Perpetual American options in a diffusion model with piecewise-linear coefficients

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We derive closed form solutions to the discounted optimal stopping problems related to the pricing of the perpetual American standard put and call options in an extension of the Black-Merton-Scholes model with piecewise-constant dividend and volatility rates. The method of proof is based on the reduction of the initial optimal stopping problems to the associated free-boundary problems and the subsequent martingale verification using a local time-space formula. We present explicit algorithms to determine the constant hitting thresholds for the underlying asset price process, which provide the optimal exercise boundaries for the options.

1 Introduction

The main aim of this paper is to present closed form solutions to the discounted optimal stopping problems of (2.3) for the process S defined in (2.1)-(2.2). These problems are related to the option pricing theory in mathematical finance and insurance, where the process S can describe the price of a risky asset (e.g. the value of a company) on a financial market. In that case, the values of (2.3) can be interpreted as fair prices of the perpetual American standard put and call options in a diffusion model with piecewise-linear coefficients. Such problems were first studied by McKean [14], who proved the optimality of the first time at which the price of

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the underlying risky asset, modelled by a geometric Brownian motion, hits a constant threshold (see also Shiryaev [21; Chapter VIII; Section 2a], Peskir and Shiryaev [19; Chapter VII; Section 25], and Detemple [7] for an extensive overview of other related results in the area). Mordecki [15]-[16], Asmussen, Avram and Pistorius [4], and Alili and Kyprianou [3] proved the optimality of the constant hitting threshold strategies for the underlying process and derived closed form expressions for the values of these optimal stopping problems in several exponential Lévy models. Some associated optimal stopping games for such processes were recently studied by Baurdoux and Kyprianou [5] among others.

The model defined in (2.1)-(2.2) is related to the framework of the so-called local models of stochastic volatility proposed by Dupire [8] and Derman and Kani [6], in which the diffusion coefficients depended on both the time and the current state of the underlying risky asset price process. Apart from easy calibration features, such extensions of the classical model with constant coefficients remained within complete market setting in which any contingent claim can be replicated by an admissible self-financing portfolio strategy, based on the underlying asset and the riskless bank account only. More recently, Ekström [9]-[10] found explicit values for the rational prices of the perpetual American options and investigated their properties in some diffusion models with time- and state-dependent volatility coefficients. The call-put duality for perpetual American options was studied by Alfonsi and Jourdain [1]-[2] within a local volatility and constant dividend yield framework. Villeneuve [22] proposed a model with both the volatility and dividend yield coefficients depending on the underlying price process and investigated sufficient conditions on the payoff functions ensuring the optimality of the constant threshold exercise strategies for the perpetual American options. Using a geometric approach, Lu [13] presented a solution of the optimal stopping problem related to the perpetual American put option in a dividend-free model with piecewise-constant volatility rate. He also studied the inverse problem of recovering the volatility rate of such type from the perpetual put option prices, initiated by Ekström and Hobson [11] within the general local volatility framework.

The purpose of this paper is to derive explicit expressions for values of one-dimensional optimal stopping problems for diffusion processes with both piecewise-linear drift and diffusion coefficients. Such values correspond to the rational prices of perpetual American standard put and call options in an extension of the Black-Merton-Scholes model for underlying dividend paying assets with both piecewise-constant dividend and volatility rates. It is assumed that these rates change their values at the times at which the underlying asset price process crosses some prescribed constant levels under the risk-neutral probability measure. Such a situation may appear in the case in which either the firm issuing the asset decides to change the dividend rate paid to stockholders or the volatility rate of the asset changes from one value to another at the times at which the market price crosses certain levels. These levels can have both statistical and psychological nature depending on the strategies of market participants. This model represents another example of local models of stochastic dividend and volatility, in which the related coefficients depend on the current state of the underlying asset price process and provides an approximation of the corresponding diffusion models with continuous coefficients studied in [9]-[10], [1]-[2], and [22]. A linear version of this diffusion model was proposed by Radner and Shepp [20] with the aim of solving some stochastic optimal impulse control problems. We present explicit algorithms to determine the constant hitting thresholds for the underlying diffusion process, which provide the optimal exercise boundaries for the options. Based on solving the associated free-boundary problems, our approach should allow to handle optimal stopping problems with more complicated payoffs than the ones of put and call options, within the general diffusion framework of both piecewise-linear drift and diffusion coefficients.

The paper is organized as follows. In Section 2, we formulate the perpetual American put and call option pricing optimal stopping problems in the diffusion model described above and their associated ordinary differential free-boundary problems. In Section 3, we derive solutions to the resulting systems of arithmetic equations equivalent to the free-boundary problems for the put and call options, separately. In Section 4, we verify that the solutions of the free-boundary problems provide the solutions of the initial optimal stopping problems.

2 Preliminaries

In this section, we present the setting and notation of the perpetual American standard put and call option optimal stopping problems in a diffusion model with piecewise-linear coefficients. We also formulate the associated ordinary differential free-boundary problems.

2.1. Formulation of the problem. Let us consider a probability space $(\Omega, \mathcal{F}, P)$ carrying a standard one-dimensional Brownian motion $B = (B_t)_{t \ge 0}$. Assume that there exists a process $S = (S_t)_{t \ge 0}$ solving the stochastic differential equation

$$dS_t = \big(r - \Delta(S_t)\big)\, S_t\, dt + \Sigma(S_t)\, S_t\, dB_t \tag{2.1}$$

with $S_0 = s$, where the functions $\Delta(s)$ and $\Sigma(s)$ are defined by

$$\Delta(s) = \sum_{i=1}^{n} \delta_i\, I(L_{i-1} < s \le L_i) \quad \text{and} \quad \Sigma(s) = \sum_{i=1}^{n} \sigma_i\, I(L_{i-1} < s \le L_i) \tag{2.2}$$

for all $s > 0$ and some $0 = L_0 < L_1 < \ldots < L_{n-1} < L_n = \infty$, $n \in \mathbb{N}$, fixed, and $I(\cdot)$ denotes the indicator function. Suppose that the process S describes the risk-neutral dynamics of the price of a risky asset (e.g. the value of an issuing firm) paying dividends. Here, $r > 0$ represents the riskless interest rate, $\sigma_i > 0$ is the volatility rate, and $\delta_i S$ such that $0 < \delta_i < r$ is the dividend rate paid to stockholders, whenever S fluctuates within the interval $(L_{i-1}, L_i]$, for every $i = 1, \ldots, n$. Note that the stochastic differential equation in (2.1) admits a unique strong solution, and hence, S is a strong Markov process with respect to its natural filtration $(\mathcal{F}_t)_{t \geq 0}$ defined by $\mathcal{F}_t = \sigma(S_u \mid 0 \leq u \leq t)$, for all $t \geq 0$ (see, e.g. [23; Theorem 4], [12; Chapter 5] or [17; Chapter VII, Section 2]). A linear diffusion model with piecewise-constant coefficients was considered in [20].

The main purpose of this paper is to compute the value functions of the optimal stopping problems

$$V^*(s) = \sup_\tau E\big[e^{-r\tau}\,(K_1 - S_\tau) \vee 0\big] \quad \text{or} \quad V^*(s) = \sup_\tau E\big[e^{-r\tau}\,(S_\tau - K_2) \vee 0\big] \tag{2.3}$$

where the suprema are taken over all stopping times τ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Such values represent the rational (or no-arbitrage) prices of the perpetual American put and call options with strike prices $K_1, K_2 > 0$, respectively. Here, the expectations are taken with respect to the equivalent martingale measure, under which the dynamics of S started at $s > 0$ are given by (2.1), and we further denote $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$, for any $x, y \in \mathbb{R}$. The left-hand problem of (2.3) was recently studied in [13] within the model of (2.1)-(2.2), under the assumption that $\Delta(s) = 0$.

2.2. Structure of the optimal stopping times. It follows from the general theory of optimal stopping for Markov processes (see, e.g. [19; Chapter I, Section 2]) that the optimal stopping times in the problems of (2.3) are given by

$$\tau^* = \inf\{t \geq 0 \,|\, V^*(S_t) = (K_1 - S_t) \vee 0\} \quad \text{or} \quad \tau^* = \inf\{t \geq 0 \,|\, V^*(S_t) = (S_t - K_2) \vee 0\} \tag{2.4}$$

whenever they exist. The latter fact means that the process S should be stopped at the first times at which it exits certain open intervals called the continuation regions. In this view, we further search for optimal stopping times of the problems of (2.3) in the form

$$\tau^* = \inf\{t \geq 0 \,|\, S_t \leq a^*\} \quad \text{or} \quad \tau^* = \inf\{t \geq 0 \,|\, S_t \geq b^*\} \tag{2.5}$$

for some $0 < a^* \leq K_1$ and $b^* \geq K_2$ to be determined. We also assume that the optimal stopping boundaries satisfy the conditions $L_{j-1} < a^* \leq L_j$ and $L_{m-1} < b^* \leq L_m$, for certain $j, m = 1, \ldots, n$ to be specified.

2.3. The free-boundary problems. It can be shown by means of standard arguments (see, e.g. [12; Chapter V, Section 5.1] or [17; Chapter VII, Section 7.3]) that the infinitesimal operator $\mathbb{L}$ of the process S acts on an arbitrary twice continuously differentiable locally bounded function $F(s)$ according to the rule

$$(\mathbb{L}F)(s) = (r - \delta_i)\, s\, F'(s) + \frac{\sigma_i^2}{2}\, s^2\, F''(s) \quad \text{for} \quad L_{i-1} < s \le L_i \tag{2.6}$$

where we set $F'(L_i) = F'(L_i-)$ and $F''(L_i) = F''(L_i-)$, for every $i = 1, \ldots, n$. In order to find explicit expressions for the unknown value functions $V^*(s)$ from (2.3) and the unknown boundaries a^* or b^* from (2.5), we may use the results of the general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [19; Chapter IV, Section 8]). We formulate the associated free-boundary problems

$$(\mathbb{L}V)(s) = rV(s) \quad \text{for} \quad s > a \quad \text{or} \quad s < b \quad \text{and such that} \quad s \neq L_i, \quad i = j, \ldots, m-1 \tag{2.7}$$

$$V(a+) = K_1 - a \quad \text{or} \quad V(b-) = b - K_2 \quad (\textit{instantaneous stopping}) \tag{2.8}$$

$$V'(a+) = -1 \quad \text{or} \quad V'(b-) = 1 \quad (\textit{smooth fit}) \tag{2.9}$$

$$V(s) = K_1 - s \quad \text{for} \quad s < a \quad \text{or} \quad V(s) = s - K_2 \quad \text{for} \quad s > b \tag{2.10}$$

$$V(s) > (K_1 - s) \vee 0 \quad \text{for} \quad s > a \quad \text{or} \quad V(s) > (s - K_2) \vee 0 \quad \text{for} \quad s < b \tag{2.11}$$

$$(\mathbb{L}V)(s) < rV(s) \quad \text{for} \quad s < a \quad \text{or} \quad s > b \tag{2.12}$$

for some $0 < a \le K_1$ or $b \ge K_2$ fixed, in the case of put or call option, respectively. Here, the conditions of (2.8) and (2.9) are used to specify the solutions of the free-boundary problems which are related to the optimal stopping problems in (2.3).

3 Solution of the free-boundary problem

In this section, we derive solutions to the free-boundary problems formulated above for the cases of put and call option, separately, and prove the uniqueness of solutions of the related arithmetic equations for optimal stopping boundaries.

3.1. The equivalent system of arithmetic equations. We first note that the general solution of the second order ordinary differential equation in (2.7) is given by

$$V(s) = \sum_{i=1}^{n} \Big(C_i^+\, s^{\gamma_i^+} + C_i^-\, s^{\gamma_i^-} \Big)\, I(L_{i-1} < s \le L_i) \tag{3.1}$$

where C_i^+ and C_i^- are some arbitrary constants, and define

$$\gamma_i^{\pm} = \frac{1}{2} - \frac{r - \delta_i}{\sigma_i^2} \pm \sqrt{\left(\frac{1}{2} - \frac{r - \delta_i}{\sigma_i^2}\right)^2 + \frac{2r}{\sigma_i^2}} \tag{3.2}$$

so that $\gamma_i^- < 0 < 1 < \gamma_i^+$ holds for every $i = 1, \ldots, n$. Hence, applying the instantaneous-stopping and smooth-fit conditions from (2.8)-(2.9) to the function in (3.1) and using the fact that the value function $V^*(s)$ is continuously differentiable for $s < a$ or $s > b$ in the case of put or call option, respectively, we get that the equalities

$$C_j^+ \, a^{\gamma_j^+} + C_j^- \, a^{\gamma_j^-} = K_1 - a \quad \text{or} \quad C_m^+ \, b^{\gamma_m^+} + C_m^- \, b^{\gamma_m^-} = b - K_2 \tag{3.3}$$

$$C_j^+ \, \gamma_j^+ \, a^{\gamma_j^+} + C_j^- \, \gamma_j^- \, a^{\gamma_j^-} = -a \quad \text{or} \quad C_m^+ \, \gamma_m^+ \, b^{\gamma_m^+} + C_m^- \, \gamma_m^- \, b^{\gamma_m^-} = b \tag{3.4}$$

$$C_{i-1}^+ \, L_{i-1}^{\gamma_{i-1}^+} + C_{i-1}^- \, L_{i-1}^{\gamma_{i-1}^-} = C_i^+ \, L_{i-1}^{\gamma_i^+} + C_i^- \, L_{i-1}^{\gamma_i^-} \quad \text{for} \quad i = j+1, \ldots, m \tag{3.5}$$

$$C_{i-1}^+ \, \gamma_{i-1}^+ \, L_{i-1}^{\gamma_{i-1}^+} + C_{i-1}^- \, \gamma_{i-1}^- \, L_{i-1}^{\gamma_{i-1}^-} = C_i^+ \, \gamma_i^+ \, L_{i-1}^{\gamma_i^+} + C_i^- \, \gamma_i^- \, L_{i-1}^{\gamma_i^-} \quad \text{for} \quad i = j+1, \ldots, m \tag{3.6}$$

hold for some $L_{j-1} < a \le L_j \wedge K_1$ or $K_2 \vee L_{m-1} < b \le L_m$. It thus follows that the function

$$V(s; a, b) \tag{3.7}$$
$$= \sum_{i=j}^{m} \Big(C_i^+(a, b, L_j, \ldots, L_{m-1}) \, s^{\gamma_i^+} + C_i^-(a, b, L_j, \ldots, L_{m-1}) \, s^{\gamma_i^-} \Big) \, I(L_{i-1} < s \le L_i)$$

satisfies the system in (2.7)-(2.9) with some $C_i^+(a, b, L_j, \ldots, L_{m-1})$ and $C_i^-(a, b, L_j, \ldots, L_{m-1})$ to be specified by the system in (3.3)-(3.6), for some $L_{j-1} < a \le L_j \wedge K_1$ or $K_2 \vee L_{m-1} < b \le L_m$.

3.2. Solution for the case of put option. Observe that we should also have $C_n^+ = 0$ in (3.1) when the left-hand part of the system in (2.7)-(2.12) is realised with $m = n$, since otherwise $V(s) \to \pm\infty$, that must be excluded by virtue of the obvious fact that the value function in (2.3) is bounded under $s \uparrow \infty$. In this case, solving the system of equations in the left-hand part of (3.3)-(3.4), we get that its solution is given by

$$C_j^+(a) = \frac{I_j^+(a)}{\gamma_j^+ - \gamma_j^-} \quad \text{and} \quad C_j^-(a) = \frac{I_j^-(a)}{\gamma_j^+ - \gamma_j^-} \tag{3.8}$$

with

$$I_j^+(a) = \frac{(\gamma_j^- - 1)a - \gamma_j^- K_1}{a^{\gamma_j^+}} \quad \text{and} \quad I_j^-(a) = \frac{(1 - \gamma_j^+)a + \gamma_j^+ K_1}{a^{\gamma_j^-}} \tag{3.9}$$

for all $L_{j-1} < a \le L_j \wedge K_1$. Then, solving the system of equations in (3.5)-(3.6), we get the recursive expressions

$$C_i^+ L_i^{\gamma_i^+} \equiv C_i^+ L_{i-1}^{\gamma_i^+} \Big(\frac{L_i}{L_{i-1}} \Big)^{\gamma_i^+} = \bigg[C_{i-1}^+ L_{i-1}^{\gamma_{i-1}^+} \, \frac{\gamma_{i-1}^+ - \gamma_i^-}{\gamma_i^+ - \gamma_i^-} + C_{i-1}^- L_{i-1}^{\gamma_{i-1}^-} \, \frac{\gamma_{i-1}^- - \gamma_i^-}{\gamma_i^+ - \gamma_i^-} \bigg] \Big(\frac{L_i}{L_{i-1}} \Big)^{\gamma_i^+} \tag{3.10}$$

and

$$C_i^- L_i^{\gamma_i^-} \equiv C_i^- L_{i-1}^{\gamma_i^-} \Big(\frac{L_i}{L_{i-1}} \Big)^{\gamma_i^-} = \bigg[C_{i-1}^+ L_{i-1}^{\gamma_{i-1}^+} \, \frac{\gamma_i^+ - \gamma_{i-1}^+}{\gamma_i^+ - \gamma_i^-} + C_{i-1}^- L_{i-1}^{\gamma_{i-1}^-} \, \frac{\gamma_i^+ - \gamma_{i-1}^-}{\gamma_i^+ - \gamma_i^-} \bigg] \Big(\frac{L_i}{L_{i-1}} \Big)^{\gamma_i^-} \tag{3.11}$$

for any $i = j+1, \ldots, n-1$. Hence, using the expressions in (3.8), we obtain that the expressions

$$C_i^+ = \frac{\mathrm{sgn}(\gamma_i^+)}{\gamma_i^+ - \gamma_i^-} \sum I_j^\pm(a) \frac{L_j^{\gamma_j^\pm}}{L_{i-1}^{\gamma_i^+}} \frac{\gamma_{i-1}^\pm - \gamma_i^-}{\gamma_{i-1}^+ - \gamma_{i-1}^-} \prod_{k=j+1}^{i-1} \mathrm{sgn}(\gamma_k^\pm) \frac{\gamma_{k-1}^\pm - \gamma_k^\mp}{\gamma_{k-1}^+ - \gamma_{k-1}^-} \Big(\frac{L_k}{L_{k-1}} \Big)^{\gamma_k^\pm} \tag{3.12}$$

and

$$C_i^- = \frac{\mathrm{sgn}(\gamma_i^-)}{\gamma_i^+ - \gamma_i^-} \sum I_j^\pm(a) \frac{L_j^{\gamma_j^\pm}}{L_{i-1}^{\gamma_i^-}} \frac{\gamma_{i-1}^\pm - \gamma_i^+}{\gamma_{i-1}^+ - \gamma_{i-1}^-} \prod_{k=j+1}^{i-1} \mathrm{sgn}(\gamma_k^\pm) \frac{\gamma_{k-1}^\pm - \gamma_k^\mp}{\gamma_{k-1}^+ - \gamma_{k-1}^-} \Big(\frac{L_k}{L_{k-1}} \Big)^{\gamma_k^\pm} \tag{3.13}$$

hold for any $i = j+1, \ldots, n-1$, while using the equalities in (3.12)-(3.13), we also get from (3.5) that the expression

$$C_n^- = \frac{1}{\gamma_{n-1}^+ - \gamma_{n-1}^-} \sum I_j^\pm(a) \frac{L_j^{\gamma_j^\pm}}{L_{n-1}^{\gamma_n^-}} \prod_{i=j+1}^{n-1} \mathrm{sgn}(\gamma_i^\pm) \frac{\gamma_{i-1}^\pm - \gamma_i^\mp}{\gamma_{i-1}^+ - \gamma_{i-1}^-} \Big(\frac{L_i}{L_{i-1}} \Big)^{\gamma_i^\pm} \tag{3.14}$$

holds. The sums in (3.12)-(3.14) as well as in (3.18)-(3.19) below should be read according to the rule

$$\begin{aligned} &\sum G(I_j^\pm(a), \gamma_j^\pm, \gamma_j^\mp, \gamma_{j+1}^\pm, \gamma_{j+1}^\mp, \ldots, \gamma_n^\pm, \gamma_n^\mp) \\ &\equiv G(I_j^+(a), \gamma_j^+, \gamma_j^-, \gamma_{j+1}^+, \gamma_{j+1}^-, \ldots, \gamma_n^+, \gamma_n^-) + G(I_j^-(a), \gamma_j^-, \gamma_j^+, \gamma_{j+1}^+, \gamma_{j+1}^-, \ldots, \gamma_n^+, \gamma_n^-) \\ &+ G(I_j^+(a), \gamma_j^+, \gamma_j^-, \gamma_{j+1}^-, \gamma_{j+1}^+, \ldots, \gamma_n^+, \gamma_n^-) + G(I_j^-(a) \gamma_j^-, \gamma_j^+, \gamma_{j+1}^-, \gamma_{j+1}^+, \ldots, \gamma_n^+, \gamma_n^-) + \cdots \\ &\cdots + G(I_j^+(a), \gamma_j^+, \gamma_j^-, \gamma_{j+1}^+, \gamma_{j+1}^-, \ldots, \gamma_n^-, \gamma_n^+) + G(I_j^-(a), \gamma_j^-, \gamma_j^+, \gamma_{j+1}^+, \gamma_{j+1}^-, \ldots, \gamma_n^-, \gamma_n^+) \\ &+ G(I_j^+(a), \gamma_j^+, \gamma_j^-, \gamma_{j+1}^-, \gamma_{j+1}^+, \ldots, \gamma_n^-, \gamma_n^+) + G(I_j^-(a), \gamma_j^-, \gamma_j^+, \gamma_{j+1}^-, \gamma_{j+1}^+, \ldots, \gamma_n^-, \gamma_n^+) \end{aligned} \tag{3.15}$$

for any measurable function $G(I_j^\pm(a), \gamma_j^\pm, \gamma_j^\mp, \gamma_{j+1}^\pm, \gamma_{j+1}^\mp, \ldots, \gamma_n^\pm, \gamma_n^\mp)$. Thus, taking into account the fact that $C_n^+ = 0$, we obtain from the left-hand part of the system in (3.5)-(3.6) that the equality

$$C_{n-1}^+ \, (\gamma_n^- - \gamma_{n-1}^+) \, L_{n-1}^{\gamma_{n-1}^+} = C_{n-1}^- \, (\gamma_{n-1}^- - \gamma_n^-) \, L_{n-1}^{\gamma_{n-1}^-} \tag{3.16}$$

is satisfied. Using the expressions in (3.12)-(3.13), we can therefore conclude that the equation in (3.16) takes the form

$$I_j^+(a) L_j^{\gamma_j^+} Q_j^+ = I_j^-(a) L_j^{\gamma_j^-} Q_j^- \tag{3.17}$$

for $L_{j-1} < a \le L_j \wedge K_1$, with

$$Q_j^+ = \mathrm{sgn}(\gamma_j^+) \sum \frac{(\gamma_j^+ - \gamma_{j+1}^\mp)(\gamma_{n-1}^\pm - \gamma_n^-)}{\gamma_{n-1}^\pm - \gamma_n^\mp} \prod_{i=j+1}^{n-1} \mathrm{sgn}(\gamma_i^\pm)(\gamma_i^\pm - \gamma_{i+1}^\mp) \Big(\frac{L_i}{L_{i-1}} \Big)^{\gamma_i^\pm} \tag{3.18}$$

and

$$Q_j^- = \mathrm{sgn}(\gamma_j^-) \sum \frac{(\gamma_j^- - \gamma_{j+1}^\mp)(\gamma_{n-1}^\pm - \gamma_n^-)}{\gamma_{n-1}^\pm - \gamma_n^\mp} \prod_{i=j+1}^{n-1} \mathrm{sgn}(\gamma_i^\pm)(\gamma_i^\pm - \gamma_{i+1}^\mp) \Big(\frac{L_i}{L_{i-1}} \Big)^{\gamma_i^\pm} \tag{3.19}$$

for every $j = 1, \ldots, n-2$, while $Q_{n-1}^{+} = \gamma_{n-1}^{+} - \gamma_n^{-}$, $Q_{n-1}^{-} = \gamma_n^{-} - \gamma_{n-1}^{-}$, $Q_n^{+} = \gamma_n^{+} - \gamma_n^{-}$, and $Q_n^{-} = 0$.

In order to prove the uniqueness of solution of the equation in (3.17), we observe that the derivatives of the functions in (3.9) are given by the expressions

$$I_j^{+\prime}(a) = \frac{(\gamma_j^{+} - 1)(\gamma_j^{-} - 1)(\overline{K}_{1,j} - a)}{a^{\gamma_j^{+}+1}} < 0, \quad I_j^{-\prime}(a) = \frac{(\gamma_j^{+} - 1)(\gamma_j^{-} - 1)(a - \overline{K}_{1,j})}{a^{\gamma_j^{-}+1}} > 0 \tag{3.20}$$

for all $0 < L_{j-1} < a \leq L_j \wedge K_1 < \overline{K}_{1,j}$, with

$$\overline{K}_{1,j} = \frac{\gamma_j^{+} \gamma_j^{-} K_1}{(\gamma_j^{+} - 1)(\gamma_j^{-} - 1)} \equiv \frac{r K_1}{\delta_j} > K_1 \tag{3.21}$$

so that the function $I_j^{+}(a)$ decreases and the function $I_j^{-}(a)$ increases on the interval $(L_{j-1}, L_j \wedge K_1]$. Hence, the equation in (3.17) admits a unique solution if and only if the inequalities

$$\frac{I_j^{+}(L_{j-1}) L_j^{\gamma_j^{+}}}{Q_j^{-}} > \frac{I_j^{-}(L_{j-1}) L_j^{\gamma_j^{-}}}{Q_j^{+}} \quad \text{and} \quad \frac{I_j^{+}(L_j \wedge K_1) L_j^{\gamma_j^{+}}}{Q_j^{-}} \leq \frac{I_j^{-}(L_j \wedge K_1) L_j^{\gamma_j^{-}}}{Q_j^{+}} \tag{3.22}$$

hold with Q_j^{+} and Q_j^{-} given by the expressions in (3.18)-(3.19).

In order to prove the inequalities in (3.22) above, we first assume that $L_{j-1} < L_j < K_1$ holds. Then, it can be verified by means of the induction principle that the inequalities $Q_j^{+} > 0$, $\gamma_j^{+} Q_j^{-} < -\gamma_j^{-} Q_j^{+}$ and $\gamma_j^{+} Q_j^{-} L_{j-1}^{\gamma_j^{+} - \gamma_j^{-}} < -\gamma_j^{-} Q_j^{+} L_j^{\gamma_j^{+} - \gamma_j^{-}}$ are satisfied for every $j = 1, \ldots, n$. Hence, it is shown using straightforward computations that there exists a unique solution a_j^* of the equation in (3.17) such that $L_{j-1} < a_j^* \leq L_j$ if and only if the relationship $\mu_{j-1} L_{j-1} \vee L_j < K_1 \leq \mu_j L_j$ holds with

$$\mu_j = \frac{(\gamma_j^{+} - 1)\, Q_j^{-} + (\gamma_j^{-} - 1)\, Q_j^{+}}{\gamma_j^{+}\, Q_j^{-} + \gamma_j^{-}\, Q_j^{+}} > 1 \tag{3.23}$$

for every $j = 1, \ldots, n$, with Q_j^{+} and Q_j^{-} given by (3.18)-(3.19). Thus, the assumption $L_{j-1} < a_j^* \leq L_j$ can equivalently be replaced by the property $\mu_{j-1} L_{j-1} \vee L_j < K_1 \leq \mu_j L_j$. Observe that the latter inequalities can hold for K_1 if either $\mu_{j-1} L_{j-1} \leq L_j$, or $L_{j-1} < L_j < \mu_{j-1} L_{j-1}$ when $Q_j^{-} \geq 0$, or $L_{j-1} < \mu_{j-1} L_{j-1} / \mu_j < L_j < \mu_{j-1} L_{j-1}$ when $Q_j^{-} < 0$. Note that the property $\mu_{j-1} L_{j-1} \vee L_j < K_1 \leq \mu_j L_j$ does not hold, when $L_{j-1} < L_j \leq \mu_{j-1} L_{j-1} / \mu_j < \mu_{j-1} L_{j-1}$ and $Q_j^{-} < 0$, in which case there is no solution a_j^* of the equation in (3.17) in the interval $(L_{j-1}, L_j]$.

Let us now assume that $L_{j-1} < K_1 \leq L_j$ holds. In this case, it can be checked by means of the induction principle that the inequality $-Q_j^{-} < Q_j^{+}$ is satisfied for every $j = 1, \ldots, n$. Hence, it is shown by means of straightforward computations and using the relationships between Q_j^{+} and Q_j^{-} referred above that the equation in (3.17) admits a unique solution a_j^* such that $L_{j-1} < a_j^* \leq K_1$ if and only if the relationship $\mu_{j-1} L_{j-1} < K_1 \leq L_j$ holds with μ_j given by

(3.23). Thus, the assumption $L_{j-1} < a_j^* \le K_1$ can equivalently be replaced by the property $\mu_{j-1} L_{j-1} < K_1 \le L_j$. Note that when the latter inequalities fail to hold, there is no solution a_j^* of the equation in (3.17) in the interval $(L_{j-1}, K_1]$.

Summarising the facts proved above, we can therefore formulate the following algorithm to specify the location interval $(L_{j-1}, L_j]$ for the solution a^* of the equation in (3.17), based on the corresponding relationships between K_1, L_i and μ_j for $i, j = 1, \ldots, n$ referred above. Without loss of generality, let us thus assume that the strike price satisfies $L_{k-1} < K_1 \le L_k$, so that there exist k possible intervals in which the solution a^* can be located. We can therefore start the following forward procedure started with $j = 1$:

(1) (searching for a solution in the interval $(L_0, L_1]$):

(a) if $K_1 \le \mu_1 L_1$ holds, then there exists a solution $0 = L_0 < a_1^* \le L_1$ of the equation in (3.17) for $j = 1$, proceed with checking whether $Q_i^- < 0$ and $\mu_i L_i < K_1$ holds for some $i = 2, \ldots, k-1$, and in the latter case, continue with step **(i+1)**,

(b) if $\mu_1 L_1 < K_1$ holds, then continue with step **(2)**;

⋮

(j) (searching for a solution in the interval $(L_{j-1}, L_j]$, for $j = 2, \ldots, k-1$):

(a) if $K_1 \le \mu_j L_j$ holds, then there exist a solution $L_{j-1} < a_j^* \le L_j$ of the equation in (3.17), proceed with checking whether $Q_i^- < 0$ and $\mu_i L_i < K_1$ holds for some $i = j+1, \ldots, k-1$, and in that case, continue with step **(i+1)**,

(b) if $\mu_j L_j < K_1$ holds, then continue with step **(j+1)**;

⋮

(k) (searching for a solution in the interval $(L_{k-1}, K_1]$):

in this case, $K_1 \le L_k$ holds by assumption, and thus, there exist a solution $L_{k-1} < a_k^* \le K_1$ of the equation in (3.17) for $j = k$.

Note that, after finding a solution $L_{j-1} < a_j^* \le L_j$ of the equation in (3.17) at step **(j)**, part (a), for some $j = 1, \ldots, k-2$, we can get another solution $L_{i-1} < a_i^* \le L_i$ only if $\mu_l L_l < \mu_{l-1} L_{l-1}$ holds for some $l = j+1, \ldots, k-1$ and $l < i$. Such a situation can occur at part (b) of any step while searching for a solution in the appropriate interval. However, these facts do not make any impact on the procedure described above, which establishes the existence of at least one solution $L_{j-1} < a_j^* \le L_j \wedge K_1$ of the equation in (3.17), for a certain $j = 1, \ldots, k$. We further denote by a^* the minimum over such solutions a_j^*, $j = 1, \ldots, k$, whenever they exist, and construct the corresponding solution $V(s; a^*)$ of the form in (3.7), which will dominate the other possible solutions of the second-order ordinary differential equation in (2.7), satisfying the conditions in (2.8)-(2.9) with the boundaries a_j^*, $j = 1, \ldots, k$. The latter fact can be shown by means of the arguments similar to the ones used in [19; Chapter VI, Remark 23.2] and [19; Chapter VI, Theorem 24.1], or by verifying directly.

3.3. Solution for the case of call option. Observe that we should also have $C_1^- = 0$ in (3.1) when the right-hand part of the system in (2.7)-(2.12) is realised with $j = 1$, since $V(s) \to \pm\infty$ otherwise, that must be excluded by virtue of the obvious fact that the value function in (2.3) is bounded under $s \downarrow 0$. In this case, solving the system of equations in the right-hand part of (3.3)-(3.4), we get that its solution is given by

$$C_m^+(b) = \frac{J_m^+(b)}{\gamma_m^+ - \gamma_m^-} \quad \text{and} \quad C_m^-(b) = \frac{J_m^-(b)}{\gamma_m^+ - \gamma_m^-} \tag{3.24}$$

with

$$J_m^+(b) = \frac{(1-\gamma_m^-)b + \gamma_m^- K_2}{b^{\gamma_m^+}} \quad \text{and} \quad J_m^-(b) = \frac{(\gamma_m^+ - 1)b - \gamma_m^+ K_2}{b^{\gamma_m^-}} \tag{3.25}$$

for all $K_2 \vee L_{m-1} < b \le L_m$. Then, solving the system of equations in (3.5)-(3.6), we obtain the recursive expressions

$$C_i^+ L_{i-1}^{\gamma_i^+} \equiv C_i^+ L_i^{\gamma_i^+} \Big(\frac{L_{i-1}}{L_i}\Big)^{\gamma_i^+} = \bigg[C_{i+1}^+ L_i^{\gamma_{i+1}^+} \frac{\gamma_{i+1}^+ - \gamma_i^-}{\gamma_i^+ - \gamma_i^-} + C_{i+1}^- L_i^{\gamma_{i+1}^-} \frac{\gamma_{i+1}^- - \gamma_i^-}{\gamma_i^+ - \gamma_i^-}\bigg] \Big(\frac{L_{i-1}}{L_i}\Big)^{\gamma_i^+} \tag{3.26}$$

and

$$C_i^- L_{i-1}^{\gamma_i^-} \equiv C_i^- L_i^{\gamma_i^-} \Big(\frac{L_{i-1}}{L_i}\Big)^{\gamma_i^-} = \bigg[C_{i+1}^+ L_i^{\gamma_{i+1}^+} \frac{\gamma_i^+ - \gamma_{i+1}^+}{\gamma_i^+ - \gamma_i^-} + C_{i+1}^- L_i^{\gamma_{i+1}^-} \frac{\gamma_i^+ - \gamma_{i+1}^-}{\gamma_i^+ - \gamma_i^-}\bigg] \Big(\frac{L_{i-1}}{L_i}\Big)^{\gamma_i^-} \tag{3.27}$$

for any $i = 2, \ldots, m-1$. Hence, using the expressions in (3.24), we obtain that the expressions

$$C_i^+ = \frac{\mathrm{sgn}(\gamma_i^+)}{\gamma_i^+ - \gamma_i^-} \sum J_m^\pm(b) \frac{L_{m-1}^{\gamma_m^\pm}}{L_i^{\gamma_i^+}} \frac{\gamma_{i+1}^\pm - \gamma_i^-}{\gamma_{i+1}^+ - \gamma_{i+1}^-} \prod_{k=i+1}^{m-1} \mathrm{sgn}(\gamma_k^\pm) \frac{\gamma_{k+1}^\pm - \gamma_k^\mp}{\gamma_{k+1}^+ - \gamma_{k+1}^-} \Big(\frac{L_{k-1}}{L_k}\Big)^{\gamma_k^\pm} \tag{3.28}$$

and

$$C_i^- = \frac{\mathrm{sgn}(\gamma_i^-)}{\gamma_i^+ - \gamma_i^-} \sum J_m^\pm(b) \frac{L_{m-1}^{\gamma_m^\pm}}{L_i^{\gamma_i^-}} \frac{\gamma_{i+1}^\pm - \gamma_i^+}{\gamma_{i+1}^+ - \gamma_{i+1}^-} \prod_{k=i+1}^{m-1} \mathrm{sgn}(\gamma_k^\pm) \frac{\gamma_{k+1}^\pm - \gamma_k^\mp}{\gamma_{k+1}^+ - \gamma_{k+1}^-} \Big(\frac{L_{k-1}}{L_k}\Big)^{\gamma_k^\pm} \tag{3.29}$$

hold for any $i = 2, \ldots, m-1$, while using the equalities in (3.28)-(3.29), we also get from (3.5) that the expression

$$C_1^+ = \frac{1}{\gamma_2^+ - \gamma_2^-} \sum J_m^\pm(b) \frac{L_{m-1}^{\gamma_m^\pm}}{L_1^{\gamma_1^+}} \prod_{i=2}^{m-1} \mathrm{sgn}(\gamma_i^\pm) \frac{\gamma_{i+1}^\pm - \gamma_i^\mp}{\gamma_{i+1}^+ - \gamma_{i+1}^-} \Big(\frac{L_{i-1}}{L_i}\Big)^{\gamma_i^\pm} \tag{3.30}$$

holds. The sums in (3.28)-(3.30) as well as in (3.34)-(3.35) below should be read according to the rule

$$\sum H(J_m^{\pm}(b), \gamma_m^{\pm}, \gamma_m^{\mp}, \gamma_{m-1}^{\pm}, \gamma_{m-1}^{\mp}, \ldots, \gamma_1^{\pm}, \gamma_1^{\mp}) \tag{3.31}$$
$$\equiv H(J_m^{+}(b), \gamma_m^{+}, \gamma_m^{-}, \gamma_{m-1}^{+}, \gamma_{m-1}^{-}, \ldots, \gamma_1^{+}, \gamma_1^{-}) + H(J_m^{-}(b), \gamma_m^{-}, \gamma_m^{+}, \gamma_{m-1}^{+}, \gamma_{m-1}^{-}, \ldots, \gamma_1^{+}, \gamma_1^{-})$$
$$+ H(J_m^{+}(b), \gamma_m^{+}, \gamma_m^{-}, \gamma_{m-1}^{-}, \gamma_{m-1}^{+}, \ldots, \gamma_1^{+}, \gamma_1^{-}) + H(J_m^{-}(b), \gamma_m^{-}, \gamma_m^{+}, \gamma_{m-1}^{-}, \gamma_{m-1}^{+}, \ldots, \gamma_1^{+}, \gamma_1^{-}) + \cdots$$
$$\cdots + H(J_m^{+}(b), \gamma_m^{+}, \gamma_m^{-}, \gamma_{m-1}^{+}, \gamma_{m-1}^{-}, \ldots, \gamma_1^{-}, \gamma_1^{+}) + H(J_m^{-}(b), \gamma_m^{-}, \gamma_m^{+}, \gamma_{m-1}^{+}, \gamma_{m-1}^{-}, \ldots, \gamma_1^{-}, \gamma_1^{+})$$
$$+ H(J_m^{+}(b), \gamma_m^{+}, \gamma_m^{-}, \gamma_{m-1}^{-}, \gamma_{m-1}^{+}, \ldots, \gamma_1^{-}, \gamma_1^{+}) + H(J_m^{-}(b), \gamma_m^{-}, \gamma_m^{+}, \gamma_{m-1}^{-}, \gamma_{m-1}^{+}, \ldots, \gamma_1^{-}, \gamma_1^{+})$$

for any measurable function $H(J_m^{\pm}(b), \gamma_m^{\pm}, \gamma_m^{\mp}, \gamma_{m-1}^{\pm}, \gamma_{m-1}^{\mp}, \ldots, \gamma_1^{\pm}, \gamma_1^{\mp})$. Thus, taking into account the fact that $C_1^- = 0$, we obtain from the right-hand part of the system in (3.5)-(3.6) that the equality

$$C_2^{+}(\gamma_1^{+} - \gamma_2^{+}) L_1^{\gamma_2^{+}} = C_2^{-}(\gamma_2^{-} - \gamma_1^{+}) L_1^{\gamma_2^{-}} \tag{3.32}$$

is satisfied. Using the expressions in (3.28)-(3.29), we can therefore conclude that the equation in (3.32) takes the form

$$J_m^{+}(b)\, L_{m-1}^{\gamma_m^{+}}\, R_m^{+} = J_m^{-}(b)\, L_{m-1}^{\gamma_m^{-}}\, R_m^{-} \tag{3.33}$$

for $K_2 \vee L_{m-1} < b \le L_m$, with

$$R_m^{+} = \mathrm{sgn}(\gamma_m^{+}) \sum \frac{(\gamma_m^{+} - \gamma_{m-1}^{\mp})(\gamma_2^{\pm} - \gamma_1^{+})}{\gamma_2^{\pm} - \gamma_1^{\mp}} \prod_{i=2}^{m-1} \mathrm{sgn}(\gamma_i^{\pm})(\gamma_i^{\pm} - \gamma_{i-1}^{\mp}) \Big(\frac{L_{i-1}}{L_i} \Big)^{\gamma_i^{\pm}} \tag{3.34}$$

and

$$R_m^{-} = \mathrm{sgn}(\gamma_m^{-}) \sum \frac{(\gamma_m^{-} - \gamma_{m-1}^{\mp})(\gamma_2^{\pm} - \gamma_1^{+})}{\gamma_2^{\pm} - \gamma_1^{\mp}} \prod_{i=2}^{m-1} \mathrm{sgn}(\gamma_i^{\pm})(\gamma_i^{\pm} - \gamma_{i-1}^{\mp}) \Big(\frac{L_{i-1}}{L_i} \Big)^{\gamma_i^{\pm}} \tag{3.35}$$

for every $m = 3, \ldots, n$, while $R_2^- = \gamma_1^+ - \gamma_2^-$, $R_2^+ = \gamma_2^+ - \gamma_1^+$, $R_1^- = \gamma_1^+ - \gamma_1^-$, and $R_1^+ = 0$.

In order to prove the uniqueness of solution of the equation in (3.33), we observe that the derivatives of the functions in (3.25) are given by the expressions

$$J_m^{+\prime}(b) = \frac{(\gamma_m^{+} - 1)(\gamma_m^{-} - 1)(b - \overline{K}_2)}{b^{\gamma_m^{+}+1}} < 0, \quad J_m^{-\prime}(b) = \frac{(\gamma_m^{+} - 1)(\gamma_m^{-} - 1)(\overline{K}_2 - b)}{b^{\gamma_m^{-}+1}} > 0 \tag{3.36}$$

for all $0 < \overline{K}_{2,m} \vee L_{m-1} < b \le L_m$, with

$$\overline{K}_{2,m} = \frac{\gamma_m^{+} \gamma_m^{-} K_2}{(\gamma_m^{+} - 1)(\gamma_m^{-} - 1)} \equiv \frac{r K_2}{\delta_m} > K_2 \tag{3.37}$$

so that the function $J_m^+(b)$ decreases and the function $J_m^-(b)$ increases on the interval $(\overline{K}_{2,m} \vee L_{m-1}, L_m]$. Hence, the equation in (3.33) admits a unique solution if and only if the inequalities

$$\frac{J_m^{+}(\overline{K}_{2,m} \vee L_{m-1}) L_{m-1}^{\gamma_m^{+}}}{R_m^{-}} > \frac{J_m^{-}(\overline{K}_{2,m} \vee L_{m-1}) L_{m-1}^{\gamma_m^{-}}}{R_m^{+}}, \quad \frac{J_m^{+}(L_m) L_{m-1}^{\gamma_m^{+}}}{R_m^{-}} \le \frac{J_m^{-}(L_m) L_{m-1}^{\gamma_m^{-}}}{R_m^{+}} \tag{3.38}$$

hold with R_m^+ and R_m^- given by the expressions in (3.34)-(3.35).

In order to prove the inequalities in (3.38) above, we first assume that $\overline{K}_{2,m} \le L_{m-1} < L_m$ holds. Then, it can be verified by means of the induction principle that the inequalities $R_m^- > 0$, $\gamma_m^+ R_m^- > -\gamma_m^- R_m^+$ and $\gamma_m^+ R_m^- L_m^{\gamma_m^+ - \gamma_m^-} > -\gamma_m^- R_m^+ L_{m-1}^{\gamma_m^+ - \gamma_m^-}$ are satisfied for every $m = 1, \ldots, n$. Hence, it is shown using straightforward computations that there exists a unique solution b_m^* of the equation in (3.33) such that $L_{m-1} < b_m^* \le L_m$ if and only if the relationship $\lambda_m L_{m-1} < K_2 \le \lambda_{m+1} L_m \wedge \delta_m L_{m-1}/r$ holds with

$$\lambda_m = \frac{(\gamma_m^+ - 1)\, R_m^- + (\gamma_m^- - 1)\, R_m^+}{\gamma_m^+\, R_m^- + \gamma_m^-\, R_m^+} < 1 \tag{3.39}$$

for every $m = 1, \ldots, n$, with R_m^+ and R_m^- given by (3.34)-(3.35). Thus, the assumption $L_{m-1} < b_m^* \le L_m$ can equivalently be replaced by the property $\lambda_m L_{m-1} < K_2 \le \lambda_{m+1} L_m \wedge \delta_m L_{m-1}/r$. Observe that the latter inequalities can hold for K_2 if either $L_m \le \delta_m L_{m-1}/(\lambda_{m+1} r)$ when $\xi_m \le 0$, or $\lambda_m L_{m-1}/\lambda_{m+1} < L_m \le \delta_m L_{m-1}/(\lambda_{m+1} r)$ when $0 < \xi_m < 1$, or $\delta_m L_{m-1}/(\lambda_{m+1} r) < L_m$ when $\xi_m < 1$, where ξ_m is given by

$$\xi_m = -\frac{\gamma_m^-(\gamma_m^- - 1) R_m^+}{\gamma_m^+(\gamma_m^+ - 1) R_m^-} \tag{3.40}$$

for every $m = 1, \ldots, n$. However, the property $\lambda_m L_{m-1} < K_2 \le \lambda_{m+1} L_m \wedge \delta_m L_{m-1}/r$ does not hold when either $L_{m-1} < L_m \le \lambda_m L_{m-1}/\lambda_{m+1}$ and $0 < \xi_m < 1$, or $\xi_m \ge 1$ holds, therefore there is no solution b_m^* of the equation in (3.33) in the interval $(L_{m-1}, L_m]$.

Let us now assume that $L_{m-1} < \overline{K}_{2,m} < L_m$ holds. In this case, it is shown by means of straightforward computations and using the relationships between R_m^+ and R_m^- referred above that the equation in (3.33) admits a unique solution b_m^* such that $\overline{K}_{2,m} < b_m^* \le L_m$ if and only if the relationship

$$\frac{\delta_m L_{m-1}}{r} \vee \frac{\delta_m \nu_m L_{m-1}}{r} < K_2 \le \lambda_{m+1} L_m \wedge \frac{\delta_m L_m}{r} \tag{3.41}$$

holds with λ_m given by (3.39) and $\nu_m = {\xi_m}^{1/(\gamma_m^+ - \gamma_m^-)} I(\xi_m > 0)$, for every $m = 1, \ldots, n$, where ξ_m has the form of (3.40). We also observe that the inequalities in (3.41) can hold for K_2 if either $\delta_m L_{m-1}/(\lambda_{m+1} r) < L_m$ when $\xi_m \le 1$, or $\delta_m \nu_m L_{m-1}/(\lambda_{m+1} r) < L_m$ when $\xi_m > 1$. However, the property of (3.41) does not hold if either $L_{m-1} < L_m \le \delta_m L_{m-1}/(\lambda_{m+1} r)$ when $\xi_m \le 1$, or $\nu_m L_{m-1} < L_m \le \delta_m \nu_m L_{m-1}/(\lambda_{m+1} r)$ when $\xi_m > 1$, or $L_m \le \nu_m L_{m-1}$ when $\xi_m > 1$ holds. Note that the last two cases are separated due to the fact that the property $\lambda_{m+1} L_m < \delta_m \nu_m L_{m-1}/r$ excludes $\delta_m L_m / r < \delta_m \nu_m L_{m-1}/r$ and vice versa.

Summarising the facts proved above, we can therefore formulate the following algorithm to specify the location interval $(L_{m-1}, L_m]$ for the solution b^* of the equation in (3.33), based on the corresponding relationships between K_2, r, δ_i, L_i, λ_m, ξ_m, and ν_m for $i, m = 1, \ldots, n$.

Without loss of generality, let us thus assume that the strike price satisfies $L_{k-1} < K_2 \leq L_k$, so that there exists $n-k+1$ possible intervals in which the solution b^* can be located. We can therefore start the following backward procedure started with $m = n$:

(n) (searching for a solution in the interval $(L_{n-1}, L_n]$):

(I) if $\delta_n L_{n-1}/r < K_2$ holds, then we look for a solution b_n^* in the smaller interval $(\overline{K}_{2,n}, L_n]$, when

(a) $\xi_n \leq 1$ holds, that yields the existence of a solution $\overline{K}_{2,n} < b_n^* \leq L_n$ of the equation in (3.33) for $m = n$, proceed with checking whether $\xi_i > 0$ and $K_2 \leq \lambda_i L_{i-1}$ holds for some $i = n-1, \dots, k+1$, and in that case, continue with step **(i-1)**,

(b) $\xi_n > 1$ and $\delta_n \nu_n L_{n-1}/r < K_2$ hold, that yields the existence of a solution $\overline{K}_{2,n} < b_n^* \leq L_n$ of the equation in (3.33) for $m = n$, proceed with checking whether $\xi_i > 0$ and $K_2 \leq \lambda_i L_{i-1}$ hold for some $i = n, \dots, k+1$, and in that case, continue with step **(i-1)**,

(c) $\xi_n > 1$ and $K_2 \leq \delta_n \nu_n L_{n-1}/r$ holds, proceed with checking whether $\xi_i > 0$ and $K_2 \leq \lambda_i L_{i-1}$ hold for some $i = n, \dots, k+1$, and in that case, continue with step **(i-1)**,

(II) if $K_2 \leq \delta_n L_{n-1}/r$ holds, then we observe that if

(a) $\lambda_n L_{n-1} < K_2$ holds, then there exist a solution $\overline{K}_{2,n} < b_n^* \leq L_n$ of the equation in (3.33) for $m = n$, then proceed with checking whether $\xi_i > 0$ and $K_2 \leq \lambda_i L_{i-1}$ holds for some $i = n-1, \dots, k+1$, and in that case, continue with step **(i-1)**,

(b) $K_2 \leq \lambda_n L_{n-1}$ holds, then continue with step **(n-1)**;

⋮

(m) (searching for a solution in the interval $(L_{m-1}, L_m]$, for $m = n-1, \dots, k+1$):

(I) if $\delta_m L_m/r < K_2$ holds, then the interval $(L_{m-1}, L_m]$ belongs to the continuation region, and we proceed further, when

(a) $\lambda_m L_{m-1} < K_2$ holds, with checking whether $\xi_i > 0$ and $K_2 \leq \lambda_i L_{i-1}$ holds for some $i = m-1, \dots, k+1$, and in that case, continue with step **(i-1)**,

(b) $K_2 \leq \lambda_m L_{m-1}$ holds, continue with step **(m-1)**,

(II) if $\delta_m L_{m-1}/r < K_2 \leq \delta_m L_m/r$ holds, then we check for a solution b_m^* in the smaller interval $(\overline{K}_{2,m}, L_m]$, when

(a) $\xi_m \leq 1$ holds, that yields the existence of a solution $\overline{K}_{2,m} < b_m^* \leq L_m$ of the equation in (3.33), proceed with checking whether $\xi_i > 0$ and $K_2 \leq \lambda_i L_{i-1}$ hold for some $i = m-1, \dots, k+1$, and in that case, continue with step **(i-1)**,

(b) $\xi_m > 1$ and $\delta_m \nu_m L_{m-1}/r < K_2$ holds, that yields the existence of a solution $\overline{K}_{2,m} < b_m^* \leq L_m$ of the equation in (3.33), then proceed with checking whether

$\xi_i > 0$ and $K_2 \le \lambda_i L_{i-1}$ holds for some $i = m, \dots, k+1$, and in that case, continue with step **(i-1)**,

(c) $\xi_m > 1$ and $K_2 \le \delta_m \nu_m L_{m-1}/r$ holds, proceed with checking whether $\xi_i > 0$ and $K_2 \le \lambda_i L_{i-1}$ hold for some $i = m, \dots, k+1$, and in that case, continue with step **(i-1)**,

(III) if $K_2 \le \delta_m L_{m-1}/r$ holds, then observe that if

(a) $\lambda_m L_{m-1} < K_2$ holds, then there exist a solution $L_{m-1} < b^*_m \le L_m$ of the equation in (3.33), proceed with checking whether $\xi_i > 0$ and $K_2 \le \lambda_i L_{i-1}$ hold for some $i = m-1, \dots, k+1$, and in that case, continue with step **(i-1)**,

(b) $K_2 \le \lambda_m L_{m-1}$ holds, then continue with step **(m-1)**;

⋮

(k) (searching for a solution in the interval $(\overline{K}_{2,k}, L_k]$):

(I) if $\delta_k L_k / r < K_2$ holds, then the interval $(K_2, L_k]$ belongs to the continuation region,

(II) if $K_2 \le \delta_k L_k / r$ holds, then observe that if

(a) either $\xi_k \le 1$ or $\xi_k > 1$ with $\delta_k \nu_k L_{k-1}/r < K_2$ holds, then there exist a solution $\overline{K}_{2,k} < b^*_k \le L_k$ of the equation in (3.33) for $m = k$,

(b) $\xi_k > 1$ with $K_2 \le \delta_k \nu_k L_{k-1}/r$ holds, then there is no solution in the interval $(\overline{K}_{2,k}, L_k]$.

Note that, after finding a solution $L_{m-1} < b^*_m \le L_m$ of the equation in (3.33) at one of the parts of step **(m)**, for some $m = n, \dots, k+2$, we can get another solution $L_{i-1} < b^*_i \le L_i$ only if $\xi_l > 0$ and $K_2 \le \lambda_l L_{l-1}$ holds for some $l = m-1, \dots, k+1$ and $l > i$. However, these facts do not make any impact on the procedure described above, whenever we search for solutions $\overline{K}_{2,m} \vee L_{m-1} < b^*_m \le L_m$ of the equation in (3.33), for certain $m = n, \dots, k$. Moreover, we observe that the algorithm presented above shows explicitly that there exist possible situations in which there does not exist any solution of the equation in (3.33) in the interval $(\overline{K}_{2,m} \vee L_{m-1}, L_m]$, for any $m = n, \dots, k$. For instance, such a situation can occur at part (I)(c) of step **(n)**, under the conditions $\lambda_n L_{n-1} < K_2$ and $\xi_i < 0$, for all $i = n-1, \dots, k+1$. We further denote by b^* the maximum over such solutions b^*_m, $m = n, \dots, k$, whenever they exist, and set $b^* = \infty$ otherwise. We then construct the corresponding solution $V(s; b^*)$ of the form in (3.7), which will dominate the other possible solutions of the second-order ordinary differential equation in (2.7), satisfying the conditions in (2.8)-(2.9) with b^*_m, $m = n, \dots, k$.

4 Main results and proof

Taking into account the facts proved above, let us now formulate the main assertions of the paper.

Theorem 1 *Suppose that the price process S of the underlying risky asset is defined by (2.1)-(2.2), and let $0 = L_0 < L_1 < \ldots < L_{n-1} < L_n = \infty$, $n \in \mathbb{N}$, be some prescribed levels. Then, in the optimal stopping problems of (2.3), related to the perpetual American put and call options with strike prices $K_1, K_2 > 0$, the value functions are given by*

$$V^*(s) = \begin{cases} K_1 - s, & \text{if } s \le a^* \\ V(s; a^*), & \text{if } s > a^* \end{cases} \quad or \quad V^*(s) = \begin{cases} V(s; b^*), & \text{if } s < b^* \\ s - K_2, & \text{if } s \ge b^* \end{cases} \tag{4.1}$$

where the functions $V(s;a)$ and $V(s;b)$ and the optimal exercise time τ^* have the form of (3.7) and (2.5), respectively, and the optimal stopping boundaries a^* and b^* are specified as follows:

(i) in the put option case, the boundary a^* satisfies $L_{j-1} < a^* \le L_j \wedge K_1$ for a certain $j = 1, \ldots, n$, and it is specified as the minimal solution of the arithmetic equation in (3.17);

(ii) in the call option case, either the boundary b^* satisfies $\overline{K}_{2,m} \vee L_{m-1} < b^* \le L_m$ for a certain $m = 1, \ldots, n$, and it is specified as the maximal solution of the arithmetic equation in (3.33), or we have $m = n$ and $b^* = \infty$ and thus there is no optimal stopping boundary.

Since both parts of the assertion formulated above are proved in a similar way, we only give a proof for the problem related to the more complicated case of the perpetual American call option.

Proof of part (ii). In order to verify the assertion stated above, it remains to show that the function $V^*(s)$ defined in the right-hand part of (4.1) coincides with the value function in the right-hand part of (2.3), and that the stopping time τ^* in the right-hand part of (2.5) is optimal with b^* either being the maximal solution of the equation in (3.33) or $b^* = \infty$. For this, let us denote by $V(s)$ the right-hand side of the right-hand expression in (4.1). Then, applying the local time-space formula from [18] (see also [19; Chapter II, Section 3.5] for a summary of the related results as well as further references) and taking into account the smooth-fit condition in the right-hand part of (2.9), we get that the expression

$$\begin{aligned} e^{-rt} V(S_t) &= V(s) + M_t \\ &\quad + \int_0^t e^{-ru} (\mathbb{L}V - rV)(S_u)\, I(S_u \neq L_i, i = 1, \ldots, n-1, S_u \neq b^*)\, du \end{aligned} \tag{4.2}$$

holds, where the process $M = (M_t)_{t \ge 0}$ defined by

$$M_t = \int_0^t e^{-ru} V'(S_u)\, \Sigma(S_u)\, S_u\, dB_u \tag{4.3}$$

is a continuous square integrable martingale with respect to the probability measure P. The latter fact can easily be observed, since the derivative $V'(s)$ and $\Sigma(s)$ are bounded functions.

By means of straightforward calculations, similar to those of the previous section, it can be verified that the conditions in the right-hand parts of (2.11) and (2.12) hold with b^* either being the maximal solution of the equation in (3.33) or $b^* = \infty$. It is also shown using the comparison arguments for solutions of second-order ordinary differential equations that, in the former case, $V(s)$ represents the maximal solution of the equation in (2.7) satisfying the conditions in the right-hand parts of (2.8)-(2.9). These facts together with the condition in the right-hand part of (2.10) yield that $(\mathbb{L}V - rV)(s) \le 0$ holds for all $s \ne L_i$, $i = 1, \dots, n-1$, and $s \ne b^*$, as well as $V(s) \ge (s - K_2) \vee 0$ is satisfied for all $s > 0$. Moreover, since the time spent by the process S at the boundary b^* as well as at the levels L_i, $i = 1, \dots, n-1$, is of Lebesgue measure zero, the indicator which appears in the integral of (4.2) can be ignored. Hence, it follows from the expression in (4.2) that the inequalities

$$e^{-r(\tau \wedge t)} \, (S_{\tau \wedge t} - K_2) \vee 0 \le e^{-r(\tau \wedge t)} \, V(S_{\tau \wedge t}) \le V(s) + M_{\tau \wedge t} \tag{4.4}$$

hold for any stopping time τ of the process S started at $s > 0$. Then, taking the expectation with respect to P in (4.4), we get by means of Doob's optional sampling theorem (see, e.g. [12; Chapter I, Theorem 3.22]) that the inequalities

$$E\big[e^{-r(\tau \wedge t)} \, (S_{\tau \wedge t} - K_2) \vee 0\big] \le E\big[e^{-r(\tau \wedge t)} \, V(S_{\tau \wedge t})\big] \le V(s) + E\big[M_{\tau \wedge t}\big] = V(s) \tag{4.5}$$

hold for all $s > 0$. Thus, letting t go to infinity and using Fatou's lemma, we obtain

$$E\big[e^{-r\tau} \, (S_\tau - K_2) \vee 0\big] \le E\big[e^{-r\tau} \, V(S_\tau)\big] \le V(s) \tag{4.6}$$

for any stopping time τ and all $s > 0$. By virtue of the structure of the stopping time τ^* in the right-hand part of (2.5), it is readily seen that the equality in (4.6) holds with τ^* instead of τ when $s \ge b^*$.

It remains to show that the equality is attained in (4.6) when τ^* replaces τ for $s < b^*$. By virtue of the fact that the function $V(s; b^*)$ and the boundary b^* satisfy the conditions in the right-hand parts of (2.7) and (2.8), it follows from the expression in (4.2) and the structure of the stopping time τ^* in the right-hand part of (2.5) that the equality

$$e^{-r(\tau^* \wedge t)} \, V(S_{\tau^* \wedge t}) = V(s) + M_{\tau^* \wedge t} \tag{4.7}$$

is satisfied for all $s < b^*$, where the process M is defined in (4.3). Observe that the variable $e^{-r\tau^*}(S_{\tau^*} - K_2) \vee 0$ is equal to zero on the event $\{\tau^* = \infty\}$ (P-a.s.), and the process $(M_{\tau^* \wedge t})_{t \ge 0}$ is a uniformly integrable martingale. Therefore, taking the expectations with respect to P and letting t go to infinity, we can apply the Lebesgue dominated convergence for the expression in (4.7) to obtain the equalities

$$E\big[e^{-r\tau^*} \, (S_{\tau^*} - K_2) \vee 0\big] = E\big[e^{-r\tau^*} \, V(S_{\tau^*})\big] = V(s) \tag{4.8}$$

for all $s < b^*$. The latter, together with the inequality in (4.6), implies the fact that $V(s)$ coincides with the function $V^*(s)$ from the right-hand part of (2.3), and τ^* from the right-hand part of (2.5) is an optimal stopping time. □

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References

[1] Alfonsi, A. and Jourdain, B. (2008). General duality for perpetual American options. *International Journal of Theoretical and Applied Finance* **11** (545–566).

[2] Alfonsi, A. and Jourdain, B. (2009). A call-put duality for perpetual American options. *Nonlinear Differential Equations and Applications* **16** (523–554).

[3] Alili, L. and Kyprianou, A. E. (2005). Some remarks on first passage of Lévy processes, the American put and pasting principles. *Annals of Applied Probability* **15** (2062–2080).

[4] Asmussen, S., Avram, F. and Pistorius, M. (2003). Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and Applications* **109** (79–111).

[5] Baurdoux, E. J. and Kyprianou, A. E. (2008). The McKean stochastic game driven by a spectrally negative Lévy process. *Electronic Journal of Probability* **13**(8) (173–197).

[6] Derman, E. and Kani, I. (1994). Riding on a smile. *Risk* **7** (32–39).

[7] Detemple, J. (2006). *American-Style Derivatives: Valuation and Computation.* Chapman and Hall/CRC, Boca Raton.

[8] Dupire, B. (1994). Pricing with a smile. *Risk* **7** (18–20).

[9] Ekström, E. (2003). Perpetual American put options in a level-dependent volatility model. *Journal of Applied Probability* **40** (783–789).

[10] Ekström, E. (2004). Properties of American option prices. *Stochastic Processes and their Applications* **114** (265–278).

[11] Ekström, E. and Hobson, D. (2011). Recovering a time-homogeneous stock price process from perpetual option prices. *Annals of Applied Probability* **21**(3) (1102–1135).

[12] Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus.* (Second Edition) Springer, New York.

[13] Lu, B. (2010). Recovering a pecewise constant volatility from perpetual put option prices. *Journal of Applied Probability* **47** (680–692).

[14] McKean, H. P. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Industrial Management Review* **6** (32–39).

[15] Mordecki, E. (1999). Optimal stopping for a diffusion with jumps. *Finance and Stochastics* **3** (227–236).

[16] Mordecki, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance and Stochastics* **6**(4) (473–493).

[17] Øksendal, B. (1998). *Stochastic Differential Equations. An Introduction with Applications.* (Fifth Edition) Springer, Berlin.

[18] Peskir, G. (2005). A change-of-variable formula with local time on curves. *Journal of Theoretical Probability* **18** (499–535).

[19] Peskir, G. and Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Birkhäuser, Basel.

[20] Radner, R. and Shepp, L. A. (1996). Risk vs. profit potential: A model for corporate strategy. *Journal of Economic Dynamics and Control* **20** (1373–1393).

[21] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance.* World Scientific, Singapore.

[22] Villeneuve, S. (2007). On threshold strategies and the smooth-fit principle for optimal stopping problems. *Journal of Applied Probability* **44**(1) (181–198).

[23] Zvonkin, A. K. (1974). A transformation of the phase space of a diffusion process that removes the drift. *Mathematics of the USSR - Sbornik* **22**(1) (129–149).