Follower Payoffs in Symmetric Duopoly Games

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Abstract

This paper compares the leader and follower payoff in a duopoly game, as they arise in sequential play, with the Nash payoff in simultaneous play. If the game is symmetric, has a unique symmetric Nash equilibrium, and players' payoffs are monotonic in the opponent's choice along their own best reply function, then the follower payoff is either higher than the leader payoff, or even lower than in the simultaneous game. This gap for the possible follower payoff had not been observed in earlier duopoly models of endogenous timing.

Keywords: Cournot, duopoly game, endogenous timing, follower, leader, Stackelberg, strategic complements, strategic substitutes.

JEL Classification Numbers: C72, D43, L13.

1 Introduction

The classic duopoly model of quantity competition by Cournot (1838) is a game between two firms that simultaneously choose quantities, with Cournot's solution as the unique Nash equilibrium. The "leadership game" of von Stackelberg (1934) uses the same payoff functions, but where one firm, the leader, moves first, assuming a best reply of the second-moving firm, the follower. The Stackelberg solution is then a subgame perfect equilibrium of this sequential game.

Many recent papers are concerned with endogenizing the "timing" in the sequential game, that is, the order of play which determines the roles of leader and follower. In a much-cited paper, Hamilton and Slutsky (1990) take a given duopoly game and let players decide to act in one of two periods. If one player moves in the first period and the other in the second, they become leader and follower, respectively. If they move in the same period, their payoffs are as in simultaneous play. The leader-follower outcome is a Nash equilibrium of the two-period game only if the follower's payoff is not smaller than her Nash payoff in the simultaneous game. In that case, there are typically two pure Nash equilibria, with either order of play; van Damme and Hurkens (1999; 2004) use risk dominance to select one of these equilibria. If the follower would suffer compared to simultaneous play, both players act in the first period, using their equilibrium strategies from the simultaneous game.

These papers and others (for example, Amir (1995)) compare explicitly the follower payoff to the payoff the player would get as a leader or in simultaneous play. The point of the present paper is a simple observation which so far, apparently, has not been made explicitly: If the game is symmetric and certain standard assumptions hold, then the follower gets either less than in the simultaneous game, or more than the leader. That is, the seemingly natural case that both players profit from sequential play as compared to simultaneous play, but the leader more so than if he was follower, can only occur in non-symmetric games.

Our assumptions about the duopoly game are designed to be general while allowing for a very simple proof. Apart from symmetry, we assume intervals as strategy spaces, unique best replies, a unique symmetric Nash equilibrium in the simultaneous game, and monotonicity of payoffs in the other player's strategy along the own best reply function.

These assumptions encompass many duopoly models of quantity or price competition. Hamilton and Slutsky (1990) make similar assumptions. Gal-Or (1985) compares leader and follower payoffs for identical firms with differentiable payoff functions. Dowrick (1986) assumes specific functional forms of quantity competition or price competition with heterogeneous goods, and also looks at simultaneous play.

The following papers on endogenous timing differ from our setup, and give further references, in particular to applied work in industrial organization. Boyer and Moreaux (1987) allow firms to choose both prices and quantities. Deneckere and Kovenock (1992) study duopolies with price setting and capacity constraints. Amir and Grilo (1999) and Amir, Grilo, and Jin (1999) allow for multiple Nash equilibria in the simultaneous game and use the theory of supermodular games (see also Vives (1999)). Tasnádi (2003) considers price setting with homogeneous goods.

Leadership in mixed extensions of finite games is analyzed by von Stengel and Zamir (2004), with an example (in Section 7) of a symmetric game where the follower payoff can be arbitrary relative to leader payoff and simultaneous payoff. In this example, each player's strategy set is not an interval but a two-dimensional mixed strategy simplex. When considering mixed strategies, best replies are not unique. To keep the present study short, we do not consider best reply correspondences instead of functions.

In Section 2, we state and discuss our assumptions in detail, and state and prove the main Theorem 1. We assume monotonicity only along the own best reply function, a property also used by Hamilton and Slutsky (1990, p. 41). Best reply functions do not have to be monotonic.

However, as discussed in Section 3, monotonic best replies determine the follower payoff. If the best reply function increases, then the follower profits from sequential play, and if it decreases, she suffers. For increasing best reply functions, this has been observed by Gal-Or (1985) and van Damme and Hurkens (2004, p. 405). For decreasing best reply functions, Gal-Or compares only follower and leader payoff, and does not consider the simultaneous game. Games with increasing or decreasing best reply functions are often called games with *strategic complements* or *substitutes*, respectively.

In Section 4, we give examples showing that the main assumptions of symmetry and monotonicity cannot be weakened.

2 Assumptions and theorem

The duopoly games considered here are assumed to fulfill the following conditions.

(a) The players' strategy sets are (not necessarily compact) real intervals X and Y, with payoff $a(x,y)$ to player I and $b(x,y)$ to player II for player I's strategy x in X and II's strategy y in Y.

(b) The best reply $r(y)$ to y is always unique, $a(r(y),y) = \max_{x\in X} a(x,y)$, and so is the best reply $s(x)$ to x, with $b(x,s(x)) = \max_{y\in Y} b(x,y)$.

(c) The payoffs $a(r(y),y)$ and $b(x,s(x))$ are (not necessarily strictly) monotonic in y respectively x.

(d) For some x_L in X and y_L in Y, the payoffs $a_L = a(x_L,s(x_L)) = \max_{x\in X} a(x,s(x))$ and $b_L = b(r(y_L),y_L) = \max_{y\in Y} b(r(y),y)$ exist, which are the payoffs to player I and II when the respective player is a leader. Moreover, x_L and y_L are unique. The follower payoffs are denoted $b_F = b(x_L,s(x_L))$ and $a_F = a(r(y_L),y_L)$.

(e) The game is symmetric, that is, $X = Y$ and $a(x,y) = b(y,x)$, and for some y_N in Y,

$$r(y) \begin{array}{c} > \\ = \\ < \end{array} y \quad \text{for} \quad y \begin{array}{c} < \\ = \\ > \end{array} y_N. \qquad (1)$$

Condition (a) is, for example, fulfilled for $X = Y = [0,\infty)$. The payoff functions are typically continuous, but we do not require this. Condition (b) is strong but often made. Condition (c) states that a player always prefers a higher or lower choice of the opponent along the own best reply function. Hamilton and Slutsky (1990, p. 41) assume condition (c) for their Theorem VI.

Condition (d) holds when payoffs are continuous and strategy sets are compact. Without compactness, it may fail, for example in the symmetric game where $x,y \geq 0$ and

$$a(x,y) = b(y,x) = 4y - \frac{(y+3)^2}{4(x+1)} - x \tag{2}$$

where $r(y) = s(y) = (y+1)/2$, condition (c) holds since $a(r(y),y) = 3y-2$, and which has a unique Nash equilibrium at $x = y = 1$, but where the leader payoff $a(x,s(x))$ exceeds $15x/16-2$ and is therefore unbounded.

Generically, player I as leader has a unique payoff-maximizing strategy x_L. If the leader's strategy is not unique, the follower payoff depends on which leader strategy is chosen. We assume uniqueness of x_L and y_L for simplicity. Otherwise, Theorem 1 below would still apply, but then the follower payoffs have to be defined depending on the choice of the leader strategy.

When the game is symmetric as stated in (e), then obviously $s(x) = r(x)$, and the game has a unique symmetric Nash equilibrium (x_N,y_N) where $x_N = y_N$. Conversely, if payoff functions are continuous and the strategy sets are compact intervals, then (1) holds when the game has only one symmetric Nash equilibrium (to see this, consider the best reply function at the endpoints of the interval). Note that non-symmetric Nash equilibria (x,y) with $x = r(y)$ and $y = s(x)$ and $x \neq y$ may exist. One may consider uniqueness of the Nash equilibrium as an alternative to (e) when the game is not symmetric. However, example (7) below shows that our theorem fails in this case.

Theorem 1. *Under conditions* (a)–(e)*, consider the leader payoff* $a_L = a(x_L,s(x_L)) = b_L$*, follower payoff* $b_F = b(x_L,s(x_L))$*, and Nash payoff* $a_N = b_N = b(x_N,y_N)$*, where* $x_N = y_N$*. Then* $b_F > b_L$ *or* $b_F \leq b_N$*.*

Proof. If $b_L = b_N$, the claim is trivial. Player I as leader can always get at least the Nash payoff a_N by choosing x_N. If $x_L = x_N$, then, since x_L is unique by (d), $a_L = a_N$, that is, $b_L = b_N$, so we can assume $a_L > a_N$ and thus $x_L \neq x_N$.

We can assume that $a(r(y),y)$ is increasing in y, since if $a(r(y),y)$ is decreasing in y we can reverse the order on Y and X (by replacing y by $-y$ and x by $-x$, say), so that (1) continues to hold. It may be useful to consider the example in Figure 1, explained after this proof, for the following argument.

If $x_L < x_N$, then, since $s(x) = r(x)$,

$$b_F = b(x_L,s(x_L)) = a(r(x_L),x_L) \leq a(r(x_N),x_N) = b_N\,.$$

If $x_L > x_N = y_N$, then $r(x_L) < x_L$ by (1). Thus,

$$b_L = a_L = a(x_L,r(x_L)) < a(r(r(x_L)),r(x_L)) \leq a(r(x_L),x_L) = b(x_L,s(x_L)) = b_F\,.$$

The first inequality holds since $r(r(x_L))$ is the unique best reply to $r(x_L)$, which is different from x_L, since otherwise $r(r(x_L)) = x_L > r(x_L)$ and thus $r(x_L) < x_N$ by (1), giving

$$a_L = a(x_L, r(x_L)) \le a(r(r(x_L)), r(x_L)) \le a(r(x_N), x_N) = a_N \tag{3}$$

which we have excluded; so $r(r(x_L)) \neq x_L$ and the inequality is strict. $\square$

The proof shows that if condition (c) is strengthened so that $a(r(y), y)$ is strictly monotonic in y, then the follower payoff b_F is strictly less than the Nash payoff b_N if it is not greater than the leader payoff (unless all these payoffs coincide).

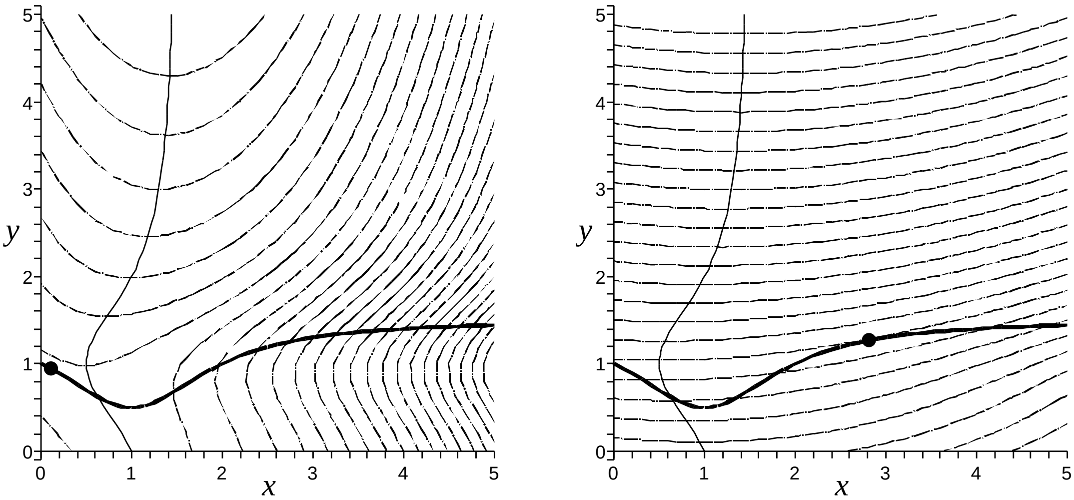

Figure 1 Contour lines of $a(x,y)$ in (4) for $\alpha = 0.7$ (left) and $\alpha = 10$ (right). The thick line is player II's best reply function $s(x) = r(x)$ with $x_L = 0.121$ (left) and $x_L = 2.808$ (right), with $(x_L, s(x_L))$ indicated by a big dot.

If the best reply function is monotonic, then the game has strategic substitutes or complements, where Theorem 1 presents a familiar results; we discuss this relationship in the next section. The following example shows that Theorem 1 holds even if the best reply function is not monotonic: Let $x, y \ge 0$ and consider the function, which will be the best reply function of player I,

$$r(y) = \frac{3}{2} - \frac{1}{(y-1)^2 + 1}.$$

This function decreases for $y \in [0, 1]$, with $r(0) = 1$, and increases from its minimum at $y = 1$ for $y \ge 1$. We consider the game with payoff functions

$$a(x,y) = b(y,x) = -\frac{x^2}{2} + x \cdot r(y) + \alpha \cdot y \tag{4}$$

for $\alpha = 0.7$ and $\alpha = 10$. Then $\frac{d}{dx}a(x,y) = -x + r(y)$, which is zero if and only if $x = r(y)$, and $r(y)$ is indeed player I's best reply to y. Figure 1 shows the contour lines of $a(x,y)$, with $r(y)$ shown as the thin curve defined by the points where the contours have a horizontal tangent. Moreover, $a(r(y),y)$ is strictly increasing in y. Player II's best reply is given by $s(x) = r(x)$, shown as a thick curve in Figure 1. The symmetric Nash equilibrium (x_N, x_N) at the intersection of the best reply curves is obtained for approximately $x_N = 0.634$. In the leadership game, player I maximizes $a(x, s(x))$ on player II's curve with the resulting value $x_L = 0.121 < x_N$ for $\alpha = 0.7$ and $x_L = 2.808 > x_N$ for $\alpha = 10$, indicated by a dot; the two values for α correspond to the two cases in the proof of Theorem 1.

3 Strategic complements and substitutes

Players' strategies are called *strategic substitutes* if the best reply to "more aggressive" behavior is "less aggressive" behavior, and *strategic complements* if the best reply to "more aggressive" behavior is "more aggressive" behavior. We use this terminology, in terms of best replies, following Mas-Colell, Whinston, and Green (1995, p. 415). Assume that "aggressive behavior" is an order on the strategy set (here an interval) representing the negative preference of the other player. For example, in quantity competition, player I typically prefers a lower quantity y of the other firm, as "less aggressive" behavior, because $a(x,y)$ is decreasing in y. In price competition, players typically prefer a higher price of the opponent as "less aggressive". Then strategic substitutes correspond to decreasing best reply functions, and strategic complements to increasing best reply functions. This does not depend on the chosen order on the interval as long as it is the same for both players.

If $a(x,y)$ is monotonic in y, the same monotonicity in y holds for $a(r(y),y)$, as in assumption (c):

Lemma 2. *Given* (a) *and* (b), *if* $a(x,y)$ *is (strictly or non-strictly) increasing or decreasing in* y, *then so is* $a(r(y),y)$.

Proof. For $y, y' \in Y$ and $y < y'$, and $a(x,y)$ strictly increasing in y, we have

$$a(r(y),y) < a(r(y),y') \leq a(r(y'),y'). \tag{5}$$

If $a(x,y)$ is strictly decreasing in y, we conclude (5) from $y > y'$. For non-strict monotonicity, replace $<$ by $\leq$ in (5). □

As mentioned, Hamilton and Slutsky (1990, p. 41) assume condition (c) for their Theorem VI. Amir (1995) notes that this condition is also necessary for their Theorem V, although he uses the stronger assumption that $a(x,y)$ is monotonic in y.

Monotonicity of $a(r(y),y)$ in y is strictly weaker than monotonicity of $a(x,y)$ in y. In the following example with $x, y \geq 0$ and

$$a(x,y) = (2x - (y+1)) \cdot (y + 1 - x), \tag{6}$$

where $a(x,y) \geq 0$ for $2x-1 \geq y \geq x-1$, we have $r(y) = 3(y+1)/4$, which is a linearly increasing best reply function. Here, $a(r(y),y) = (y+1)^2/8$, which is strictly increasing in y, but $a(x,y)$ is not monotonic in y. If (6) defines a symmetric game with $a(x,y) = b(y,x)$, then Theorem 1 applies with $x_N = 3$, $x_L = 4.2$, $s(x_L) = 3.9$, and $b_N = 2$, $b_L = 2.45$, $b_F = 3.38$.

Strategic complements and substitutes mean that $r(y)$ increases or decreases, respectively. Even when only $a(r(y),y)$ increases in y (but not generally $a(x,y)$ in y), this can be reasonably interpreted as a unique preference of player I for larger values of y as "less aggressive behavior". Then strategic complements and substitutes give rise to the two cases $b_F > b_L$ and $b_F \leq b_N$, respectively, in Theorem 1. We exclude the trivial case $b_L = b_N$, which arises, for example, when there is no strategic interaction.

Proposition 3. *Assume conditions* (a)–(e) *and the notation in Theorem 1, and let* $b_L > b_N$. *If* $r(y)$ *is increasing in* y, *then* $b_F > b_L$, *so that in a game with strategic complements the follower is better off than the leader. If* $r(y)$ *is decreasing in* y, *then* $b_F \leq b_N$, *so that in a game with strategic substitutes the follower is worse off than in the simultaneous game.*

Proof. As in the proof Theorem 1, we can assume that $a(r(y),y)$, which is equal to $b(y,r(y))$, is increasing in y, if necessary by reversing the order on both X and Y. This does not affect whether $r\colon X \to Y$ is increasing or decreasing.

Suppose that $r(y)$ is increasing in y. Then $y_L \leq y_N$ implies $r(y_L) \leq r(y_N) = y_N$ and therefore (3) which contradicts $b_L > b_N$. This excludes the first case in the proof of Theorem 1, so the second case $y_L > y_N$ applies, where $b_F > b_L$.

If $r(y)$ is decreasing in y, then $y_L \geq y_N$ implies $r(y_L) \leq r(y_N) = y_N$, which gives the same contradiction, so that the first case $y_L < y_N$ in the proof of Theorem 1 applies, that is, $b_F \leq b_N$. □

4 Symmetry and monotonicity are necessary

Theorem 1 is stated in such a way that it still makes sense for non-symmetric games, namely that player II prefers being follower to being leader (or is worse off than in the Nash equilibrium), rather than just stating "the follower is better off than the leader".

The following example shows that the symmetry condition (e) is necessary. Consider the game with $x,y \geq 0$ and payoff functions

$$\begin{aligned} a(x,y) &= x \cdot (\frac{4}{3} + \frac{2}{3}y - x), \\ b(x,y) &= y \cdot (\frac{4}{3} + \frac{2}{3}x - y) + 4x \end{aligned} \tag{7}$$

which has the (symmetric and linear) best reply functions $r(y) = (2+y)/3$ and $s(x) = (2+x)/3$. Moreover, $a(x,y)$ is increasing in y and $b(x,y)$ is increasing in x. The unique Nash equilibrium is $(1,1)$ with payoffs $a_N = 1$ to player I and $b_N = 5$ to player II.

When player I in (7) is a leader, the function $a(x,s(x))$ is maximized for $x_L = 8/7$ with payoff $a_L = a(x_L, s(x_L)) = 1 + 1/63$ to player I as leader and payoff $b_F = b(x_L, s(x_L)) = 5 + 65/147$ to player II as follower. However, when player II is the leader, her function $b(r(y),y)$ is maximized for $y_L = 2$ with payoff $b_L = b(r(y_L), y_L) = 5 + 7/9$, and payoff $a_F = a(r(y_L), y_L) = 1 + 7/9$ to player I as follower. Note that $b_L > b_F > b_N$, so the conclusion of Theorem 1 does not apply. Here, player II prefers being a leader to being a follower, whereas player I prefers following to leading. This agrees with Dowrick (1986, p. 255, Proposition 2): "If both firms have upward-sloping reaction functions, then if one prefers to lead, the other must prefer to be the von Stackelberg follower." All assumptions by Dowrick are met in (7), writing (for $y > 0$) $b(x,y) = y \cdot (4/3 + 2x/3 + 4x/y - y)$ where the second factor has negative derivative with respect to y and positive derivative with respect to x. Dowrick (1986, p. 257, Proposition 3) notes that both firms prefer to be followers when they "face similar cost and demand structures", which however is not made precise. Boyer and Moreaux (1987) quantify this distinction in terms of the "cost differential" between the firms, for a specific payoff function.

Without the monotonicity condition (c), it may happen that $b_L > b_F > b_N$, even when the game is symmetric. Consider the symmetric game with $x, y \geq 0$ and payoff

$$a(x,y) = b(y,x) = (0.72x - 0.125y - 0.785)(6.16 - y - 0.72x) \tag{8}$$

which has the (linear) best reply function

$$r(y) = \max(\frac{1389 - 175y}{288}, 0) \approx \max(4.823 - 0.608y, 0).$$

The unique Nash equilibrium is $(x_N, y_N) = (3,3)$ and has payoff $b_N = b(3,3) = 1$. The leader payoff is $b_L = a_L = a(x_L, s(x_L)) \approx a(6.822, 0.678) \approx 2.306$ and the follower payoff is $b_F = b(x_L, s(x_L)) \approx 1.322$, with $b_L > b_F > b_N$. Here, $a(r(y), y) = ((43 - 9y)/16)^2$ as long as $r(y) > 0$, that is, $y < 7.937$. This function is not monotonic, but has a minimum for $y = 43/9 \approx 4.778$.

The function in the example (8) does not make too much sense from an economic viewpoint since the follower payoff b_F is obtained as a product of two negative terms (unlike the payoff in the Nash equilibrium), which is crucial for this particular construction.

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