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On the sequential testing problem for some diffusion processes

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We study the Bayesian problem of sequential testing of two simple hypotheses about the drift rate of an observable diffusion process. The optimal stopping time is found as the first time at which the posterior probability of one of the hypotheses exits a region restricted by two stochastic boundaries depending on the current observations. The proof is based on an embedding of the initial problem into a two-dimensional optimal stopping problem and the analysis of the associated parabolic-type free-boundary problem. We also show that the problem admits a closed form solution under certain nontrivial relations between the coefficients of the observable diffusion.

1. Introduction

The problem of sequential testing of two simple hypotheses about the drift rate coefficient $\mu(x)$ of an observable diffusion process is to determine as soon as possible and with minimal error probabilities if the true rate is either $\mu_0(x)$ or $\mu_1(x)$. This problem admits two different

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formulations. In the Bayesian formulation, it is assumed that the drift rate $\mu(x)$ has an *a priori* given distribution. The variational formulation does not involve any probabilistic assumption about the unknown coefficient $\mu(x)$. In this paper, we only study the Bayesian formulation.

By means of the Bayesian approach, Wald and Wolfowitz [24]-[25] proved the optimality of the classical sequential probability ratio test (SPRT) with constant stopping boundaries in the variational formulation of the problem, for sequences of i.i.d. observations. Dvoretzky, Kiefer and Wolfowitz [5] pointed out that if the (continuous time) likelihood-ratio process has stationary independent increments, then the SPRT with constant boundaries remains optimal in the variational problem. Mikhalevich [15] and Shiryaev [21] (see also [22; Chapter IV] or [19; Chapter VI]) obtained an explicit solution of the Bayesian problem of testing hypotheses about the constant drift rate of an observable Wiener process. The initial optimal stopping problem for the posterior probability of one of the hypotheses was reduced to the associated free-boundary problem for an ordinary differential operator. A complete proof of the statement of [5] (under some mild assumptions) was given by Irle and Schmitz [10], for the case in which the log-likelihood ratio has stationary independent increments. Peskir and Shiryaev [18] derived an explicit solution of the Bayesian problem of testing hypotheses about the constant intensity rate of an observable Poisson process. The associated free-boundary problem for a differential-difference operator was solved by means of the conditions of smooth and continuous fit. More recently, Dayanik and Sezer [4], and then Dayanik, Poor and Sezer [3] provided a solution of the Bayesian sequential (multi-)hypotheses testing problem for a general compound Poisson process. A finite time horizon version of the Wiener sequential testing problem was studied in Gapeev and Peskir [8].

In the present paper, we make an embedding of the initial Bayesian problem into an extended optimal stopping problem for a two-dimensional Markov diffusion process having the posterior probability of one of the hypotheses and the observations as its state space components. We show that the optimal stopping time is expressed as the first time at which the posterior probability process exits a region restricted by two stochastic boundaries depending on the current state of the observation process. This remark leads to the fact that the stopping boundaries for the associated SPRT, which turns out to be optimal in the corresponding variational formulation of the problem, can be no longer constant and depend on the current observations. We verify that the value function and the optimal stopping boundaries in the Bayesian formulation are characterised by means of the associated free-boundary problem for a second order partial differential operator. The latter turns out to be of parabolic type, since the observation process is a one-dimensional diffusion. We also derive a closed form solution of the resulting free-boundary problem for a special nontrivial subclass of observable diffusions.

The paper is organised as follows. In Section 2, for the initial Bayesian sequential testing problem, we construct a two-dimensional optimal stopping problem and formulate the associated free-boundary problem. We reduce the resulting parabolic-type partial differential operator to the normal form which is amenable for further considerations. In Section 3, applying the change-of-variable formula with local time on surfaces obtained by Peskir [17], we verify that the solution of the free-boundary problem, which satisfies certain additional conditions, provides the solution of the initial optimal stopping problem. In Section 4, we show that the value function admits an explicit representation in terms of the optimal stopping boundaries, which are uniquely determined by a coupled system of transcendental equations, under certain non-trivial relations between the coefficients of the observable diffusion. We also give some remarks on the optimality of the SPRT with stochastic boundaries depending on the current observations, in the corresponding variational formulation of the problem. The main result of the paper is stated in Theorem 3.2.

2. Preliminaries

In this section, we give the Bayesian formulation of the problem (see [22; Chapter IV, Section 2] or [19; Chapter VI, Section 21] for the case of Wiener processes) in which it is assumed that one observes a sample path of the diffusion process $X = (X_t)_{t \geq 0}$ with drift rate $\mu_0(x) + \theta(\mu_1(x) - \mu_0(x))$, where the random parameter θ may be 1 or 0 with probability π or $1 - \pi$, respectively. We also formulate the associated free-boundary problem.

2.1. Formulation of the problem. For a precise probabilistic formulation of the Bayesian sequential testing problem, suppose that all the considerations take place on a probability space $(\Omega, \mathcal{F}, P_\pi)$ where the probability measure P_π has the structure:

$$P_\pi = \pi P_1 + (1 - \pi) P_0 \tag{2.1}$$

for any $\pi \in [0,1]$. Let θ be a random variable taking two values 1 and 0 with probabilities $P_\pi(\theta=1)=\pi$ and $P_\pi(\theta=0)=1-\pi$, and let $W=(W_t)_{t\ge 0}$ be a standard Wiener process started at zero under P_π. It is assumed that θ and W are independent.

Suppose that we observe a continuous process $X=(X_t)_{t\ge 0}$ solving the stochastic differential equation:

$$dX_t = \big(\mu_0(X_t) + \theta\, (\mu_1(X_t) - \mu_0(X_t))\big)\, dt + \sigma(X_t)\, dW_t \quad (X_0 = x) \tag{2.2}$$

where $\mu_i(x)$, $i=0,1$, and $\sigma(x)>0$ are some continuously differentiable functions on $(0,\infty)$. For simplicity of exposition, we assume the state space of the process X to be the positive half line $(0,\infty)$, since that is the case in the examples considered below. It thus follows from [14; Theorem 4.6] that the equation in (2.2) admits a unique strong solution under $\theta=i$, and hence, $P_\pi(X\in\,\cdot\,|\,\theta=i)=P_i(X\in\,\cdot\,)$ is the distribution law of a time-homogeneous diffusion process started at some $x>0$, with diffusion coefficient $\sigma^2(x)$ and drift rate $\mu_i(x)$, for every $i=0,1$. Let π and $1-\pi$ play the role of prior probabilities of the simple statistical hypotheses:

$$H_1 : \theta = 1 \qquad \text{and} \qquad H_0 : \theta = 0 \tag{2.3}$$

respectively.

Being based upon the continuous observation of X, our task is to test sequentially the hypotheses H_1 and H_0 with a minimal loss. For this, we consider a sequential decision rule (τ,d) where τ is a stopping time of the observation process X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t=\sigma(X_s\,|\,0\le s\le t)$ of the process X, for $t\ge 0$) and d is an $\mathcal{F}_\tau$-measurable function taking on values 0 and 1. After stopping the observations at time τ, the terminal decision function d indicates which hypothesis should be accepted according to the following rule: if $d=1$ we accept H_1, and if $d=0$ we accept H_0. The problem consists of computing the Bayesian risk function:

$$V_*(\pi) = \inf_{(\tau,d)} \big(E_\pi \tau + aP(d=0,\theta=1) + bP(d=1,\theta=0)\big) \tag{2.4}$$

and finding the optimal decision rule (τ_*,d_*), called the π-Bayes decision rule, at which the infimum is attained in (2.4). Here, $E_\pi\tau$ is the average cost of the observations, and $aP_\pi(d=0,\theta=1)+bP_\pi(d=1,\theta=0)$ is the average loss due to a wrong terminal decision, where $a>0$ and $b>0$ are some given constants.

2.2. The posterior probability and innovation process. By means of the standard arguments in [22; pages 166-167], one can reduce the Bayesian problem of (2.4) to the optimal stopping problem:

$$V_*(\pi) = \inf_\tau E_\pi \big[\tau + G_{a,b}(\Pi_\tau) \big] \tag{2.5}$$

for the *posterior probability* process $\Pi = (\Pi_t)_{t \ge 0}$ defined by $\Pi_t = P_\pi(\theta = 1 \,|\, \mathcal{F}_t)$, for $t \ge 0$, with $P_\pi(\Pi_0 = \pi) = 1$. Here, $G_{a,b}(\pi) = a\pi \wedge b(1-\pi)$ for $\pi \in [0,1]$, and the optimal decision function is given by $d_* = 1$ if $\Pi_{\tau_*} \ge c$, and $d_* = 0$ if $\Pi_{\tau_*} < c$, where we set $c = b/(a+b)$.

Using the arguments based on Bayes' formula (see, e.g. [14; Theorem 7.23]), it is shown in [22; pages 180-181] that the posterior probability Π can be expressed as:

$$\Pi_t = \left(\frac{\pi}{1-\pi} \, L_t \right) \Big/ \left(1 + \frac{\pi}{1-\pi} \, L_t \right) \tag{2.6}$$

for any $\pi \in (0,1)$ fixed, where the *likelihood ratio* process $L = (L_t)_{t \ge 0}$ is defined as the Radon-Nikodým derivative:

$$L_t = \frac{d(P_1 \,|\, \mathcal{F}_t)}{d(P_0 \,|\, \mathcal{F}_t)} \tag{2.7}$$

for all $t \ge 0$. By means of Girsanov's theorem for diffusion-type processes [14; Theorem 7.19], we get that the process L admits the representation:

$$L_t = \exp\left(\int_0^t \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma^2(X_s)} \, dX_s - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_s) - \mu_0^2(X_s)}{\sigma^2(X_s)} \, ds \right) \tag{2.8}$$

for all $t \ge 0$. Then, applying Itô's formula (see, e.g. [14; Theorem 4.4] or [20; Chapter IV, Theorem 3.3]), we obtain that the process Π solves the stochastic differential equation:

$$d\Pi_t = \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)} \, \Pi_t (1 - \Pi_t) \, d\overline{W}_t \quad (\Pi_0 = \pi) \tag{2.9}$$

where the *innovation* process $\overline{W} = (\overline{W}_t)_{t \ge 0}$ defined by:

$$\overline{W}_t = \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \left(\frac{\mu_0(X_s)}{\sigma(X_s)} + \Pi_s \, \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \right) ds \tag{2.10}$$

is a standard Wiener process under the measure P_π with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$, according to P. Lévy's characterisation theorem (see, e.g. [14; Theorem 4.1] or [20; Chapter IV, Theorem 3.6]). It therefore follows from (2.10) that the process X admits the representation:

$$dX_t = \big(\mu_0(X_t) + \Pi_t \left(\mu_1(X_t) - \mu_0(X_t) \right) \big) \, dt + \sigma(X_t) \, d\overline{W}_t \quad (X_0 = x). \tag{2.11}$$

By virtue of Remark to [14; Theorem 4.6] (see also [16; Theorem 5.2.1]), we thus conclude that the process (Π, X) turns out to be a unique strong solution of the two-dimensional system of stochastic differential equations in (2.9) and (2.11). Hence, according to [16; Theorem 7.2.4], (Π, X) is a (time-homogeneous strong) Markov process with respect to its natural filtration, which obviously coincides with $(\mathcal{F}_t)_{t \ge 0}$.

2.3. The extended optimal stopping problem. For the problem of (2.5), let us consider the following extended optimal stopping problem:

$$V_*(\pi, x) = \inf_{\tau} E_{\pi,x}\big[\tau + G_{a,b}(\Pi_\tau)\big] \tag{2.12}$$

where $P_{\pi,x}$ is a measure of the diffusion process (Π, X) started at some $(\pi, x) \in [0,1] \times (0, \infty)$ and solving the two-dimensional system of equations in (2.9) and (2.11). The infimum in (2.12) is therefore taken over all stopping times τ of (Π, X) being a *Markovian sufficient statistic* in the problem (see [22; Chapter II, Section 15] for an explanation of this notion). By means of the results of the general theory of optimal stopping (see, e.g. [22; Chapter III] or [19; Chapter I, Section 2.1]), it follows from the structure of the reward functional in (2.12) that the optimal stopping time is given by:

$$\tau_* = \inf\{t \ge 0 \,|\, V_*(\Pi_t, X_t) = G_{a,b}(\Pi_t)\} \tag{2.13}$$

whenever $E_{\pi,x}\tau_* < \infty$ holds, so that the continuation region has the form:

$$C_* = \{(\pi, x) \in [0,1] \times (0, \infty) \,|\, V_*(\pi, x) < G_{a,b}(\pi)\}. \tag{2.14}$$

2.4. Structure of the continuation and stopping regions. In order to specify the structure of the stopping time in (2.13), let us further assume that:

$$\text{either} \quad \mu_0(x) < \mu_1(x) \quad \text{or} \quad \mu_0(x) > \mu_1(x) \quad \text{holds for all} \quad x > 0. \tag{2.15}$$

Let us now proceed with an extension of the arguments from [8; Subsection 2.5], by setting $d = G_{a,b}(c)$ and denoting by $\pi_0 = \pi_0(\delta)$ and $\pi_1 = \pi_1(\delta)$ the unique points $0 < \pi_0 < c < \pi_1 < 1$ which satisfy $G_{a,b}(\pi_0) = G_{a,b}(\pi_1) = d - \delta$, for some $\delta \in (0, d)$. For each $x > 0$ fixed, let us then choose $\varepsilon > 0$ such that $\rho(x) > \varepsilon$, where $\rho(x)$ is the so-called signal/noise ratio function defined by:

$$\rho(x) = \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma(x)}\right)^2 \tag{2.16}$$

for any $x > 0$. Hence, taking into account the condition of (2.15), for the earliest of the first passage times:

$$\tau_\delta = \inf\{t \ge 0 \,|\, \Pi_t \notin (\pi_0, \pi_1)\} \quad \text{and} \quad \zeta_\varepsilon = \inf\{t \ge 0 \,|\, \rho(X_t) \le \varepsilon\} \tag{2.17}$$

we conclude that the inequalities:

$$E_{c,x}\big[\tau_\delta \wedge \zeta_\varepsilon\big] \le \int_{\pi_0}^{c} \frac{2(\pi_1 - c)(u - \pi_0)}{\varepsilon(\pi_1 - \pi_0)u^2(1-u)^2}\, du + \int_{c}^{\pi_1} \frac{2(\pi_1 - u)(c - \pi_0)}{\varepsilon(\pi_1 - \pi_0)u^2(1-u)^2}\, du \le \frac{K\delta^2}{\varepsilon} \tag{2.18}$$

hold for some $K > 0$ large enough, not depending on δ, and any $x > 0$. It thus follows that the inequalities:

$$\begin{aligned} &E_{c,x}\big[\tau_\delta \wedge \zeta_\varepsilon + G_{a,b}(\Pi_{\tau_\delta \wedge \zeta_\varepsilon})\big] \\ &\le (d - \delta)\, P_{c,x}\big(G_{a,b}(\Pi_{\tau_\delta \wedge \zeta_\varepsilon}) \le d - \delta\big) + d\, P_{c,x}\big(G_{a,b}(\Pi_{\tau_\delta \wedge \zeta_\varepsilon}) > d - \delta\big) + \frac{K\delta^2}{\varepsilon} \\ &\le d - \delta + \delta\, P_{c,x}\big(G_{a,b}(\Pi_{\tau_\delta \wedge \zeta_\varepsilon}) > d - \delta\big) + \frac{K\delta^2}{\varepsilon} \end{aligned} \tag{2.19}$$

are satisfied for all $\delta > 0$, where the probability of the event in the last line converges to zero under $\delta \downarrow 0$, for each $\varepsilon > 0$ fixed. Choosing $\delta > 0$ in (2.19) small enough, we therefore see that the property:

$$E_{c,x}\big[\tau_\delta \wedge \zeta_\varepsilon + G_{a,b}(\Pi_{\tau_\delta \wedge \zeta_\varepsilon})\big] < d \tag{2.20}$$

holds for any $\varepsilon > 0$ and $x > 0$ fixed. This fact implies that it is never optimal to stop the process (Π, X) at (c, x), whenever the condition of (2.15) is satisfied. These arguments, together with the easily proved concavity of the function $\pi \mapsto V_*(\pi, x)$ on $[0, 1]$ (see, e.g. [13] or [22; pages 168-169]), show that there exists a couple of functions $g_i^*(x)$, $i = 0, 1$, such that $0 < g_0^*(x) < c < g_1^*(x) < 1$ for $x > 0$, and the continuation region in (2.14) for the optimal stopping problem of (2.12) takes the form:

$$C_* = \big\{(\pi, x) \in [0, 1] \times (0, \infty) \,\big|\, \pi \in \big(g_0^*(x), g_1^*(x)\big)\big\} \tag{2.21}$$

and thus, the corresponding stopping region is the closure of the set:

$$D_* = \big\{(\pi, x) \in [0, 1] \times (0, \infty) \,\big|\, \pi \in \big[0, g_0^*(x)\big) \cup \big(g_1^*(x), 1\big]\big\}. \tag{2.22}$$

2.5. Behaviour of the optimal stopping boundaries. In order to characterise the behaviour of the boundaries $g_i^*(x)$, $i = 0, 1$, in (2.21)-(2.22), we apply the Itô-Tanaka formula (see, e.g. [20; Chapter VI, Theorem 1.5]) to the function $G_{a,b}(\pi) = a\pi \wedge b(1-\pi)$ to get:

$$G_{a,b}(\Pi_t) = G_{a,b}(\pi) + \frac{1}{2} \int_0^t \Delta_\pi \frac{\partial G_{a,b}}{\partial \pi}(\Pi_s)\, I(\Pi_s = c)\, d\ell_s^c(\Pi) + M_t^c \tag{2.23}$$

where $\Delta_\pi((\partial G_{a,b})/(\partial \pi))(\pi) = -b - a$, the process $\ell^c(\Pi) = (\ell_t^c(\Pi))_{t \ge 0}$ is the local time of Π at the point c given by:

$$\ell_t^c(\Pi) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(c - \varepsilon < \Pi_s < c + \varepsilon) \left(\frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \right)^2 \Pi_s^2 (1 - \Pi_s)^2 \, ds \tag{2.24}$$

as a limit in probability. Here, the process $M^c = (M_t^c)_{t \ge 0}$ defined by:

$$M_t^c = \int_0^t \frac{\partial G_{a,b}}{\partial \pi}(\Pi_s)\, I(\Pi_s \ne c)\, \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)}\, \Pi_s (1 - \Pi_s)\, d\overline{W}_s \tag{2.25}$$

is a continuous local martingale under $P_{\pi,x}$, and $I(\cdot)$ denotes the indicator function.

Let us now fix some (π, x) from the continuation region C_* in (2.21) and denote by $\tau_* = \tau_*(\pi, x)$ the optimal stopping time in the problem of (2.12). Then, assuming that the process $(M_{\tau_* \wedge t}^c)_{t \ge 0}$ is a uniformly integrable martingale (as it turns out to be the case in the proof of Lemma 3.1 below) and applying Doob's optional sampling theorem (see, e.g. [14; Theorem 3.6] or [20; Chapter II, Theorem 3.2]), we get using the expression in (2.23) that:

$$V_*(\pi, x) = E_{\pi,x}\big[\tau_* + G_{a,b}(\Pi_{\tau_*})\big] = G_{a,b}(\pi) + E_{\pi,x}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\Pi)\right] \tag{2.26}$$

holds for all $(\pi, x) \in [0,1] \times (0, \infty)$. It is also seen from the expression in (2.26) that the initial problem of (2.12) is equivalent to an optimal stopping problem for the local time $\ell^c(\Pi)$ of the process Π at the point c. By means of the general optimal stopping theory for Markov processes (see, e.g. [22; Chapter III] or [19; Chapter I, Section 2.2]), we thus conclude that:

$$V_*(\pi, x) - G_{a,b}(\pi) = E_{\pi,x}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\Pi)\right] < 0 \tag{2.27}$$

holds. Let us then take $x' > 0$ such that $x < x'$ when either $\pi < c$ or $\pi > c$. Hence, using the facts that (Π, X) is a time-homogeneous Markov process and $\tau_* = \tau_*(\pi, x)$ does not depend on x', taking into account the comparison results for solutions of stochastic differential equations in [23], we obtain from the expression in (2.23) and the structure of the process $\ell^c(\Pi)$ in (2.24) that:

$$\begin{aligned} V_*(\pi, x') - G_{a,b}(\pi) &\le E_{\pi,x'}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\Pi)\right] \\ &\le E_{\pi,x}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\Pi)\right] = V_*(\pi, x) - G_{a,b}(\pi) \end{aligned} \tag{2.28}$$

holds, whenever the *signal/noise ratio* function $\rho(x)$ given by (2.16) is an increasing function on $(0,\infty)$. By virtue of the inequality in (2.27), we may therefore conclude that $(\pi,x')\in C_*$, so that the boundary $g_0^*(x)$ is decreasing (increasing) and the boundary $g_1^*(x)$ is increasing (decreasing) in (2.21)-(2.22) whenever $\rho(x)$ is increasing (decreasing) on $(0,\infty)$, respectively.

Summarising the facts proved above, we are now ready to formulate the following assertion.

Lemma 2.1. *Suppose that $\mu_i(x)$, $i=0,1$, and $\sigma(x)>0$ are continuously differentiable functions on $(0,\infty)$ in (2.2). Assume that the condition of (2.15) holds and the process $(M^c_{\tau_*\wedge t})_{t\ge 0}$ from (2.25) is a uniformly integrable martingale. Then, the optimal Bayesian sequential decision rule in the problem of (2.4) of testing the hypotheses in (2.3) has the structure:*

$$\tau_* = \inf\big\{t\ge 0 \,\big|\, \Pi_t \notin (g_0^*(X_t), g_1^*(X_t))\big\} \tag{2.29}$$

and

$$d_* = \begin{cases} 1, & \text{if } \Pi_{\tau_*} = g_1^*(X_{\tau_*}) \\ 0, & \text{if } \Pi_{\tau_*} = g_0^*(X_{\tau_*}) \end{cases} \tag{2.30}$$

whenever $E_{\pi,x}\tau_*<\infty$ holds for all $(\pi,x)\in[0,1]\times(0,\infty)$, and $\tau_*=0$ otherwise. Here, for the couple of functions $g_i^*(x)$, $i=0,1$, the properties:

$$g_0^*(x):(0,\infty)\to(0,c) \quad \text{is decreasing/increasing} \tag{2.31}$$
$$\text{if} \quad \rho(x) \quad \text{is increasing/decreasing}$$
$$g_1^*(x):(0,\infty)\to(c,1) \quad \text{is increasing/decreasing} \tag{2.32}$$
$$\text{if} \quad \rho(x) \quad \text{is increasing/decreasing}$$

hold with $\rho(x)$ defined in (2.16), for all $x>0$.

2.6. The free-boundary problem. By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}_{(\Pi,X)}$ of the process (Π,X) from (2.9) and (2.11) has the structure:

$$\mathbb{L}_{(\Pi,X)} = \frac{1}{2}\left(\frac{\mu_1(x)-\mu_0(x)}{\sigma(x)}\right)^2 \pi^2(1-\pi)^2\,\frac{\partial^2}{\partial\pi^2} + (\mu_1(x)-\mu_0(x))\,\pi(1-\pi)\,\frac{\partial^2}{\partial\pi\partial x} \tag{2.33}$$
$$+ \big(\mu_0(x)+\pi\,(\mu_1(x)-\mu_0(x))\big)\,\frac{\partial}{\partial x} + \frac{1}{2}\,\sigma^2(x)\,\frac{\partial^2}{\partial x^2}$$

for all $(\pi,x)\in[0,1]\times(0,\infty)$.

In order to find analytic expressions for the unknown value function $V_*(\pi, x)$ from (2.12) (with $G_{a,b}(\pi) = a\pi \wedge b(1-\pi)$) and the boundaries $g_i^*(x)$, $i = 0, 1$, from (2.21)-(2.22), we use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [9], [22; Chapter III, Section 8] and [19; Chapter IV, Section 8]) to formulate the associated *free-boundary problem*:

$$(\mathbb{L}_{(\Pi,X)}V)(\pi, x) = -1 \quad \text{for} \quad (\pi, x) \in C \tag{2.34}$$

$$V(\pi, x)\big|_{\pi=g_0(x)+} = ag_0(x), \quad V(\pi, x)\big|_{\pi=g_1(x)-} = b(1 - g_1(x)) \tag{2.35}$$

$$V(\pi, x) = G_{a,b}(\pi) \quad \text{for} \quad (\pi, x) \in D \tag{2.36}$$

$$V(\pi, x) < G_{a,b}(\pi) \quad \text{for} \quad (\pi, x) \in C \tag{2.37}$$

where C and D are defined as C_* and D_* in (2.21) and (2.22) with $g_i(x)$, $i = 0, 1$, instead of $g_i^*(x)$, $i = 0, 1$, and the *instantaneous-stopping* conditions in (2.35) are satisfied for all $x > 0$.

Note that the superharmonic characterisation of the value function (see [6], [22; Chapter III, Section 8] and [19; Chapter IV, Section 9]) implies that $V_*(\pi, x)$ from (2.12) is the largest function satisfying (2.34)-(2.37) with the boundaries $g_i^*(x)$, $i = 0, 1$.

Remark 2.2. Observe that, since the system in (2.34)-(2.37) admits multiple solutions, we need to find some additional conditions which would specify the appropriate solution providing the value function and the optimal stopping boundaries for the initial problem of (2.12). In order to derive such conditions, we shall reduce the operator in (2.33) to the normal form. We also note that the fact that the stochastic differential equations for the posterior probability and the observation process in (2.9) and (2.11) are driven by the same (one-dimensional) innovation Wiener process yields the property that the infinitesimal operator in (2.33) turns out to be of parabolic type.

2.7. The change of variables. In order to find the normal form of the operator in (2.33) and formulate the associated optimal stopping and free-boundary problem, we use the one-to-one correspondence transformation of processes proposed by A.N. Kolmogorov in [11]. For this, let us define the process $Y = (Y_t)_{t \geq 0}$ by:

$$Y_t = \log \frac{\Pi_t}{1 - \Pi_t} - \int_z^{X_t} \frac{\mu_1(w) - \mu_0(w)}{\sigma^2(w)} \, dw \tag{2.38}$$

for all $t \ge 0$ and any $z > 0$ fixed. Then, taking into account the assumption that the functions $\mu_i(x)$, $i = 0, 1$, and $\sigma(x)$ are continuously differentiable on $(0, \infty)$, by means of Itô's formula, we get that the process Y admits the representation:

$$dY_t = -\frac{\sigma^2(X_t)}{2} \left[\frac{\mu_1^2(X_t) - \mu_0^2(X_t)}{\sigma^4(X_t)} + \frac{\partial}{\partial x} \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma^2(x)} \right) \bigg|_{x = X_t} \right] dt \quad (Y_0 = y) \tag{2.39}$$

with

$$y = \log \frac{\pi}{1 - \pi} - \int_z^x \frac{\mu_1(w) - \mu_0(w)}{\sigma^2(w)} \, dw \tag{2.40}$$

for any $z > 0$ fixed. It is seen from the equation in (2.39) that the process Y started at $y \in \mathbb{R}$ is of bounded variation. Moreover, under the assumption of (2.15), it follows from the relation in (2.38) that there exists a one-to-one correspondence between the processes (Π, X) and (Π, Y). Hence, for any $z > 0$ fixed, the value function $V_*(\pi, x)$ from (2.12) is equal to the one of the optimal stopping problem:

$$U_*(\pi, y) = \inf_{\tau} E_{\pi,y} \big[\tau + G_{a,b}(\Pi_\tau) \big] \tag{2.41}$$

where the supremum is taken over all stopping times τ with respect to the natural filtration of (Π, Y), which clearly coincides with $(\mathcal{F}_t)_{t \ge 0}$. Here, $E_{\pi,y}$ denotes the expectation under the assumption that the two-dimensional Markov process (Π, Y) from (2.6) and (2.38) starts at some $(\pi, y) \in (0, 1) \times \mathbb{R}$. It thus follows from (2.29) that there exists a couple of functions $h_i^*(y)$, $i = 0, 1$, such that $0 < h_0^*(y) < c < h_1^*(y) < 1$ for $y \in \mathbb{R}$, and the optimal stopping time in the problem of (2.41) has the structure:

$$\tau_* = \inf\{ t \ge 0 \,|\, \Pi_t \notin (h_0^*(Y_t), h_1^*(Y_t)) \} \tag{2.42}$$

whenever $E_{\pi,y} \tau_* < \infty$, and $\tau_* = 0$ otherwise.

2.8. The equivalent free-boundary problem. Standard arguments then show that the infinitesimal operator $\mathbb{L}_{(\Pi,Y)}$ of the process (Π, Y) from (2.9) and (2.39) has the structure:

$$\begin{aligned} \mathbb{L}_{(\Pi,Y)} = {} & \frac{1}{2} \left(\frac{\mu_1(x(\pi,y)) - \mu_0(x(\pi,y))}{\sigma(x(\pi,y))} \right)^2 \pi^2 (1 - \pi)^2 \, \frac{\partial^2}{\partial \pi^2} \\ & - \frac{\sigma^2(x(\pi,y))}{2} \left[\frac{\mu_1^2(x(\pi,y)) - \mu_0^2(x(\pi,y))}{\sigma^4(x(\pi,y))} + \frac{\partial}{\partial x} \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma^2(x)} \right) \bigg|_{x = x(\pi,y)} \right] \frac{\partial}{\partial y} \end{aligned} \tag{2.43}$$

for all $(\pi, y) \in (0,1) \times \mathbb{R}$. Here, by virtue of the assumption in (2.15), the expression for $x(\pi, y) \equiv x(\pi, y; z)$ is uniquely determined by the relation in (2.40), for any $z > 0$.

We are now ready to formulate the associated free-boundary problem for the unknown value function $U_*(\pi, y) \equiv U_*(\pi, y; z)$ from (2.41) and the boundaries $h_i^*(y) \equiv h_i^*(y; z)$, $i = 0, 1$, from (2.42):

$$(\mathbb{L}_{(\Pi,Y)}U)(\pi, y) = -1 \quad \text{for} \quad h_0(y) < \pi < h_1(y) \tag{2.44}$$

$$U(\pi, y)\big|_{\pi=h_0(y)+} = a h_0(y), \quad U(\pi, y)\big|_{\pi=h_1(y)-} = b(1 - h_1(y)) \tag{2.45}$$

$$U(\pi, y) = G_{a,b}(\pi) \quad \text{for} \quad \pi < h_0(y) \quad \text{and} \quad \pi > h_1(y) \tag{2.46}$$

$$U(\pi, y) < G_{a,b}(\pi) \quad \text{for} \quad h_0(y) < \pi < h_1(y) \tag{2.47}$$

where the *instantaneous-stopping* conditions of (2.45) are satisfied for all $y \in \mathbb{R}$. Moreover, we assume that the *smooth-fit* conditions:

$$\frac{\partial U}{\partial \pi}(\pi, y)\bigg|_{\pi=h_0(y)+} = a, \quad \frac{\partial U}{\partial \pi}(\pi, y)\bigg|_{\pi=h_1(y)-} = -b \tag{2.48}$$

hold and the one-sided derivatives:

$$\frac{\partial U}{\partial y}(\pi, y)\bigg|_{\pi=h_0(y)+}, \quad \frac{\partial U}{\partial y}(\pi, y)\bigg|_{\pi=h_1(y)-} \quad \text{exist} \tag{2.49}$$

for all $y \in \mathbb{R}$ and any $z > 0$ fixed.

We further search for solutions of the parabolic-type free-boundary problem in (2.44)-(2.47) satisfying the conditions of (2.48) and (2.49) and such that the resulting boundaries are continuous and of bounded variation. Since such free-boundary problems cannot, in general, be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the related variational inequalities and their connection with the optimal stopping problems have been extensively studied in the literature (see, e.g. [7], [1], [12] or [16]).

3. Main results

In this section, we formulate and prove the main assertions of the paper concerning the Bayesian sequential testing problem for diffusion processes.

3.1. Verification lemma. We begin with the following verification lemma related to the free-boundary problem in (2.44)-(2.49).

Lemma 3.1. *Suppose that $\mu_i(x)$, $i = 0, 1$, and $\sigma(x) > 0$ are continuously differentiable functions on $(0, \infty)$ in (2.2). Assume that the function $U(\pi, y; h_0^*(y), h_1^*(y))$ and the couple of continuous boundaries of bounded variation $h_i^*(y)$, $i = 0, 1$, form a unique solution of the free-boundary problem in (2.44)-(2.47) satisfying the conditions of (2.48) and (2.49). Then, the value function of the optimal stopping problem in (2.41) takes the form:*

$$U_*(\pi, y) = \begin{cases} U(\pi, y; h_0^*(y), h_1^*(y)), & \text{if } \pi \in (h_0^*(y), h_1^*(y)) \\ G_{a,b}(\pi), & \text{if } \pi \in [0, h_0^*(y)] \cup [h_1^*(y), 1] \end{cases} \tag{3.1}$$

and the couple $h_i^*(y)$, $i = 0, 1$, provides the optimal stopping boundaries for (2.42) whenever $E_{\pi,y}\tau_* < \infty$ holds, for all $(\pi, y) \in (0, 1) \times (0, \infty)$.

Proof. Let us denote by $U(\pi, y)$ the right-hand side of the expression in (3.1). Hence, applying the change-of-variable formula with local time on surfaces from [17] to $U(\pi, y)$ and $h_i^*(y)$, $i = 0, 1$, and taking into account the smooth-fit conditions in (2.48), we obtain:

$$U(\Pi_t, Y_t) = U(\pi, y) + \int_0^t (\mathbb{L}_{(\Pi,Y)}U)(\Pi_s, Y_s)\, I(\Pi_s \neq h_0^*(Y_s), \Pi_s \neq h_1^*(Y_s))\, ds + M_t \tag{3.2}$$

where the process $M = (M_t)_{t \geq 0}$ defined by:

$$M_t = \int_0^t \frac{\partial U}{\partial \pi}(\Pi_s, Y_s)\, \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)}\, \Pi_s(1 - \Pi_s)\, d\overline{W}_s \tag{3.3}$$

is a continuous local martingale under $P_{\pi,y}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$.

It follows from the equation in (2.44) and the conditions of (2.46)-(2.47) that the inequality $(\mathbb{L}_{(\Pi,Y)}U)(\pi, y) \geq -1$ holds for any $(\pi, y) \in (0, 1) \times \mathbb{R}$ such that $\pi \neq h_i^*(y)$ for $i = 0, 1$, as well as $U(\pi, y) \leq G_{a,b}(\pi)$ is satisfied for all $(\pi, y) \in (0, 1) \times \mathbb{R}$. Recall the assumption that the boundaries $h_i^*(y)$, $i = 0, 1$, are continuous and of bounded variation and the fact that the process Y from (2.38) is of bounded variation too. We thus conclude from the assumption of continuous differentiability of the functions $\mu_i(x)$, $i = 0, 1$, and $\sigma(x)$ that the time spent by the process Π at the boundaries $h_i^*(Y)$, $i = 0, 1$, is of Lebesgue measure zero, so that the indicator which appears in (3.2) can be ignored. Hence, the expression in (3.2) yields that the inequalities:

$$\tau + G_{a,b}(\Pi_\tau) \geq \tau + U(\Pi_\tau, Y_\tau) \geq U(\pi, y) + M_\tau \tag{3.4}$$

hold for any stopping time τ of the process (Π, Y) started at $(\pi, y) \in (0,1) \times \mathbb{R}$.

Let $(\tau_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the processes M. Taking the expectations with respect to the probability measure $P_{\pi,y}$ in (3.4), by means of Doob's optional sampling theorem, we get that the inequalities:

$$
\begin{aligned}
E_{\pi,y}\big[\tau \wedge \tau_n + G_{a,b}(\Pi_{\tau\wedge\tau_n})\big] &\geq E_{\pi,y}\big[\tau \wedge \tau_n + U(\Pi_{\tau\wedge\tau_n}, Y_{\tau\wedge\tau_n})\big] \\
&\geq U(\pi, y) + E_{\pi,y}\big[M_{\tau\wedge\tau_n}\big] = U(\pi, y)
\end{aligned} \tag{3.5}
$$

hold for all $(\pi, y) \in (0,1) \times \mathbb{R}$. Hence, letting n go to infinity and using Fatou's lemma, we obtain:

$$
E_{\pi,y}\big[\tau + G_{a,b}(\Pi_\tau)\big] \geq E_{\pi,y}\big[\tau + U(\Pi_\tau, Y_\tau)\big] \geq U(\pi, y) \tag{3.6}
$$

for any stopping time τ and all $(\pi, y) \in (0,1) \times \mathbb{R}$. By virtue of the structure of the stopping time in (2.42), it is readily seen that the inequalities in (3.6) hold with τ_* instead of τ when either $\pi \leq h_0^*(y)$ or $\pi \geq h_1^*(y)$.

It remains to show that the equalities are attained in (3.6) when τ_* replaces τ, for $(\pi, y) \in (0,1) \times \mathbb{R}$ such that $h_0^*(y) < \pi < h_1^*(y)$. By virtue of the fact that the function $U(\pi, y)$ and the boundaries $h_i^*(y)$, $i = 0, 1$, satisfy the conditions in (2.44) and (2.45), it follows from the expression in (3.2) and the structure of the stopping time in (2.42) that the equalities:

$$
\tau_* \wedge \tau_n + U(\Pi_{\tau_*\wedge\tau_n}, Y_{\tau_*\wedge\tau_n}) = U(\pi, y) + M_{\tau_*\wedge\tau_n} \tag{3.7}
$$

hold for all $(\pi, y) \in (0,1) \times \mathbb{R}$ and any localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ of M. Hence, taking into account the assumption $E_{\pi,y}\tau_* < \infty$ together with the fact that $0 \leq U(\pi, y) \leq ab/(a+b)$ holds, we conclude from the expression in (3.7) that the process $(M_{\tau_*\wedge t})_{t \geq 0}$ is a uniformly integrable martingale. Therefore, taking the expectations in (3.7) and letting n go to infinity, we apply the Lebesgue dominated convergence theorem to obtain the equalities:

$$
E_{\pi,y}\big[\tau_* + G_{a,b}(\Pi_{\tau_*})\big] = E_{\pi,y}\big[\tau_* + U(\Pi_{\tau_*}, Y_{\tau_*})\big] = U(\pi, y) \tag{3.8}
$$

for all $(\pi, y) \in (0,1) \times \mathbb{R}$, which together with the inequalities in (3.6) directly imply the desired assertion. $\square$

3.2. Solution of the problem. We are now in a position to formulate the main assertion of the paper, which follows from a straightforward combination of Lemmas 2.1 and 3.1 above and the standard change-of-variable arguments. More precisely, after obtaining the solution $U_*(\pi, y) \equiv U_*(\pi, y; z)$ with $h_i^*(y) \equiv h_i^*(y; z)$, $i = 0, 1$, of the free-boundary problem in (2.44)-(2.47), which satisfies the conditions in (2.48) and (2.49), we put $y = y(\pi, x; z)$ and $z = x$, in order to get the solution of the initial Bayesian sequential testing problem stated in (2.4).

Theorem 3.2. *Suppose that the assumptions of Lemmas 2.1 and 3.1 hold including the property in (2.15). Then, in the sequential testing problem of (2.4) and (2.12) for the observation process X from (2.2), the Bayesian risk function takes the form $V_*(\pi, x) = U_*(\pi, y(\pi, x)) \equiv U_*(\pi, y(\pi, x; x); x)$ and the optimal stopping boundary $g_i^*(x)$ in (2.29)-(2.30) satisfying (2.31)-(2.32) is uniquely determined by the equation $g_i(x) = h_i^*(y(g_i(x), x)) \equiv h_i^*(y(g_i(x), x; x); x)$, for each $x > 0$ fixed and every $i = 0, 1$. Here, the function $U_*(\pi, y) \equiv U_*(\pi, y; z)$ and the couple of continuous boundaries of bounded variation $h_i^*(y) \equiv h_i^*(y; z)$, $i = 0, 1$, form a unique solution of the free-boundary problem in (2.44)-(2.49), and the expression for $y(\pi, x) \equiv y(\pi, x; z)$ is explicitly determined by the relation in (2.40), for all $(\pi, y) \in (0, 1) \times \mathbb{R}$ and any $z > 0$ fixed.*

3.3. Some upper and lower bounds. Let us finally give a short note concerning the case of bounded signal/noise ratio function $\rho(x)$ defined in (2.16).

Remark 3.3. Suppose that there exist some $0 < \underline{\rho} < \overline{\rho} < \infty$ such that $\underline{\rho} \leq \rho(x) \leq \overline{\rho}$ holds for all $x > 0$. Let us denote by $\underline{V}_*(\pi, x)$ with $\underline{g}_i^*(x)$, $i = 0, 1$, and by $\overline{V}_*(\pi, x)$ with $\overline{g}_i^*(x)$, $i = 0, 1$, the solution of the Bayesian sequential testing problem with $\rho(x) \equiv \underline{\rho}$ and $\rho(x) \equiv \overline{\rho}$, respectively. In those cases, the problem of (2.12) degenerates into an optimal stopping problem for the one-dimensional Markov process Π, and the value functions $\underline{V}_*(\pi, x) \equiv \underline{V}_*(\pi)$ and $\overline{V}_*(\pi, x) \equiv \overline{V}_*(\pi)$ with the couples of stopping boundaries $\underline{g}_0^*(x) \equiv \underline{A}$, $\underline{g}_1^*(x) \equiv \underline{B}$ and $\overline{g}_0^*(x) \equiv \overline{A}$, $\overline{g}_1^*(x) \equiv \overline{B}$ are given by the expressions in (4.70) and (4.85) of [22; Chapter IV, Section 2]. Taking into account the properties of the couple $g_i^*(x)$, $i = 0, 1$, in (2.31)-(2.32) and the fact that $V_*(\pi, x) = G_{a,b}(\pi)$ for all $0 \leq \pi \leq g_0^*(x)$ and $g_1^*(x) \leq \pi \leq 1$, we therefore conclude by standard comparison arguments that the inequalities $\overline{V}_*(\pi) \leq V_*(\pi, x) \leq \underline{V}_*(\pi)$ and thus $0 < \overline{A} \leq g_0^*(x) \leq \underline{A} < c < \underline{B} \leq g_1^*(x) \leq \overline{B} < 1$ hold for all $(\pi, x) \in [0, 1] \times (0, \infty)$.

4. Conclusions

In this section, we consider some particular cases of observable diffusions and give some hints to the solution of the sequential testing problem in the variational formulation.

4.1. Some particular cases. In order to pick up some special cases in which the free-boundary problem in (2.44)-(2.49) can admit a closed form solution, for the rest of the paper, we assume that the property:

$$\mu_i(x) = \frac{\eta_i \sigma^2(x)}{x} \quad \text{for some} \quad \eta_i \in \mathbb{R},\, i = 0, 1, \tag{4.1}$$
$$\text{such that} \quad \eta_0 \neq \eta_1 \quad \text{and} \quad \eta_0 + \eta_1 = 1$$

holds for all $x > 0$. Moreover, we assume that the diffusion coefficient $\sigma(x)$ satisfies:

$$\sigma(x) \sim C_0\, x^{\alpha} \quad \text{as} \quad x \downarrow 0 \qquad \text{and} \qquad \sigma(x) \sim C_\infty\, x^{\beta} \quad \text{as} \quad x \uparrow \infty \tag{4.2}$$

with some $C_0, C_\infty > 0$ as well as $\alpha, \beta \in \mathbb{R}$ such that $(1-\alpha)\eta \le 0$ and $(1-\beta)\eta \ge 0$ holds, where we set $\eta = 1/(\eta_1 - \eta_0)$. Then, the process $Y = (Y_t)_{t \ge 0}$ takes the form:

$$Y_t = \log \frac{\Pi_t}{1 - \Pi_t} - \frac{1}{\eta} \log \frac{z}{X_t} \tag{4.3}$$

for any $z > 0$ fixed. It is easily seen from the structure of the expression in (4.3) that the one-to-one correspondence between the processes (Π, X) and (Π, Y) remains true in this case. Getting the expression for X_t from (4.3) and substituting it into the equation of (2.9), we obtain:

$$d\Pi_t = \frac{\sigma(ze^{-\eta Y_t}[\Pi_t/(1-\Pi_t)]^{\eta})}{\eta z e^{-\eta Y_t}[\Pi_t/(1-\Pi_t)]^{\eta}}\, \Pi_t(1-\Pi_t)\, d\overline{W}_t \quad (\Pi_0 = \pi) \tag{4.4}$$

for any $z > 0$ fixed. Applying Itô's formula to the expression in (4.3) and taking into account the representations in (2.9) and (2.11) as well as the assumption of (4.1), we get:

$$Y_t = \log \frac{\pi}{1 - \pi} \tag{4.5}$$

for all $t \ge 0$. It thus follows that the infinitesimal operator $\mathbb{L}_{(\Pi, Y)}$ from (2.43) takes the form:

$$\mathbb{L}_{(\Pi,Y)} = \frac{1}{2}\, \frac{\sigma^2(ze^{-\eta y}[\pi/(1-\pi)]^{\eta})}{\eta^2 z^2 e^{-2\eta y}[\pi/(1-\pi)]^{2\eta}}\, \pi^2 (1-\pi)^2 \frac{\partial^2}{\partial \pi^2} \tag{4.6}$$

for all $(\pi, y) \in (0,1) \times \mathbb{R}$ and any $z > 0$ fixed.

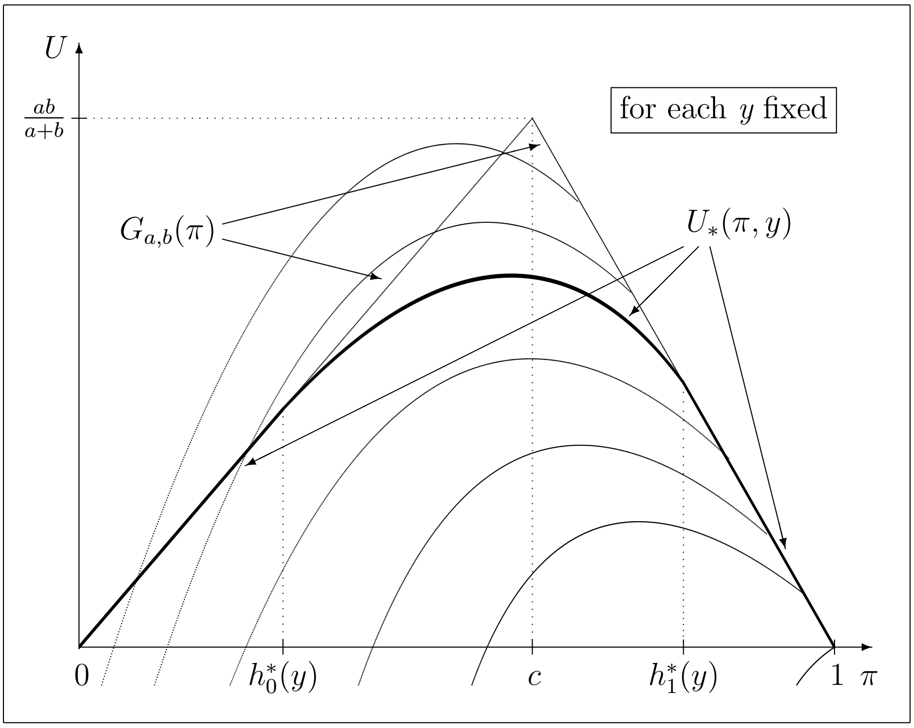

Figure 4.1. A computer drawing of the value function $U_*(\pi, y)$ and the optimal stopping boundaries $h_0^*(y)$ and $h_1^*(y)$ in Corollary 4.1.

4.2. The proof of existence and uniqueness. Let us now follow the arguments of [18; Section 3] (see also [19; Chapter VI, Section 21]) and integrate the equation in (2.44) with the ordinary operator from (4.6), for any $h_1(y) \in (c, 1)$ and each $y \in \mathbb{R}$ fixed. Taking into account the continuous differentiability of the function $\sigma(x) > 0$ and using the boundary conditions from (2.45) and (2.48) at the point $h_1(y)$, we obtain:

$$U(\pi, y; h_1(y)) = b(1 - \pi) - \int_{\pi}^{h_1(y)} \int_{u}^{h_1(y)} \frac{2\eta^2 z^2 e^{-2\eta y} v^{2(\eta-1)}}{\sigma^2(ze^{-\eta y}[v/(1-v)]^{\eta})(1-v)^{2(\eta+1)}} \, dv du \tag{4.7}$$

for all $\pi \in (0, h_1(y)]$ and any $z > 0$ fixed. It is easily seen from (4.7) that the function $\pi \mapsto U(\pi, y; h_1(y))$ is concave on $(0, h_1(y))$, and hence, the inequality $U(\widetilde{h}_1(y), y; \widehat{h}_1(y)) < U(\widetilde{h}_1(y), y; \widetilde{h}_1(y))$ holds for $0 < \widetilde{h}_1(y) < \widehat{h}_1(y) < 1$ and each $y \in \mathbb{R}$ fixed. This means that, for different $\widetilde{h}_1(y)$ and $\widehat{h}_1(y)$, the curves $\pi \mapsto U(\pi, y; \widetilde{h}_1(y))$ and $\pi \mapsto U(\pi, y; \widehat{h}_1(y))$ have no points of intersection on the whole interval $(0, \widetilde{h}_1(y)]$. By virtue of the assumptions of (4.2) with $(1-\alpha)\eta \le 0$ and $(1-\beta)\eta \ge 0$, it also follows from (4.7) that $U(\pi, y; h_1(y)) \to -\infty$ as $\pi \downarrow 0$ and $\pi \uparrow 1$, for any $h_1(y) \in (c, 1)$, and $U(\pi, y; 1-) < 0$ holds for all $\pi \in (0, 1)$, as well as $U(1-, y; 1-) = 0$. In this case, for some $h_1(y) \in (c, 1)$, the curve $\pi \mapsto U(\pi, y; h_1(y))$ intersects the line $\pi \mapsto a\pi$ at some point $h_0(y) \in (0, c)$. Since the curves $\pi \mapsto U(\pi, y; \widetilde{h}_1(y))$

do not intersect each other on the intervals $(0, \widetilde{h}_1(y))$, for different $\widetilde{h}_1(y) \in (c, 1)$, we may conclude that there exists a unique point $h_1^*(y)$ which is obtained by moving the point $\widetilde{h}_1(y)$ from $h_1(y)$ and such that the boundary conditions from (2.45) and (2.48) hold at some point $h_0^*(y) \in (0, c)$ (see Figure 4.1 above). It thus follows that the boundaries $h_i^*(y)$, $i = 0, 1$, are uniquely determined by the coupled system:

$$b + a = \int_{h_0(y)}^{h_1(y)} \frac{2\eta^2 z^2 e^{-2\eta y} u^{2(\eta-1)}}{\sigma^2(ze^{-\eta y}[u/(1-u)]^\eta)(1-u)^{2(\eta+1)}} \, du \tag{4.8}$$

$$b(1 - h_1(y)) = a h_0(y) - \int_{h_0(y)}^{h_1(y)} \int_u^{h_1(y)} \frac{2\eta^2 z^2 e^{-2\eta y} v^{2(\eta-1)}}{\sigma^2(ze^{-\eta y}[v/(1-v)]^\eta)(1-v)^{2(\eta+1)}} \, dv du \tag{4.9}$$

for each $y \in \mathbb{R}$ and any $z > 0$ fixed. It is seen from the regular structure of the integrands in (4.8) and (4.9) that the boundaries $h_i^*(y) \equiv h_i^*(y; z)$, $i = 0, 1$, are continuous and of bounded variation, for each $y \in \mathbb{R}$ and any $z > 0$ fixed. Moreover, it follows from the concavity of the function $\pi \mapsto U(\pi, y; h_1^*(y))$ on $[h_0^*(y), h_1^*(y)]$ that the condition of (2.47) holds in this case.

4.3. The average observation time. By means of standard arguments based on integrating of the Green measure of the one-dimensional diffusion in (4.4) with respect to its speed measure (see, e.g. [2; Chapter II] or [19; Chapter IV, Section 13]), it is shown that:

$$\begin{aligned} E_{\pi,y}\tau_* = &\int_{h_0^*(y)}^{\pi} \frac{(h_1^*(y) - \pi)(u - h_0^*(y))}{h_1^*(y) - h_0^*(y)} \frac{2\eta^2 z^2 e^{-2\eta y} u^{2(\eta-1)}}{\sigma^2(ze^{-\eta y}[u/(1-u)]^\eta)(1-u)^{2(\eta+1)}} \, du \\ &+ \int_{\pi}^{h_1^*(y)} \frac{(h_1^*(y) - u)(\pi - h_0^*(y))}{h_1^*(y) - h_0^*(y)} \frac{2\eta^2 z^2 e^{-2\eta y} u^{2(\eta-1)}}{\sigma^2(ze^{-\eta y}[u/(1-u)]^\eta)(1-u)^{2(\eta+1)}} \, du \end{aligned} \tag{4.10}$$

holds for each $h_0^*(y) < \pi < h_1^*(y)$ and $y \in \mathbb{R}$, while $E_{\pi,y}\tau_* = 0$ otherwise, for any $z > 0$ fixed. By virtue of the continuous differentiability of the function $\sigma(x)$, we can therefore conclude that $E_{\pi,y}\tau_* < \infty$ is satisfied for all $(\pi, y) \in (0, 1) \times \mathbb{R}$ in this case.

4.4. Some remarks and examples. Summarising the facts proved above, let us formulate the following assertion.

Corollary 4.1. *Suppose that $\mu_i(x)$, $i = 0, 1$, and $\sigma(x) > 0$ are continuously differentiable functions on $(0, \infty)$ such that the conditions of (4.1) and (4.2) hold with $(1 - \alpha)\eta \leq 0$ and $(1 - \beta)\eta \geq 0$, where $\eta = 1/(\eta_1 - \eta_0)$. Then, the value function $U_*(\pi, y) \equiv U_*(\pi, y; z)$ from*

(2.41) admits the representation:

$$U_*(\pi, y) = \begin{cases} U(\pi, y; h_1^*(y)), & \text{if } \pi \in (h_0^*(y), h_1^*(y)) \\ G_{a,b}(\pi), & \text{if } \pi \in [0, h_0^*(y)] \cup [h_1^*(y), 1] \end{cases} \tag{4.11}$$

where $U(\pi, y; h_1(y)) \equiv U(\pi, y; z; h_1(y; z))$ is given by (4.7), and the continuous boundaries of bounded variation $h_i^*(y) \equiv h_i^*(y; z)$, $i = 0, 1$, from (2.42) are uniquely determined by the system of (4.8) and (4.9), for all $(\pi, y) \in (0, 1) \times \mathbb{R}$ and any $z > 0$ fixed.

Remark 4.2. It follows from the assumptions of Corollary 4.1 that the boundaries $h_i^*(y)$, $i = 0, 1$, from the system in (4.8)-(4.9) are continuously differentiable on $(0, \infty)$. In this case, by means of straightforward computations applied to the expression in (4.7), it is shown using (4.8)-(4.9) that the smooth-fit conditions:

$$\frac{\partial U_*}{\partial y}(\pi, y)\bigg|_{\pi = h_0^*(y)+} = 0, \quad \frac{\partial U_*}{\partial y}(\pi, y)\bigg|_{\pi = h_1^*(y)-} = 0 \tag{4.12}$$

hold for all $y \in \mathbb{R}$. Hence, taking into account the expression in (2.48) which holds for $U_*(\pi, y)$ at $h_i^*(y)$, $i = 0, 1$, and the one-to-one correspondence given by (2.40), for any $z > 0$ fixed, we conclude from (4.12) that the smooth-fit conditions:

$$\frac{\partial V_*}{\partial x}(\pi, x)\bigg|_{\pi = g_0^*(x)+} = 0, \quad \frac{\partial V_*}{\partial x}(\pi, x)\bigg|_{\pi = g_1^*(x)-} = 0 \tag{4.13}$$

are satisfied for all $x > 0$. This property can be explained by the fact that the continuous process Π intersects the continuous boundaries $h_i^*(Y)$ and thus $g_i^*(X)$, $i = 0, 1$, with a positive probability.

Example 4.3. Suppose that we have $\sigma(x) = x$ in (4.1), for all $x > 0$, and some $\eta_i \in \mathbb{R}$, $i = 0, 1$, where the restriction $\eta_0 + \eta_1 = 1$ is omitted. In this case, the process X in (2.2) is a geometric Brownian motion under $\theta = i$, for every $i = 0, 1$. It is easily seen that the initial problem of (2.4) is then equivalent to the Bayesian sequential testing problem for the observable Wiener process $\log X$ with the unknown constant drift rate $\eta_0 + \theta(\eta_1 - \eta_0) - 1/2$. The latter problem was reduced to (2.12) and solved as an optimal stopping problem for a one-dimensional Markov process Π in [21] (see also [22; Chapter IV, Section 2] or [19; Chapter VI, Section 21]).

Remark 4.4. We finally note that the corresponding variational formulation of the problem can be considered following the structure of arguments similar to the one used in [18; Section 3] (see also [19; Chapter VI, Section 21]). Those arguments are based on the embedding of the latter problem into the corresponding Bayesian one, and then the specifying of the appropriate sequential decision rule for the admissible error probabilities of the first and second kind given. Such arguments particularly lead to the fact that in the cases in which the process Y is constant, the sequential probability ratio test turns out to be optimal with constant boundaries which may only depend on the starting point of the observation process.

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