$ decreases in $\lambda$. The intuition behind these properties is that the deviation $Z$ is likely to be much smaller when the weighted average $\lambda Y$ is mainly taken from recent values (i.e. when $\lambda$ is large). In that case, the solution $X$ of equation (1) converges to zero more slowly, and we should await a lower optimal deviation level $B_*(\lambda)$ before exercising the option in view of the discounted payoff in (8).

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