Two switching multiple disorder problems for Brownian motions

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The multiple disorder problem seeks to determine a sequence of stopping times which are as close as possible to the unknown times of disorders at which the observation process changes its probability characteristics. We derive closed form solutions in two formulations of the multiple disorder problem for an observable Brownian motion with switching constant drift rates. The method of proof is based on the reduction of the initial problems to appropriate optimal switching problems and the analysis of the associated coupled free-boundary problems. We also describe the sequential switching multiple disorder detection procedures resulting from these formulations.

1. Introduction

Suppose that at time $t = 0$ we begin to observe a sample path of some continuous process $X = (X_t)_{t \geq 0}$ with probability characteristics changing at some unknown disorder times $(\eta_n)_{n \in \mathbb{N}}$ at which an unobservable two-state process $\Theta = (\Theta_t)_{t \geq 0}$ switches between one state and the other. The switching multiple disorder problem is to decide at which time instants $(\tau_n)_{n \in \mathbb{N}}$ one should give alarm signals to indicate the occurrence of changes in the current state of the

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process Θ as close as possible to the initial disorder times $(\eta_n)_{n\in\mathbb{N}}$. Such disorder (or change-point) detection problems have originally arisen and still play a prominent role in quality control, where one observes the output of a production line and wishes to detect deviations from acceptable levels. After the introduction of the original control charts by Shewhart [29], various modifications of the disorder problem have been recognized (see, e.g. Pages [23]) and implemented in a number of applied sciences (see, e.g. Carlstein, Müller and Siegmund [12]).

The problem of detecting a single change in the constant drift rate of a Brownian motion (Wiener process) was formulated and explicitly solved by Shiryaev [30]-[31] and [34]-[35] (see also Shiryaev [36; Chapter IV] and Peskir and Shiryaev [24; Chapter VI, Section 22] for further references). The optimal time of alarm was sought as a stopping time minimising a linear combination of the false alarm probability and the average time delay in detecting of the disorder correctly. Shiryaev [30] and [32] also proposed another formulation of the problem in which the occurrence of a single change should be preceded by a long period of observations under which a stationary regime has been established. The resulting optimal *multistage* detection procedure consisted in searching for a sequence of stopping times minimising the average time delay given that the mean time between two false alarms is fixed. More recently, Feinberg and Shiryaev [16] derived an explicit solution of the quickest detection problem in the generalized Bayesian formulation and proved the asymptotic optimality of the associated detection procedure for the related minimax formulation. Extensive overviews of these and other related sequential quickest change-point detection methods were provided in Shiryaev [37] and Poor and Hadjiliadis [26].

In the present paper, we formulate and solve the switching multiple disorder problem for an observed Wiener process X changing its drift rate from μ_j to μ_{1-j} when Θ changes its state from j to $1-j$, for every $j = 0, 1$. In contrast to the problem of detecting a *single* change, in the *switching multiple* disorder problem, one looks for an infinite sequence of the alarm times $(\tau_n)_{n\in\mathbb{N}}$ minimising a series of linear combinations of *discounted* average losses due to false alarms and delay penalties in detecting of the disorder times $(\eta_n)_{n\in\mathbb{N}}$ correctly. We propose two different formulations of the problem based on a specification of dynamics of the process Θ. In the first formulation, Θ is assumed to be a continuous time Markov chain of intensity λ, the dynamics of which are not influenced by the alarm times $(\tau_n)_{n\in\mathbb{N}}$. In the second formulation, it is assumed that the subsequent time η_n, at which Θ changes its state, can only occur after the previous alarm is sounded at τ_{n-1}. Moreover, it is assumed that the differences $(\eta_n - \tau_{n-1})_{n\in\mathbb{N}}$ form a sequence of (conditionally) independent exponential random variables.

Apart from other possible areas of application, such a situation usually happens in models of liquid financial markets having trading investors of different kinds. It is natural to assume that the *small* investors can only influence little fluctuations of the market prices of risky assets, while the *large* investors can affect the pricing trends as well, by means of either buying or selling substantial amounts of assets. More precisely, the pricing trends should either rise up or fall down at some random times, after essential amounts of assets are bought or sold, respectively. We can thus consider a model of a financial market of such kind in which the dynamics (of the logarithms) of the asset prices are described by a Brownian motion with switching drift rates. We may further assume that our model allows for an infinite number of transactions (free on charge) on the infinite time interval and use an exponential constant discounting rate r, which can be chosen equal to the riskless short rate of a bank account. The problem of detecting of a single change in the probability characteristics of accessible financial data, which is associated with the appearance of arbitrage opportunities in the market, was considered by Shiryaev [37].

In the present paper, we reduce the initial multiple disorder problems to appropriate *optimal switching problems* for filtering estimates of the current state of the unobservable drift rate of a Brownian motion. The use of exponential discounting makes our problem well connected to the problem of single disorder detection with exponential delay penalty costs studied by Poor [25], Beibel [8], and Bayraktar and Dayanik [3]. We show that the optimal switching times can be expressed as the first times at which the appropriate posterior probability processes hit certain constant boundaries. We derive closed form expressions for the resulting Bayesian risk functions and the optimal switching boundaries by means of solving the associated *coupled free-boundary problems* for ordinary differential operators. We also construct sequential switching multiple disorder detection procedures resulting from the two formulations.

Optimal switching problems represent extensions of stopping problems and games in which one looks for an infinite sequence of optimal stopping times. A general approach for studying such problems was developed in Bensoussan and Friedman [9]-[10] and Friedman [17] (see also Friedman [18; Chapter XVI]). This investigation was continued by Brekke and Øksendal [11], Duckworth and Zervos [14], Yushkevich and Gordienko [40], and Hamadène and Jeanblanc [19] among others for the continuous time case, and by Yushkevich [38]-[39] for the discrete time case. Other optimal switching and impulse control problems involving hidden Markov chains in the observable jump processes were recently studied by Bayraktar and Ludkovski [6]-[7].

The paper is organized as follows. In Section 2, for the initial multiple disorder problems, we construct the appropriate optimal switching problems and reduce the latter to their equivalent coupled optimal stopping problems. In Section 3, we derive closed form solutions of the associated coupled free-boundary problems, which are expressed in terms of Heun's double confluent functions and Kummer's confluent hypergeometric functions. In Section 4, we verify that the solutions of the coupled free-boundary problems provide the solutions of the initial optimal switching problems, and describe the resulting sequential switching multiple disorder detection procedures. The main results of the paper are stated in Theorems 4.1 and 4.2. The optimal sequential detecting schemes are displayed more explicitly in Remark 4.3.

2. Formulation of the problems

In this section, we give two formulations of the switching multiple disorder problem for an observed Brownian motion (see, e.g. [36; Chapter IV, Section 4] or [24; Chapter VI, Section 22] for the single disorder case). In these formulations, it is assumed that one observes a sample path of the Brownian motion X with the drift rate switching between μ_0 and μ_1 at some random times $(\eta_n)_{n \in \mathbb{N}}$.

2.1. (The setting.) Let us assume that all the considerations take place on a probability space $(\Omega, \mathcal{G}, P_\pi)$ with a standard Brownian motion (Wiener process) $B = (B_t)_{t \ge 0}$ started at zero under P_π. Suppose that there exists a right-continuous process Θ with two states 0 and 1, having the initial distribution $\{1-\pi, \pi\}$ under P_π, for $\pi \in [0,1]$. It is assumed that the process Θ is unobservable, so that the switching times $\eta_n = \inf\{t \ge \eta_{n-1} \,|\, \Theta_t \neq \Theta_{\eta_{n-1}}\}$, for $n \in \mathbb{N}$, with $\eta_0 = 0$, at which Θ changes its state from j to $1-j$, for every $j = 0, 1$, are unknown, that is, they cannot be observed directly.

Suppose that we observe a continuous process $X = (X_t)_{t \ge 0}$ solving the stochastic differential equation:

$$dX_t = \big(\mu_0 + (\mu_1 - \mu_0)\, \Theta_t\big)\, dt + \sigma\, dB_t \quad (X_0 = 0) \tag{2.1}$$

where $\mu_0 \neq \mu_1$ and $\sigma > 0$ are some given constants. Being based upon the continuous observation of X, our task is to find among non-decreasing sequences of stopping times $(\tau_n)_{n \in \mathbb{N}}$ of X (i.e., stopping times with respect to the natural filtration $\mathcal{F}_t = \sigma(X_s \,|\, 0 \le s \le t)$ of the process X, for $t \ge 0$) at which the alarms should be sounded *as close as possible* to the unknown switching times of the process Θ. More precisely, the *switching multiple disorder* *problem* consists of computing the Bayesian risk functions:

$$V_0^*(\pi) = \inf_{(\tau_{0,n})} \sum_{k=1}^{\infty} E_\pi \Big[a\, e^{-r\tau_{0,2k-1}}\, I(\Theta_{\tau_{0,2k-1}} = 1) + b\, e^{-r\tau_{0,2k}}\, I(\Theta_{\tau_{0,2k}} = 0) \qquad (2.2)$$

$$+ \sum_{j=0}^{1} \int_{\tau_{0,2k-2+j}}^{\tau_{0,2k-1+j}} e^{-rt}\, I(\Theta_t = j)\, dt \Big]$$

$$V_1^*(\pi) = \inf_{(\tau_{1,n})} \sum_{k=1}^{\infty} E_\pi \Big[b\, e^{-r\tau_{1,2k-1}}\, I(\Theta_{\tau_{1,2k-1}} = 0) + a\, e^{-r\tau_{1,2k}}\, I(\Theta_{\tau_{1,2k}} = 1) \qquad (2.3)$$

$$+ \sum_{j=0}^{1} \int_{\tau_{1,2k-2+j}}^{\tau_{1,2k-1+j}} e^{-rt}\, I(\Theta_t = 1-j)\, dt \Big]$$

and finding the non-decreasing sequences of optimal stopping times $(\tau^*_{i,n})_{n\in\mathbb{N}}$ such that $\tau^*_{i,0} = 0$, $i = 0, 1$, at which the infima in (2.2) and (2.3) are attained, respectively, where $I(\cdot)$ denotes the indicator function. Note that the function $V_i^*(\pi)$ expresses the Bayesian risk of the whole sequence $(\tau_{i,n})_{n\in\mathbb{N}}$ in the case in which the process Θ starts at $\Theta_0 = 1-i$, for every $i = 0, 1$ fixed, and all $\pi \in [0, 1]$. We therefore see that $E_\pi\big[e^{-r\tau_{i,n}}\, I(\Theta_{\tau_{i,n}} = j)\big]$ is the average discounted loss due to a false alarm, and $E_\pi\big[\int_{\tau_{i,n-1}}^{\tau_{i,n}} e^{-rt}\, I(\Theta_t = 1-j)\, dt\big]$ expresses the average discounted loss due to a delay in detecting of the time at which Θ changes its state from j to $1-j$ correctly, for every $i, j = 0, 1$ and any $n \in \mathbb{N}$. In this case, $a, b > 0$ are costs of false alarms and $r > 0$ is a discounting rate.

Using the fact that $(\tau_{i,n})_{n\in\mathbb{N}}$ is a non-decreasing sequence of stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, by means of standard arguments, which are similar to those presented in [36; pages 195-197], we get that:

$$E_\pi\big[e^{-r\tau_{i,n}}\, I(\Theta_{\tau_{i,n}} = j)\big] = E_\pi\big[E_\pi\big[e^{-r\tau_{i,n}}\, I(\Theta_{\tau_{i,n}} = j)\,\big|\,\mathcal{F}_{\tau_{i,n}}\big]\big] = E_\pi\big[e^{-r\tau_{i,n}}\, P_\pi(\Theta_{\tau_{i,n}} = j\,|\,\mathcal{F}_{\tau_{i,n}})\big] \qquad (2.4)$$

and

$$E_\pi\Big[\int_{\tau_{i,n-1}}^{\tau_{i,n}} e^{-rt}\, I(\Theta_{\tau_{i,n}} = j)\, dt\Big] = E_\pi\Big[\int_0^\infty e^{-rt}\, I(\tau_{i,n-1} \leq t, \Theta_t = j, t < \tau_{i,n})\, dt\Big] \qquad (2.5)$$

$$= E_\pi\Big[\int_0^\infty E_\pi\big[e^{-rt}\, I(\tau_{i,n-1} \leq t, \Theta_t = j, t < \tau_{i,n})\,\big|\,\mathcal{F}_t\big]\, dt\Big] = E_\pi\Big[\int_{\tau_{i,n-1}}^{\tau_{i,n}} e^{-rt}\, P_\pi(\Theta_t = j\,|\,\mathcal{F}_t)\, dt\Big]$$

holds for every $i, j = 0, 1$ and any $n \in \mathbb{N}$.

We further consider two different formulations of the problem, depending on the specified dynamics of the process Θ. The first formulation does not involve any influence of the alarm times $\tau_{i,n}$ on the times of changes η_n. In the second formulation, it is assumed that the change at η_n can occur only after the previous alarm is sounded at $\tau_{i,n-1}$, for every $i = 0, 1$ and any $n \in \mathbb{N}$.

2.2. (The first formulation.) Suppose that Θ is a continuous time Markov chain which is independent of the Brownian motion B and has the initial distribution $\{1-\pi, \pi\}$ under P_π. Assume that Θ has the transition-probability matrix $\{e^{-\lambda t}, 1-e^{-\lambda t}; 1-e^{-\lambda t}, e^{-\lambda t}\}$ and the intensity-matrix $\{-\lambda, \lambda; \lambda, -\lambda\}$, for all $t \geq 0$ and some $\lambda > 0$ fixed. In other words, the Markov chain Θ changes its state at exponentially distributed times of intensity λ, which are independent of the dynamics of the Brownian motion B. Such a process Θ is called *telegraphic signal* of intensity λ in the literature (see, e.g. [21; Chapter IX, Section 4] or [15; Chapter VIII]).

It thus follows from [21; Chapter IX, Theorem 9.1] (see also [21; Chapter IX, Example 3]) that the *posterior probability* process $\Pi = (\Pi_t)_{t\geq 0}$ defined by $\Pi_t = P_\pi(\Theta_t = 1 \,|\, \mathcal{F}_t)$ solves the stochastic differential equation:

$$d\Pi_t = \lambda(1-2\Pi_t)\,dt + \frac{\mu_1-\mu_0}{\sigma}\,\Pi_t(1-\Pi_t)\,d\overline{B}_t \quad (\Pi_0 = \pi) \tag{2.6}$$

where the innovation process $\overline{B} = (\overline{B}_t)_{t\geq 0}$ defined by:

$$\overline{B}_t = \frac{1}{\sigma}\Big(X_t - \int_0^t \big(\mu_0 + (\mu_1-\mu_0)\,\Pi_s\big)\,ds \Big) \tag{2.7}$$

is a standard Brownian motion according to P. Lévy's characterization theorem (see, e.g. [21; Chapter IV, Theorem 4.1]). It is also seen from (2.6) that Π is a (time-homogeneous strong) Markov process with respect to its natural filtration, which obviously coincides with $(\mathcal{F}_t)_{t\geq 0}$.

Taking into account the expressions in (2.4) and (2.5), we therefore conclude that the Bayesian risk functions from (2.2) and (2.3) admit the representations:

$$\begin{aligned} V_0^*(\pi) = \inf_{(\tau_{0,n})} \sum_{k=1}^{\infty} E_\pi \Big[& a\, e^{-r\tau_{0,2k-1}}\, \Pi_{\tau_{0,2k-1}} + \int_{\tau_{0,2k-2}}^{\tau_{0,2k-1}} e^{-rt}\,(1-\Pi_t)\,dt \\ & + b\, e^{-r\tau_{0,2k}}\,(1-\Pi_{\tau_{0,2k}}) + \int_{\tau_{0,2k-1}}^{\tau_{0,2k}} e^{-rt}\,\Pi_t\,dt \Big] \end{aligned} \tag{2.8}$$

$$\begin{aligned} V_1^*(\pi) = \inf_{(\tau_{1,n})} \sum_{k=1}^{\infty} E_\pi \Big[& b\, e^{-r\tau_{1,2k-1}}\,(1-\Pi_{\tau_{1,2k-1}}) + \int_{\tau_{1,2k-2}}^{\tau_{1,2k-1}} e^{-rt}\,\Pi_t\,dt \\ & + a\, e^{-r\tau_{1,2k}}\, \Pi_{\tau_{1,2k}} + \int_{\tau_{1,2k-1}}^{\tau_{1,2k}} e^{-rt}\,(1-\Pi_t)\,dt \Big] \end{aligned} \tag{2.9}$$

where the infima are taken over all sequences of stopping times $(\tau_{i,n})_{n \in \mathbb{N}}$, $i = 0, 1$, of the process Π. By virtue of the strong Markov property of the process Π, we can reduce the system of (2.8) and (2.9) to the following *coupled optimal stopping problem*:

$$V_0^*(\pi) = \inf_{\tau_0} E_\pi \left[a \, e^{-r\tau_0} \, \Pi_{\tau_0} + \int_0^{\tau_0} e^{-rt} \, (1 - \Pi_t) \, dt + V_1^*(\Pi_{\tau_0}) \right] \tag{2.10}$$

$$V_1^*(\pi) = \inf_{\tau_1} E_\pi \left[b \, e^{-r\tau_1} \, (1 - \Pi_{\tau_1}) + \int_0^{\tau_1} e^{-rt} \, \Pi_t \, dt + V_0^*(\Pi_{\tau_1}) \right] \tag{2.11}$$

where the infima are taken over all stopping times τ_i, $i = 0, 1$, of the process Π with $P_\pi(\Pi_0 = \pi) = 1$. We further search for optimal stopping times in (2.10) and (2.11) of the form:

$$\tau_0^* = \inf\{t \geq 0 \, | \, \Pi_t \leq g_*\} \quad \text{and} \quad \tau_1^* = \inf\{t \geq 0 \, | \, \Pi_t \geq h_*\} \tag{2.12}$$

for some $0 < g_* < h_* < 1$, where g_* is the largest and h_* is the smallest number π from $[0, 1]$ such that $V_0^*(\pi) = a\pi + V_1^*(\pi)$ and $V_1^*(\pi) = b(1 - \pi) + V_0^*(\pi)$ holds, respectively. This fact implies that the sequences of stopping times $(\tau_{i,n}^*)_{n \in \mathbb{N}}$ given by:

$$\tau_{i,2k-1+i}^* = \inf\{t \geq \tau_{i,2k-2+i}^* \, | \, \Pi_t \leq g_*\} \quad \text{and} \quad \tau_{i,2k-i}^* = \inf\{t \geq \tau_{i,2k-1-i}^* \, | \, \Pi_t \geq h_*\} \tag{2.13}$$

for every $i = 0, 1$ and any $k \in \mathbb{N}$, are optimal in the problems of (2.8) and (2.9).

2.3. (The second formulation.) As that is the case in the previous formulation, for every $i = 0, 1$, let us denote by $(\zeta_{i,2k-i})_{k \in \mathbb{N}}$ and $(\zeta_{i,2k-1+i})_{k \in \mathbb{N}}$ the sequences of alarm times sounded to indicate that the state of Θ has been changed from 0 to 1 or from 1 to 0, respectively. Let us now assume that the switching time η_n of the process Θ can only occur after the previous alarm is sounded at $\zeta_{i,n-1}$, for any $n \in \mathbb{N}$. Suppose that $(\xi_{i,n})_{n \in \mathbb{N}}$ defined by $\xi_{i,n} = \eta_n - \tau_{i,n-1}$ forms a sequence of (conditionally) independent non-negative random variables such that $\xi_{i,n}$ is independent of the Brownian motion B on the time interval $[\zeta_{i,n-1}, \zeta_{i,n}]$. Moreover, we assume that the properties $P_\pi(\eta_n = \zeta_{i,n-1} \, | \, \mathcal{F}_{\zeta_{i,n-1}}) = \Pi_{\zeta_{i,n-1}}$ and $P_\pi(\eta_n > t \, | \, \eta_n > \zeta_{i,n-1}, \mathcal{F}_{\zeta_{i,n-1}}) = e^{-\lambda(t - \zeta_{i,n-1})}$ hold for all $t \geq \zeta_{i,n-1}$ and some $\lambda > 0$ fixed. In other words, the process Θ changes its state in the exponential time $\xi_{i,n} = \eta_n - \zeta_{i,n-1}$ of intensity λ after the time of the previous alarm $\zeta_{i,n-1}$, where $\xi_{i,n}$ does not depend on the subsequent fluctuations of the process B.

It thus follows from [21; Chapter IX, Theorem 9.1] (see also [21; Chapter IX, Example 2] or [15; Chapter VIII]) that the posterior probability process Π solves the stochastic differential equation:

$$d\Pi_t^{(0)} = -\lambda \Pi_t^{(0)} \, dt + \frac{\mu_1 - \mu_0}{\sigma} \, \Pi_t^{(0)} (1 - \Pi_t^{(0)}) \, d\overline{B}_t \quad (\Pi_{\zeta_{i,2k-2+i}}^{(0)} = \Pi_{\zeta_{i,2k-2+i}}^{(1)}) \tag{2.14}$$

for $\zeta_{i,2k-2+i} \le t \le \zeta_{i,2k-1+i}$ and

$$d\Pi_t^{(1)} = \lambda(1-\Pi_t^{(1)})\,dt + \frac{\mu_1-\mu_0}{\sigma}\,\Pi_t^{(1)}(1-\Pi_t^{(1)})\,d\overline{B}_t \quad (\Pi_{\zeta_{i,2k-1-i}}^{(1)} = \Pi_{\zeta_{i,2k-1-i}}^{(0)}) \tag{2.15}$$

for $\zeta_{i,2k-1-i} \le t \le \zeta_{i,2k-i}$, where the process $\overline{B}$ is defined in (2.7) and turns out to be a standard Brownian motion on the time intervals $[\zeta_{i,n-1}, \zeta_{i,n}]$, for every $i = 0, 1$ and any $k, n \in \mathbb{N}$.

Taking into account the expressions in (2.4) and (2.5), we may conclude that the Bayesian risk functions in this formulation are given by:

$$U_0^*(\pi) = \inf_{(\zeta_{0,n})} \sum_{k=1}^{\infty} E_\pi \Big[a\,e^{-r\zeta_{0,2k-1}}\,\Pi_{\zeta_{0,2k-1}}^{(0)} + \int_{\zeta_{0,2k-2}}^{\zeta_{0,2k-1}} e^{-rt}\,(1-\Pi_t^{(0)})\,dt \tag{2.16}$$
$$+ b\,e^{-r\zeta_{0,2k}}\,(1-\Pi_{\zeta_{0,2k}}^{(1)}) + \int_{\zeta_{0,2k-1}}^{\zeta_{0,2k}} e^{-rt}\,\Pi_t^{(1)}\,dt \Big]$$

$$U_1^*(\pi) = \inf_{(\zeta_{1,n})} \sum_{k=1}^{\infty} E_\pi \Big[b\,e^{-r\zeta_{1,2k-1}}\,(1-\Pi_{\zeta_{1,2k-1}}^{(1)}) + \int_{\zeta_{1,2k-2}}^{\zeta_{1,2k-1}} e^{-rt}\,\Pi_t^{(1)}\,dt \tag{2.17}$$
$$+ a\,e^{-r\tau_{1,2k}}\,\Pi_{\zeta_{1,2k}}^{(0)} + \int_{\zeta_{1,2k-1}}^{\zeta_{1,2k}} e^{-rt}\,(1-\Pi_t^{(0)})\,dt \Big]$$

where the infima are taken over all sequences of stopping times $(\zeta_{i,n})_{n\in\mathbb{N}}$ of the processes $\Pi^{(i)} = (\Pi_t^{(i)})_{t\ge 0}$, $i = 0, 1$, solving the stochastic differential equations in (2.14) and (2.15), respectively. By virtue of the strong Markov property of the processes $\Pi^{(i)}$, $i = 0, 1$, we can reduce the system of (2.16) and (2.17) to the following coupled optimal stopping problem:

$$U_0^*(\pi) = \inf_{\zeta_0} E_\pi \left[a\,e^{-r\zeta_0}\,\Pi_{\zeta_0}^{(0)} + \int_0^{\zeta_0} e^{-rt}\,(1-\Pi_t^{(0)})\,dt + U_1^*(\Pi_{\zeta_0}^{(0)}) \right] \tag{2.18}$$

$$U_1^*(\pi) = \inf_{\zeta_1} E_\pi \left[b\,e^{-r\zeta_1}\,(1-\Pi_{\zeta_1}^{(1)}) + \int_0^{\zeta_1} e^{-rt}\,\Pi_t^{(1)}\,dt + U_0^*(\Pi_{\zeta_1}^{(1)}) \right] \tag{2.19}$$

where the infima are taken over all stopping times ζ_i of the processes $\Pi^{(i)}$, $i = 0, 1$, respectively. We further search for the optimal stopping times in (2.18) and (2.19) of the form:

$$\zeta_0^* = \inf\{t \ge 0 \,|\, \Pi_t^{(0)} \le p_*\} \quad \text{and} \quad \zeta_1^* = \inf\{t \ge 0 \,|\, \Pi_t^{(1)} \ge q_*\} \tag{2.20}$$

for some $0 < p_* < q_* < 1$, where p_* is the largest and q_* is the smallest number π from $[0,1]$ such that $U_0^*(\pi) = a\pi + U_1^*(\pi)$ and $U_1^*(\pi) = b(1-\pi) + U_0^*(\pi)$ holds, respectively. This fact implies that the sequences of stopping times $(\zeta_{i,n}^*)_{n\in\mathbb{N}}$ given by:

$$\zeta_{i,2k-1+i}^* = \inf\{t \ge \zeta_{i,2k-2+i}^* \,|\, \Pi_t^{(0)} \le p_*\} \quad \text{and} \quad \zeta_{i,2k-i}^* = \inf\{t \ge \zeta_{i,2k-1-i}^* \,|\, \Pi_t^{(1)} \ge q_*\} \tag{2.21}$$

for every $i = 0, 1$ and any $k \in \mathbb{N}$, are optimal in the problems of (2.16) and (2.17).

Remark 2.1. Recall that $(\zeta_{i,n})_{n \in \mathbb{N}}$ is a non-decreasing sequence of stopping times with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Then, by virtue of the assumption that $\eta_n \geq \zeta_{i,n-1}$ holds, standard arguments show that the equalities:

$$E_\pi \left[\int_{\zeta_{i,n-1}}^{\zeta_{i,n}} e^{-rt} \, I(\Theta_t = j) \, dt \right] = E_\pi \left[\int_0^\infty e^{-rt} \, I(\zeta_{i,n-1} \leq t, \eta_n \leq t, t < \zeta_{i,n}) \, dt \right] \tag{2.22}$$
$$= E_\pi \left[I(\eta_n < \zeta_{i,n}) \int_{\eta_n}^{\zeta_{i,n}} e^{-rt} \, dt \right] = \frac{1}{r} \, E_\pi \left[e^{-r\zeta_{i,n}} \left(e^{r(\zeta_{i,n} - \eta_n)^+} - 1 \right) \right]$$

are satisfied for every $i, j = 0, 1$ and any $n \in \mathbb{N}$. This fact builds a connection between the introduction of exponential discounting into the switching multiple disorder problem of (2.16)-(2.17) and the consideration of single disorder detection problems with exponential delay penalty costs studied in [25], [8] and [3].

2.4. (Coupled free-boundary problems.) Standard arguments based on an application of Itô's formula (see, e.g. [20; Chapter V, Section 5.1] or [22; Chapter VII, Section 7.3]) imply that the infinitesimal operator $\mathbb{L}$ of the process Π from (2.6) acts on an arbitrary twice continuously differentiable (locally) bounded function $F(\pi)$ according to the rule:

$$(\mathbb{L}F)(\pi) = \lambda \, (1 - 2\pi) \, F'(\pi) + \frac{1}{2} \left(\frac{\mu_1 - \mu_0}{\sigma} \right)^2 \pi^2 (1 - \pi)^2 \, F''(\pi) \tag{2.23}$$

for all $\pi \in (0, 1)$. In order to find the unknown value functions $V_0^*(\pi)$ and $V_1^*(\pi)$ from (2.18) and (2.19) as well as the unknown boundaries g_* and h_* from (2.12), we may use the results of the general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [36; Chapter III, Section 8] and [24; Chapter IV, Section 8]). More precisely, we formulate the associated *coupled free-boundary problem*:

$$(\mathbb{L}V_0 - rV_0)(\pi) = -(1 - \pi) \quad \text{for} \quad \pi > g, \quad (\mathbb{L}V_1 - rV_1)(\pi) = -\pi \quad \text{for} \quad \pi < h \tag{2.24}$$

$$V_0(g+) = a \, g + V_1(g+), \quad V_1(h-) = b \, (1 - h) + V_0(h-) \tag{2.25}$$

$$V_0'(g+) = a + V_1'(g+), \quad V_1'(h-) = -b + V_0'(h-) \tag{2.26}$$

$$V_0(\pi) = a \, \pi + V_1(\pi) \quad \text{for} \quad \pi < g, \quad V_1(\pi) = b \, (1 - \pi) + V_0(\pi) \quad \text{for} \quad \pi > h \tag{2.27}$$

$$V_0(\pi) < a \, \pi + V_1(\pi) \quad \text{for} \quad \pi > g, \quad V_1(\pi) < b \, (1 - \pi) + V_0(\pi) \quad \text{for} \quad \pi < h \tag{2.28}$$

$$(\mathbb{L}V_0 - rV_0)(\pi) > -(1 - \pi) \quad \text{for} \quad \pi < g, \quad (\mathbb{L}V_1 - rV_1)(\pi) > -\pi \quad \text{for} \quad \pi > h \tag{2.29}$$

with $0 < g < h < 1$, where the *instantaneous-stopping* and *smooth-fit* conditions of (2.25) and (2.26) are satisfied at g_* and h_*. Note that the superharmonic characterisation of the value function (see, e.g. [36; Chapter III, Section 8] and [24; Chapter IV, Section 9]) implies that $V_0^*(\pi)$ from (2.10) and $V_1^*(\pi)$ from (2.11) are the largest functions satisfying the expressions in (2.24)-(2.25) and (2.27)-(2.28) with the boundaries g_* and h_*.

Furthermore, standard arguments show that the infinitesimal operator $\mathbb{L}_i$ of the process $\Pi^{(i)}$ from (2.14)-(2.15) acts on an arbitrary twice continuously differentiable (locally) bounded function $F(\pi)$ according to the rule:

$$(\mathbb{L}_0 F)(\pi) = -\lambda\,\pi\,F'(\pi) + \frac{1}{2}\left(\frac{\mu_1-\mu_0}{\sigma}\right)^2 \pi^2(1-\pi)^2\,F''(\pi) \tag{2.30}$$

$$(\mathbb{L}_1 F)(\pi) = \lambda\,(1-\pi)\,F'(\pi) + \frac{1}{2}\left(\frac{\mu_1-\mu_0}{\sigma}\right)^2 \pi^2(1-\pi)^2\,F''(\pi) \tag{2.31}$$

for all $\pi \in (0,1)$ and every $i = 0,1$. In order to find the unknown value functions $U_0^*(\pi)$ and $U_1^*(\pi)$ from (2.18) and (2.19) as well as the unknown boundaries p_* and q_* from (2.12), we formulate the associated coupled free-boundary problem:

$$(\mathbb{L}_0 U_0 - rU_0)(\pi) = -(1-\pi) \quad \text{for} \quad \pi > p, \quad (\mathbb{L}_1 U_1 - rU_1)(\pi) = -\pi \quad \text{for} \quad \pi < q \tag{2.32}$$

$$U_0(p+) = a\,p + U_1(p+), \quad U_1(q-) = b\,(1-q) + U_0(q-) \tag{2.33}$$

$$U_0'(p+) = a + U_1'(p+), \quad U_1'(q-) = -b + U_0'(q-) \tag{2.34}$$

$$U_0(\pi) = a\,\pi + U_1(\pi) \quad \text{for} \quad \pi < p, \quad U_1(\pi) = b\,(1-\pi) + U_0(\pi) \quad \text{for} \quad \pi > q \tag{2.35}$$

$$U_0(\pi) < a\,\pi + U_1(\pi) \quad \text{for} \quad \pi > p, \quad U_1(\pi) < b\,(1-\pi) + U_0(\pi) \quad \text{for} \quad \pi < q \tag{2.36}$$

$$(\mathbb{L}_0 U_0 - rU_0)(\pi) > -(1-\pi) \quad \text{for} \quad \pi < p, \quad (\mathbb{L}_1 U_1 - rU_1)(\pi) > -\pi \quad \text{for} \quad \pi > q \tag{2.37}$$

with $0 < p < q < 1$, where the *instantaneous-stopping* and *smooth-fit* conditions of (2.33) and (2.34) are satisfied at p_* and q_*. The superharmonic characterisation of the value function implies that $U_0^*(\pi)$ from (2.18) and $U_1^*(\pi)$ from (2.19) are the largest functions satisfying the expressions in (2.32)-(2.33) and (2.35)-(2.36) with the boundaries p_* and q_*.

3. Solutions of the coupled free-boundary problems

In this section we solve the systems of (2.24)-(2.29) and (2.32)-(2.37) and prove the existence and uniqueness of solutions of those coupled free-boundary problems associated to the corresponding formulations of the switching multiple disorder problem.

3.1. (Existence in the first formulation.) The general solutions of the second order ordinary differential equations in (2.24) are given by:

$$V_i(\pi) = C_{i0}\, Q_{i0}(\pi) + C_{i1}\, Q_{i1}(\pi) + \frac{\lambda}{r(2\lambda + r)} + \frac{i\pi}{2\lambda + r} + \frac{(1-i)(1-\pi)}{2\lambda + r} \tag{3.1}$$

where C_{ij}, $j = 0, 1$, are some arbitrary constants, and the functions $Q_i(\pi)$, $i = 0, 1$, are given by:

$$Q_i(\pi) = \sqrt{\pi(1-\pi)}\, \exp\left(\frac{i2\lambda}{\rho(1-\pi)} + \frac{(1-i)2\lambda}{\rho\pi}\right) H_i\left((-1)^{i+1}\varphi, \psi, 0, \xi; \frac{1}{1-2\pi}\right) \tag{3.2}$$

for all $\pi \in (0, 1)$ with

$$\rho = \left(\frac{\mu_1 - \mu_0}{\sigma}\right)^2, \quad \varphi = \frac{8\lambda}{\rho}, \quad \psi = \frac{\varphi^2}{4} + \varphi - \frac{8r}{\rho} - 1 \quad \text{and} \quad \xi = 4\varphi - \psi. \tag{3.3}$$

Here, the functions $H_i(\alpha, \beta, \gamma, \delta; x)$, $i = 0, 1$, are two positive fundamental solutions (i.e. non-trivial linearly independent particular solutions) of Heun's double confluent ordinary differential equation:

$$H''(x) + \frac{2x^5 - \alpha x^4 - 4x^3 + 2x + \alpha}{(x-1)^3(x+1)^3}\, H'(x) + \frac{\beta x^2 + (2\alpha + \gamma)x + \delta}{(x-1)^3(x+1)^3}\, H(x) = 0 \tag{3.4}$$

with the boundary conditions $H(0) = 1$ and $H'(0) = 0$. Note that the series expansion of the solution of the equation in (3.4) converges under all $-1 < x < 1$, and the appropriate analytic continuation can be obtained through the identity $H(\alpha, \beta, \gamma, \delta; x) = H(-\alpha, -\delta, -\gamma, -\beta; 1/x)$. The (irregular) singularities at -1 and 1 of the equation in (3.4) are of unit rank and can be transformed into that of a confluent hypergeometric equation (see, e.g. [13] and [28] for an extensive overview and further details). According to the results from [27; Chapter V, Section 50], we can specify the positive (strictly) convex functions $Q_i(\pi)$, $i = 0, 1$, as (strictly) decreasing and increasing on the interval $(0, 1)$ and having singularities at 0 and 1, respectively.

Taking into account the fact that the function $V_0(\pi)$ should be bounded as $\pi \uparrow 1$ while the function $V_1(\pi)$ should be bounded at $\pi \downarrow 0$, we must put $C_{01} = C_{10} = 0$ in (3.1). Then, applying the instantaneous-stopping and smooth-fit conditions from (2.25) and (2.26) to the function in (3.1), we get that the equalities:

$$C_{11}\, Q_1(g) - C_{00}\, Q_0(g) = R_0(g) \quad \text{and} \quad C_{11}\, Q_1(h) - C_{00}\, Q_0(h) = R_1(h) \tag{3.5}$$

$$C_{11}\, Q_1'(g) - C_{00}\, Q_0'(g) = R_0'(g) \quad \text{and} \quad C_{11}\, Q_1'(h) - C_{00}\, Q_0'(h) = R_1'(h) \tag{3.6}$$

hold for some $0 < g < h < 1$, where we set:

$$R_0(\pi) = -a\,\pi + \frac{1-2\pi}{2\lambda + r} \quad \text{and} \quad R_1(\pi) = b\,(1-\pi) + \frac{1-2\pi}{2\lambda + r} \tag{3.7}$$

for all $\pi \in [0,1]$. Solving the left-hand part of the system in (3.5)-(3.6), we obtain:

$$\widehat{C}_{00}(g) = \frac{R_0(g)Q_1'(g) - R_0'(g)Q_1(g)}{Q_1(g)Q_0'(g) - Q_1'(g)Q_0(g)} \quad \text{and} \quad \widehat{C}_{11}(g) = \frac{R_0(g)Q_0'(g) - R_0'(g)Q_0(g)}{Q_1(g)Q_0'(g) - Q_1'(g)Q_0(g)} \tag{3.8}$$

and the solution of the right-hand part there gives:

$$\widetilde{C}_{00}(h) = \frac{R_1(h)Q_1'(h) - R_1'(h)Q_1(h)}{Q_1(h)Q_0'(h) - Q_1'(h)Q_0(h)} \quad \text{and} \quad \widetilde{C}_{11}(h) = \frac{R_1(h)Q_0'(h) - R_1'(h)Q_0(h)}{Q_1(h)Q_0'(h) - Q_1'(h)Q_0(h)} \tag{3.9}$$

so that the system in (3.5)-(3.6) is equivalent to:

$$\widehat{C}_{00}(g) = \widetilde{C}_{00}(h) \quad \text{and} \quad \widehat{C}_{11}(g) = \widetilde{C}_{11}(h) \tag{3.10}$$

for $0 < g < h < 1$. It thus follows that the functions:

$$V_0(\pi; g) = \widehat{C}_{00}(g)\, Q_0(\pi) + \frac{\lambda + r(1-\pi)}{r(2\lambda + r)} \quad \text{and} \quad V_1(\pi; h) = \widetilde{C}_{11}(h)\, Q_1(\pi) + \frac{\lambda + r\pi}{r(2\lambda + r)} \tag{3.11}$$

provide a solution of the system in (2.24)-(2.26) for any $0 < g < h < 1$ fixed.

3.2. (Uniqueness in the first formulation.) Let us now show that the system in (3.10) with (3.8)-(3.9) admits a unique solution g_* and h_*. For this, using the standard comparison arguments for solutions of the second order ordinary differential equations in (2.24), we conclude that the resulting curves $\pi \mapsto V_0(\pi; g)$ and $\pi \mapsto V_1(\pi; h)$ from (3.11) do not intersect each other on the intervals $[g, 1)$ and $(0, h]$, respectively, for different $0 < g < h < 1$ fixed. We also observe by virtue of the properties of the functions $Q_i(\pi)$, $i = 0, 1$, in (3.2) that $V_0(\pi; g)$ and $V_1(\pi; h)$ are bounded and concave on $[g, 1)$ and $(0, h]$, respectively, and such that $V_0'(\pi; g) \to \infty$ as $\pi \downarrow 0$ and $V_1'(\pi; h) \to -\infty$ as $\pi \uparrow 1$. On the other hand, using the conditions in (2.27), we obtain by means of straightforward computations that the inequalities in (2.29) are satisfied whenever $0 < g < \overline{g}$ and $\overline{h} < h < 1$, where we set:

$$\overline{g} = \frac{1 + \lambda a}{2 + a(2\lambda + r)} \quad \text{and} \quad \overline{h} = \frac{1 + b(\lambda + r)}{2 + b(2\lambda + r)} \tag{3.12}$$

and note that $0 < \overline{g} < 1/2 < \overline{h} < 1$ holds. Hence, we may conclude that if the conditions:

$$V_0'(\overline{g}+; \overline{g}) < a + V_1'(\overline{g}+; \overline{h}) \quad \text{and} \quad V_1'(\overline{h}-; \overline{h}) > -b + V_0'(\overline{h}-; \overline{g}) \tag{3.13}$$

are satisfied, then the boundaries g_* and h_* belong to the intervals $(0, \overline{g})$ and $(\overline{h}, 1)$, respectively. In other words, the assumptions in (3.13) describe the set of all admissible parameters $a, b > 0$ for which the free-boundary problem of (2.24)-(2.29) admits a unique solution, so that the optimal stopping and switching times are given by (2.12) and (2.13), respectively.

3.3. (Existence in the second formulation.) The general solutions of the second order ordinary differential equations in (2.32) have the form:

$$U_i(\pi) = D_{i0}\, G_{i0}(\pi) + D_{i1}\, G_{i1}(\pi) + \frac{\lambda}{r(\lambda+r)} + \frac{i\pi}{\lambda+r} + \frac{(1-i)(1-\pi)}{\lambda+r} \tag{3.14}$$

where D_{ij} are some arbitrary constants and the functions $G_{ij}(\pi)$, $i,j=0,1$, are given by:

$$G_{00}(\pi) = (1-\pi)\left(\frac{\pi}{1-\pi}\right)^{\gamma_+} \Psi\left(\gamma_+ - 1, \gamma_+ - \gamma_- + 1; \frac{2\lambda\pi}{\rho(1-\pi)}\right) \tag{3.15}$$

$$G_{01}(\pi) = (1-\pi)\left(\frac{\pi}{1-\pi}\right)^{\gamma_+} \Phi\left(\gamma_+ - 1, \gamma_+ - \gamma_- + 1; \frac{2\lambda\pi}{\rho(1-\pi)}\right) \tag{3.16}$$

and

$$G_{10}(\pi) = \pi\left(\frac{1-\pi}{\pi}\right)^{\gamma_+} \Phi\left(\gamma_+ - 1, \gamma_+ - \gamma_- + 1; \frac{2\lambda(1-\pi)}{\rho\pi}\right) \tag{3.17}$$

$$G_{11}(\pi) = \pi\left(\frac{1-\pi}{\pi}\right)^{\gamma_+} \Psi\left(\gamma_+ - 1, \gamma_+ - \gamma_- + 1; \frac{2\lambda(1-\pi)}{\rho\pi}\right) \tag{3.18}$$

with

$$\rho = \left(\frac{\mu_1-\mu_0}{\sigma}\right)^2 \quad \text{and} \quad \gamma_\pm = \frac{1}{2} + \frac{\lambda}{\rho} \pm \sqrt{\left(\frac{1}{2}+\frac{\lambda}{\rho}\right)^2 + \frac{2r}{\rho}} \tag{3.19}$$

for all $\pi \in (0,1)$. Here, we denote by:

$$\Phi(\alpha,\beta;x) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k}{(\beta)_k}\,\frac{x^k}{k!} \tag{3.20}$$

$$\Psi(\alpha,\beta;x) = \frac{\pi}{\sin(\pi\beta)}\left(\frac{\Phi(\alpha,\beta;x)}{\Gamma(1+\alpha-\beta)\Gamma(\beta)} - x^{1-\beta}\,\frac{\Phi(1+\alpha-\beta,2-\beta;x)}{\Gamma(\alpha)\Gamma(2-\beta)}\right) \tag{3.21}$$

Kummer's confluent hypergeometric functions of the first and second kind, respectively, for $\beta \neq 0,-1,-2,\ldots$ and $(\beta)_k = \beta(\beta+1)\cdots(\beta+k-1)$, $k \in \mathbb{N}$, where the series in (3.20) converges under all $x>0$ (see, e.g. [1; Chapter XIII] and [2; Chapter VI]), and Γ denotes Euler's Gamma function. According to the results from [27; Chapter V, Section 50], we can specify the positive (strictly) convex functions $G_{i0}(\pi)$, $i=0,1$, and $G_{i1}(\pi)$, $i=0,1$, as (strictly) decreasing and increasing on the interval $(0,1)$ with singularities at 0 and 1, respectively.

Taking into account the fact that the function $U_0(\pi)$ should be bounded as $\pi \uparrow 1$ while the function $U_1(\pi)$ should be bounded at $\pi \downarrow 0$, we must put $D_{01} = D_{10} = 0$ in (3.14). Then, applying the instantaneous-stopping and smooth-fit conditions from (2.33) and (2.34) to the function in (3.14), we get that the equalities:

$$D_{11}\, G_{11}(p) - D_{00}\, G_{00}(p) = S_0(p) \quad \text{and} \quad D_{11}\, G_{11}(q) - D_{00}\, G_{00}(q) = S_1(q) \tag{3.22}$$

$$D_{11}\, G'_{11}(p) - D_{00}\, G'_{00}(p) = S'_0(p) \quad \text{and} \quad D_{11}\, G'_{11}(q) - D_{00}\, G'_{00}(q) = S'_1(q) \tag{3.23}$$

hold for some $0 < p < q < 1$, where we set:

$$S_0(\pi) = -a\,\pi + \frac{1-2\pi}{\lambda + r} \quad \text{and} \quad S_1(\pi) = b\,(1-\pi) + \frac{1-2\pi}{\lambda+r} \tag{3.24}$$

for all $\pi \in [0,1]$. Solving the left-hand part of the system in (3.22)-(3.23), we obtain:

$$\widehat{D}_{00}(p) = \frac{S_0(p)G'_{11}(p) - S'_0(p)G_{11}(p)}{G_{11}(p)G'_{00}(p) - G'_{11}(p)G_{00}(p)} \quad \text{and} \quad \widehat{D}_{11}(p) = \frac{S_0(p)G'_{00}(p) - S'_0(p)G_{00}(p)}{G_{11}(p)G'_{00}(p) - G'_{11}(p)G_{00}(p)} \tag{3.25}$$

and the solution of the right-hand part there gives:

$$\widetilde{D}_{00}(q) = \frac{S_1(q)G'_{11}(q) - S'_1(q)G_{11}(q)}{G_{11}(q)G'_{00}(q) - G'_{11}(q)G_{00}(q)} \quad \text{and} \quad \widetilde{D}_{11}(q) = \frac{S_1(q)G'_{00}(q) - S'_1(q)G_{11}(q)}{G_{11}(q)G'_{00}(q) - G'_{11}(q)G_{00}(q)} \tag{3.26}$$

so that the system in (3.22)-(3.23) is equivalent to:

$$\widehat{D}_{00}(p) = \widetilde{D}_{00}(q) \quad \text{and} \quad \widehat{D}_{11}(p) = \widetilde{D}_{11}(q) \tag{3.27}$$

for $0 < p < q < 1$. It thus follows that the functions:

$$U_0(\pi;p) = \widehat{D}_{00}(p)\,G_{00}(\pi) + \frac{\lambda + r(1-\pi)}{r(\lambda+r)} \quad \text{and} \quad U_1(\pi;q) = \widetilde{D}_{11}(q)\,G_{11}(\pi) + \frac{\lambda + r\pi}{r(\lambda+r)} \tag{3.28}$$

provide a solution of the system in (2.32)-(2.34) for any $0 < p < q < 1$ fixed.

3.4. (Uniqueness in the second formulation.) Let us finally follow the schema of arguments above, to prove that the system of equations in (3.27) with (3.25)-(3.26) admits a unique solution p_* and q_*. For this, we use the standard comparison arguments for solutions of the second order ordinary differential equations in (2.32) to conclude that the curves $\pi \mapsto U_0(\pi;p)$ and $\pi \mapsto U_1(\pi;q)$ from (3.28) do not intersect each other on the intervals $[p,1)$ and $(0,q]$, respectively, for different $0 < p < q < 1$ fixed. We also observe by virtue of the properties of the functions $G_{ii}(\pi)$, $i = 0,1$, in (3.15) and (3.18) that $U_0(\pi;p)$ and $U_1(\pi;q)$ are bounded and concave on $[p,1)$ and $(0,q]$, respectively, and such that $U'_0(\pi;p) \to \infty$ as $\pi \downarrow 0$ and $U'_1(\pi;q) \to -\infty$ as $\pi \uparrow 1$. Moreover, using the conditions in (2.35), we obtain by means of straightforward computations that the inequalities in (2.37) are equivalent to:

$$(2 + a(\lambda+r))p - \frac{r}{\lambda+r} < -\lambda \widetilde{D}_{11}(q)G'_{11}(p) \tag{3.29}$$

$$(2 + b(\lambda+r))(1-q) - \frac{r}{\lambda+r} < \lambda \widehat{D}_{00}(p)G'_{00}(q) \tag{3.30}$$

for $0 < p < q < 1$. Note that since the derivative $G'_{11}(\pi)$ is positive and increasing from zero to infinity, while the derivative $G'_{00}(\pi)$ is negative and increasing from minus infinity to zero, it is shown by means of standard arguments that the inequalities in (3.29) and (3.30) hold whenever $0 < p < \overline{p}$ and $\overline{q} < q < 1$, where the numbers $\overline{p}$ and $\overline{q}$ are set by:

$$\overline{p} = \widehat{p} \wedge \frac{r}{(\lambda + r)(2 + a(\lambda + r))} < \frac{1}{2} \quad \text{and} \quad \overline{q} = \widehat{q} \vee \frac{\lambda + (\lambda + r)(1 + b(\lambda + r))}{(\lambda + r)(2 + b(\lambda + r))} > \frac{1}{2}. \tag{3.31}$$

Here, the couple $\widehat{p}$ and $\widehat{q}$ is determined as a unique solution of the corresponding equations instead of the inequalities in (3.29) and (3.30) whenever it exists, and $\widehat{p} = \widehat{q} = 1/2$ otherwise. Hence, we may conclude that if the conditions:

$$U_0'(\overline{p}+; \overline{p}) < a + U_1'(\overline{p}+; \overline{q}) \quad \text{and} \quad U_1'(\overline{q}-; \overline{q}) > -b + U_0'(\overline{q}-; \overline{p}) \tag{3.32}$$

hold, then the system in (3.27) admits a unique solution p_* and q_* such that $0 < p_* < \overline{p}$ and $\overline{q} < q_* < 1$. Therefore, the assumptions in (3.32) describe the set of all admissible parameters $a, b > 0$ for which the free-boundary problem of (2.32)-(2.37) admits a unique solution, so that the optimal stopping and switching times are given by (2.20) and (2.21), respectively.

4. Main results and proofs

Taking into account the facts proved above, we are now ready to formulate and prove the main assertions of the paper.

Theorem 4.1. *Assume that the conditions in (3.13) are satisfied with $\overline{g}$ and $\overline{h}$ defined in (3.12). Then, in the switching multiple disorder problem of (2.8)-(2.9) and (2.10)-(2.11) for the process X from (2.1), the Bayesian risk functions $V_i^*(\pi)$, $i = 0, 1$, take the form:*

$$V_0^*(\pi) = \begin{cases} V_0(\pi; g_*), & \text{if } g_* < \pi \leq 1 \\ a\,\pi + V_1(\pi; h_*), & \text{if } 0 \leq \pi \leq g_* \end{cases} \tag{4.1}$$

$$V_1^*(\pi) = \begin{cases} V_1(\pi; h_*), & \text{if } 0 \leq \pi < h_* \\ b\,(1 - \pi) + V_0(\pi; g_*), & \text{if } h_* \leq \pi \leq 1 \end{cases} \tag{4.2}$$

and the optimal switching times $(\tau_{i,n}^*)_{n \in \mathbb{N}}$, $i = 0, 1$, have the structure of (2.13). Here, the functions $V_0(\pi; g)$ and $V_1(\pi; h)$ are given by (3.11), and the optimal stopping boundaries g_* and h_*, such that $0 < g_* < \overline{g} < 1/2 < \overline{h} < h_* < 1$, are uniquely determined by the coupled system of the equations in (3.10) with $\widehat{C}_{ii}(g)$ and $\widetilde{C}_{ii}(h)$ given by (3.8)-(3.9), where the functions $Q_i(\pi)$ and $R_i(\pi)$, $i = 0, 1$, are defined in (3.2) and (3.7), respectively.

Theorem 4.2. *Assume that the conditions in (3.32) are satisfied with* $\overline{p}$ *and* $\overline{q}$ *defined by (3.31), where* $\widehat{p}$ *and* $\widehat{q}$ *is a unique solution of the system of equations replacing the inequalities in (3.29)-(3.30) whenever it exists, and* $\widehat{p} = \widehat{g} = 1/2$ *otherwise. Then, in the switching multiple disorder problem of (2.16)-(2.17) and (2.18)-(2.19) for the process* X *from (2.1), the Bayesian risk functions* $U_i^*(\pi)$, $i = 0, 1$, *take the form:*

$$U_0^*(\pi) = \begin{cases} U_0(\pi; p_*), & \text{if } p_* < \pi \le 1 \\ a\,\pi + U_1(\pi; q_*), & \text{if } 0 \le \pi \le p_* \end{cases} \tag{4.3}$$

$$U_1^*(\pi) = \begin{cases} U_1(\pi; q_*), & \text{if } 0 \le \pi < q_* \\ b\,(1-\pi) + U_0(\pi; p_*), & \text{if } q_* \le \pi \le 1 \end{cases} \tag{4.4}$$

and the optimal switching times $(\zeta_{i,n}^*)_{n\in\mathbb{N}}$, $i = 0, 1$, *have the structure of (2.21). Here, the functions* $U_0(\pi; p)$ *and* $U_1(\pi; q)$ *are given by (3.28), and the optimal stopping boundaries* p_* *and* q_*, *such that* $0 < p_* < \overline{p} < 1/2 < \overline{q} < q_* < 1$, *are uniquely determined by the coupled system of the equations in (3.27) with* $\widehat{D}_{ii}(p)$ *and* $\widetilde{D}_{ii}(q)$, $i = 0, 1$, *given by (3.25)-(3.26), where the functions* $G_{ii}(\pi)$ *and* $S_i(\pi)$, $i = 0, 1$, *are defined in (3.15)-(3.18) and (3.24), respectively.*

Proof. Since the verification of the assertions stated above can be done using similar ways of arguments, we present the proof of the second one only. Namely, we show that the functions in (4.3) and (4.4) coincide with the value functions in (2.18) and (2.19), respectively, and the stopping times ζ_i^*, $i = 0, 1$, from (2.20) and thus the switching times $(\zeta_{i,n}^*)_{n\in\mathbb{N}}$ from (2.21) are optimal with the boundaries p_* and q_* specified above. For this, let us denote by $U_0(\pi)$ and $U_1(\pi)$ the right-hand sides of the expressions in (4.3) and (4.4), respectively. Hence, applying Itô's formula to $e^{-rt}U_i(\Pi_t^{(i)})$, $i = 0, 1$, and taking into account the smooth-fit conditions in (2.34), we obtain:

$$e^{-rt}\,U_i(\Pi_t^{(i)}) = U_i(\pi) + \int_0^t e^{-rs}\,(\mathbb{L}_i U_i - rU_i)(\Pi_s^{(i)})\,I(\Pi_s^{(i)} \neq p_*, \Pi_s^{(i)} \neq q_*)\,ds + M_t^{(i)} \tag{4.5}$$

where the processes $M^{(i)} = (M_t^{(i)})_{t\ge0}$ defined by:

$$M_t^{(i)} = \int_0^t e^{-rs}\,U_i'(\Pi_s^{(i)})\,\frac{\mu_1-\mu_0}{\sigma}\,\Pi_s^{(i)}(1-\Pi_s^{(i)})\,d\overline{B}_s \tag{4.6}$$

are continuous square integrable martingales under the probability measure P_π with respect to the filtration $(\mathcal{F}_t)_{t\ge0}$, for every $i = 0, 1$. The latter fact can easily be observed, since the derivatives $U_i'(\pi)$, $i = 0, 1$, are bounded functions.

Taking into account the assumptions in (3.32), it is shown by means straightforward computations and using the properties of the functions $U_i(\pi)$, $i = 0, 1$, that the conditions of (2.36) and (2.37) hold with $0 < p_* < \overline{p}$ and $\overline{q} < q_* < 1$. These facts together with the conditions in (2.32)-(2.33) and (2.35) yield that the inequalities $(\mathbb{L}_0 U_0 - r U_0)(\pi) \ge -(1-\pi)$ and $(\mathbb{L}_1 U_1 - r U_1)(\pi) \ge -\pi$ hold for all $\pi \in [0,1]$ such that $\pi \neq p_*$ and $\pi \neq q_*$, as well as $U_0(\pi) \le a\pi + U_1(\pi)$ and $U_1(\pi) \le b(1-\pi) + U_0(\pi)$ are satisfied for all $\pi \in [0,1]$. It also follows from the regularity of the diffusion processes $\Pi^{(i)}$, $i = 0, 1$, solving the stochastic differential equations in (2.14) and (2.15), that the indicator which appears in the formula (4.5) can be ignored. We therefore obtain from the expression in (4.5) that the inequalities:

$$
\begin{aligned}
& a\, e^{-r\zeta_0}\, \Pi^{(0)}_{\zeta_0} + \int_0^{\zeta_0} e^{-rs}\, (1 - \Pi^{(0)}_s)\, ds + e^{-r\zeta_0}\, U_1(\Pi^{(0)}_{\zeta_0}) \\
& \ge e^{-r\zeta_0}\, U_0(\Pi^{(0)}_{\zeta_0}) + \int_0^{\zeta_0} e^{-rs}\, (1 - \Pi^{(0)}_s)\, ds \ge U_0(\pi) + M^{(0)}_{\zeta_0}
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
& b\, e^{-r\zeta_1}\, (1 - \Pi^{(1)}_{\zeta_1}) + \int_0^{\zeta_1} e^{-rs}\, \Pi^{(1)}_s\, ds + e^{-r\zeta_1}\, U_0(\Pi^{(1)}_{\zeta_1}) \\
& \ge e^{-r\zeta_1}\, U_1(\Pi^{(1)}_{\zeta_1}) + \int_0^{\zeta_1} e^{-rs}\, \Pi^{(1)}_s\, ds \ge U_1(\pi) + M^{(1)}_{\zeta_1}
\end{aligned}
\tag{4.8}
$$

hold for any stopping times ζ_i of the processes $\Pi^{(i)}$, $i = 0, 1$, respectively.

For every $i = 0, 1$, let $(\varkappa_{i,n})_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the processes $M^{(i)}$. Then, taking the expectations with respect to the probability measure P_π in (4.7)-(4.8), by means of the optional sampling theorem (see, e.g. [21; Theorem 3.6] or [20; Chapter I, Theorem 3.22]), we get:

$$
\begin{aligned}
& E_\pi \Big[a\, e^{-r(\zeta_0 \wedge \varkappa_{0,n})}\, \Pi^{(0)}_{\zeta_0 \wedge \varkappa_{0,n}} + \int_0^{\zeta_0 \wedge \varkappa_{0,n}} e^{-rs}\, (1 - \Pi^{(0)}_s)\, ds + e^{-r(\zeta_0 \wedge \varkappa_{0,n})}\, U_1(\Pi^{(0)}_{\zeta_0 \wedge \varkappa_{0,n}}) \Big] \\
& \ge E_\pi \Big[e^{-r(\zeta_0 \wedge \varkappa_{0,n})}\, U_0(\Pi^{(0)}_{\zeta_0 \wedge \varkappa_{0,n}}) + \int_0^{\zeta_0 \wedge \varkappa_{0,n}} e^{-rs}\, (1 - \Pi^{(0)}_s)\, ds \Big] \ge U_0(\pi) + E_\pi \big[M^{(0)}_{\zeta_0 \wedge \varkappa_{0,n}} \big] = U_0(\pi)
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
& E_\pi \Big[b\, e^{-r(\zeta_1 \wedge \varkappa_{1,n})}\, (1 - \Pi^{(1)}_{\zeta_1 \wedge \varkappa_{1,n}}) + \int_0^{\zeta_1 \wedge \varkappa_{1,n}} e^{-rs}\, \Pi^{(1)}_s\, ds + e^{-r(\zeta_1 \wedge \varkappa_{1,n})}\, U_0(\Pi^{(1)}_{\zeta_1 \wedge \varkappa_{1,n}}) \Big] \\
& \ge E_\pi \Big[e^{-r(\zeta_1 \wedge \varkappa_{1,n})}\, U_1(\Pi^{(1)}_{\zeta_1 \wedge \varkappa_{1,n}}) + \int_0^{\zeta_1 \wedge \varkappa_{1,n}} e^{-rs}\, \Pi^{(1)}_s\, ds \Big] \ge U_1(\pi) + E_\pi \big[M^{(1)}_{\zeta_1 \wedge \varkappa_{1,n}} \big] = U_1(\pi)
\end{aligned}
\tag{4.10}
$$

for all $\pi \in [0,1]$. Thus, letting n go to infinity and using Fatou's lemma, we obtain that the inequalities:

$$E_\pi\left[a\,e^{-r\zeta_0}\,\Pi^{(0)}_{\zeta_0}+\int_0^{\zeta_0}e^{-rs}\,(1-\Pi^{(0)}_s)\,ds+e^{-r\zeta_0}\,U_1(\Pi^{(0)}_{\zeta_0})\right]\geq U_0(\pi) \tag{4.11}$$

$$E_\pi\left[b\,e^{-r\zeta_1}\,(1-\Pi^{(1)}_{\zeta_1})+\int_0^{\zeta_1}e^{-rs}\,\Pi^{(1)}_s\,ds+e^{-r\zeta_1}\,U_0(\Pi^{(1)}_{\zeta_1})\right]\geq U_1(\pi) \tag{4.12}$$

are satisfied for any stopping times ζ_i, $i=0,1$, and all $\pi\in[0,1]$. By virtue of the structure of the stopping times in (2.20), it is readily seen that the equalities in (4.11) and (4.12) hold with ζ_i^* instead of ζ_i, $i=0,1$, when either $\pi\leq p_*$ or $\pi\geq q_*$, respectively.

It remains to show that the equalities are attained in (4.11) and (4.12) when ζ_i^* replaces ζ_i, $i=0,1$, for $p_*<\pi<q_*$. By virtue of the fact that the functions $U_i(\pi)$, $i=0,1$, with the boundaries p_* and q_* satisfy the conditions in (2.32) and (2.33), it follows from the expression in (4.5) and the structure of the stopping times in (2.20) that the equalities:

$$e^{-r(\zeta_0^*\wedge\varkappa_{0,n})}\,U_0(\Pi^{(0)}_{\zeta_0^*\wedge\varkappa_{0,n}})+\int_0^{\zeta_0^*\wedge\varkappa_{0,n}}e^{-rs}\,(1-\Pi^{(0)}_s)\,ds=U_0(\pi)+M^{(0)}_{\zeta_0^*\wedge\varkappa_{0,n}} \tag{4.13}$$

$$e^{-r(\zeta_1^*\wedge\varkappa_{1,n})}\,U_1(\Pi^{(1)}_{\zeta_1^*\wedge\varkappa_{1,n}})+\int_0^{\zeta_1^*\wedge\varkappa_{1,n}}e^{-rs}\,\Pi^{(1)}_s\,ds=U_1(\pi)+M^{(1)}_{\zeta_1^*\wedge\varkappa_{1,n}} \tag{4.14}$$

are satisfied for all $\pi\in[0,1]$. Observe that the integrals here are finite (P_π-a.s.) as well as the processes $(M^{(i)}_{\zeta_i^*\wedge t})_{t\geq0}$, $i=0,1$, are uniformly integrable martingales. Therefore, taking the expectations in (4.13) and (4.14) and letting n go to infinity, we can apply the Lebesgue dominated convergence theorem to obtain the equalities:

$$E_\pi\left[a\,e^{-r\zeta_0^*}\,\Pi^{(0)}_{\zeta_0^*}+\int_0^{\zeta_0^*}e^{-rs}\,(1-\Pi^{(0)}_s)\,ds+e^{-r\zeta_0^*}\,U_1(\Pi^{(0)}_{\zeta_0^*})\right]=U_0(\pi) \tag{4.15}$$

$$E_\pi\left[b\,e^{-r\zeta_1^*}\,(1-\Pi^{(1)}_{\zeta_1^*})+\int_0^{\zeta_1^*}e^{-rs}\,\Pi^{(1)}_s\,ds+e^{-r\zeta_1^*}\,U_0(\Pi^{(1)}_{\zeta_1^*})\right]=U_1(\pi) \tag{4.16}$$

for all $\pi\in[0,1]$. The latter, together with the inequalities in (4.11) and (4.12), directly imply the desired assertion. □

Remark 4.3. The results formulated above show that the following sequential procedure is optimal. Being based on the observations of X, we construct the posterior probability process Π and stop the latter for the first time as soon as it exits either the region (g_*,h_*) or (p_*,q_*), appropriately, and then conclude that the process Θ has switched either from 0 to 1 or from 1 to 0, respectively. Then, we continue to observe the process Π which is currently located either in the regions $[0,g_*]$ and $[h_*,1]$ or in $[0,p_*]$ and $[q_*,1]$, and stop the observations as soon as it comes to the opposite region. We may thus conclude that Θ should have switched either from 0 to 1 or from 1 to 0, respectively, and continue the procedure from the beginning.

Remark 4.4. Taking into account the results obtained above, we may also conclude that the appropriate *minimal* Bayesian risk functions take the form:

$$V^*(\pi) = \min\{V_0^*(\pi), V_1^*(\pi)\} \quad \text{and} \quad U^*(\pi) = \min\{U_0^*(\pi), U_1^*(\pi)\} \tag{4.17}$$

for $\pi \in [0,1]$, where the functions $V_i^*(\pi)$ and $U_i^*(\pi)$, $i = 0, 1$, are defined in (2.8)-(2.9) and (2.18)-(2.19), respectively. It is also seen that if either $V^*(\pi) = V_i^*(\pi)$ or $U^*(\pi) = U_i^*(\pi)$ holds for any $\pi \in [0,1]$ fixed, then the sequences $(\tau_{i,n}^*)_{n \in \mathbb{N}}$ or $(\zeta_{i,n}^*)_{n \in \mathbb{N}}$ given by (2.13) and (2.21) are optimal in (4.17), appropriately.

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