# Robust replication in $H$-self-similar Gaussian market models under model uncertainty

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We consider the robust hedging problem in the framework of model uncertainty, where the log-returns of the stock price are Gaussian and $H$-self-similar with $H \in (1/2, 1)$. These assumptions lead to two natural but mutually exclusive hypotheses, both being self-contained to fix the probabilistic model for the stock price. Namely, the investor may assume that either the market is efficient, i.e. the stock price process is a semimartingale, or that the centred log-returns are stationary. We show that to be able to super-hedge a European contingent claim with a convex payoff robustly, the investor must assume that the markets are efficient. If it turns out that the stationarity hypothesis is true, then the investor can actually super-hedge the option and thereby receive some net profit.

## 1 Introduction

In the classical Black-Scholes model of financial markets, the logarithm of the stock price is modeled by a Brownian motion with linear drift. Due to the increasing demand of research on various market properties, an extensive number of generalizations of that model have been proposed so far to improve its shortcomings indicated in the resent literature. For instance, some known recent statistical studies of the real financial data conclude that the centred log-returns of the stock prices may exhibit the long-range dependence property (see, e.g. [25, Chapter IV]). This observation generates an intention to extend the driving Brownian motion with independent increments to a Gaussian process either having the so-called long memory or

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at least enjoying the *$H$-self-similarity* of its sample paths. The latter property, which is in many cases taken to be evidence for the long-range dependence (when $H$ belongs to the open interval $(1/2, 1)$).

A natural candidate for the new driving process is the *fractional Brownian motion*, which is a Gaussian process characterized by being self-similar and having stationary increments. This, what we further call hypothesis (H1), will result in a market model that contains arbitrage opportunities (see, e.g. [10, 21, 25, 26]). Another possible candidate for the extension of the initial driving Brownian motion is an $H$-self-similar Gaussian martingale having independent increments, which would still be in the realm of self-similarity, but would not generate arbitrage in the model. This, what we further call hypothesis (H2), does not exhibit long-range dependence, but resembles it statistically, at least through the $H$-self-similarity property (see, e.g. [23] for an extensive overview of the results on self-similar processes). Of course, there are many cases in which the $H$-self-similarity with $H \in (1/2, 1)$ is not related to the long-range dependence. For instance, $\alpha$-stable Lévy processes are self-similar with independent increments. However, since the latter are purely jump processes (except the Gaussian case), the modeling of underlying assets using such processes leads to incomplete markets in which only a few contingent claims can be replicated, so the analysis provided below cannot be applied.

In the present paper, we consider a model of financial markets in which an investor issuing a European-type contingent claim assumes that the centred log-prices of the underlying risky asset are jointly Gaussian and self-similar with parameter $H$ from the interval $(1/2, 1)$, that corresponds to the case of long-range dependence. We assume that the investor is not sure about the fact which one of the hypotheses, (H1) or (H2), is actually realized. Then, she looks for a so-called *robust hedging strategy* with possible consumption, that allows to super-hedge the given contingent claim independently of the fact which one of the hypotheses is actually true. We define the *robust hedging price* of a contingent claim as the minimal initial wealth required to construct the robust hedging strategy. It turns out that, in order to hedge a European contingent claim with a convex payoff robustly, the investor should assume that (H2) is true. So that the *robust hedging price* is the wealth of a perfect hedge under the hypothesis (H2). In the case in which the hypothesis (H1) is realized, the investor could additionally consume a *net hedging profit*.

The problem of robust pricing and hedging was studied by Avellaneda, Levy and Parás [1] who obtained pricing and hedging bounds within the framework of an extension of the Black-Scholes model with restricted and uncertain volatility. El Karoui, Jeanblanc and Shreve

[12] proved the fact that if the misspecified (local) volatility dominates the true one from the market, then the wealth of the corresponding hedging strategy exceeds the payoff of a European contingent claim with a convex payoff at expiration. The proof of robustness of the resulting delta hedge was further simplified by Hobson [16] using the stochastic coupling arguments. Lyons [19] applied an analytic approach based on the pathwise Itô calculus due to Föllmer [13] for the uncertain volatility case. Further comparisons results for option prices were derived by Henderson [14] for passport options, and Henderson and Hobson [15] for jump-diffusion models.

More recently, Bergenthum and Rüschendorf [4]-[7] generalized these comparison results to the case of underlying multi-dimensional exponential semimartingales by means of introducing directionally convex functions. They also developed a new approach to establish the propagation of convexity property for path-dependent options and several underlying multivariate processes. This approach was taken further by Schied and Stadje [24] who proved the robustness of delta hedging for a larger class of path-dependent options using the pathwise integration approach in local volatility models. Another problem of finding optimal consumption strategies in incomplete semimartingale market models under model uncertainty was studied by Burgert and Rüschendorf [8]. In the present paper, we study the robust hedging problem in the (possibly non-semimartingale) framework of model uncertainty in the driving Gaussian $H$-self-similar process for the dynamics of the underlying price of the risky asset.

The paper is organized as follows. In Section 2, we define the Gaussian market models and fix the notation. There, we give a short note on $H$-self-similar Gaussian processes and recall the notions of self-financing strategies and arbitrage. Since we need stochastic integrals to define the self-financing condition, but cannot use the classical Itô calculus in the models with fractional Brownian motion, we have to use the forward (or pathwise) integrals. After a short review of forward integration, we reformulate the uncertainty setting with the two competing hypothesis (H1) and (H2) in terms of the driving Gaussian $H$-self-similar process. In Section 3, which is the core of the paper, we formulate and solve the robust hedging problem for European options with convex payoffs under the model uncertainty, and comment on the role of the Gaussianity assumption for the uncertainty setting. We conclude the paper with some remarks and discussion in Section 4. There, we comment how the robust option pricing problem can be viewed through the concept related to the so-called *average risk-neutral measure*. We also observe the connection between the robust hedging and the Wick-Itô-Skorohod approach for pricing of contingent claims.

## 2 Gaussian market models with uncertainty

### Gaussian self-similar market models

In this section, we consider the classical pricing model with two assets: the riskless bond, or money account, with the value $S^{(0)} = (S_t^{(0)})_{t\in[0,T]}$, and the risky stock with the price $S = (S_t)_{t\in[0,T]}$. Here, $T > 0$ is the maturity time for the contingent claims. Without loss of generality, we assume that the stock price is already discounted, that is $S_t^{(0)} \equiv 1$.

Suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ is a filtered probability space satisfying the usual conditions of completeness and right continuity of the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The stock price process $S$ is driven by an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted centred Gaussian process $X = (X_t)_{t\in[0,T]}$ which is normalized by $X_0 = 0$ and $\mathrm{Var}[X_1] = 1$, that is

$$S_t = S_0 \exp\left(\int_0^t \mu(u)\,\mathrm{d}u - \frac{\sigma^2}{2}\mathrm{Var}[X_t] + \sigma X_t\right), \tag{2.1}$$

where $\sigma > 0$ is a model parameter, the *volatility* of the stock, and $\mu$ is an absolutely continuous function integrable with respect to the Lebesgue measure. In addition to the normalization $X_0 = 0$ and $\mathrm{Var}[X_1] = 1$, we assume that the Gaussian process $X$ is $H$-self-similar with an exponent $H \in (1/2, 1)$, that is

$$(X_t)_{t\in[0,T]} \stackrel{d}{=} (a^{-H} X_{at})_{t\in[0,T/a]},$$

where $\stackrel{d}{=}$ means the equality of finite-dimensional distributions (cf., e.g. [23, Section 7.1]). Since $X$ is centred and Gaussian with $X_0 = 0$, the $H$-self-similarity means that

$$\mathrm{Cov}\,[X_t, X_s] = a^{-2H}\,\mathrm{Cov}\,[X_{at}, X_{as}]$$

for each $a > 0$ fixed and any $t, s \geq 0$.

If $X$ has stationary increments, then we denote it by $B$ and note that

$$\mathrm{Cov}\,[B_t, B_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

So we see that $B$ is the the *fractional Brownian motion*, which was introduced in [18] and given its name in [20] (see also [23, Section 7.2]). On the other hand, if $X = M$ is a Gaussian martingale, then

$$\mathrm{Cov}[M_t, M_s] = E[M_t M_s] = E\big[M_s\, E[M_t \,|\, \mathcal{F}_s]\big] = E[M_s^2] = s^{2H}$$

holds for all $s \leq t$. Therefore, the process $M$ is uniquely defined in this case too.

## Self-financing strategies and arbitrage

In order to define the notion of arbitrage, let us now fix some notation and recall some basic concepts of a financial market model (see, e.g. [25, Chapter VII]).

A *trading strategy* is a two-dimensional process $\pi$ with $\pi_t = (\beta_t, \gamma_t)$, $t \in [0, T]$, where $\beta_t$ denotes the number of bonds and $\gamma_t$ denotes the number of stocks owned by the investor at time $t$. The process $\pi$ is supposed to be adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ which is assumed to be generated by the stock price process $S$. We will further assume that there exist some smooth functions $\beta(t, s)$ and $\gamma(t, s)$ from the class $\mathcal{C}^{1,1}([0, T) \times \mathbb{R}_+)$ such that the representations $\beta_t = \beta(t, S_t)$ and $\gamma_t = \gamma(t, S_t)$ hold for all $t \in [0, T]$. The *wealth* process $V(\pi)$ associated with the trading strategy $\pi$ is

$$V_t(\pi) = \beta_t + \gamma_t S_t, \qquad t \in [0, T].$$

We assume that $\pi$ is *admissible*, i.e. $V(\pi)$ is bounded from below by some deterministic constant, that rules out doubling strategies and hence the constructing of artificially cheap strategies. Being based on the idea of the budget constraint on the changes of the position in the portfolio on the time interval $[t, t + \Delta t]$, we assume that trading strategies are *self-financing*, that means

$$V_{t+\Delta t}(\pi) \;=\; V_t(\pi) + \gamma_t(S_{t+\Delta t} - S_t).$$

From this it follows that the trading strategy is self-financing if its wealth satisfies

$$V_t(\pi) = V_0(\pi) + \int_0^t \gamma_u \, \mathrm{d}S_u, \tag{2.2}$$

where the type of the integral is explained below.

An (admissible self-financing) trading strategy $\pi$ realizes an *arbitrage opportunity* if for $V_0(\pi) = 0$ we have $V_T(\pi) \geq 0$ ($P$-a.s.) and $V_T(\pi) > 0$ holds with a positive $P$-probability. A (perfect) *hedge* of a contingent claim with a non-negative $\mathcal{F}_T$-measurable payoff $G$ is a (self-financing) trading strategy $\pi$ that replicates the claim, that is $V_T(\pi) = G$ ($P$-a.s.). A (super-) *hedge* of a claim with the payoff $G$ is a trading strategy $\pi$ such that $V_T(\pi) \geq G$ ($P$-a.s.). The (no-arbitrage) *rational price* $P(G)$ of the claim $G$ is then the *minimal* initial capital needed to super-hedge it, that means

$$P(G) = \inf\left\{\beta_0 + \gamma_0 S_0\,;\ \text{there is a } \pi \text{ such that } V_T(\pi) \geq G\ \ (P\text{-a.s.})\right\}. \tag{2.3}$$

## Forward integration

Since we will be working with a non-semimartingale model, the classical Itô integration theory is not at our disposal. However, there is an economically meaningful notion of integral, viz. the *forward integral*, that can be applied for non-semimartingales and, in particular, to the definition of (2.2). Actually, there are several slightly different versions of the forward integral. Here, we use a simple approach introduced in [13]. For different kinds of forward integrals, we refer to [22] and [29].

Let $\Pi_n = \{0 = t_{n,0} < \cdots < t_{n,K(n)} = T\}$ be a partition of the interval $[0,T]$. The sequence of partitions $(\Pi_n)_{n=1}^{\infty}$ of $[0,T]$ is called *refining* if

$$\mathrm{mesh}(\Pi_n) := \max_{t_{n,k}\in\Pi_n} |t_{n,k} - t_{n,k-1}| \to 0$$

as $n \to \infty$. In what follows, $(\Pi_n)$ will be a fixed refining sequence of partitions of $[0,T]$, that is omitted in the notation.

Further, we cannot assume that the processes are properly integrable over the entire interval $[0,T]$. Thus, we define the integrals over the sub-intervals $[0,t]$ for each $t < T$. The integral over the interval $[0,T]$ will then be interpreted as an improper forward integral.

**Definition 2.1** Let $Z = (Z_u)_{u\in[0,T]}$ be a continuous process and let $t < T$. The *forward integral* of a process $Y = (Y_u)_{u\in[0,T]}$ with respect to $Z$ *along the sequence* $(\Pi_n)$ is defined by

$$\int_0^t Y_u \,\mathrm{d}Z_u := \lim_{n\to\infty} \sum_{\substack{t_{n,k}\in\Pi_n \\ t_{n,k}\le t}} Y_{t_{n,k-1}} \left(Z_{t_{n,k}} - Z_{t_{n,k-1}}\right),$$

where the limit is assumed to exist ($P$-a.s.). The forward integral over the whole interval $[0,T]$ is the improper forward integral defined by

$$\int_0^T Y_u \,\mathrm{d}Z_u := \lim_{t\uparrow T} \int_0^t Y_u \,\mathrm{d}Z_u,$$

where the limit is again understood in the ($P$-a.s.) sense.

A priori, there is nothing that ensures the existence of the forward integral. However, we can show that if the integrator is a process of finite quadratic variation and the integrand is a smooth function of the integrator, then the forward integral exists.

**Definition 2.2** A process $Z = (Z_t)_{t\in[0,T]}$ is a process of *finite quadratic variation along the sequence* $(\Pi_n)_{n=1}^\infty$ if the limit

$$\langle Z\rangle_t := \lim_{n\to\infty} \sum_{\substack{t_{n,k}\in\Pi_n \\ t_{n,k}\le t}} \left(Z_{t_{n,k}} - Z_{t_{n,k-1}}\right)^2$$

exists and is finite ($P$-a.s.), for all $t \le T$, and is continuous in $t$.

**Examples 2.3** (i) For a standard Brownian motion $W$, we have $\mathrm{d}\langle W\rangle_t = \mathrm{d}t$. This fact follows from the Borel-Cantelli lemma.

(ii) For a fractional Brownian motion $B$ with $H \in (1/2, 1)$, we have $\mathrm{d}\langle B\rangle_t = 0$. This fact follows from the Hölder continuity of sample paths of the fractional Brownian motion.

(iii) If $A$ is a continuous process with zero quadratic variation along the sequence $(\Pi_n)$ and $X$ is a continuous process of finite quadratic variation process along $(\Pi_n)$, then $\mathrm{d}\langle Z + A\rangle_t = \mathrm{d}\langle Z\rangle_t$. This fact follows from the Cauchy-Schwartz inequality.

(iv) If $X$ is a process of finite quadratic variation along the sequence $(\Pi_n)$ and $f \in \mathcal{C}^1(\mathbb{R})$, then the process $f \circ Z$ is of finite quadratic variation along $(\Pi_n)$ too. Indeed, by [13, p. 148],

$$\mathrm{d}\langle f\circ Z\rangle_t = f'(Z_t)\,\mathrm{d}\langle Z\rangle_t.$$

The following Itô formula for the forward integral is a simple generalization of the theorem from [13, p. 144]. The proof is based on a second order two-dimensional Taylor expansion and is, essentially, the same as in the semimartingale case.

**Lemma 2.4 (The Itô formula)** *Let $Z$ be a continuous process of finite quadratic variation. Suppose that $f \in \mathcal{C}^{1,2}([0,T)\times\mathbb{R})$. Then the expression*

$$\begin{aligned} f(t, Z_t) &= f(s, Z_s) \\ &+ \int_s^t \frac{\partial}{\partial t} f(u, Z_u)\,\mathrm{d}u + \frac{1}{2}\int_s^t \frac{\partial^2}{\partial z^2} f(u, Z_u)\,\mathrm{d}\langle Z\rangle_u + \int_s^t \frac{\partial}{\partial z} f(u, Z_u)\,\mathrm{d}Z_u \end{aligned}$$

*holds for $s \le t < T$. In particular, the forward integral exists and has a modification which is continuous in $t$.*

**Remarks 2.5** (i) If the process $Z$ has zero quadratic variation, then the Itô formula coincides with the classical change-of-variable formula.

(ii) In the remainder of the paper, we choose continuous modifications for forward integrals.

(iii) The forward integral with a non-semimartingale integrator does not satisfy a dominated convergence theorem.

## The uncertainty setting

Assuming $X$ to be centered normalized continuous $H$-self-similar Gaussian process $X$ with an exponent $H \in (1/2, 1)$ is not sufficient to fix the probabilistic model for the risky asset $S$. But either one of the following natural but mutually exclusive assumptions is:

(H1) the centred log-returns are stationary, i.e. $X$ has stationary increments;

(H2) the market is efficient in the sense that there are no arbitrage opportunities.

**Remarks 2.6** (i) Recall that the *Fundamental Theorem of Asset Pricing* argues that a model of financial market does not admit arbitrage opportunities in a weaker sense called *free lunch with vanishing risk* if and only if there exists a probability measure being equivalent to the original one and such that the stock price process becomes a local martingale (see [11]). From this fact, it follows that the hypothesis (H2) is equivalent to the assumption that the $H$-self-similar Gaussian process $X$ with an exponent $H \in (1/2, 1)$ is a continuous local martingale, and thus a square integrable martingale. On the other hand, we recall that an $H$-self-similar Gaussian process with $H \in (1/2, 1)$ having stationary increments must be a fractional Brownian motion. So that the hypothesis (H1) is equivalent to the fact that *the driving process $X$ is the fractional Brownian motion $B$*, and the hypothesis (H2) is the same as the property of *the driving process $X$ to be the Gaussian martingale $M$*.

(ii) By virtue of the Kolmogorov continuity criterion, we see that both $B$ and $M$ are continuous processes.

(iii) The Gaussian martingale $M$ can be realized by means of an integral representation with respect to a standard Brownian motion $W$:

$$M_t = \sqrt{2H} \int_0^t u^{H-1/2} \, \mathrm{d}W_u.$$

In particular, this fact yields that the hypothesis (H2) is realized, and means that we are dealing with the time-inhomogeneous Black-Scholes model with

$$\frac{\mathrm{d}S_t}{S_t} = \mu(t)\,\mathrm{d}t + \sigma\sqrt{2H}t^{H-1/2}\,\mathrm{d}W_t \tag{2.4}$$

so that we have

$$\mathrm{d}\langle M\rangle_t = 2Ht^{2H-1}\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}\langle S\rangle_t = \sigma^2 2Ht^{2H-1}S_t^2\,\mathrm{d}t.$$

We also note that the market model under (H2) is complete in the sense that any claim can be hedged perfectly by an admissible self-financing strategy.

(iv) Let us also observe that the hypothesis (H1), corresponds to the model with

$$\frac{\mathrm{d}S_t}{S_t} = \big(\mu(t) - \sigma^2 Ht^{2H-1}\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t.$$

The reason for the local drift to be $\mu(t) - \sigma^2 Ht^{2H-1}$ is that the fractional Brownian motion with $H \in (1/2, 1)$ has zero quadratic variation. Hence, the corresponding Itô formula takes a form of the classical change-of-variable formula. Moreover, $S$ fails to be a semimartingale in that case, and thus, arbitrage opportunities can be presented explicitly under (H1) using the fact that the sample paths of $S$ have zero quadratic variation (see, e.g. [10] and [25, Chapter VII, Section 2c]).

(v) Note that the resulting volatility coefficient $\sigma\sqrt{2H}t^{H-1/2}$ is equal to zero at time $t = 0$ in the related time-inhomogeneous Black-Scholes model (2.4) while it is strictly positive at $t > 0$. This property is consistent with the fact that the initial stock price $S_0$ is constant. The consideration of analogues of more general stochastic local volatility models in which the related coefficient $\sigma(S_t)$ is a suitable function of the stock price is possible whenever the appropriate forward integrals are well defined.

# 3 Robust replication

## The problem

In this section, we consider the robust hedging problem for an investor, who does not know whether (H1) or (H2) is true, but who must hedge a European-type contingent claim. The robust pricing and hedging problem was studied in [1, 19, 12, 16, 14, 15, 24] among others within the uncertain volatility framework and in [4]-[7] within more general semimartingale setting.

**Definition 3.1** (i) An admissible self-financing strategy $\pi$ is a *robust hedge* for a contingent claim with a non-negative payoff $G$ under the uncertainty (H1) versus (H2) if it super-hedges the claim under the both hypotheses (H1) and (H2).

(ii) The *robust hedging price* $R(G)$ of the claim with the payoff $G$ is

$$R(G) = \inf\{\beta_0 + \gamma_0 S_0\,;\ \text{there is a } \pi \text{ which is a robust hedge for } G\}. \tag{3.1}$$

(iii) A robust hedging strategy $\pi$ is *minimal* if it is a perfect hedge under (H2).

We now find the solution to the robust hedging problem in the case in which the contingent claim $G$ is a European-type option, and its payoff is of the form $G = F(S_T)$ for some non-negative convex function $F$ on $\mathbb{R}_+$ having bounded one-sided derivatives

$$|F'(s\pm)| \le K \tag{3.2}$$

and thus being of a linear growth for all $s > 0$ and some constant $K > 0$ fixed (cf. [12, Definition 2.5]). Such a condition is particularly satisfied for payoff functions of (non-path-dependent) vanilla-type options.

## The solution

Let us first consider the case in which the hypothesis (H2) is true. It is well-known how contingent claims can be replicated in such a non-homogeneous Black-Scholes model (see, e.g. [25, Chapter VIII, Section 1]). Indeed, let $v(t, S_t)$ be the price of the option with the payoff $F(S_T)$ at time $t$ and assume that $v \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R}_+)$. Applying the Itô formula we get

$$\begin{aligned} v(t, S_t) &= v(0, S_0) \\ &+ \int_0^t \frac{\partial}{\partial s} v(u, S_u)\, \mathrm{d}S_u + \int_0^t \left( \frac{\partial}{\partial t} v(u, S_u) + \sigma^2 H u^{2H-1} S_u^2 \frac{\partial^2}{\partial s^2} v(u, S_u) \right) \mathrm{d}u \end{aligned} \tag{3.3}$$

for $t \le T$. Thus, we see that the function $v$ satisfies the boundary value problem for *fractional-type backward Black-Scholes partial differential equation*

$$\frac{\partial}{\partial t} v(t, s) + \sigma^2 H t^{2H-1} s^2 \frac{\partial^2}{\partial s^2} v(t, s) = 0; \tag{3.4}$$

$$v(T, s) = F(s); \tag{3.5}$$

for all $t < T$ and $s > 0$. Hence, applying the *Feynman-Kac formula*, we obtain that the non-negative solution of system (3.4)-(3.5) takes the form

$$v(t, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\left(s \exp\left(\sigma y \sqrt{T^{2H} - t^{2H}} - \frac{\sigma^2}{2}(T^{2H} - t^{2H})\right)\right) e^{-\frac{y^2}{2}} \, \mathrm{d}y \tag{3.6}$$

for all $t \leq T$ and $s > 0$, where the integral is positive and finite by virtue of the assumptions (3.2). It therefore follows that the rational price $P(F(S_T))$ from (2.3) is equal to

$$v(0, S_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\left(S_0 \exp\left(\sigma y T^H - \frac{\sigma^2}{2} T^{2H}\right)\right) e^{-\frac{y^2}{2}} \, \mathrm{d}y \tag{3.7}$$

and the admissible self-financing strategy

$$\pi_t = \left(v(t, S_t) - S_t \frac{\partial}{\partial s} v(t, S_t), \frac{\partial}{\partial s} v(t, S_t)\right) \tag{3.8}$$

is a (minimal) perfect hedge for the option with the payoff $F(S_T)$ under the hypothesis (H2) for $t \leq T$, where we set $F'(s) = (F'(s+) - F'(s-))/2$ at the points $s > 0$ at which the derivative does not exist. So that the robust hedging price $R(F(S_T))$ from (3.1) must be at least $v(0, S_0)$ that is given by (3.7). Note that the value (3.7) represents the initial wealth of the *minimal* hedge under the hypothesis (H2) in the sense that the strategy starting with a wealth strictly greater than $v(0, S_0)$ obviously realizes an arbitrage opportunity.

Let us now consider the case in which the hypothesis (H1) is true. We shall show that the strategy (3.8) with the initial wealth (3.7) is actually a super-hedge. For this, using the arguments of [12, Section 3] and taking into account assumption (3.2), we see that

$$\begin{aligned} \frac{\partial}{\partial s} v(t, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} & \exp\left(\sigma y \sqrt{T^{2H} - t^{2H}} - \frac{\sigma^2}{2}(T^{2H} - t^{2H})\right) \\ & \times F'\left(s \exp\left(\sigma y \sqrt{T^{2H} - t^{2H}} - \frac{\sigma^2}{2}(T^{2H} - t^{2H})\right)\right) e^{-\frac{y^2}{2}} \, \mathrm{d}y \end{aligned} \tag{3.9}$$

holds, where the derivative $F'$ is defined almost everywhere and is bounded and nondecreasing, since $F$ is a non-negative and convex function satisfying (3.2). It thus follows that expression (3.9) is bounded and non-decreasing in $s$, so that

$$\frac{\partial^2}{\partial s^2} v(t, s) \geq 0 \tag{3.10}$$

holds for all $t < T$ and $s > 0$. Hence, inequality (3.10) together with equation (3.4) yields that

$$\frac{\partial}{\partial t} v(t, s) \leq 0 \tag{3.11}$$

holds for all $t < T$ and $s > 0$.

Recall now that under the hypothesis (H1) we have $\mathrm{d}\langle S\rangle_t = 0$. Consequently, applying the Itô formula to the function $v$ from (3.6), we obtain

$$v(t, S_t) = v(0, S_0) + \int_0^t \frac{\partial}{\partial t} v(u, S_u)\,\mathrm{d}u + \int_0^t \frac{\partial}{\partial s} v(u, S_u)\,\mathrm{d}S_u \tag{3.12}$$

for $t \leq T$. The value $v(0, S_0)$ and the latter integral in (3.12) would represent a replication of $F(S_T) = v(T, S_T)$, while the former integral in (3.12) gives a *consumption* process

$$C_t := -\int_0^t \frac{\partial}{\partial t} v(u, S_u)\,\mathrm{d}u \tag{3.13}$$

which is positive and increasing for $t \leq T$, by virtue of property (3.11). Therefore, once we have noticed that formula (3.12) with equation (3.5) yields that the strategy $\pi$ defined in (3.8) represents hedging with consumption, or super-hedging, we may conclude that the following result holds:

**Theorem 3.2** *Suppose that the non-negative payoff function $F$ of a contingent claim is convex and such that condition (3.2) holds for all $s > 0$ and some $K > 0$. Then, the robust hedging price $R(F(S_T))$ from (3.1) is given by (3.7), and the corresponding minimal robust hedging strategy has the form (3.8). More precisely, if the hypothesis (H2) is true, then the investor hedges the claim perfectly, while if the hypothesis (H1) is true, then she super-hedges the claim. Moreover, in the latter case, the investor could consume her net hedging profit $C_T$ given by (3.13) at time $T > 0$ at which the option is exercised.*

**Remark 3.3** It is interesting to observe that, by deriving Theorem 3.2, we did not actually rely upon the fact that the logarithm of the stock price process is Gaussian. The resulting Gaussian pricing function in (3.6) was due to the Feynman-Kac formula, but the pricing argument was derived directly from the Itô formula and the properties of quadratic variation. Thus, the assertion of Theorem 3.2 remains true under the following more general uncertainty setting:

The driving centred $H$-self-similar process $X$ with an exponent $H \in (1/2, 1)$ in (2.1) is continuous and such that $X_0 = 0$ and $\mathrm{Var}[X_1] = 1$, and there is an uncertainty between the following hypotheses:

(H1′) $\mathrm{d}\langle X\rangle_t = 0$;

(H2′) $\mathrm{d}\langle X\rangle_t = 2Ht^{2H-1}\,\mathrm{d}t$.

Note that, under the hypothesis (H1′), the market model admits arbitrage opportunities. Under the hypothesis (H2′), the market model may still admit arbitrage opportunities, but at least any European vanilla-type options can be hedged (see [3]). Moreover, the arguments of the proof can be extended to the local volatility setup $\sigma(S_t)$ whenever the corresponding forward integrals are well defined.

**Remark 3.4** In the previous literature cited above, the robust pricing and hedging problem under model uncertainty was considered for the cases in which the dynamics of prices of the underlying risky assets are described by semimartingales, that yields the efficiency of the models of financial markets. The present paper develops the same approach for the framework in which the process driving the dynamics of the price of the underlying asset is Gaussian and $H$-self-similar with $H \in (1/2, 1)$. The model uncertainty is expressed by two separate hypotheses, one of which is based on the stationarity of increments of the driving process, that leads to the appearance of arbitrage opportunities due to the zero quadratic variation of the price of the underlying asset. The other hypothesis assumes that the model is efficient due to the fact that the driving process is a martingale. In particular, we observe that the resulting *tracking error* process defined in [12, Definition 2.6] takes the form of (3.13) and represents a non-negative value which the investor issuing the contingent claim could consume, whenever the stationarity hypothesis is realized.

**Example 3.5** Let $\tau$ be a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and specify the $H$-self-similar process $X$ with an exponent $H \in (1/2, 1)$ by

$$X_t = \begin{cases} B_t, & \text{if} \quad t < \tau; \\ B_\tau + (M_t - M_\tau), & \text{if} \quad \tau \leq t \leq T. \end{cases} \tag{3.14}$$

In other words, $\tau$ is the time at which the market becomes efficient. Let us then consider the model in which the stock price process $S$ is given by (2.1) with an $H$-self-similar Gaussian process $X$ from (3.14). In this case, we get from Theorem 3.2 that, for this model, the strategy (3.8) is a perfect hedge when $\tau = 0$, and a super-hedge otherwise. Furthermore, one can clearly consider several stopping times $\tau_j$, $j = 1, \ldots, m$, such that $0 \leq \tau_1 < \cdots < \tau_m \leq T$, for some $m \in \mathbb{N}$, at which the market changes from an efficient to an arbitrage one and vice versa. Therefore, we get again from Theorem 3.2 that the strategy (3.8) is a robust hedge.

# 4 Concluding remarks and discussion

The pricing of options and arbitrage in the *fractional Black-Scholes model*, i.e. under the hypothesis (H1), has been studied in [2, 3, 9, 17, 25, 27] among others. Since the fractional Black-Scholes model based on the forward (pathwise) integration admits arbitrage opportunities, there is no risk-neutral measure to be used for pricing. The most likely analogue to the risk-neutral measure is the so-called *average risk-neutral measure*. Since, in this case, it is not appropriate to ask for an equivalent probability measure under which $S$ is a martingale, one asks merely for an equivalent measure $Q$ such that $S_t$ is log-normal with

$$E_Q[S_t] = S_0$$

for all $t \leq T$. Such a probability measure $Q$ exists and is unique. This measure was introduced in [28]. Another approach is to use the so-called *Wick-Itô-Skorohod integrals* to define the wealth processes of admissible self-financing strategies and to compute the prices of contingent claims at times $0 < t < T$ in terms of quasi-conditional expectations. This was the approach taken in [17]. The connection of that and the more economically meaningful forward integration approach was investigated in [27].

Being economically not well-justified, both the Wick-Itô-Skorohod approach and the approach based on the forward integration and using the average risk-neutral-measure surprisingly give the same pricing (and hedging) formulas as the hypothesis (H2) does. So that they actually correspond to the case in which the stock price process is driven by an $H$-self-similar Gaussian martingale. Let us also note that the consumption process (3.13) is the difference between the wealth values of the corresponding replicating self-financing strategies, obtained by means of the forward and Wick-Itô-Skorohod integration approaches studied in [27].

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