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CLOSURE PROPERTIES OF INDEPENDENCE CONCEPTS FOR CONTINUOUS UTILITIES

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This paper examines nine independence concepts for ordinal and expected utilities: utility and preference independence, weak separability (a uniqueness of nonstrict conditional preferences), as well as generalizations that allow for complete indifference or reversal of preferences. Some of these conditions are closed under set-theoretic operations, which simplifies the verification of such assumptions in a practical decision analysis. Preference independence is closed under union without assumptions of strict essentiality, which are however necessary for closure under differences of preference independence and its mentioned generalizations. The well-known additive representation of a utility function is used, in a corrected form with a self-contained proof, to show closure under symmetric difference. This generalizes a classical result of Gorman (1968), and supplements its proof. Generalized utility independence has all, whereas weak separability has no closure properties. An independent set of any kind is utility or preference independent if this holds for a subset. Counterexamples are given throughout to show that the results are as strong as possible. The approach consistently uses conditional functions instead of preference relations, based on simple topological notions like connectivity and continuity.

1. Introduction. Notions of independence or "separability" have been investigated in a number of economic settings. They are important for the analysis of decisions under certainty and uncertainty (cf. Fishburn [11], Keeney and Raiffa [17]) and for measuring preferences (cf. Krantz et al. [19]), or for problems of aggregation and decentralization (cf. Blackorby, Primont and Russell [4]). Closure properties, in particular with respect to the set-theoretic operations union, intersection and difference applied to overlapping sets of essential variables, are of practical importance if independence assumptions have to be verified, for instance in order to use a specific representation of the utility function (cf. [10], [17]). From a theoretical point of view, closure under the mentioned operations leads to a unique hierarchy of "independent attributes" with a corresponding functional decomposition. This hierarchy can be denoted by a "utility tree" for utility functions (Gorman [14, pp. 373ff]), which can also be applied to expected-utility functions (von Stengel [27]), and which is, more generally, given by a "composition tree" for quite a number of settings in combinatorial optimization (Möhring and Radermacher [23]). (Interestingly, a typical, rather simple argument taken from general topology—Lemma 10 below—can in the present context be used to establish this composition tree; the closure under symmetric difference is not needed in the general model [23, p. 329], and its proof requires more complicated methods; cf. Theorems 21 and 22 below.) We focus here on the closure properties alone; the main results are indicated in the abstract.

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Most of the concepts are defined in Fishburn and Keeney [12], whose terminology we use. The functional equations that define the independence concepts are given in §2; representations with conditional functions and the use of continuity are stated in §3. Predominantly technical material has mostly been deferred to appendices. We systematically investigate all set-theoretic operations for each single concept in §4, §5 and §6, and the relationship between any two concepts for the special case of *comparable* sets in §7. We have not surveyed all operations between sets of any two concepts, although most theorems and counterexamples have been designed to capture as many results in this direction as possible. §7 will be of help to the reader interested in this, and the spirit of the counterexamples (which are somewhat similar to the examples of Fishburn and Keeney [12, pp. 304ff]) should be clear. They have been constructed with the aid of the proofs (and vice versa, as usual), which suggests that the latter are rather canonical; for instance, Counterexample 24 is a near miss of a disproof of Theorem 19.

The present paper also relates very closely to Gorman [14] and seems to give the first correct proof of a main result of that paper. The following connections to [14] should be stressed: the closure under union for the condition of preference independence as well as its "weaker" form WPI (s.b.) holds without strict essentiality. It can be proved without the additive representation and the "Cauchy-equation" used in [14, p. 372], [4, p. 132], and furthermore holds even if this representation does not exist (Counterexample 23 below). So this closure condition conveniently replaces one of the three main steps in Gorman's proof; the closure under differences, given strict essentiality, is here proved as there. The third step for the symmetric difference, in [14] also needed for the union, uses a theorem about associative functions, however in a *stronger* form than actually proved by Aczél [1, p. 312], which is cited in [14, p. 371], [4, p. 129] as the source for this theorem. This is pointed out in detail in Appendix E; the reason is that the respective functions need to be bijective in each argument ("invertible" in [1]), but are here only injective (cancellative), not necessarily surjective. An example is a utility function that is a sum of three real variables ranging in finite intervals. Nevertheless, Gorman's theorem is true, actually if and only if the result ascribed to Aczél holds, and it is proved in full in Appendix E. Nearly all of the technical aspects needed to establish the additive utility function are given there, which may be of interest to some readers.

A few general mathematical methods are used rather uniformly in the proofs. We employ functional representations with conditional functions (Lemma 4), which are convenient when applying several independence assumptions. In fact, Lemma 6 states that a functional representation in terms of a "subutility" function with *arbitrary* reference vector *entails* the constraints of strict monotonicity given by preference independence (compare [4, p. 56f]). We should like to mention that, for general multiplace functions, decompositions solely into suitable conditional functions lead naturally to closure properties as considered here, but under certain stronger conditions. This theory is described in von Stengel [28] and sketched further following Proposition 16 below. Conditional preference *relations* are used only in Appendix A to provide a simplified version of weak separability.

The assumptions about the *domain* of the utility function can be kept very general. Instead of finite-dimensional Euclidean spaces [12] or visualizations with "indifference curves" or "isoquants" [20], we use mere topological connectivity (like Debreu [8], [9]), which suffices throughout and is necessary in the important cases (cf. Counterexamples 13 and 27). This is also more general than the arcconnectivity assumed by Gorman [14, p. 367]. An exception is the assumption of topological separability for the existence of a utility function (cf. Appendix A), which is however not used in proofs. This condition is essentially implicit if one assumes that the given preference relation is represented by a real-valued utility function; compare Wakker [29, p. 106]. Only the basic notions of general topology (cf., e.g., Kelley [18]) are employed; there is no need to use arcs, sequences, limits, convexity or even differentiability. Instead, almost every argument can be carried out considering the *ranges*, generally intervals, of real-valued functions in suitable representations. The most involved notion is that of *compactness* of a real interval in the proofs of Lemma 10 and Theorem 21.

The investigation of the assumption of strict essentiality is somewhat incomplete. This condition is treated here mainly because corresponding statements have been made in the literature, in particular [14, Theorem 1]. In Mak [21], strict essentiality for all variables is crucial in deriving closure properties for weak separability [21, p. 259]; see also [22]. In comparison, our assumptions are much weaker, and possible reversals of preferences are also considered, which however do not lead to greater generality (compare Counterexamples 14 and 17 and the second paragraph of §6). A useful tool in using strict essentiality seems to be Lemma 10, as demonstrated in the proof of Theorem 19.

Also, it is open whether separate continuity of the utility function in its individual variables would be a sufficient general condition, as mentioned in the beginning of §3. Conceivable counterexamples to joint continuity would have been too arcane, though. Finally, the fact that most of the counterexamples are fairly simple might suggest even more general independence concepts with new closure properties. Since these counterexamples employ functions that are piecewise affine in each variable, such an approach would however involve divisions of domains not only into subspaces, but into subsets of coordinate axes, which is clearly beyond the scope of this paper.

As concerns notation, we have tried to use the letters of the alphabet and not the popular hats, stars and overbars to distinguish between different objects, with an attempt to use at most one prime or subscript. Instead of using superscripts, fixed elements are denoted by numerals $0, 1, 2$, etc. (as in [14, p. 367]), since these are immediately recognized as constants. They are elements of abstract sets, and should be easily distinguishable from numbers that are used at the same time to count (in subscripts of $y_0, y_1, \ldots$, say). Furthermore, primed variables tend to be more fixed than naked ones, but this will always be indicated; note also the remark following Definition 1. A dot as in $g[\cdot, y]$ stands anonymously for a single argument of a function that results from a function (here g) of several variables if all variables but this one are fixed (at y), as stated in Definition 1.

2. Definitions and summary of closure properties.

We assume a finite set M is given, whose elements, which we simply call *coordinates*, may be used to index, for instance, given attributes, goods or time periods. For each $i \in M$ let S_i be a nonempty, topologically connected set (e.g., a real interval), whose elements may correspondingly denote possible attribute levels or amounts of consumption. Given a set A of coordinates, define $S_A = \prod_{i \in A} S_i$ to be the space spanned by the corresponding "coordinate axes" S_i, $i \in A$, equipped with the usual product topology and therefore also topologically connected. No particular ordering of the coordinates shall be assumed, so that they may be arranged into groups as it is suitable: given a partition $\mathscr{P}$ of M (that is, a set of nonempty, pairwise disjoint sets whose union is M), we can postulate

$$S_M = \prod_{B \in \mathscr{P}} S_B.$$

As a special case, if M is the disjoint union of the sets A and B with $\mathscr{P} = \{A, B\}$, this postulate says $S_M = S_A \times S_B$.

The set S_M may be interpreted, for instance, as the "outcome space" of a decision. It is here the domain of a so-called *utility function* [4], [14]

$$f: S_M \to \mathbb{R}$$

(also called ordinal utility function or "value function" [17]) that represents a *preference relation* for sure prospects. The concepts defined below apply to this preference relation, which we will however not consider further: it is merely important to note that a utility function is unique only up to strictly increasing transformations, that is, any concept defined in terms of f must also apply to the function mapping $x \in S_m$ to $\phi[f(x)]$, where $\phi: F \to \mathbb{R}$ is a strictly increasing function on the image F of f. We assume that a utility function exists that, additionally, is also *continuous*: conditions for its existence are given in Debreu [8] (cf. also Appendix A). The image of the continuous function $f: S_M \to \mathbb{R}$ is an interval since S_M is connected by assumption. We also assume usually that all coordinates are *essential* (with respect to f), that is, for $i \in M$, $B = M - \{i\}$ there are elements $y \in S_B$ and $x, x' \in S_i$ such that $f(x, y) \neq f(x', y)$ (arranging coordinates such that $S_M = S_i \times S_B$); this is a nugatory assumption since, for inessential coordinates i, the set $S_B = S_{M-\{i\}}$ can serve as a domain for f instead of S_M (i.e., inessential variables of f can be dropped). S_i is called *strictly essential* [14] if *for all* $y \in S_B$ there are $x, x' \in S_i$ with $f(x, y) \neq f(x', y)$.

Some concepts require the more restrictive interpretation of f as a so-called *expected-utility function*. Then a preference relation is given on a set of probability distributions over outcomes, and the expectation of f represents preferences. An expected-utility function is unique up to strictly increasing *affine* transformations. Usually, properties of an expected-utility function that are implied by given conditions of so-called "utility independence" (s.b.) are purely algebraic in character and do not require topological conditions (for instance, Theorems 12 and 22 below; this point is stressed further in [27]; for a survey of decompositions of expected-utility functions cf. Farquhar [10]). Here, we also suppose continuity in regarding utility independence in connection with a concept that also applies to sure-prospect decisions (cf. Theorem 25 below). Conditions for the existence of a continuous expected-utility function on a connected domain are given in Debreu [9, p. 18].

For a partition $\{A, B\}$ of M, S_A and S_B are topologically connected sets. For brevity, let $X = S_A$, $Y = S_B$, thus $S_M = S_A \times S_B = X \times Y$. We say X fulfills a certain *independence* concept (where it is understood that X is independent of "of Y" with respect to the given function $f: X \times Y \to \mathbb{R}$) provided a specific functional representation of $f(x, y)$ exists in terms of $y \in Y$ and a real value $h(x)$ for $x \in X$, for some function h of X.

DEFINITION 1. Let $f: X \times Y \to \mathbb{R}$. In the following definitions, it is assumed that functions $h: X \to \mathbb{R}$ and $g: H \times Y \to \mathbb{R}$, where H is the image of h, can be chosen suitably such that

$$f(x, y) = g[h(x), y]$$

holds for $x \in X$, $y \in Y$ and such that g fulfills certain properties with respect to its

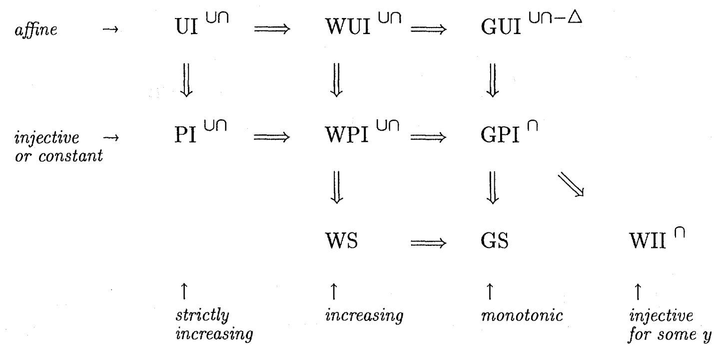

FIGURE 1. Implication Diagram of Independence Concepts, with Restrictions on $g[\cdot, y]$ in $f(x, y) = g[h(x), y]$.

first argument. By $g[\cdot, y]$ we denote the function $H \to \mathbb{R}$ that maps $h \in H$ to $g[h, y]$. Throughout, y is any element of Y. Then X is called

PI (*preference independent*) if $g[\cdot, y]$ is strictly increasing,

WPI (*weakly preference independent*) if $g[\cdot, y]$ is strictly increasing or constant,

GPI (*generalized preference independent*) if $g[\cdot, y]$ is strictly monotonic (i.e., strictly increasing or decreasing) or constant,

WII (*weakly indifference independent*) if $g[\cdot, 0]$ is injective for some element 0 of Y.

WS (*weakly separable*) if $g[\cdot, y]$ is (not necessarily strictly) increasing,

GS (*generalized separable*) if $g[\cdot, y]$ is monotonic,

GUI (*generalized utility independent*) if $g[\cdot, y]$ is affine, that is, $g[h, y] = a(y) \cdot h + b(y)$ for some $a, b\colon Y \to \mathbb{R}$,

WUI (*weakly utility independent*) if thereby always $a(y) \geqslant 0$,

UI (*utility independent*) if thereby $a(y) > 0$ holds.

(As notational convention, the ranges for variables or images of functions denoted by lower case letters, irrespective of primes or subscripts, shall be those denoted by the corresponding upper case letters, if not indicated otherwise.)

Definition 1 makes sense even if f is not assumed to be continuous on a connected domain; we will shortly investigate the implications of this additional hypothesis. In Figure 1 we have depicted the various implications that hold between the defined independence concepts, which are more or less obvious. To observe that GPI implies WII, note that, in $f(x, y) = g[h(x), y]$, the function $g[\cdot, y]$ cannot always be constant if x is essential (if x is inessential, we can choose h constant, so that $g[\cdot, y]$ is trivially injective on the one-point set H; this case shall nevertheless be excluded). The prefix "W" (except for "WII", which we named following [12]) indicates that $g[\cdot, y]$ has to be increasing (in the weaker sense) and can therefore also be constant, whereas the prefix "G" allows for a change of *orientation*, that is, $g[\cdot, y]$ can be increasing as well as decreasing for different y.

In Figure 1 we have indicated with superscripts under which set-theoretic operations the respective independence concept is closed, in the following sense: given $f\colon S_M \to \mathbb{R}$ and a set $A \subseteq M$ of coordinates, a specific independent concept either applies to S_A or not, viz., if a respective functional decomposition for $f\colon S_A \times S_{M-A} \to \mathbb{R}$ exists or not. Since S_A is uniquely identified by A, we also say "A is independent" if S_A is, and thus investigate whether with independent sets A, B of coordinates also $A \cup B$, $A \cap B$, $A - B$ or $A \Delta B = (A - B) \cup (B - A)$ are independent. This will generally only hold for *overlapping* (i.e., intersecting, noncomparable) sets A, B, if all coordinates in M are essential (which, as mentioned, can be assumed without loss of generality). These closure properties are shown in Figure 1 under the *continuity* hypotheses (which can be omitted for UI, WUI and GUI), and will be proved below where they are new. We will also give counterexamples for the indicated cases that closure properties do not apply, in particular for the concepts WS, GS of weak and generalized separability.

The above concepts have been defined explicitly in the literature (with the exception, perhaps, of WUI, WPI and GS), usually with respect to the given preference relation the utility function represents. The "utility independence" concepts UI, WUI, GUI make sense only for an *expected-utility* function f since they are not invariant under strictly increasing, but only under positive-affine transformations of f; for a definition in terms of the represented preference relation over gambles, cf. (for example) Fishburn and Keeney [12, p. 297f]. All other concepts apply to utility functions and, of course, the more specific expected-utility functions. Preference independence of X, also called (strict) separability (Blackorby, Primont and Russell [4], Gorman [14]), asserts an independence of the "induced preference relation" on the subspace X; cf. Debreu [9, p. 21], Fishburn and Keeney [12, p. 299], Gorman [14, p. 367]. For GPI (and WPI in a similar way), cf. Fishburn and Keeney [12, p. 299f]; the equivalence of WII with the concept used there will be explained shortly.

The concept WS of weak separability has been introduced by Bliss [5, pp. 147–151] and is named here following Blackorby, Primont and Russell [4, pp. 42–60]. A simplified account of their concept is given in Appendix A.

3. Reference vectors and continuous subutilities. It is well known that if $f\colon X \times Y \to \mathbb{R}$ is a continuous function on a connected domain, and X is preference independent, say, then g and h in the representation $f(x, y) = g[h(x), y]$ can also be chosen as continuous functions. In the proofs below, it however suffices to use the continuity of the function $g[\cdot, y]\colon H \to \mathbb{R}$ for $y \in Y$, on which the restriction (e.g., "strictly increasing") is imposed to define the particular type of independence (e.g., PI). This *separate continuity* of g (in its first variable) corresponds to the assumption that $f(x, y)$ is continuous in x, as will be shown shortly. Obviously, $f(x, y)$ is continuous in y (for fixed x) iff $g[h(x), y]$ is. The separate continuity of $f(x, y)$ in x and y follows from joint continuity of $f\colon S_M \to \mathbb{R}$, where X may be an arbitrary subspace S_A of S_M. If connectivity of Y is not needed, Y may be equipped with the trivial topology (and thus be an arbitrary, for instance, finite set), so that $f(x, y)$ is trivially continuous in y, and continuity of f is equivalent to continuity of $f(x, y)$ in x. In many cases below, we will simply assume that f is continuous, even though separate continuity (of $f(x, y, z, t)$ in x, y and z, say) may suffice, if the argument of f is split into several parts.

According to Debreu [9, p. 22], if X is PI, with $f(x, y) = g[h(x), y]$, where $f(x, y)$ and $h(x)$ are continuous in x and X is connected, then $g[\cdot, y]\colon H \to \mathbb{R}$ is also continuous for all $y \in Y$ (Gorman [14] assumed arcconnectivity, which is unnecessarily strong; cf. also the discussion in [15]). This follows from the following, rather intuitive lemma; compare also Wakker [33, Lemma 2.1].

LEMMA 2. *Let $g\colon H \to \mathbb{R}$, where H is an interval, be monotonic. Then g is continuous iff the image G of g is an interval.*

PROOF. A monotonic function on an interval is continuous iff it leaves no gaps. The formal details are straightforward. □

We immediately obtain that, for GS and therefore (cf. Figure 1) for all the above independence concepts except WII, $g[\cdot, y]$ is continuous provided h and $f(\cdot, y)\colon X \to \mathbb{R}$ are and X is connected, since the images of these functions are intervals (note that affine functions are trivially continuous):

COROLLARY 3. *Let* f: $X \times Y \to \mathbb{R}$, *the set* X *be topologically connected and* h *and* f *be continuous, where*

$$f(x, y) = g[h(x), y].$$

Then if X *is PI, WPI, GPI, WS or GS with* g, h *as in Definition* 1, $g[\cdot, y]$: $H \to \mathbb{R}$ *is continuous for all* $y \in Y$.

Lemma 2 also shows that if ϕ is a strictly increasing transformation of a continuous utility function f: $S_M \to \mathbb{R}$, where S_M is connected, then ϕ is continuous on the image of f if (and only if) $\phi[f(\cdot)]$ is; see also Wakker [33].

The fact that, in turn, h can be chosen continuous if $f(x, y)$ is continuous in x follows for WII (and any stronger concept) from the following lemma. For WS, h in Definition 1 can be chosen continuous according to Appendix A; however, we will merely show that even if g and h (and thus f) in Definition 1 are restricted to be continuous functions on connected domains, no closure properties hold for WS or GS.

LEMMA 4. *Given* f: $X \times Y \to \mathbb{R}$, *let* 0 *be an element of* Y *and* K *be the image of the function* $f(\cdot, 0)$. *Then the following statements are equivalent*:

(a) $f(x, y) = g[h(x), y]$ *and* $g[\cdot, 0]$ *is injective for some functions* g, h,

(b) $f(x, y) = g'[f(x, 0), y]$ *for a unique function* g': $K \times Y \to \mathbb{R}$.

These statements remain equivalent if any one of the restrictions given by an independence concept in Definition 1 *is imposed on* g *and* g'.

PROOF. If the equation in (b) holds, then g' is uniquely determined since K is the image of $f(\cdot, 0)$. The condition (b) implies (a) with $g = g'$ since then $g[\cdot, 0]$ is the identity id_K on K and therefore injective. Here, any restriction imposed on g' applies obviously to g.

Conversely, suppose (a) holds. Let H be the image of h. Since $f(x, 0) = g[h(x), 0]$, the image of the injective function $g[\cdot, 0]$ on H is K. Let ϕ: $K \to H$ be the inverse of this function. Then

$$h(x) = \phi(g[h(x), 0]) = \phi(f(x, 0)) \quad \text{and}$$

$$f(x, y) = g[\phi(f(x, 0)), y] =: g'[f(x, 0), y],$$

which shows (b). If a restriction given by a concept of Definition 1 is imposed on g in (a), it applies to g' in (b) for WII since $g'[\cdot, 0] = id_K$. For GS, which subsumes the other concepts, note that in this case the function $g[\cdot, 0]$ is monotonic and therefore strictly monotonic because it is injective. If $g[\cdot, 0]$ is affine, strictly increasing or strictly monotonic, then so is its inverse ϕ, and these properties, as well as being simply increasing or monotonic or constant, carry over, correspondingly, from $g[\cdot, y]$ to $g[\phi(\cdot), y] = g'[\cdot, y]$ for all y. □

If X is WII, the asserted injectivity of $g[\cdot, 0]$ thus implies that $0 \in Y$ is a suitable *reference vector*, so that the *conditional* function $f(\cdot, 0)$ (which is called "subutility" function in [14, p. 368]) can be used instead of h in Definition 1. The equation in Lemma 4(b) also shows the equivalence to Fishburn and Keeney's definition of WII [12, p. 297]. If $f(x, y)$ is continuous (it suffices: in x), $f(\cdot, 0)$ is continuous, too. For WII itself, this does not yield continuity of $g[\cdot, y]$ since no monotonicity constraint is imposed on this function; neither will such a condition be necessary in the application of WII (in particular Theorem 25 below). However, this holds for the next stronger concept GPI:

COROLLARY 5. *Let f: $X \times Y \to \mathbb{R}$ be continuous, where X is topologically connected. Let X be GPI and let $0 \in Y$ be such that the function $f(\cdot, 0)$ is not constant; that is, its image is a nondegenerate interval K. Then*

$$f(x, y) = g[f(x, 0), y]$$

holds with a function g: $K \times Y \to \mathbb{R}$, where $g[\cdot, y]$: $K \to \mathbb{R}$ is continuous and either strictly monotonic or constant, respectively.

PROOF. With $f(x, y) = g[h(x), y]$ as in Definition 1 where X is GPI, the function $g[\cdot, 0]$ is not constant if $f(\cdot, 0)$ is not constant and thus strictly monotonic and thereby injective. The claim follows from Lemma 4 and Corollary 3. □

Lemma 4 can in general not be applied to the concepts WS or GS, since if $g[\cdot, y]$: $H \to \mathbb{R}$ is always monotonic, and even if for any $h, h' \in H$ with $h < h'$ there exists $y \in Y$ with $g[h, y] < g[h', y]$, then $g[\cdot, y]$ may nonetheless never be injective, that is, no suitable reference vector exists (otherwise, these concepts would imply WII).

Applied to preference independence PI, Lemma 4 shows that an *arbitrary* reference vector can be chosen (cf. also [4, p. 56]). Let this vector be denoted by a new variable $y' \in Y$ instead of 0. In Lemma 4(b), the function g' depends in general on the choice of y'; this can be expressed by adding y' as an additional argument to a new, single function, call it g again:

$$f(x, y) = g[f(x, y'), y', y].$$

Whenever the variable y' is fixed (note that it does not appear on the left-hand side), the old equation of Lemma 4(b) is given. The domain of g is here a subset of $F \times Y \times Y$, where F is the image of f: if y' is fixed, $f(\cdot, y')$ does not necessarily take all the values of f. So let $g[k, y', y]$ be defined for all $y, y' \in Y$ and all $k \in F$ such that $k = f(x, y')$ for some $x \in X$. Interestingly enough, this functional equation is already *sufficient* for X to be PI, provided X and Y are topologically connected and $f(x, y)$ is continuous in x and y. In other words, given these topological conditions,

$$\text{PI} \Leftrightarrow \text{WII} \quad \text{with } \textit{arbitrary} \text{ reference vector } 0$$

(compare this also with the assertion "GPI ⇔ II" in Fishburn and Keeney [12, p. 300f] for *convex* X, Y). This is asserted by the following lemma which is proved in Appendix B.

LEMMA 6. *Let X and Y be topologically connected and f: $X \times Y \to \mathbb{R}$ be continuous. Then*

$$f(x, y) = g[f(x, y'), y', y]$$

holds for some function g if and only if X is PI. This equation implies that the function $g[\cdot, y', y]$: $K(y') \to \mathbb{R}$ is continuous and strictly increasing, where $K(y')$ is the image of $f(\cdot, y')$, and that $g[\cdot, y, y]$ is the identity on $K(y)$; if X is UI, then, in addition, $g[\cdot, y', y]$ is affine.

4. Closure under union. Let f: $S_M \to \mathbb{R}$ be the given utility function and assume that all coordinates in M are essential. Then, for different sets $A, B \subseteq M$ of coordinates, the independence of S_A and S_B (for a given concept) may entail the

$f(x,y)$		$x=-1$	$x=0$	$x=1$
y	0	1	0	1
	1	2	2	3

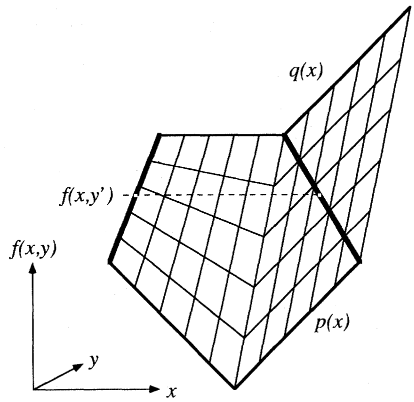

FIGURE 2

independence of the spaces corresponding to the three coordinate sets $A - B$, $A \cap B$, $B - A$ and their unions $A \Delta B$, $A \cup B$. The only case with interesting consequences will be given if all these three sets are nonempty (and thus contain essential coordinates), i.e., if A and B *overlap*. For shorthand, we write $X = S_{A-B}$, $Y = S_{A \cap B}$, $Z = S_{B-A}$ and $T = S_{M-(A \cup B)}$ and thus consider $f(x, y, z, t)$ with essential variables x, y, z. The set T may contain only the empty vector ( ) if $T = S_\varnothing$, i.e., if $A \cup B = M$; otherwise, t is also an essential variable of $f(x, y, z, t)$.

To show that indeed only overlapping independent sets A, B of coordinates can have closure properties, we consider the alternative cases that A, B are comparable or disjoint. The respective counterexamples we consider in some detail, since other, more complicated ones further below are constructed by similar methods. Assume B is contained in A, i.e., $Z = S_\varnothing$ above. So, let $f\colon X \times Y \times T \to \mathbb{R}$ and $X \times Y$ and Y be independent; the only relevant question is whether $X = S_{A-B}$ is also independent. We show that even given the strongest independence assumption UI (cf. Figure 1) for $X \times Y$ and Y, neither WII nor GS has to hold for X, so no closure properties apply to comparable independent sets; in particular, no independence concept is closed under *complements*. We let T be inessential, so $f\colon X \times Y \to \mathbb{R}$, and $X \times Y$ is trivially UI since $f(x, y) = g[f(x, y), (\,)]$, where $g[\cdot, (\,)]$ is the identity id_F on the image F of f and thus positive-affine.

COUNTEREXAMPLE 7. *There is a continuous function $f\colon X \times Y \to \mathbb{R}$ with intervals X, Y, such that Y is UI but X is not WII.*

PROOF. Consider a function with values as indicated in Figure 2.

Since only two values for y are given, we can use linear interpolation to obtain a function that is affine in $y \in [0, 1] = Y$. The two rows of the table we can extend to two continuous functions p, q of $x \in [-1, 1] = X$, e.g., $p(x) = |x|$ and

$$q(x) = \begin{cases} 2 & \text{for } x \leqslant 0, \\ 2 + x & \text{for } x \geqslant 0 \end{cases},$$

(piecewise definitions we usually give in such a way that they are defined *twice* on the border point, here $x = 0$, with the same value, here $q(0) = 2$, thus automatically obtaining continuity). The resulting function is $f(x, y) = p(x) + (q(x) - p(x)) \cdot y$ (cf. Figure 2), and since the coefficient of y is given by

$$q(x) - p(x) = \begin{cases} 2 - |x| & \text{for } -1 \leqslant x \leqslant 0, \\ 2 & \text{for } 0 \leqslant x \leqslant 1, \end{cases}$$

i.e., is always positive, it is assured that Y is UI. The idea is that $f(x, y)$ cannot be expressed in terms of $f(x, y')$ for any fixed $y' \in Y$, i.e., no representation

$$f(x, y) = g[f(x, y'), y] \tag{1}$$

exists, which would be the case if X were WII by Lemma 4, for the following reason: for any two different x, x' there exists y with $f(x, y) \neq f(x', y)$, so that $f(\cdot, y')$ would have to be injective if (1) were to hold. But this is not the case for any y', as is easily seen from Figure 2: to be precise, observe that

$$f(-1, y') = 1 + (2 - 1) \cdot y' = 1 - y' + 2y' = f(1 - y', y')$$

(cf. the bold lines in Figure 2), so with (1) we would obtain

$$\begin{aligned} 2 = q(-1) = f(-1, 1) &= g[f(-1, y'), 1] \\ &= g[f(1 - y', y'), 1] = f(1 - y', 1) = q(1 - y') = 3 - y' \end{aligned}$$

or $y' = 1$; but this choice of y' in (1) is certainly not possible since then $f(\cdot, y') = f(\cdot, 1) = q$ and q is constant for negative arguments. Thus X is not WII. The counterexample is constructed to obtain certain *global* properties of the function: $p(x)$ and $q(x)$ could have been chosen as polynomials to complete the above table (with somewhat more complicated computations to show a specific contradiction), so conditions of differentiability would not change the matter. □

COUNTEREXAMPLE 8. *There is a continuous function f: $X \times Y \to \mathbb{R}$ with intervals X, Y, such that Y is UI but X is not GS.*

PROOF. The scheme is similar: take the table

$f(x,y)$		x = −1	0	1
y	0	0	1	2
	1	3	4	3

interpolate with $p(x) = x + 1 < q(x) = 4 - x^2$ for $-1 \leqslant x \leqslant 1$ so that $f(x, y) = p(x) + (q(x) - p(x)) \cdot y$ for $0 \leqslant y \leqslant 1$, where Y is UI. The table suffices to see that X is not GS: $f(\cdot, 0)$ increases but $f(\cdot, 1)$ partly increases *and* decreases. So, with $f(x,y) = g[h(x), y]$ and $h(-1) < h(0)$, say ($h(-1) \neq h(0)$ is necessary), the function $g[\cdot, 0]$—assumed to be monotonic—would be increasing, thus $h(0) < h(1)$; but the inequality $f(-1, 1) < f(0, 1)$ shows that $g[\cdot, 1]$ also increases, with $f(0, 1) \leqslant f(1, 1)$ in consequence, which is not true. □

If the sets of coordinates A and B are *disjoint* and S_A and S_B are independent, it is merely interesting whether the space $S_{A \cup B}$ that corresponds to their union is independent, too, since there are no additional sets given by their intersection, differences or symmetric difference. The intersection of two disjoint coordinate sets corresponds to the space $S_\varnothing$ with a single element ( ) which can be regarded as the value for an additional inessential variable. Similarly, the converse holds: if f: $X \times Y \times Z \times T \to \mathbb{R}$ is given and y is inessential with respect to $f(x, y, z, t)$, the independence of $X \times Y$ and $Y \times Z$ is equivalent to that of X and Z, regarding $f(x, z, t)$ only, since y can be dropped. So any overlap must be regarded with respect to essential coordinates. For no type of independence, any closure properties hold for disjoint independent sets:

COUNTEREXAMPLE 9. *There is a continuous function* f: $X \times Z \times T \to \mathbb{R}$ *with intervals* X, Z, T, *such that* X *and* Z *are UI but* $X \times Z$ *is neither GS nor WII.*

PROOF. Consider the function f: $X \times Z \times T \to \mathbb{R}$ with $X = Z = \mathbb{R}$ and $T = [1, 2]$ given by

$$f(x, z, t) = t \cdot x + (3 - t) \cdot z.$$

Here, X and Z are certainly UI. For $x, z = 0, 1$ and $t = 1, 2$ we have

$f(x,z,t)$		$t=1$, $x=0$	$t=1$, $x=1$	$t=2$, $x=0$	$t=2$, $x=1$
z	0	0	1	0	2
	1	2	3	1	3

This shows $X \times Z$ is not GS, since no total order on $X \times Z$ can be established that is independent of $t \in T$: if $f(x, z, t) = g[h(x, z), t]$ with monotonic $g[\cdot, t]$, then $g[\cdot, 1]$ and $g[\cdot, 2]$ would have to have the same orientation since both $f(0, 0, 1) < f(1, 1, 1)$ and $f(0, 0, 2) < f(1, 1, 2)$ hold; but, in contrast, $f(0, 1, 1) > f(1, 0, 1)$ and $f(0, 1, 2) < f(1, 0, 2)$.

To show that $X \times Z$ is not WII, assume, to the contrary, that for some fixed $t' \in T$, $f(x, z, t) = g[f(x, z, t'), t]$ according to Lemma 4. Geometrically regarded, the graph of the function $f(x, z, t')$ of x, z is a plane that is tilted—depending on the value of t'—around the line given by $\{f(x, x, t') | x \in X\}$. The solutions of $f(x, z, t') \equiv$ *constant* in $X \times Z$ are lines which depend on t': with *constant* $= 1$, for instance, we obtain $f(1/t', 0, t') = f(0, 1/(3 - t'), t')$, which would imply $f(1/t', 0, t) = f(0, 1/(3 - t'), t)$ for *all* t, or

$$t\frac{1}{t'} = (3 - t)\frac{1}{3 - t'} \quad \text{resp.} \quad \frac{3 - t'}{t'} = \frac{3 - t}{t},$$

which is not true for $t \neq t'$. So $X \times Z$ is not WII. □

We now proceed to investigate closure under union of overlapping sets of coordinates for the different independence concepts. First, we will prove it for PI and WPI, using the following lemma, which has additional applications further below.

LEMMA 10. *Let* Y, Z *be nonempty topologically connected sets,* k: $Y \times Z \to \mathbb{R}$, $k(y,z)$ *be continuous in* y *and* z *and* A, B *be two elements of the image* K *of* k, $A \leqslant B$. *Then* K *is an interval, and at least one of the following cases holds*:

(a) *there is an element* z_0 *of* Z *such that the function* $k(\cdot, z_0)$ *takes a constant value* C, *where* $A \leqslant C \leqslant B$, *or*

(b) *with* $I(z)$ *defined as the interval that is the image of the continuous function* $k(\cdot, z)$ *for* $z \in Z$, A *and* B *are linked by a finite chain of such intervals, more precisely*: *there are elements* $z_0, z_1, \ldots, z_n$ *of* Z $(n \geqslant 0)$ *such that* $A \in I(z_0)$, $B \in I(z_n)$ *and*

$$I(z_i) \cap I(z_{i+1}) \quad \textit{has nonempty interior for } 0 \leqslant i < n.$$

The proof is given in Appendix C. Lemma 10 is concerned with "partial images", that is, with the images of the conditional functions $k(\cdot, z)$ for a separately continuous function k of two variables that are given if one of the variables is fixed (at z). These images $I(z)$ exhaust the image of k, since $k(y, z) \in I(z)$. In case (a), $I(z)$ is for some z degenerated to a one-point set $\{C\}$. Case (b) states that if this is not the case, finitely many of successively *intersecting* such images connect two arbitrary images A and B under k; the fact that the intersections contain at a time not only one point but a whole neighborhood of it (and therefore several different points) will be useful later.

The following theorem has been given under stronger hypotheses by Gorman [14, p. 372], in particular with strict essentiality of z for $f(x, y, z, t)$, using an additive representation like in Theorem 21 below that depends on t. Our proof is independent of such a representation, which may not exist under the weaker hypotheses; an example is the function $f(x, y, z) + t$ for $t \in \mathbb{R}$ with f as in Counterexample 23 below. Gorman [14, p. 367] also assumed arcconnectivity, which is not necessary; also, this condition does not seem to produce additional positive results since in the counterexamples below, coordinate axes are intervals which are arcconnected and even convex. A second assumption in [14] is topological separability, which is here essentially implicit in considering a real-valued utility function instead of a preference relation [29, p. 106], as mentioned in the introduction. In the following theorems, only those variables are assumed to be essential for which this is necessary (note that Theorem 11 is of a certain interest even if z is inessential; compare Proposition 28 below).

THEOREM 11. *Let X, Y, Z be topologically connected, $f\colon X \times Y \times Z \times T \to \mathbb{R}$ be continuous, where y is essential for $f(x, y, z, t)$, and let $X \times Y$ be GPI. Then if $Y \times Z$ is PI or WPI, so is $X \times Y \times Z$.*

PROOF. Let—according to Definition 1—

$$f(x, y, z, t) = g[h(x, y), z, t] \tag{1}$$

$$= j[x, k(y, z), t],$$

where $g[\cdot, z, t]$ is strictly monotonic or constant, and $j[x, \cdot, t]$ is strictly increasing or constant, the latter only if $Y \times Z$ is WPI but not PI. By Corollary 5, g and h can be chosen such that, for suitable $0 \in T$, $1 \in Z$,

$$h(x, y) = f(x, y, 1, 0), \tag{2}$$

where $f(x, y, 1, 0)$ is not a constant function of x, y. In particular, since y is essential, $f(2, \cdot, 1, 0)$ is not constant for suitable $2 \in X$, and, again by Corollary 5, k and j can be chosen such that

$$k(y, z) = f(2, y, z, 0). \tag{3}$$

With these choices, each of the functions g, h, j, k is, in its first two arguments, separately continuous on a connected domain; this holds for the real-valued arguments because of Corollary 5, and otherwise because of the properties of f: for instance, the function $g[h(x, y), \cdot, t]$ is continuous because it equals $f(x, y, \cdot, t)$.

Our goal is, first, to show for all t

$$f(x, y, z, 0) = f(x', y', z', 0) \Rightarrow f(x, y, z, t) = f(x', y', z', t) \tag{4}$$

by means of the specialized versions with $z = z'$ and $x = x'$, viz.,

(5) $$f(x, y, z, 0) = f(x', y', z, 0) \Rightarrow f(x, y, z, t) = f(x', y', z, t),$$

(6) $$f(x, y, z, 0) = f(x, y', z', 0) \Rightarrow f(x, y, z, t) = f(x, y', z', t).$$

These last implications hold for the following reason: let $f(x, y, z, 0) = f(x', y', z, 0)$. If $g[\cdot, z, 0]$ is strictly monotonic, it is injective, i.e., $h(x, y) = h(x', y')$ by (1) and therefore also $f(x, y, z, t) = f(x', y', z, t)$. For constant $g[\cdot, z, 0]$, $f(2, \cdot, z, 0)$ is constant by (1), which is equal to $k(\cdot, z)$ by (3), so $j[2, k(\cdot, z), t] = f(2, \cdot, z, t) = g[h(2, \cdot), z, t]$ and $g[\cdot, z, t]$ is constant (it would be injective otherwise) because $2 \in X$ was chosen such that $h(2, \cdot) = f(2, \cdot, 1, 0)$ is a nonconstant function on Y; so in this case, (5) holds, too, irrespective of the values of $h(x, y)$ and $h(x', y')$. Similarly, because $Y \times Z$ is also GPI, (6) holds, too.

To show (4), let $f(x, y, z, 0) = f(x', y', z', 0)$ and $A = k(y, z)$, $B = k(y', z')$, where w.l.o.g. $A \leqslant B$. The assumptions of Lemma 10 are met. We first treat the case that (a) holds in this lemma, i.e., $k(\cdot, z_0) \equiv C$ for some $z_0 \in Z$, $A \leqslant C \leqslant B$. Then, since $j[x', \cdot, 0]$ and $j[x, \cdot, 0]$ are increasing,

$$j[x', C, 0] \leqslant j[x', B, 0] = f(x', y', z', 0)$$
$$= f(x, y, z, 0) = j[x, A, 0] \leqslant j[x, C, 0].$$

Because $j[\cdot, C, 0] = f(\cdot, y_0, z_0, 0)$ (for some arbitrary y_0), which is a continuous function on the connected set X, $j[x_0, C, 0] = f(x, y, z, 0)$ holds for some $x_0 \in X$. We have actually proved—and will use again—:

$$A \leqslant k(y_0, z_0) \leqslant B \Rightarrow$$

(7) $$f(x, y, z, 0) = f(x_0, y_0, z_0, 0)$$
$$= f(x', y', z', 0) \quad \text{for some } x_0 \in X.$$

Since $k(\cdot, z_0) = g[h(2, \cdot), z_0, 0]$ is constant, $g[\cdot, z_0, 0]$ is constant, so we note

$$f(x, y, z, 0) = f(x, y, z_0, 0) = f(x', y', z_0, 0) = f(x', y', z', 0),$$

and every single equality holds also with t instead of 0, because of (6), (5), (6), that is, $f(x, y, z, t) = f(x', y', z', t)$.

In the situation of Lemma 10(b), consider first the simplest case $n = 0$, i.e., A and B are in the common image $I(z_0)$ of $k(\cdot, z_0)$ for some $z_0 \in Z$ (in particular for $A = B$): let $k(y, z) = k(a, z_0)$, $k(y', z') = k(b, z_0)$ for suitable $a, b \in Y$. Then obviously

$$f(x, y, z, 0) = f(x, a, z_0, 0),$$
$$f(x', y', z', 0) = f(x', b, z_0, 0),$$

thus

$$f(x, a, z_0, 0) = f(x', b, z_0, 0),$$

and these equations carry over from 0 to t, by (6), (6), (5), i.e., $f(x, y, z, t) = f(x', y', z', t)$. If $A = k(a, z_0)$, $B = k(b, z_n)$ for $n > 0$, let C be a common point of the intervals $I(z_0)$, $I(z_1)$, which can obviously be chosen such that $A \leqslant C \leqslant B$: let $C = k(y_0, z_0) = k(y_1, z_1)$. By (7), there exists $x_0 \in X$ such that

$$f(x,y,z,0) = f(x,a,z_0,0) = f(x_0,y_0,z_0,0) = f(x_0,y_1,z_1,0),$$

which implies

$$f(x,y,z,t) = f(x_0,y_1,z_1,t) \tag{8}$$

by (6), (5), (6). The proof of (4) is now easily completed by induction: if $n = 1$, then $C = k(y_1, z_1)$ and B belong to a common interval $I(z_1)$, and $f(x_0, y_1, z_1, t) = f(x', y', z', t)$ as before, which shows $f(x, y, z, t) = f(x', y', z', t)$ by (8). Otherwise, C and B are linked by a chain of n intervals where A and B are linked by $n + 1$ many ones, and the reasoning can be repeated (generalizing (8) to $f(x, y, z, t) = f(x_{i-1}, y_i, z_i, t)$ for $1 \leqslant i \leqslant n$), to finally prove (4).

With (4) thus shown, it is possible to define unambiguously a function q: $P \times T \to \mathbb{R}$ with

$$q[f(x,y,z,0),t] = f(x,y,z,t), \tag{9}$$

where P is the appropriate subset of the image F of f. We will show, without difficulty, that $q[\cdot, t]$ is either strictly increasing or constant. The values $t \in T$ such that $q[\cdot, t]$ is constant are given as follows: suppose $j[2, \cdot, t]$ is constant, which is only possible if $Y \times Z$ is WPI. Then, for some $c \in \mathbb{R}$,

$$\begin{aligned} c &= f(2,y,z,t) && \text{for all } y, z, \\ &= g[h(2,y),z,t] && \text{for all } y, z, \end{aligned}$$

which implies $g[\cdot, z, t]$ is constant because $h(2, \cdot\,)$ is a nonconstant function on Y. Thus we can continue

$$\begin{aligned} c &= g[h(2,y),z,t] && \text{for all } z, \\ &= g[h(x,y),z,t] && \text{for all } x, y, z, \\ &= f(x,y,z,t) && \text{for all } x, y, z, \end{aligned}$$

or $q[\cdot, t] \equiv c$ in (9).

Let therefore, for the rest of the proof, $t \in T$ be given such that $j[2, \cdot, t]$ is strictly increasing; this is true for all t if $Y \times Z$ is PI. Then if $g[\cdot, z, 0]$ is strictly increasing or decreasing, $g[\cdot, z, t]$ has the same orientation: for, let, for some $a, b \in Y$, $h(2, a) < h(2, b)$ hold (note $h(2, \cdot\,)$ is not constant), and assume $g[\cdot, z, 0]$ is increasing, i.e., $g[h(2, a), z, 0] < g[h(2, b), z, 0]$. This entails the following inequalities:

$$\begin{aligned} f(2,a,z,0) &< f(2,b,z,0) && \text{by (1)}, \\ k(a,z) &< k(b,z) && \text{by (3)}, \\ j[2,k(a,z),t] &< j[2,k(b,z),t] && \text{by assumption on } j[2, \cdot, t], \\ g[h(2,a),z,t] &< g[h(2,b),z,t] && \text{by (1)}. \end{aligned}$$

So $g[\cdot, z, t]$, which is strictly monotonic or constant, is indeed increasing; for strictly decreasing $g[\cdot, z, 0]$, the same reasoning with $<$ reversed to $>$ shows that $g[\cdot, z, t]$ is strictly decreasing, too.

With $z = 1$, this implies $g[\cdot, 1, t]$ is strictly increasing since $g[\cdot, 1, 0]$ is the identity according to (1), (2). The former reasoning—dually, with g and j, x and z and 2 and 1 interchanged—then shows that $j[x, \cdot, t]$ strictly increases if $j[x, \cdot, 0]$ does. In total, we obtain

$$(10)\qquad f(x,y,z,0) < f(x',y',z,0) \Rightarrow f(x,y,z,t) < f(x',y',z,t),$$

$$(11)\qquad f(x,y,z,0) < f(x,y',z',0) \Rightarrow f(x,y,z,t) < f(x,y',z',t),$$

that is, in analogy to (5) and (6), inequalities for fixed z or x carry over from 0 to t.

To show that $q[\cdot, t]$ is strictly increasing, let $f(x, y, z, 0) < f(x', y', z', 0)$. If $k(y, z) \leqslant k(y', z')$, we distinguish two cases according to the value of $f(x', y, z, 0)$. If

$$f(x,y,z,0) < f(x',y,z,0) \leqslant f(x',y',z',0)$$

(note $j[x', \cdot, 0]$ is increasing), these inequalities carry over from 0 to t. If

$$f(x',y,z,) \leqslant f(x,y,z,0) < f(x',y',z',0),$$

$f(x, y, z, 0) = f(x', y_0, z_0, 0)$ holds for some $y_0 \in Y$, $z_0 \in Z$ by continuity of $j[x', \cdot, 0]$, so by (4) and (11),

$$f(x,y,z,t) = f(x',y_0,z_0,t) < f(x',y',z',t).$$

If $k(y, z) > k(y', z')$, then

$$f(x,y,z,0) < f(x',y',z',0) \leqslant f(x',y,z,0),$$

so $f(x', y', z', 0) = f(x_1, y_1, z, 0)$ for some $x_1 \in X$, $y_1 \in Y$ by continuity of $g[\cdot, z, 0]$, and we can analogously infer $f(x, y, z, t) < f(x', y', z', t)$.

We have thus shown that $q[\cdot, t]$ is strictly increasing or constant depending on the respective behavior of $j[2, \cdot, t]$, which proves the theorem. □

The initial reasoning applied in the proof of Theorem 11 can be used to prove very easily closure under union of overlapping sets of coordinates for UI, WUI and GUI. For UI, this has been proven earlier by Keeney and Raiffa [17, p. 317] and for GUI by Fishburn and Keeney [13, p. 931].

THEOREM 12. *Let X, Y, Z, T be nonempty sets, f: $X \times Y \times Z \times T \to \mathbb{R}$ such that y is essential for $f(x, y, z, t)$, and let $X \times Y$ be GUI. Then if $Y \times Z$ is UI, WUI or GUI, so is $X \times Y \times Z$.*

PROOF. Consider the proof of Theorem 11 up to equation (3). The assertions there only rely on the fact that $X \times Y$ and $Y \times Z$ are GPI, which holds here, and not on topological properties. Let the equations (1), (2), (3) there be given, where $g[\cdot, z, t]$ and $j[x, \cdot, t]$ are affine according to Lemma 4, and $j[x, \cdot, t]$ is increasing or strictly increasing if $Y \times Z$ is WUI or UI. Substituting (2), (3) and (1) into (1) yields

$$(4)\qquad f(x,y,z,t) = g[f(x,y,1,0),z,t]$$

$$= j[x, g[f(2,y,1,0),z,0],t],$$

which for $x = 2$ implies

$$g[f(2, y, 1, 0), z, t] = j[2, g[f(2, y, 1, 0), z, 0], t].$$

This shows that

$$g[\cdot, z, t] = j[2, g[\cdot, z, 0], t], \tag{5}$$

since both are affine functions that are uniquely determined if two distinct arguments are given, which are provided by the values of the nonconstant function $f(2, y, 1, 0)$ of y (y is essential and appears only once, and in this term, in (4)). So (5) asserts an identity of functions defined for all real numbers (which is not true under the hypotheses of Theorem 11). In consequence,

$$\begin{aligned} f(x, y, z, t) &= g[f(x, y, 1, 0), z, t] \\ &= j[2, g[f(x, y, 1, 0), z, 0], t] \\ &= j[2, f(x, y, z, 0), t], \end{aligned}$$

and $X \times Y \times Z$ shares the independence properties given for $Y \times Z$ by the restrictions imposed on $j[2, \cdot, t]$. □

The conditions of topological connectivity in Theorem 11 apply precisely to the coordinate axes that belong to an independent subspace ($X \times Y$ or $Y \times Z$), not to the remaining coordinates corresponding to T. None of these conditions can be omitted, in contrast to Theorem 12 which deals with utility independence concepts. This is demonstrated by the following counterexample, where the preference independence assumptions about $X \times Y$, $Y \times Z$ and $X \times Z$ are symmetric in X, Y and Z, so that the missing connectivity of X is representative for any one of these sets.

COUNTEREXAMPLE 13. *There is a continuous function f: $X \times Y \times Z \times T \to \mathbb{R}$, where Y, Z, T are intervals, all variables are essential, $X \times Y$, $Y \times Z$ and $X \times Z$ are PI, but $X \times Y \times Z$ is not even GS, since X is not topologically connected (or X is connected but $f(x, y, z, t)$ is only continuous in y, z, t but not in x).*

PROOF. Let $Y = Z = T = [0, 1]$, $X = \{0, 1\}$ and consider the function f defined by

$$f(x, y, z, t) = (t + 1)(2y + z) + (t + 4)x,$$

which has the following table:

$f(x,y,z,t)$		$x=0$, $y=0$	$x=0$, $y=1$	$x=1$, $y=0$	$x=1$, $y=1$
$t=0$	$z=0$	0	2	$\boxed{4}$	6
	$z=1$	1	$\boxed{3}$	5	7
$t=1$	$z=0$	0	4	$\boxed{5}$	9
	$z=1$	2	$\boxed{6}$	7	11

Obviously, $Y \times Z$ is UI and therefore PI, which can be seen from the table from the fact that the four squares—given by different values for x and t—look alike in their ordering; one can also think of these squares as pieces of planes, for continuous choices of y and z in $Y \times Z = [0,1]^2$. Likewise, the four rows of the table—given by different values for z and t—show a unique ordering, strictly increasing in x (irrespective of the value for y) and in y (for fixed x). So $X \times Y$ is PI, and, since the same argument applies if z is regarded instead of y, $X \times Z$ is PI, too. (Also, X, Y and Z are PI, which can be seen immediately or according to Proposition 16 below, and so x, y, z are even strictly essential; f could also be modified such that subspaces involving T are PI.) Nevertheless, $X \times Y \times Z$ is not GS since for different values of t —cf. the indicated places in the table—there are partial changes in the ordering, for example $f(0,1,1,0) < f(1,0,0,0)$ but $f(0,1,1,1) > f(1,0,0,1)$, whereas, for all t, $f(0,0,0,t) < f(1,1,1,t)$.

If X is replaced by the interval $[0,1]$, the same holds if, in the definition of f, x is replaced by

$$u(x) = \begin{cases} \epsilon x & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ \epsilon x + (1-\epsilon) & \text{for } \frac{1}{2} < x \leqslant 1, \end{cases}$$

with $\epsilon = 1/10$, say, which produces the same table, with only small changes if x is varied in $[0, \frac{1}{2}]$ or $(\frac{1}{2}, 1]$; of course, $u(x)$ and $f(x,y,z,t)$ are discontinuous functions of x in this case. □

In this section, we have shown closure under union of overlapping sets of coordinates for all independence concepts stronger than GPI. The next counterexample shows that this does not hold for GPI, WII or GS.

COUNTEREXAMPLE 14. *There is a continuous function f: $X \times Y \times Z \times T \to \mathbb{R}$, where X, Y, Z, T are intervals, y is essential for $f(x,y,z,t)$, $X \times Y$ and $Y \times Z$ are GPI, but $X \times Y \times Z$ is neither WII nor GS.*

PROOF. Let $X = Z = \mathbb{R}$ and $Y = T = (0, \infty)$, and define f according to $f(x,y,z,t) = g[x \cdot y, z, t]$ with

$$g[h,z,t] = \begin{cases} h \cdot z & \text{for } z \leqslant 0 \text{ or } h \leqslant 0, \\ h \cdot z \cdot t & \text{for } z \geqslant 0 \text{ and } h \geqslant 0. \end{cases}$$

$g[\cdot, z, t]$ is strictly decreasing (for $z > 0$), constant (for $z = 0$) or strictly increasing (for $z > 0$), so $X \times Y$ is GPI. Since always $y > 0$ holds, the conditions $x \cdot y \leqslant 0$ and $x \cdot y \geqslant 0$ are equivalent to $x \leqslant 0$ and $x \geqslant 0$, respectively, so the entire function is given by

$$f(x,y,z,t) = \begin{cases} x \cdot y \cdot z & \text{for } x \leqslant 0 \text{ or } z \leqslant 0, \\ x \cdot y \cdot z \cdot t & \text{for } x \geqslant 0 \text{ and } z \geqslant 0, \end{cases}$$

which is symmetric in x and z, so $Y \times Z$ is also GPI. $X \times Y \times Z$ is not WII, since $f(x,y,z,t)$ cannot be expressed in terms of $f(x,y,z,t')$ (cf. Lemma 4) for any fixed $t' \in T$: this would imply $f(x,y,z,t') = f(x',y',z',t') \Rightarrow f(x,y,z,t) = f(x',y',z',t)$ for all t, in particular,

$$f(-1,t',-1,t') = t' = f(1,1,1,t') \Rightarrow f(-1,t',-1,t) = t' = f(1,1,1,t) = t,$$

which is not true for $t \neq t'$.

Furthermore, $X \times Y \times Z$ is not GS, since

$$f(1,1,-1,t) = -1 < f(-1,1,-1,t) = 1 \quad \text{for all } t,$$

but, for instance,

$$f(-1,2,-1,3) = 2 < f(1,1,1,3) = 3$$

and

$$f(-1,2,-1,1) = 2 > f(1,1,1,1) = 1. \quad \square$$

Finally, weak separability WS is also not closed under union of overlapping sets. In the literature, the closure property has been stated for PI with an additional assumption of strict essentiality for one of the variables (cf. Gorman [14, p. 369]), which by Theorem 11 is not necessary. Nevertheless, even with this assumption, the respective closure is not given for WS; the question was mentioned, for instance, in Blackorby, Primont and Russell [4, p. 134]. However, Mak [21, p. 259], [22, p. 600] states this closure property for WS if all coordinates are strictly essential; his methods are considerably more complex than our proof of Theorem 11.

COUNTEREXAMPLE 15. *There is a continuous function f: $X \times Y \times Z \times T \to \mathbb{R}$, where X, Y, Z, T are intervals, x and y are essential and z is strictly essential for $f(x, y, z, t)$ and $X \times Y$ and $Y \times Z$ are WS, but $X \times Y \times Z$ is not GS.*

PROOF. The counterexample is based on Counterexample 9, using the fact that y can be (partly) inessential if x, z range in suitable subsets of X, Z. Let $X = Z = \mathbb{R}$, $Y = (0, \infty)$ and $T = [1, 2]$, and define

$$f(x,y,z,t) = \begin{cases} -xyz & \text{for } x \leqslant 0,\ z \leqslant 0, \\ 0 & \text{for } xz \leqslant 0, \\ x^t z^{3-t} & \text{for } x \geqslant 0,\ z \geqslant 0. \end{cases}$$

The function f is continuous and depends always on z, so z is strictly essential. We have $f(x, y, z, t) = g[h(x, y), z, t]$, where

$$h(x,y) = \begin{cases} xy & \text{for } x \leqslant 0, \\ x & \text{for } x \geqslant 0 \end{cases}$$

(note $h(x, y) \leqslant 0$ is equivalent to $x \leqslant 0$), and

$$g[h,z,t] = \begin{cases} (-z) \cdot h & \text{for } h \leqslant 0,\ z \leqslant 0, \\ 0 & \text{for } hz \leqslant 0, \\ h^t z^{3-t} & \text{for } h \geqslant 0,\ z \geqslant 0. \end{cases}$$

$g[\cdot, z, t]$ is increasing for all $z \in Z$, $t \in T$, so $X \times Y$ is WS. The functions h and g are also continuous. $Y \times Z$ is WS since $f(x, y, z, t)$ is symmetric in x and z except for a necessary replacement of t by $3 - t$ (both ranging in $T = [1, 2]$). $X \times Y \times Z$ is not GS, since $f(-1, 1, -1, t) = -1 < f(1, 1, 1, t) = 1$ for all t, but (for any y) $f(1, y, 2, 1) = 4 > f(2, y, 1, 1) = 2$, whereas $f(1, y, 2, 2) = 2 < f(2, y, 1, 2) = 4$. $\square$

5. Closure under intersection. Given two sets A, B of coordinates, where S_A and S_B are independent for some given concept, $S_{A \cap B}$ may be independent as well; this is only nontrivial if A and B overlap. In this case, Lemma 4 immediately gives a positive answer for any independence concept that is at least as strong as WII (cf. Figure 1). The result is very easy and does not even rely on topological conditions. (For some cases, the property has been stated previously, as in Gorman [14, p. 370] and Keeney and Raiffa [17, p. 316].)

PROPOSITION 16. *Let $f\colon X \times Y \times Z \times T \to \mathbb{R}$. Then if $X \times Y$ and $Y \times Z$ are both UI, WUI, GUI, PI, WPI, GPI or WII, so is Y.*

PROOF. Let one of the given independence concepts hold for $X \times Y$ and $Y \times Z$. Then, according to Lemma 4(b), $f(x, y, z, t) = g[f(x, y, 1, 0), z, t]$ for suitable $1 \in Z$, $0 \in T$, and $f(x, y, z, t) = j[x, k(y, z), t]$, which by substitution yields

$$(1) \qquad f(x, y, z, t) = g[j[x, k(y, 1), 0], z, t].$$

Since the function $g[j[x, \cdot\,, 0], z, t]$ shares the restrictions imposed by GPI or any stronger concept on $g[\cdot, z, t]$ and $j[x, \cdot\,, t]$ for all $x \in X$, $z \in Z$, $t \in T$ (for instance, being affine or increasing), the respective concept holds for Y as well. As concerns WII, note that for suitable $3 \in X$, $2 \in T$, k can be chosen such that $k(y, z) = f(3, y, z, 2)$ holds, which shows that for $x = 3$, $z = 1$, $t = 2$ in (1), the function $g[j[3, \cdot\,, 0], 1, 2]$ is the identity and therefore injective. □

The preceding—rather canonical—proof "by substitution" suggests that the condition of WII, which allows this substitution using the representation with a canonical function given in Lemma 4(b), is the reason why GPI and all the stronger concepts are closed under intersection. As mentioned in the introduction, [28] describes a theory of decomposing general multiplace functions into conditional functions. Depending on the considered function, properties like continuity or monotonicity are thereby simply inherited and not imposed initially. The important new element is to request that not only the function h, but also g in Lemma 4(a) can be obtained from f by fixing variables suitably. Dually to Lemma 4(b), this is equivalent to a certain surjectivity with respect to single variables, which is here in general missing and needs to be replaced by continuity arguments. The additive representation as in Theorem 21 below is more generally given by a representation with an associative operation, whose characterization is a separate matter. A number of elements of this theory are already visible in the proofs of Theorems 12 and 21 and of Proposition 16. The main technical difficulty is to assert a certain "compatibility" of the reference vectors without restricting the considered functions too much. A number of applications in Operations Research, like decompositions of switching functions, can be obtained as special cases; see [28].

Indeed, closure under intersection is not given for WS or GS, the only concepts defined here that do not imply WII; the question was posed in Blackorby, Primont and Russell [4, p. 120]. For WS, this has been shown by Mak [20, Example 4.2]. According to the following observation, this does not change even if reversals of preferences are admitted. If all coordinates are strictly essential, however, WS is closed under intersection according to Mak [21, §5].

COUNTEREXAMPLE 17. *There is a continuous function $f\colon X \times Y \times Z \to \mathbb{R}$ with intervals X, Y, Z, such that $X \times Y$ and $Y \times Z$ are WS but Y is not GS.*

PROOF. Let $X = Y = Z = [-1, 1]$. The idea is somewhat similar to Counterexample 15: to construct a function $f(x, y, z)$ that is symmetric in x and z, and has different behavior in y according to whether x and z are positive or negative. In particular, we let $f(x, y, z)$ always be positive for the first quadrant of $X \times Z$ (i.e., $x \geqslant 0$, $z \geqslant 0$), zero for the second and fourth quadrants ($xz \leqslant 0$) and negative for the third one. With $f(1, y, 1) = |y|$ and $f(-1, y, -1) = (y - 1)/2$ for $y \in Y$, the subspace Y is not GS (as argued in detail in Counterexample 8). Let

$$f(x,y,z) = \begin{cases} \max\{x, z, (y-1)/2\} & \text{for } x \leqslant 0,\ z \leqslant 0, \\ 0 & \text{for } xz \leqslant 0, \\ \min\{x, z, |y|\} & \text{for } x \geqslant 0,\ z \geqslant 0, \end{cases}$$

which is a continuous function, symmetric in x and z. Then $f(x, y, z) = g[h(x, y), z]$ with

$$h(x,y) = \begin{cases} \max\{x, (y-1)/2\} & \text{for } x \leqslant 0, \\ \min\{x, |y|\} & \text{for } x \geqslant 0, \end{cases}$$

(note $h(x, y) \geqslant 0 \Leftrightarrow x \geqslant 0$), and

$$g[h,z] = \begin{cases} \min\{0, \max\{h, z\}\} & \text{for } z \leqslant 0, \\ \max\{0, \min\{h, z\}\} & \text{for } z \geqslant 0. \end{cases}$$

The operations $\min\{0, \cdot\,\}$ and $\max\{0, \cdot\,\}$, which are increasing, are used to cut off the respective (positive or negative) part of the definition of $h(x, y)$ that is not needed in the second ($x \geqslant 0$, $z \leqslant 0$) and fourth ($x \leqslant 0$, $z \geqslant 0$) quadrants of $X \times Z$. Since $g[\cdot, z]$ is increasing, $X \times Y$ is WS, and so is $Y \times Z$. But, as mentioned, Y is not GS. □

6. Closure under set-theoretic difference and symmetric difference. Given two overlapping independent sets A, B of coordinates, their set-theoretic differences $A - B$ and $B - A$ may be independent, too, as well as their symmetric difference $A \Delta B$. Table 1 gives a survey of these results for the different independence concepts, which will be shown in this section. As before, let X, Y, Z denote the spaces $S_{A-B}, S_{A\cap B}, S_{B-A}$. The abbreviation "s.e." (in the column for Z) means "strict essentiality", and a prefixed "$\neg$" indicates that one cannot necessarily conclude that the respective independence property holds. Probably the most interesting result in this respect applies to preference independence PI under the assumption of strict essentiality of one of the differences, which has been proved by Gorman [14, pp. 369–373]. It implies that PI holds for the other difference. We have generalized this such that—in analogy to Theorem 11—the weaker property WPI or GPI can be inferred, too, if this concept applies to one of the given independent sets of coordinates. The proof employs Lemma 10, and contains the first part of Gorman's proof. The assertion for the symmetric difference is proved, by different methods, with the aid of the important *additive representation* for utility functions that holds for overlapping PI sets. As indicated in the introductory section, some amendments of its proof in [14] are necessary, which will be given in Appendix E. (The reasoning in Blackorby, Primont and Russell [4, p. 115] is nearly identical to the proof of Gorman [14], except that the two parts for difference and symmetric differences are also presented separately.)

As already remarked in an earlier paper [27], the differences of overlapping UI sets A, B are not necessarily UI, but only GUI, since a change of orientation for multiplicatively represented expected-utility functions is possible (cf. Theorem 22 below). This holds, of course, only if strict essentiality is violated, but A and B can even be GUI in this case. In full generality, differences of overlapping PI sets can therefore at most be expected to be GPI. Since GUI has the nicest closure properties, the same might apply if GPI is regarded instead of PI; this conjecture was the starting

TABLE 1

assumptions			*conclusions*		
$X \times Y$	$Y \times Z$	Z	X	Z	$X \times Z$
UI	UI	—	GUI, ¬WUI	GUI, ¬WUI	GUI, ¬WUI
GUI	GUI	—	GUI	GUI	GUI
PI	PI	s.e.	PI	PI	PI
WPI	PI	s.e.	WPI	¬GS	¬GPI
GPI	PI	s.e.	GPI	¬GS	¬GPI
PI	PI	—	¬GS, ¬WII	¬GS, ¬WII	¬WII
WS, WII	PI	s.e.	¬GS, ¬WII	¬GS, ¬WII	¬GS, ¬WII

point for the investigations of this paper. It proved false, however: GPI is not closed under union (Counterexample 14), and differences of PI (and thus GPI) sets need neither be WII nor GS. Furthermore, even with the assumption of strict essentiality for one of the differences, closure under differences is not given for WS, GS and WII, and neither under symmetric difference, which is also not given for WPI or GPI. (The question was posed for weak separability WS in [4, p. 120].)

The closure under union for overlapping PI or GUI sets simplifies our investigations since the "outer" coordinates (here denoted throughout by the subspace T) can be dropped. This is stated in the next lemma, which asserts a certain "transitivity" of the independence concepts PI and GUI. It is used in the following theorems.

LEMMA 18. *Let X be topologically connected, f: $X \times Y \times T \to \mathbb{R}$ and $X \times Y$ be PI with $f(x, y, t) = g[f(x, y, 0), t]$, where $g[\cdot, t]$ is strictly increasing. Then X is PI/WPI/GPI if and only if there are suitable functions j, k such that*

$$f(x, y, 0) = j[k(x), y]$$

holds, and $j[\cdot, y]$ is strictly increasing/or constant/or strictly decreasing, and continuous if f is continuous. If $X \times Y$ is GUI with $g[\cdot, t]$ affine, then X is GUI iff this equation holds with affine $j[\cdot, y]$.

PROOF. The "if" part is given since strictly increasing as well as affine functions are closed under composition. The converse is immediate, since if X is PI/WPI/GPI (resp., GUI), then the equation $f(x, y, z) = j[k(x), y, t]$, with appropriate restrictions on $j[\cdot, y, t]$, holds in particular for $t = 0$. □

THEOREM 19. *Let X, Y, Z be topologically connected, f: $X \times Y \times Z \times T \to \mathbb{R}$ be continuous, where x and y are essential and z is strictly essential for $f(x, y, z, t)$, and let $Y \times Z$ be PI. Then*

(a) *if $X \times Y$ is GPI, WPI or PI, so is X, and*

(b) *it is possible that $X \times Y$ is WPI but Z is not GS.*

The proof of this theorem is somewhat technical and has therefore been deferred to Appendix D. Its hypotheses are chosen as close as possible to those of Theorem 11, but need to be stronger; the important additional assumption is that of strict essentiality of z. With this assumption, weak preference independence of $Y \times Z$ (which is admitted in Theorem 11) is equivalent to preference independence, since if $Y \times Z$ is WPI, with $f(x, y, z, t) = j[x, k(y, z), t]$, where $j[x, \cdot, t]$ is either strictly increasing or constant, the latter is not possible if z is strictly essential, since for such choices of $x \in X$, $t \in T$, z would not matter. The reader may verify that, with strict essentiality of z and (additionally) connectivity of T, $Y \times Z$ is also GPI only if it is PI, since a change of orientation of $j[x, \cdot, t]$ is only possible if this function is somewhere constant, by continuity in x and t (as proved for Lemma 6 or Theorem 26 below). In summary, Theorem 19 can be regarded as a statement of closure under set-theoretic difference for PI, WPI and GPI, if the other difference is strictly essential.

If strict essentiality of z is dropped, to account for the possibility that $Y \times Z$ is properly WPI, assertion (b) of Theorem 19—regarded with X and Z interchanged—shows that one can no longer infer that X is GS, let alone PI, even if $X \times Y$ is PI. Furthermore, it also demonstrates that the symmetric difference $X \times Z$ is not necessarily GPI, since otherwise this would hold for Z by Proposition 16. Counterexample 23 below also shows that even if both $X \times Y$ and $Y \times Z$ are PI, the respective inferences fail without strict essentiality of z.

If in Theorem 19(a), $X \times Y$ is PI, X is PI and therefore strictly essential, so that the theorem, applied again with X and Z interchanged, yields that Z is PI, too. These assertions have been stated by Gorman [14]. As mentioned, the corresponding part of his proof ([14, pp. 369–371]; cf. also [4, pp. 115–120]), except for a shortcut provided by Lemma 10, appears as part of our proof of Theorem 19 (cf. Appendix D).

COROLLARY 20. *Let X, Y, Z be topologically connected, f: $X \times Y \times Z \times T \to \mathbb{R}$ be continuous, where x and y are essential and z is strictly essential for $f(x, y, z, t)$, and let $X \times Y$ and $Y \times Z$ be PI. Then X and Z are PI.*

Furthermore, the following theorem (stated by Gorman [14, Theorem 1]) yields closure under symmetric difference for overlapping PI sets (given strict essentiality), since addition is a symmetric operation and increasing in both arguments. Gorman [14] employed slightly stronger assumptions mentioned before Theorem 11 above but, more importantly, used results on functional equations from [1] incorrectly, as mentioned in the introduction. These inaccuracies will be pointed out in the corrected proof of Theorem 21 given in Appendix E, which seems to be the first correct proof of this result.

THEOREM 21. *Let X, Y, Z be topologically connected, f: $X \times Y \times Z \times T \to \mathbb{R}$ be continuous, where x and y are essential and z is strictly essential for $f(x, y, z, t)$, and $X \times Y$ and $Y \times Z$ are PI. Then there are real-valued continuous functions a, b, c (defined on X, Y, Z, respectively) and g, such that*

$$f(x, y, z, t) = g[a(x) + b(y) + c(z), t]$$

holds, where $g[\cdot, t]$ is strictly increasing for all $t \in T$. Consequently, X, Y, Z, $X \times Z$ and $X \times Y \times Z$ are also PI.

It should be mentioned that the function g in Theorem 21 cannot necessarily be assumed to be the identity function if t is inessential, for instance if f is a product of three positive real variables. In that case, one however obtains an additively represented utility function by applying to f the inverse of g, which is admitted as a strictly increasing transformation. The result can also be applied to a larger number of given overlapping PI sets in order to obtain an additive utility function with more than three summands (cf. Gorman [14, p. 388]).

As far as the utility independence concepts UI, WUI and GUI are concerned, the function $f(x, y, z) = xyz$ with $x > 0$, $y \in \mathbb{R}$, $z > 0$ shows that if $X \times Y$ and $Y \times Z$ are UI, then X and Z are not necessarily WUI, but at most GUI. So UI and WUI are not closed under differences or symmetric difference (by Proposition 16) of overlapping sets; this holds for GUI, however. Similar to Theorem 21, the result is due to a representation with an associative and symmetric binary operation, which is here affine in both arguments.

THEOREM 22. *Let f: $X \times Y \times Z \times T \to \mathbb{R}$, where x, y and z are essential for $f(x, y, z, t)$, and let $X \times Y$ and $Y \times Z$ be GUI. Then X, Z and $X \times Z$ are GUI.*

PROOF. As before, we can assume w.l.o.g. by Lemma 18 that t is inessential and consider $f(x, y, z)$ only, since $X \times Y \times Z$ is GUI by Theorem 12. According to Fishburn and Keeney [13, Lemma 2, p. 932], $\phi[f(x, y, z)] = a(x) * b(y) * c(z)$ holds with suitable functions a: $X \to \mathbb{R}$, b: $Y \to \mathbb{R}$, c: $Z \to \mathbb{R}$, where $*$ is either addition or multiplication, and ϕ: $\mathbb{R} \to \mathbb{R}$ is an invertible affine transformation, which can be omitted if $*$ is addition. Since, in either case, $*$ is associative, and affine in both arguments, X and Z are GUI, and, since $*$ is also a *symmetric* operation, $X \times Z$ is GUI. (This is more or less the original proof by Keeney and Raiffa [17, pp. 316–318], which however disregards a possible change of orientation if $*$ is multiplication, so UI has to be replaced by GUI there; cf. also [27, Theorem 8].) □

We conclude this section with the remaining negative results listed in Table 1.

COUNTEREXAMPLE 23. *There is a continuous function f: $X \times Y \times Z \to \mathbb{R}$ with intervals X, Y, Z, such that x, y and z are essential for $f(x, y, z)$, and $X \times Y$ and $Y \times Z$ are PI, but both X and Z are neither GS nor WII.*

PROOF. Let $X = Y = Z = \mathbb{R}$, and a, b: $\mathbb{R} \to \mathbb{R}$ be defined by $a(r) = |1 - r| + 1 > 0$ and $b(r) = |1 + r| + 1 > 0$ for $r \in \mathbb{R}$. The function f defined according to

$$f(x, y, z) = \begin{cases} a(x) \cdot y \cdot a(z) & \text{for } y \leqslant 0, \\ b(x) \cdot y \cdot b(z) & \text{for } y \geqslant 0, \end{cases}$$

is continuous and symmetric in x and z. $X \times Y$ (and thus $Y \times Z$) is PI since

$$f(x, y, z) = g[h(x, y), z] \quad \text{with } h(r, s) = g[s, r] = \begin{cases} a(r) \cdot s & \text{for } s \leqslant 0, \\ b(r) \cdot s & \text{for } s \geqslant 0, \end{cases}$$

(note $h(x, y) \leqslant 0 \Leftrightarrow y \leqslant 0$), where $g[\cdot, z]$ is strictly increasing. X is not GS: $f(x, -1, 1) = -a(x)$ and $f(x, 1, -1) = b(x)$, and these two functions cannot be expressed as monotonic transformations of one single function, since both $-a(0) = -2 < -a(1) = -1$ and $b(0) = 2 < b(1) = 3$, but $-a(1) > -a(2) = -2$ and $b(1) < b(2) = 4$. If X were WII, $f(x, y, z)$ should be expressible in terms of $f(x, y', z')$—for some fixed y', z'—and y, z by Lemma 4. The term $f(x, y', z')$ is a multiple either of $a(x)$ or of $b(x)$, depending on the sign of y'. But neither function of x can be expressed in terms of the values of the other: $b(x)$ (or any nonzero multiple of it) cannot be expressed in terms of $a(x)$ for all $x \in X$ since $a(0) = a(2)$ but $b(0) \neq b(2)$, and vice versa, since $a(x) = b(-x)$ (a and b interchange meaning if x is replaced by $-x$). So X is not WII, and by symmetry, Z is also neither GS nor WII. □

The next counterexample, finally, shows that the independence assumptions in Theorem 19(a) cannot be weakened to WS or WII (which may even hold at the same time) in order to obtain any independence property for the differences X and Z or the symmetric difference $X \times Z$. That is, even under the assumptions of strict essentiality of Theorem 19 or Corollary 20, none of the concepts WS, GS or WII is closed under set-theoretic difference or symmetric difference. Mak [20, Example 4.3] shows this for WS if no strict essentiality is assumed, but states that, with strict essentiality for all coordinates, WS is closed under differences [21, pp. 259ff] and claims this also for the symmetric difference [21, p. 259]. The results of von Stengel [28] indicate that closure under symmetric difference holds because the associative operations in the considered class of functions happen to be commutative, as is here the case via Theorem 21, and also in [21, p. 261], but not in general [28, Theorem 4.10]. A "composition tree" can be constructed without this closure property [23, p. 329] (compare [21, p. 262]), so it is not essential for a unique hierarchical decomposition. A theory for decomposing monotonic multiplace functions as proposed by Mak is desirable, but in view of the complexity of the proofs in [21], [22] it would be interesting to compare it with the fairly canonical approach, without monotonicity constraints, in [28] of decompositions into conditional functions (or equivalently, functions with generalized "neutral elements" [28, Lemma 3.7]). It is not obvious whether the regularity conditions in [22, p. 598f] permit substantially more general monotonic functions.

COUNTEREXAMPLE 24. *There is a continuous function* $f\colon X \times Y \times Z \to \mathbb{R}$, *where* X, Y *and* Z *are intervals*, x *and* y *are essential and* z *is strictly essential for* $f(x, y, z)$, $Y \times Z$ *is PI and* $X \times Y$ *is WS and WII, but* X, Z *and* $X \times Z$ *are neither GS nor WII.*

PROOF. The construction is similar to Counterexample 23. There, $f(x, y, z)$ is positive or negative depending on the sign of y, and has a different respective behavior in x. The "transition" occurs for $y = 0$, in which case the value for z does not matter, such that z is not strictly essential. Here, this transition is to take place in a whole interval $[0, 1]$, where z (but not x) is still of influence on the behavior of the function.

Let $X = Y = \mathbb{R}$, $Z = [-2, \infty)$ and define $a, b\colon X \to \mathbb{R}$ similarly to Counterexample 23 by $a(x) = (|1 - x| + 1)/2 > 0$ and $b(x) = (|1 + x| + 1)/2 > 0$, where $a(0) = b(0) = 1$. The function f is so constructed as to depend only on z for $z \in [-2, -1]$ (so z is strictly essential), on y, z for $z \in (-1, 0]$, and on x, y, z for $z > 0$. For $z \in [-2, 1]$, $y \in [-1, 2]$ and, representatively, for $x = 0$, the graph of f is drawn in Figure 3. Let

$$f(x,y,z) = \begin{cases} z+2 & \in [0,1] \quad \text{for } -2 \leqslant z \leqslant -1, \\ -z & \in [0,1] \quad \text{for } -1 \leqslant z \leqslant 0,\ y \leqslant 0, \\ (y-1)(z+1)+1 & \in [0,1] \quad \text{for } -1 \leqslant z \leqslant 0,\ 0 \leqslant y \leqslant 1, \\ 1 & \in [0,1] \quad \text{for } -1 \leqslant z \leqslant 0,\ y \geqslant 1, \\ a(x)yz & \in (-\infty,0] \quad \text{for } z \geqslant 0,\ y \leqslant 0, \\ y & \in [0,1] \quad \text{for } z \geqslant 0,\ 0 \leqslant y \leqslant 1, \\ b(x)(y-1)z+1 & \in [1,\infty) \quad \text{for } z \geqslant 0,\ y \geqslant 1. \end{cases}$$

The function f is continuous. Note that f ranges in $[0, 1]$ for $z \leqslant 0$ or $z \geqslant 0$ and $y \in [0, 1]$. In fact, for $z > 0$ (e.g., $z = 1$), $f(x, y, z)$ is negative or greater than 1 iff this holds for y. So the function $f(x, \cdot, z)$ captures the case distinctions in the definition of f, as follows: $f(x, y, z) = j[x, k(y, z)] = g[h(x, y), z]$ with $k(y, z) = f(0, y, z)$ and $j[x, k] = f(x, k, 1)$ for $k \in \mathbb{R}$, $h(x, y) = f(x, y, 1)$ and $g[h, z] = f(0, h, z)$ for $h \in \mathbb{R}$; note the identity of k, j and g, h. Figure 3 shows in fact a graph of k. Since $j[x, \cdot\,]$ is strictly increasing, $Y \times Z$ is PI. By Lemma 4 ($g[\cdot, 1] = \mathrm{id}_{\mathbb{R}}$), $X \times Y$ is WII. Since $g[\cdot, z]$ is increasing (constant for $z \leqslant 0$, except for strictly increasing on $[0, 1]$ for $-1 < z \leqslant 0$, otherwise strictly increasing), $X \times Y$ is WS. Since $f(x, -1, 1) = -a(x)$ and $f(x, 2, 1) = b(x) + 1$, X is neither GS nor WII, as argued in Counterexample 23. So $X \times Z$ is not WII, since otherwise X would be WII by Proposition 16 ($X \times Y$ is WII). $X \times Z$ is not GS, either, as is easily seen from Figure 3: for $y < 0$, $f(0, y, 1) < f(0, y, -2) < f(0, y, -1)$, but, for $y > 1$, $f(0, y, -2) < f(0, y, -1) < f(0, y, 1)$, so no monotonic transformation (depending on y) of a single function of x, z can represent $f(x, y, z)$ for all y. The same values also show that Z is not GS. Z is not WII, since $f(x, y, z)$ cannot be expressed in terms of

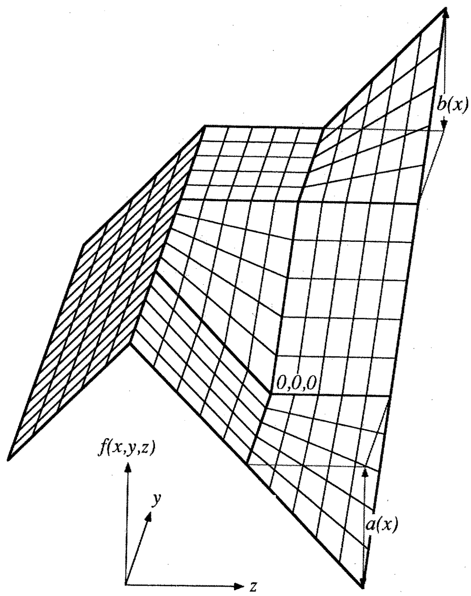

FIGURE 3

$f(x', y', z)$ for any fixed x', y': for $y' \geqslant 1$, this function of z is constant on the subset $[-1, 0]$ of Z, for $0 \leqslant y' \leqslant 1$ it is constant for $z \geqslant 0$, and for $y' \leqslant 0$, $f(x', y', -2) = 0 = f(x', y', 0)$ holds, none of which is true for all x', y', which would be the case if Z were WII, according to Lemma 4. □

7. Inheritance of stronger independence properties from subsets. In this final section we systematically investigate the relationships that hold between two sets of coordinates with *different* independence properties, for the special case that one of these sets contains the other. As pointed out just before Counterexample 7, an independence property of the larger set does not imply any of the smaller one, since the entire set M of coordinates is UI and fulfills therefore any concept of Definition 1 (cf. Figure 1), but there are functions for which some subsets of coordinates have no independence properties (e.g., in Counterexample 9). However, a strong independence assumption that holds for a small set of coordinates can "propagate up" to larger sets that have weaker properties. Indeed, utility and preference independence are *implied* by *any* concept of Definition 1 as soon as the respective property (UI or PI) is known for a smaller set of (essential) coordinates, which may be a singleton. This is of practical importance with regard to verification of these strong independence assumptions (in a decision analysis, say), since such a task is conceptually easier for smaller rather than for large sets of "attributes" (cf. Keeney and Raiffa [17, p. 311]). Another application is that certain independence concepts become indistinguishable (in the sense that the stronger one holds) as soon as the utility function has a behavior in its individual variables that is more restrictive than mere continuity: for instance, if the utility function is strictly increasing in its variables (where these are assumed to be real-valued), all singletons are PI, so the concepts PI and WS collapse into one (as mentioned in Blackorby, Primont and Russell [4, p. 50]).

Table 2 gives a summary of these results: the rows refer to the (weak) concept known for the larger set of coordinates—the corresponding subspace is referred to as $X \times Y$—, the columns to the (stronger) concept that applies to the subspace X, with entries as to what property holds for $X \times Y$ in consequence. A prefixed "$\neg$" means the respective concept cannot be inferred, "$\neg$W.." is short for "$\neg$WS, $\neg$WII", and a

TABLE 2

$X \times Y$ *(result)*		X						
		UI	WUI	GUI	PI	WPI	GPI	WS
	WUI	UI	—	—	UI	—	—	—
	GUI	UI	WUI	—	UI	WUI	—	WUI
	PI	UI	$\neg$GUI	$\neg$GUI	—	—	—	—
$X \times Y$	WPI	UI	$\neg$GUI	$\neg$GUI	PI	—	—	—
(given)	GPI	UI	WPI, $\neg$GUI	$\neg$GUI	PI	WPI	—	WPI
	WS	UI	$\neg$WII	$\neg$WII	PI	$\neg$WII	$\neg$WII	—
	GS	UI	$\neg$WS, $\neg$WII	$\neg$W..	PI	$\neg$W..	$\neg$W..	$\neg$W..
	WII	UI	$\neg$GS	$\neg$GS	PI	$\neg$GS	$\neg$GS	$\neg$GS

dash "—" indicates that $X \times Y$ has already properties as strong as X. The boxed entries imply the other ones in rather obvious fashion, as will be explained below. For instance, it can be seen that the assumption WUI for X does not "propagate up" if weaker properties (like WII or WS) are known for $X \times Y$. So in particular, if X is GPI (which is implied by WUI), no positive inferences can be drawn either, and the respective column in Table 2 has only negative entries; columns for the concepts WII and GS, which are anyhow very weak, have been omitted for that reason, as well as an "all-dash" row for UI.

The "utility independence" part of the following theorem has been stated by Fishburn and Keeney [12, pp. 312–315] for (more restrictive) *convex* domains, in a generalization of an earlier result [16, pp. 28ff], [17, pp. 311–316]. We can give here a rather short proof with the aid of Lemma 6 (the easy part) and Lemma 10.

THEOREM 25. *Let X and Y be topologically connected, $f: X \times Y \times T \to \mathbb{R}$ be continuous, where x is essential for $f(x, y, t)$, and let $X \times Y$ be WII. Then if X is UI or PI, so is $X \times Y$.*

PROOF. Let X be UI or PI: by Lemma 6, $f(x, y, t) = g[f(x, y', t'), y', t', y, t]$ holds for some function g that is strictly increasing in its first argument (whenever it is defined) and affine (if X is UI). For no choice of y', t' is $f(\cdot, y', t')$ constant, since —consider the equation with these values fixed—x would be inessential otherwise. Since $X \times Y$ is WII, $f(x, y, t) = j[f(x, y, 0), t]$ holds for some function j and some $0 \in T$ by Lemma 4. With $y' = y$ and $t' = 0$, thus

$$j[f(x,y,0),t] = g[f(x,y,0),y,0,y,t] \tag{1}$$

holds. Since the continuous function $f(\cdot, y, 0)$ on the set X (which is connected) is never constant, its image, call it $I(y)$, is a proper interval. Equation (1) shows that the function $j[\cdot, t]$ is strictly increasing (and affine if X is UI) on any $I(y)$ for $y \in Y$ (note that $g[\cdot, y, 0, y, t]$ is not defined elsewhere). To show that $j[\cdot, t]$ is not only piecewise increasing or affine but uniformly so throughout its domain K, say, Lemma 10 is used with $k(x, y) = f(x, y, 0)$ (and X, Y instead of Y, Z). According to this lemma, K is an interval, and case (b) applies since $k(\cdot, y)$ is never constant: for any fixed $1 \in Y$ and $A \in I(1)$, A is linked to any $B \in K$ by a finite chain $I(y_0), I(y_1), \dots, I(y_n)$ of intervals, with $y_0 = 1$, $B \in I(y_n)$ (if $A > B$, apply Lemma 10 with A and B interchanged and consider $y_0, y_1, \dots, y_n$ in reverse order). These intervals successively intersect: $j[\cdot, t]$ is strictly increasing on $I(y_0)$ and on $I(y_1)$, and thus on $I(y_0) \cup I(y_1)$ since the two intervals have a common point. If $j[\cdot, t]$ is furthermore piecewise affine (given X is UI), we use the fact that $I(y_0)$ and $I(y_1)$ intersect in a neighborhood, which contains two distinct points, for which the two affine functions $g[\cdot, y_0, 0, y_0, t]$ and $g[\cdot, y_1, 0, y_1, t]$ have identical values by (1); so they are equal, and $j[\cdot, t]$ is affine on $I(y_0) \cup I(y_1)$. By induction, $j[\cdot, t]$ is strictly increasing (and, in case, affine) on $I(y_0) \cup \cdots \cup I(y_i)$ for $1 \leqslant i \leqslant n$, in particular for $i = n$, and thus throughout K, since B (contained in $I(y_n)$) was chosen arbitrarily. So $X \times Y$ is PI (or UI) like X. □

The weak concept of generalized separability GS can also be used instead of WII, as the following result shows (the only positive one for this concept). A similar (but unnecessarily restrictive) theorem is stated in Blackorby, Primont and Russell [4, p. 50].

THEOREM 26. *Let X, Y, T be topologically connected sets, f: $X \times Y \times T \to \mathbb{R}$ be continuous and x be essential for $f(x, y, t)$, and let $X \times Y$ be GS. Then if X is UI or PI, so is $X \times Y$; if $X \times Y$ is WS, connectivity of T is not needed.*

PROOF. Let X be UI or PI and $X \times Y$ be GS. It suffices to show that $X \times Y$ is PI, since then $X \times Y$ is WII, and Theorem 25 shows that $X \times Y$ is UI if X is UI. Let $f(x, y, t) = j[k(x, y), t]$ hold, where $j[\cdot, t]$ is a monotonic function on the image K of k for all t. We first use the connectivity of T to show that $X \times Y$ is WS. Rather intuitively, $j[\cdot, t]$ is somewhere constant if it changes orientation, which is in contrast to the fact that x is strictly essential since it is essential and X is PI. To see this, consider the subsets I and D of T defined by

$$I = \{t \in T | j[k, t] < j[l, t] \text{ for some } k, l \in K,\ k < l\} = \bigcup_{k, l \in K,\ k < l} g_{kl}^{-1}((-\infty, 0)),$$

$$D = \{t \in T | j[k, t] > j[l, t] \text{ for some } k, l \in K,\ k < l\} = \bigcup_{k, l \in K,\ k < l} g_{kl}^{-1}((0, \infty)),$$

where the functions g_{kl}: $T \to \mathbb{R}$ for $k, l \in K$ are defined by $g_{kl}(t) = j[k, t] - j[l, t]$; they are continuous since $f(x, y, t)$ is continuous in t. Obviously, $j[\cdot, t]$ is increasing and not constant (thus not decreasing) iff $t \in I$, and decreasing and not constant iff $t \in D$ holds, so the sets I and D are disjoint, open, and they exhaust T since $j[\cdot, t]$ is never constant by strict essentiality of x. Since T is connected, either I or D is empty, and $X \times Y$ is WS, since if $j[\cdot, t]$ is always decreasing, consider instead of k, j the functions k', j' defined by $k' = -k$ and $j'[-k, t] = j[k, t]$.

In $f(x, y, t) = j[k(x, y), t]$ with increasing $j[\cdot, t]$, the function k: $X \times Y \to \mathbb{R}$ can be chosen continuous according to Appendix A. Furthermore, there it is also shown that k can be assumed to fulfill w.l.o.g.

$$(1) \qquad k(x, y) \leqslant k(x', y') \Leftrightarrow \forall t\ f(x, y, t) \leqslant f(x', y', t).$$

Let $f(x, y, t) = g[h(x), y, t]$ hold where $g[\cdot, y, t]$ is strictly increasing, using the fact that X is PI. We show that $j[\cdot, t]$ is for all y strictly increasing on the image of $k(\cdot, y)$, which is always a nondegenerate interval (otherwise x would not be strictly essential); the respective part of the proof for Theorem 25 (the one that uses Lemma 10) then shows that $X \times Y$ is PI. To this end, let $k(x, y) < k(x', y)$ hold. Then $f(x, y, t') < f(x', y, t')$ for some t' (Remark: Mak [21, p. 252] calls in this case $j[k, t]$ *parsimonious* in k), since $f(x, y, t) \geqslant f(x', y, t)$ for all t would imply $k(x, y) \geqslant k(x', y)$ by (1). Since $g[\cdot, y, t']$ is strictly increasing, $h(x) < h(x')$ holds and thus

$f(x, y, t) < f(x', y, t)$ for all t, which proves that $j[\cdot, t]$ is indeed piecewise, and thus globally, strictly increasing, which was to be shown. □

Topological connectivity is a necessary assumption to conclude that $X \times Y$ inherits the property UI or PI (or any weaker one) from X if it is WII or GS. This is demonstrated by the following counterexample.

COUNTEREXAMPLE 27. *There are continuous functions f: $X \times Y \times T \to \mathbb{R}$, where X and T are topologically connected sets, but Y (and thus $X \times Y$) is not, such that X is UI and $X \times Y$ is*

(a) *WII and WS but not GPI, or*

(b) *PI but not GUI.*

PROOF. Let $Y = \{0, 1\}$, $T = [0, 1]$ and $X = [0, 1]$ to show (a), $X = [0, 1)$ to show (b), and define

$$f(x, y, t) = \begin{cases} x & \text{for } y = 0, \\ x + 2 - t & \text{for } y = 1. \end{cases}$$

Clearly, f is continuous and X is UI (since for fixed y, only one of the two cases applies). Then $f(x, y, t) = g[h(x, y), t]$ with $h(x, y) = x + 2y$ (ranging in $[0, 1] \cup [2, 3]$ or $[0, 1) \cup [2, 3)$, respectively) and

$$g[h, t] = \begin{cases} h & \text{for } h \leqslant 1, \\ h - t & \text{for } h \geqslant 2. \end{cases}$$

The function $g[\cdot, t]$ is increasing for all t, so $X \times Y$ is WS, and in case (b) even strictly increasing, so that $X \times Y$ is PI. Since $g[\cdot, 0]$ is the identity, that is, $h(x, y) = f(x, y, 0)$, $X \times Y$ is WII, and according to Lemma 4, g would be a suitable choice in Definition 1 if $X \times Y$ were (a) GPI or (b) GUI, respectively. But $g[\cdot, 1]$ is not constant, and in case (a) not injective (and thus not strictly monotonic), since

$$g[h(1,0), 1] = g[1, 1] = 1 = g[h(0, 1), 1] = g[2, 1],$$

so $X \times Y$ is not GPI, and in case (b) not affine on its domain $[0, 1) \cup [2, 3)$, so $X \times Y$ is not GUI. □

If $X \times Y$ is GPI, that is, $f(x, y, t)$ is strictly monotonic in a suitable function of x, y, a change of orientation cannot occur if $f(x, y, t)$ is increasing in (a function of) x: if X is WS, then $X \times Y$ is WPI. The reason is, rather intuitively, that changing the orientation for a large set of coordinates would entail a change for any coordinates therein. If X is WPI, this is already stated in Theorem 11 (with inessential z), but continuity assumptions are not necessary.

PROPOSITION 28. *Let f: $X \times Y \times T \to \mathbb{R}$, where x is essential for $f(x, y, t)$, and let X be WS. Then if the subspace $X \times Y$ is GPI, it is WPI.*

PROOF. Let $f(x, y, t) = g[h(x), y, t]$ hold with increasing $g[\cdot, y, t]$ and, with $X \times Y$ GPI,

(1) $$f(x, y, t) = j[f(x, y, 0), t]$$

for some $0 \in T$ by Corollary 5, where $j[\cdot, t]$ is strictly monotonic or constant. Since x is essential, $f(x, y', t') < f(x', y', t')$ holds for some x, x', y', t', thus $h(x) < h(x')$ and

(2) $$f(x, y', t) \leqslant f(x', y', t) \quad \text{for all } t,$$

since $g[\cdot, y', t]$ is increasing. For $t = 0$, $f(x, y', 0) \neq f(x', y', 0)$ holds since otherwise $f(x, y', t') = f(x', y', t')$ by (1). Thus $f(x, y', 0) < f(x', y', 0)$, and (2) shows that $j[\cdot, t]$ is not strictly decreasing, that is, is only strictly increasing or constant. So $X \times Y$ is WPI. □

We summarize the positive results listed in Table 2: if X is UI, $X \times Y$ is UI if any independence concept holds for $X \times Y$, by Theorems 25 and 26, since WII implies any concept except WS and GS. If $X \times Y$ is GPI and X is WS (in particular, WPI or WUI), then $X \times Y$ is WPI by Proposition 28. The column for PI in Table 2 is like that for UI, where the two rows for the "affine" concepts WUI and GUI indicate UI (if X is PI) or WUI (if X is WS), since the equivalences

$$\text{UI} \Leftrightarrow \text{GUI and PI},$$

$$\text{WUI} \Leftrightarrow \text{GUI and WPI}$$

hold by Definition 1 (note the restrictions on $g[\cdot, y]$, also indicated in Figure 1). These are, however, all the positive inferences that are possible.

COUNTEREXAMPLES 29. *There are continuous functions f: $X \times Y \times T \to \mathbb{R}$ with intervals X, Y, T, such that X is WUI and $X \times Y$ is*
(a) *PI but not GUI, or*
(b) *WS but not WII, or*
(c) *GS but not WS, or*
(d) *WII but not GS.*

PROOF. In all cases, let $X = (0, \infty)$, $Y = \mathbb{R}$ and $f(x, y, t) = g[h(x, y), t]$ with the continuous function h defined by

$$h(x, y) = \begin{cases} y & \text{for } y \leqslant 0, \\ xy & \text{for } y \geqslant 0, \end{cases}$$

which fulfills $h(x, y) \leqslant 0 \Leftrightarrow y \leqslant 0$. The function $g[\cdot, t]$ will be affine and increasing for positive arguments, so that X is WUI, and behave differently, according to the particular cases, for negative arguments (where x does not matter, so no restrictions are given by the fact that X is WUI).

For (a), let $T = (0, \infty)$ and

$$g[h, t] = \begin{cases} h & \text{for } h \leqslant 0, \\ ht & \text{for } h \geqslant 0, \end{cases}$$

so that

$$f(x, y, t) = \begin{cases} y & \text{for } y \leqslant 0, \\ xyt & \text{for } y \geqslant 0. \end{cases}$$

Since $t > 0$, $X \times Y$ is PI. Because $h(x, y) = f(x, y, 1)$, g would be a suitable choice in Definition 1 if $X \times Y$ were GUI (by Lemma 4), but $g[\cdot, 2]$ is not affine, so this is not the case.

To show (b), let $T = \mathbb{R}$ and

$$g[h, t] = \begin{cases} h \cdot |t| & \text{for } ht \geqslant 0, \\ 0 & \text{for } ht \leqslant 0, \end{cases}$$

so that

$$f(x,y,t) = \begin{cases} -yt & \text{for } y \leqslant 0,\, t \leqslant 0, \\ 0 & \text{for } yt \leqslant 0, \\ xyt & \text{for } y \geqslant 0,\, t \geqslant 0. \end{cases}$$

Clearly, $g[h,t]$ is continuous, and increasing in h, so $X \times Y$ is WS. But $f(x,y,t)$ cannot be expressed in terms of $f(x,y,t')$ for any fixed $t' \in T$ since the function $f(x,\cdot,t')$ for $t' \geqslant 0$ is constant for negative arguments, and similarly for positive arguments if $t' \leqslant 0$, none of which is true for all t'. So $X \times Y$ is not WII.

Case (c) is covered by $T = \mathbb{R}$, where

$$g[h,t] = \max\{ht, 0\},$$

so that

$$f(x,y,t) = \begin{cases} yt & \text{for } y \leqslant 0,\, t \leqslant 0, \\ 0 & \text{for } yt \leqslant 0, \\ xyt & \text{for } y \geqslant 0,\, t \geqslant 0. \end{cases}$$

The function g is continuous, and $g[\cdot,t]$ is monotonic, so $X \times Y$ is GS. If $X \times Y$ were WS, $f(1,-1,1) = 0 < f(1,1,1) = 1$ would imply $f(1,-1,t) \leqslant f(1,1,t)$ for all t, but $f(1,-1,-1) = 1 > f(1,1,-1) = 0$. Incidentally, $X \times Y$ is in this example not WII, either (f is very similar to (b) above).

Finally, (d) is shown with $T = \mathbb{R}$ by

$$g[h,t] = \begin{cases} ht & \text{for } h \leqslant 0, \\ h & \text{for } h \geqslant 0, \end{cases}$$

so that

$$f(x,y,t) = \begin{cases} yt & \text{for } y \leqslant 0, \\ xy & \text{for } y \geqslant 0. \end{cases}$$

Here, $X \times Y$ is WII since $g[\cdot,1]$ is the identity. The function $g[\cdot,t]$ has no unique monotonicity behavior since t may change sign: if $X \times Y$ were GS, the fact that

$$f(1,-1,1) = -1 < f(1,0,t) = 0 < f(1,1,t) = 1$$

holds for all t would entail $f(1,-1,t) \leqslant f(1,0,t)$ for all t, but $f(1,-1,-1) = 1 > 0$. □

An example similar to (a) is given in Fishburn and Keeney [12, p. 307]. The preceding counterexamples show that if any independence concept other than UI or PI—of which WUI is the strongest (cf. Figure 1)—holds for X, no property of $X \times Y$ can be inferred other than one that is already given; the only exception is that $X \times Y$ is WPI if it is GPI by Proposition 28. If X is WUI (or GUI), the corresponding affine representation for $f(x,y,t)$ does not carry over to $X \times Y$ by (a) even if $X \times Y$ is PI; therefore the "$\neg$GUI" entries in the rows for PI, WPI, GPI in Table 2. If $X \times Y$ is WS and X is WUI, no independence property for $X \times Y$ (other than GS, which is weaker) holds, since it would imply WII, in contrast to (b); so Theorem 26 cannot be generalized. Furthermore, GS does not share with GPI an inheritance of unique orientation (that is, of being only increasing) from subsets by (c). In other words, it is not possible to generalize Proposition 28 from GPI to GS, with WS instead of WPI as a result for $X \times Y$, if X is WS (or even WUI). Finally, the independence assumptions for X in Theorem 25 cannot be weakened: given $X \times Y$ is WII, any stronger concept would imply GS, which does not necessarily hold for $X \times Y$ if X is WUI, according to (d).

Appendix A: Weak separability. We will give here an alternative definition of the concept WS of Definition 1 of weak separability, which is taken from Bliss [5] and Blackorby, Primont and Russell [4], although somewhat simplified. This will be done using induced preference relations: given a set S and a function $f\colon S \to \mathbb{R}$, the binary relation R defined on S by $sRs' \Leftrightarrow f(s) \leqslant f(s')$ is a total preorder on S (a preorder is a reflexive and transitive relation, and "total" means that sRs' or $s'Rs$ holds for any $s,s' \in S$). We say R is induced (or represented) by f. Given a "preference relation" R on S that is a total preorder, a continuous utility function that represents it exists if the following conditions hold (according to Debreu [8, Theorem I, p. 162]): S has a topology such that S is connected, topologically separable (i.e., containing a countable dense subset), and where the sets $\{s \mid sRs'\}$ and $\{s \mid s'Rs\}$ are closed for all $s' \in S$. We assume here that each coordinate axis S_i, $i \in M$, is topologically connected and topologically separable, so that this holds also for any product space $S_A = \prod_{i \in A} S_i$ for $A \subseteq M$, in particular for $S = S_M$.

Given $f\colon X \times Y \to \mathbb{R}$, where X is any subspace of S_M, and fixed $y \in Y$, the function $f(\cdot, y)\colon X \to \mathbb{R}$ induces a total preorder on X. The intersection P of these preorders (for $y \in Y$) given by

$$xPx' \Leftrightarrow \forall y\ f(x,y) \leqslant f(x',y)$$

is again a preorder. Call X weakly separable if P is also *total*, that is, in other words, if no two x, x' are "incomparable" in that $f(x, y) > f(x', y)$ and $f(x, y') < f(x', y')$ hold for some y, y'. Equivalently, Blackorby, Primont and Russell [4, pp. 43–45] and Bliss [5, p. 147f] consider the function β mapping $X \times Y$ to the power set of X defined by $\beta(x, y) := \{x' \in X \mid f(x, y) \leqslant f(x', y)\}$ and request that the image $\beta(X \times Y)$ of β is "nested", that is, totally ordered by inclusion.

We show that this definition of weak separability implies the concept of Definition 1 if the topological assumptions hold (similar proofs are found in [5, p. 149], [4, p. 59f] and [21, p. 253f]). Let X be connected and topologically separable and $f(x, y)$ be continuous in x. Then, for all $a \in X$, the sets $\{x \in X \mid xPa\}$ and $\{x \in X \mid aPx\}$ are closed in X: for instance,

$$\begin{aligned} \{x \in X \mid xPa\} &= \{x \in X \mid \forall y\ f(x,y) \leqslant f(a,y)\} \\ &= \bigcap_{y \in Y} \{x \in X \mid f(x,y) \leqslant f(a,y)\} \\ &= \bigcap_{y \in Y} f(\cdot, y)^{-1}((-\infty, f(a,y)]), \end{aligned}$$

which is an intersection of closed sets by the assumed continuity of $f(\cdot, y)$ for all y. If P is total, Debreu's theorem [8, p. 162] then yields the existence of a (continuous) function $h\colon X \to \mathbb{R}$ that represents P by $h(x) \leqslant h(x') \Leftrightarrow xPx'$. The function $g\colon H \times Y \to \mathbb{R}$ (with H as the image of h) can then be *defined* by $g[h(x), y] = f(x, y)$, where g is well defined because $h(x) = h(x') \Rightarrow f(x, y) = f(x', y)$ holds according to the definition of P, which also shows that $g[\cdot, y]$ is increasing.

The converse is rather obvious. Let X be WS according to Definition 1, that is, let $f(x, y) = g[h(x), y]$ hold with functions $h: X \to \mathbb{R}$ (with image H) and $g: H \times Y \to \mathbb{R}$ such that $g[\cdot, y]$ is increasing. Then $h(x) \leqslant h(x') \Rightarrow xPx'$ for all x, x', which implies that P is total, i.e., X is weakly separable according to the new definition.

For this direction, topological conditions are not required, but can be used to show that h can be chosen continuous, provided $f(x, y)$ is continuous in x (topological separability or connectivity of X is not necessary), as follows: if $f(\cdot, y)$ is continuous for all y, it was just shown that then the sets $\{x | xPa\}$ and $\{x | aPx\}$ for $a \in X$ are closed, or, in Debreu's terms [8], that the topology on X is "natural". Furthermore, there is a countable subset Z of X that is "order-dense" with respect to the total preorder represented by h, and thus with respect to P (Birkhoff [3, p. 201]): for each closed interval I with rational endpoints (of which there are countably many), pick an element z of X with $h(z) \in I$ if I intersects H, or otherwise, if there is a largest element l of H smaller than the lower endpoint of I, pick $z \in X$ such that $h(z) = l$; let Z be the set of these elements z, which is countable. Then, for $x, x' \in X$ with xPx' but not $x'Px$, the inequality $h(x) < h(x')$ holds, and there exists $z \in Z$ with $h(x) \leqslant h(z) \leqslant h(x')$: consider rationals p, q with $h(x) < p \leqslant q \leqslant h(x')$; if $[p, q] \cap H$ is empty and $[h(x), p) \cap H$ has no largest element, the latter set contains an element h' of H other than $h(x)$, so that $[r, p]$ intersects H, were r is a rational with $h(x) < r < h' < p$; in each case, there is an element $z \in Z$ with $h(x) \leqslant h(z) \leqslant h(x')$ and thereby xPz, zPx'. This is the claimed order-density of Z with respect to P. According to [8, Lemma II, p. 161], there is a continuous function on X that represents P, which can be used instead of h, with g defined anew accordingly, as before.

Appendix B: Proof of Lemma 6.

Proof of Lemma 6. If X is PI or UI, the existence of g and the restrictions on $g[\cdot, y', y]$ follow from Lemma 4 and Corollary 5, which proves the "if" part. Conversely, assume

$$f(x, y) = g[f(x, y'), y', y]$$

holds. Let y, y' be fixed. Obviously, for any $a, b \in X$,

$$f(a, y) = f(b, y) \Rightarrow g[f(a, y), y, y'] = g[f(b, y), y, y'],$$

which is equivalent to

$$g[f(a, y'), y', y] = g[f(b, y'), y', y] \Rightarrow f(a, y') = f(b, y').$$

In other words, the function $g[\cdot, y', y]$, call it ϕ for short, is injective (as already stated in Lemma 4(a)). Let $h = f(\cdot, y)$, and $k = f(\cdot, y')$ with image K, which is an interval since k is continuous, so $h(x) = \phi(k(x))$. We want to prove monotonicity and continuity of ϕ; this is trivial for inessential x, where K is degenerated to one point, so let this not be the case. Either one of the conditions in question implies the other, because of Lemma 2 and since continuous injective functions on intervals are strictly monotonic. Does continuity of ϕ follow from continuity of h and k? This is true if X is arcconnected (cf. Gorman [14, p. 387f] or Wakker [33, Lemma 2.5]), but not if X is just connected: consider the (discontinuous) function ϕ: $[0, 1] \to \mathbb{R}$ given by

$$\phi(r) = \begin{cases} 0 & \text{for } r = 0, \\ \sin(1/r) & \text{for } 0 < r \leqslant 1, \end{cases}$$

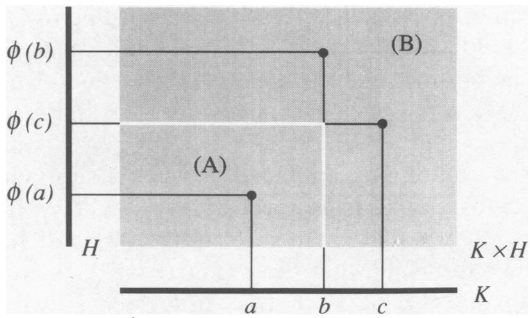

FIGURE 4

and let X be its graph, i.e., $X = \{(r, \phi(r)) | 0 \leqslant r \leqslant 1\}$. X is connected since $X - \{(0,0)\}$ is obviously arcconnected and $(0,0)$ is an accumulation point of that set. With k and h as projections, i.e., $k(r,s) = r$ and $h(r,s) = s$ for $(r,s) = x \in X$, $h(x) = \phi(k(x))$ holds, where h and k are continuous (compare also [33, Example 2.3]). This counterexample, however, gives a clue how to use the injectivity of ϕ: because the functions k, h: $X \to \mathbb{R}$ are continuous, so is the pair (k, h): $X \to K \times H$ whose image $\{(k(x), h(x)) | x \in X\}$, which is the graph of ϕ, is connected. We show that this is not the case if ϕ is not monotonic.

Assume that ϕ is not monotonic: then it is easy to see that there are three elements a, b, c of K with $a < b < c$ such that $\phi(b)$ does not lie between $\phi(a)$ and $\phi(c)$. W.l.o.g. $\phi(a) < \phi(c) < \phi(b)$ as depicted in Figure 4, otherwise change signs suitably. Then $A = k^{-1}((-\infty, b)) \cap h^{-1}((-\infty, \phi(c)))$ is an open subset of X, which is not empty since it contains x with $k(x) = a$ (a was taken from the image K of k), because $k(x) = a < b$ and $h(x) = \phi(k(x)) = \phi(a) < \phi(c)$. Similarly, the set $B = k^{-1}((b, \infty)) \cup h^{-1}((\phi(c), \infty))$ is a nonempty open subset of X, containing $k^{-1}(\{c\})$, for instance. A and B are obviously disjoint (the corresponding two regions of $K \times H$ are shown in Figure 4). They also form a partition of X: any x not belonging to A or B would have to fulfill $k(x) = b$, $h(x) \leqslant \phi(c)$ or $k(x) \leqslant b$, $h(x) = \phi(c)$ (the white lines in Figure 4). These cases are not possible, the first since it implies $h(x) = \phi(k(x)) = \phi(b) > \phi(c)$, the second because $\phi(c) = h(x) = \phi(k(x))$ implies $c = k(x) > b$ by injectivity of ϕ. But this is a contradiction to the assumption that X is connected (for a similar argument see [33, Lemmas 2.2 and 2.4]). So, ϕ is strictly monotonic and thus continuous by Lemma 2. We have now shown that X is GPI even without using the connectivity of Y.

For fixed $y' \in Y$, $g[\cdot, y', y]$ is either strictly increasing or strictly decreasing for *all* $y \in Y$: suppose, otherwise, that, for some $p, q \in Y$, $g[\cdot, y', p]$ increases:

$$g[f(a, y'), y', p] - g[f(b, y'), y', p] < 0$$

and $g[\cdot, y', q]$ decreases:

$$g[f(a, y'), y', q] - g[f(b, y'), y', q] > 0$$

for suitable $a, b \in X$, $f(a, y') < f(b, y')$. On the left-hand side are terms of the continuous function $f(a, y) - f(b, y)$ of $y \in Y$. Since Y is connected,

$$g[f(a, y'), y', y] - g[f(b, y'), y', y] = 0$$

holds for some y, which contradicts the injectivity of $g[\cdot, y', y]$. Since $g[\cdot, y', y']$ is the identity which is strictly increasing, $g[\cdot, y', y]$ is therefore strictly increasing for all y, y'. □

Appendix C: Proof of Lemma 10.

PROOF OF LEMMA 10. Let $A = k(a, z_0)$ for suitable $a \in Y$, $z_0 \in Z$. If $B = k(b, z_0)$ for some $b \in Y$, (b) holds with $n = 0$, otherwise let $B = k(b, z_n)$ for some $b \in Y$, $z_n \in Z$ for some as yet unspecified natural number $n > 0$, i.e.,

$$A \in I(z_0), \quad B \in I(z_n).$$

Let $A \leqslant C \leqslant B$ for some C. We show that C belongs to K (thus $[A, B] \subseteq K$, and, with arbitrary choices for A, B, this shows K is an interval). This is clearly the case if $C \in I(z_n)$, otherwise $k(a, z_0) \leqslant C < k(a, z_n)$, thus, since $k(a, \cdot)$ is continuous and Z connected, $C = k(a, z)$ for suitable $z \in Z$, i.e., $C \in K$.

The intervals $I(z)$, $z \in Z$, therefore cover $[A, B]$. To prove (b), we will essentially show that this also holds for the open interiors of these intervals, to apply the Heine-Borel covering theorem (i.e., to use the compactness of $[A, B]$). This does not necessarily hold if one of these intervals $I(z)$ is degenerated (which however implies assertion (a)), or if A or B is an endpoint of the interval K; in the latter case, a suitable subcompactum of $[A, B]$ will serve the purpose.

Assume therefore for the rest of the proof that (a) does not hold, i.e., a point $C \in [A, B]$ belongs to $I(z)$ for some z only if $I(z)$ is nondegenerate.

Let $A \leqslant C \leqslant B$, and assume that C does not belong to the interior of any $I(z)$, $z \in Z$ (it is an endpoint of some $I(z)$ then). Let the subsets L and U of Z be defined by

$$L = \{z \in Z | k(y, z) < C \text{ for some } y \in Y\} = \bigcup_{y \in Y} k(y, \cdot)^{-1}((-\infty, C)),$$

$$U = \{z \in Z | k(y, z) > C \text{ for some } y \in Y\} = \bigcup_{y \in Y} k(y, \cdot)^{-1}((C, \infty)).$$

The sets L and U are disjoint, since otherwise C would belong to the interior of $I(z)$ for $z \in L \cap U$, which was excluded. The union of L and U is Z, since for any z that does not belong to either set, $k(y, z) = C$ for all y, which was excluded as well. L and U are open because $k(y, \cdot)$ is continuous for $y \in Y$. Thus, since Z is connected, either L or U is empty. This is only possible if $C = A$ and A is the lower endpoint of K (then L is empty), or if $C = B$ and B is the upper endpoint of K (U is empty); both A and B could be chosen this way if K has endpoints. Otherwise, we have shown that any interior point of K in $[A, B]$ belongs to the interior of $I(z)$ for some $z \in Z$. If A (B) is the lower (upper) endpoint of K, let A' (B') be an interior point of the—by assumption—nondegenerate interval $I(z_0)$ ($I(z_n)$), otherwise let $A' = A$ ($B' = B$).

The interiors of $I(z)$, $z \in Z$, then form an open covering of the compact interval $[A', B']$. According to the definition of compactness [18, p. 135], there are finitely many of these, $I(z_0), I(z_1), \ldots, I(z_n)$, whose interiors still cover $[A', B']$. Without loss of generality, let thereby z_0, z_n retain their original meaning (so the intervals themselves in fact cover $[A, B]$), the covering be minimal subject to this constraint, and the elements $z_1, \ldots, z_{n-1}$ of Z be numbered in such a way that the lower endpoints of the intervals $I(z_1), \ldots, I(z_n)$ are in increasing order; this order is strict and, likewise, the upper endpoints of $I(z_0), \ldots, I(z_{n-1})$ are in strictly increasing order, for otherwise the covering would not be minimal. By this construction, the interiors of $I(z_i)$ and $I(z_{i+1})$ for $0 \leqslant i < n$ intersect in a nonempty open set, thus containing a neighborhood of a point, which shows (b). □

Appendix D: Proof of Theorem 19.

PROOF OF THEOREM 19. Let $X \times Y$ be GPI. By Theorem 11, $X \times Y \times Z$ is PI, so by Corollary 5 and Lemma 18 it suffices to consider $f(x, y, z, 0)$ instead of $f(x, y, z, t)$; drop 0 for simplicity of notation. We will show

(1) $$f(x, y, z) = r[u(x), y, z],$$

where $r[\cdot, y, z]$ is strictly monotonic or constant, and strictly increasing (or constant) if $X \times Y$ is PI (WPI). Since $X \times Y$ is GPI and $Y \times Z$ is PI,

(2) $$\begin{aligned} f(x, y, z) &= g[h(x, y), z] \\ &= j[x, k(y, z)], \end{aligned}$$

where $g[\cdot, z]$ is strictly monotonic or constant (and further restricted, as usual, if $X \times Y$ is WPI or PI), and $j[x, \cdot\,]$ is strictly increasing. By Corollary 5, w.l.o.g.

(3) $$h(x, y) = f(x, y, 1) \quad \text{and} \quad k(y, z) = f(3, y, z)$$

hold for suitable elements $1 \in Z$, $3 \in X$, where g, h, j, k are separately continuous functions (as above, in the proof of Theorem 11).

Since x is essential in (2), $h(\cdot, 2)$ is not constant for some $2 \in Y$; this function is, by Corollary 5, a suitable candidate for u in (1). Let $x, x' \in X$ be such that $h(x, 2) < h(x', 2)$. Then, for all z, $f(x, 2, z) \lessgtr f(x', 2, z)$ or $f(x, 2, z) = f(x', 2, z)$ by (2) depending on whether $g[\cdot, z]$ increases, decreases or is constant, which holds irrespective of the particular values chosen for x, x'. So the important part is to generalize this for all $y' \in Y$, say, instead of 2.

Let $y' \in Y$, and $A = k(2, 1)$, $B = k(y', 1)$. For the moment, assume $A \leqslant B$. We apply Lemma 10, although with Y and Z interchanged: let $I(y)$ denote the interval that is the image of $k(y, \cdot\,)$; since this function on Z is never constant—otherwise, by (2), Z would not be strictly essential—case (b) of Lemma 10 applies, that is, $k(2, 1)$ and $k(y', 1)$ are linked by a finite chain $I(y_0), I(y_1), \ldots, I(y_n)$ of these intervals, which successively intersect, where w.l.o.g. $y_0 = 2$, $y_n = y'$. (If $A > B$, interchange A and B and consider $y_0, \ldots, y_n$ in reverse order; so the choice of $y' \in Y$ is in fact arbitrary.) In other words, there are suitable elements $w_0, \ldots, w_{n-1}, z_1, \ldots, z_n$ of Z such that $k(y_i, w_i) = k(y_{i+1}, z_{i+1})$ for $0 \leqslant i < n$. If for $0 \leqslant i < n$, $g[\cdot, w_i]$ is not constant (and therefore injective), the proof is easily completed (this is in fact the case if $X \times Y$ is PI): assume that y' is given such that $w_0, \ldots, w_{n-1}$ can be chosen this way. Then, for $h(x, y_0) < h(x', y_0)$ (note $y_0 = 2$),

$$f(x, y_0, w_0) \lessgtr f(x', y_0, w_0)$$

holds by (2), depending on whether $g[\cdot, w_0]$ is strictly increasing or decreasing, thus

$$f(x, y_1, z_1) \lessgtr f(x', y_1, z_1),$$

which shows that $g[\cdot, z_1]$ is also strictly monotonic, and (depending, furthermore, on the orientation of this function)

$$h(x, y_1) \lessgtr h(x', y_1).$$

Therefore, by induction, with i, $i + 1$ $(0 \leqslant i < n)$ instead of 0, 1, $h(x, y_n) \lessgtr h(x', y_n)$ holds, i.e. (since $y_n = y'$), $h(x, y') < h(x', y')$ or $h(x, y') > h(x', y')$. Whichever of these inequalities is given for a particular choice of $x, x' \in X$ with $h(x, 2) < h(x', 2)$, it holds also for any such x, x', since the reasoning depends only on the orientation and injectivity of $g[\cdot, w_i]$ and $g[\cdot, z_{i+1}]$ for $0 \leqslant i < n$. In a similar fashion, the latter also implies

$$h(x,2) = h(x',2) \Rightarrow h(x, y') = h(x', y'),$$

so that a function s can be defined according to

$$s[h(x,2), y'] = h(x, y'). \tag{4}$$

(Note that so far, there is still a restriction on the choice of y', unless $X \times Y$ is PI). The established implications

$$h(x,2) < h(x',2) \Rightarrow h(x, y') < h(x', y')$$

or, for some y' (if $X \times Y$ is not WPI),

$$h(x,2) < h(x',2) \Rightarrow h(x, y') > h(x', y')$$

show that $s[\cdot, y']$ is strictly monotonic, and thus $g[s[\cdot, y'], z] =: r[\cdot, y', z]$ is strictly monotonic or constant for any z. With $u = h(\cdot, 2)$, equation (1) holds by (4) for $y = y'$. If no restriction is imposed on $y' \in Y$, then X is GPI, and, according to the above remarks, PI or WPI if this holds for $X \times Y$.

What remains to consider, in order to prove (a), is the case that $A = k(2, 1)$ and $B = k(y', 1)$ are linked by a chain of intervals such that an intersection $I(y_i) \cap I(y_{i+1})$ for some i, $0 \leqslant i < n$, contains only values $k(y_i, w_i)$ where $g[\cdot, w_i]$ is constant. Assume this case holds; we will show $h(\cdot, y')$ is constant so (4) can also be applied. Then, because of (3), $k(\cdot, w_i)$ is constant, and furthermore even

$$g[\cdot, z] \text{ is constant} \Leftrightarrow k(\cdot, z) \text{ is constant} \tag{5}$$

for $z \in Z$, since $g[h(3, \cdot), z] = k(\cdot, z)$ holds, where $h(3, \cdot)$ is a nonconstant function on Y (y is essential) and $g[\cdot, z]$ is injective if not constant. Let Z_0 be the set of these elements of Z, i.e., $Z_0 = \{z \in Z | k(\cdot, z) \text{ is constant}\}$, K_0 be the corresponding set of values of k, i.e., $K_0 = \{k(y, z) | y \in Y,\ z \in Z_0\} = \{c | k(\cdot, z) \equiv c \text{ for some } z\}$, and $Z_+ = Z - Z_0$. By definition, K_0 is a subset of $I(y)$ for any y. The above intersection $I(y_i) \cap I(y_{i+1})$ is a proper interval, since it contains a neighborhood, as stated in Lemma 10(b). It furthermore contains K_0, and in fact equals K_0, since any other element of the image of $k(y_i, \cdot)$ is given by $k(y_i, w_i)$ with nonconstant $k(\cdot, w_i)$ and thus injective $g[\cdot, w_i]$, which was excluded from this intersection. So K_0 is a proper interval, and the chain has only one link: the intervals $I(2)$ and $I(y')$ both contain K_0 and can be assumed to intersect in that set (if they have other points in common, we can proceed as before).

Z_0 is closed: let, for some $4, 5 \in Y$, $h(3, 4) \neq h(3, 5)$ hold. Then, for $z \in Z$, $g[\cdot, z]$ is constant (i.e., $z \in Z_0$) iff $f(3, 4, z) = g[h(3, 4), z] = g[h(3, 5), z] = f(3, 5, z)$, that is, according to (3), iff $k(4, z) = k(5, z)$. Therefore, with ϕ: $Z \to \mathbb{R}$ defined by

$\phi(z) = k(4, z) - k(5, z)$, which is a continuous function, $Z_0 = \phi^{-1}(\{0\})$, and Z_0 is closed since $\{0\}$ is closed. (We have used the fact that Y is GPI; a general argument is also possible.)

The image $I(2)$ of $k(2, \cdot\,)$ is the disjoint union of the set $\{k(2, z)|z \in Z_+\}$—which by (3) contains $k(2, 1)$—and K_0: For, $h(\cdot, 2)$ is not constant, but

$$(6) \qquad k(y, z) \in K_0,\ z \in Z_+ \Rightarrow h(\cdot\,, y) \text{ is constant.}$$

This holds because for invertible $g[\cdot, z]$ and constant $g[\cdot, z_0]$ (that is, cf. (5), for $z \in Z_+$ and $z_0 \in Z_0$), the equation $k(y, z) = k(y_0, z_0)$ (for any y_0) implies the following identities of constant functions:

$$g[h(\cdot\,, y), z] = f(\cdot\,, y, z) = f(\cdot\,, y_0, z_0) = g[h(\cdot\,, y_0), z_0];$$

thus $h(\cdot, y)$ is constant.

Since we have assumed that $k(2, 1) \leqslant k(y', 1)$ holds, and $k(2, 1) \notin K_0 = I(2) \cap I(y')$, $k(2, 1)$ is a (proper) lower bound for K_0 and $P = \inf K_0 \in I(2)$ exists. (If $k(2, 1) > k(y', 1)$, consider $P = \sup K_0$ instead, and proceed mutatis mutandis.) There is an element p of $Q = k(2, \cdot\,)^{-1}(\{P\})$ that is an accumulation point of Z_+ and Z_0, since if otherwise, for each $q \in Q$ there were an open neighborhood N_q of q contained in Z_+ or Z_0, then Z_+ and Z_0 would both be open and form a partition of Z, which is connected, however: namely, in this case,

$$Z_+ = k(2, \cdot)^{-1}((-\infty, P)) \cup \bigcup_{q \in Q \cap Z_+} N_q$$

and

$$Z_0 = k(2, \cdot)^{-1}((P, \infty)) \cup \bigcup_{q \in Q \cap Z_0} N_q.$$

Since Z_0 is closed, $p \in Z_0$ and thus $k(2, p) = P \in K_0$. Since $I(2) \cap I(y') = K_0$, $k(y', z) \geqslant P$ for all z. K_0 is a proper interval, so let $[P, P + \epsilon) \subseteq K_0$ for some $\epsilon > 0$, which is an open set relative to the image $I(y')$ of $k(y', \cdot\,)$. Then $k(y', \cdot\,)^{-1}([P, P + \epsilon))$ is an open subset of Z containing p (since $k(y', p) = P$ because $k(\cdot, p)$ is constant), which intersects Z_+ by construction of p. Thus, $k(y', z) \in [P, P + \epsilon) \subseteq K_0$ for some $z \in Z_+$, and $h(\cdot, y')$ is constant by (6). So for this case, (4) can also be used, where correspondingly $s[\cdot, y']$ is constant, and $r[\cdot, y', z]$ is also constant. So (1) holds for all y, z, which finally proves assertion (a) of the theorem.

To show (b), consider the function f: $X \times Y \times Z \to \mathbb{R}$ with $X = (0, \infty)$, $Y = Z = \mathbb{R}$ defined by

$$f(x, y, z) = \begin{cases} z & \text{for } z \leqslant 0, \\ yz & \text{for } y \leqslant 0,\ z \geqslant 0, \\ xyz & \text{for } y \geqslant 0,\ z \geqslant 0, \end{cases}$$

where z is obviously strictly essential. Then

$$f(x, y, z) = j[x, k(y, z)] = g[h(x, y), z] \quad \text{with}$$

$$k(r, s) = j[r, s] = h(r, s) = g[r, s] = \begin{cases} s & \text{for } s \leqslant 0, \\ r \cdot s & \text{for } s \geqslant 0, \end{cases}$$

for $r, s \in \mathbb{R}$; note the restriction $r > 0$ for $r = x$ in $j[r, s]$: this implies that $j[x, \cdot\,]$ is strictly increasing, that is, $Y \times Z$ is PI. Since $g[\cdot, z]$ is strictly increasing or constant, $X \times Y$ is WPI. However, $f(1, 1, z) = z$ and $f(1, -1, z) = -|z|$, so Z is not GS. □

Appendix E: Proof of Theorem 21. In this appendix, we prove Theorem 21 and thereby the closure under symmetric difference for PI sets via an additive representation, as stated in Gorman [14, Theorem 1]. The closure under union is given by Theorem 11, under intersection by Proposition 16 and under differences by Theorem 19, and these properties will be used below. In the proof, the limitations of the old proof in [14, p. 371f] will be indicated, which are due to the fact that certain solvability conditions stated in [1, p. 311] do not generally hold here. The result in Aczél [1, p. 312], used by Gorman [14, p. 371], is however true without these conditions, and follows from our reasoning, or directly from the statement of Theorem 21.

These particular restrictions do not apply to Debreu's construction [9, pp. 22–25] of the additive utility function that relies on certain local geometrical regularities given by the so-called "Thomsen condition" (cf. also Radó [24]). However, a crucial additional hypothesis is there that the symmetric difference is PI [9, p. 23]. It is possible to only show this condition and then use Debreu's result. However, we found that such a proof has all ingredients of showing the additive representation directly, including a finer and finer grained use of the Thomsen condition to construct an additive function, and a local-global extension (compare Wakker [31]). Below, it is also indicated that topological or geometric conditions cannot be replaced by entirely abstract "algebraic" ones (cf. Radó [25], Taylor [26]): the associative operation considered here reduces to addition and is therefore commutative (note [1, p. 267]) since it can be continuously "generated" by "powers"and "roots" of a single element, like a group that is generated by one element where the group operation reduces to addition of exponents. So the following proof seems to be the least complex one that is more or less self-contained.

PROOF OF THEOREM 21. By Theorem 11, $X \times Y \times Z$ is PI. Call a function *well behaved* [14, p. 370] if it is defined on a real interval and strictly increasing and continuous. Let f be denoted by f', and consider the continuous function f: $X \times Y \times Z \to \mathbb{R}$ (with image F) such that

$$f'(x, y, z, t) = g'[f(x, y, z), t]$$

holds with well-behaved $g'[\cdot, t]$, according to Corollary 5. With respect to f' as well as to f, the sets X, $X \times Y$, $Y \times Z$ and Z are PI by Corollary 20 and Lemma 18. We will prove: there is a well-behaved function ϕ: $F \to \mathbb{R}$ such that

$$(1) \qquad \phi[f(x, y, z)] = \phi[f(x, 1, 2)] + \phi[f(0, y, 2)] + \phi[f(0, 1, z)]$$

holds, with suitable elements $0 \in X$, $1 \in Y$, $2 \in Z$. Then the theorem is proved, with $g[\cdot, t] = g'[\phi^{-1}(\cdot), t]$ and, for instance, $\phi[f(\cdot, 1, 2)]$ in place of the function a: $X \to \mathbb{R}$. We will establish (1) in several steps.

(i) *Representation with a partial associative operation*. Consider any elements $0 \in X$, $1 \in Y$, $2 \in Z$, and define the functions u: $X \to \mathbb{R}$, v: $Y \to \mathbb{R}$, w: $Z \to \mathbb{R}$, with images U, V, W, respectively, by

$$u(x) = f(x, 1, 2), \quad v(y) = f(0, y, 2), \quad w(z) = f(0, 1, z).$$

Let furthermore A denote the image of the function $f(x, y, 2)$ of x, y, and similarly let $B = \{f(0, y, z) | y \in Y,\ z \in Z\}$, where obviously $U, V \subseteq A$ and $V, W \subseteq B$. Since

$Y \times Z$ is PI, there is a function j: $X \times B \to \mathbb{R}$, well behaved in its second argument, such that $f(x, y, z) = j[x, f(0, y, z)]$ holds, by Corollary 5. Similarly, $f(x, y, z) = k[u(x), y, z]$ for a suitable function k, where $k[\cdot, y, z]$ is well behaved, since X is PI. These two equations show that a function l: $U \times B \to \mathbb{R}$ can be unambiguously defined by

$$l[u(x), f(0, y, z)] = f(x, y, z), \tag{2}$$

where l is well behaved in both arguments. Similarly,

$$f(x, y, z) = m[f(x, y, 2), w(z)] \tag{3}$$

holds for a function m: $A \times W \to \mathbb{R}$ that is also well behaved in both arguments, since $X \times Y$ and Z are PI. Then, by (3), $f(0, y, z) = m[f(0, y, 2), w(z)] = m[v(y), w(z)]$, and $f(x, y, 2) = l[u(x), v(y)]$ by (2), and $f(x, 1, z) = l[u(x), w(z)] = m[u(x), w(z)]$. The latter shows that l and m are identical for common arguments, where the first argument is, in order to be admissible for l, of the form $u(x)$ for some $x \in X$, and the second of the form $w(z)$, to be admissible for m. Thus both functions can be regarded as restrictions of a single binary operation $\cdot$, well behaved in both arguments, with domain $(U \times B) \cup (A \times W)$. Substitution of the former equations into (2) and (3) yields

$$\begin{aligned} f(x, y, z) &= u(x) \cdot [v(y) \cdot w(z)] \\ &= [u(x) \cdot v(y)] \cdot w(z). \end{aligned} \tag{4}$$

The "product" symbol $\cdot$ shall be omitted for simplicity. Furthermore, with $e = f(0, 1, 2) = u(0) = v(1) = w(2)$, the identities

$$ue = u, \quad ve = v = ev, \quad w = ew \tag{5}$$

hold for $u \in U$, $v \in V$, $w \in W$ (the letters u, v, w shall be used both for the functions defined above as well as for variables ranging in U, V, W, respectively). In other words, $\cdot$ is a binary operation defined for uv, vw, $u(vw)$ and $(uv)w$, that is "associative" in the sense that $u(vw) = (uv)w$ holds by (4) and that has unit $e \in U \cap V \cap W$, since also $(uv)e = u(ve) = uv$ and $e(vw) = (ev)w = vw$ hold by (5); note that A and B are the sets of possible values for uv and vw, respectively.

The operation $\cdot$ is a total function $F \times F \to F$ if $U = W = F$ (since $U \subseteq A \subseteq F$, $W \subseteq B \subseteq F$), and it is fully associative if also $V = F$ holds. Thus, if $0, 1, 2$ can be chosen so that the functions u, v, w defined by $f(\cdot, 1, 2)$, $f(0, \cdot, 2)$ and $f(0, 1, \cdot)$ are surjective with respect to the image F of the unrestricted function f, (4) gives a fully associative representation. This is stated in Aczél [1, p. 311]; all that is needed are these surjectivity assumptions plus the cancellativity conditions asserted by Lemma 4(a) that are given by (2) and (3) (cf. also Taylor [26, p. 26f]). The additional topological condition that $\cdot$ is well behaved in both arguments shows in this case that it is a continuous group operation, which can be represented additively [1, p. 57], resulting in (1) (cf. Aczél [1, p. 312], cited in [14, p. 371]).

However, these surjectivity assumptions do not always hold: with $X = Y = Z = [0, 1]$ and $f(x, y, z) = x + y + z$, the function f is well behaved but not surjective in its individual variables nor in the sum of any two of them; the above representation (4) is given, for instance, with $e = 0 = f(0, 0, 0)$, $u = v = w = \mathrm{id}$ and $A = B = [0, 2]$, where $\cdot$ is addition with domain $([0, 1] \times [0, 2]) \cup ([0, 2] \times [0, 1])$. So it is necessary to provide a more general reasoning (in Radó [25, p. 323], the functions l and m of (2) and (3) are also assumed to be separately bijective in each variable, or "quasigroup" operations). An approach would be to extend the partial operation in (4) to a fully associative operation. To this end, it is necessary to employ the good behavior of $\cdot$ in its arguments: without continuity, the associativity condition $u(vw) = (uv)w$ may hold for selected choices $u \in U$, $v \in V$, $w \in W$, but not in general (if U is a proper subset of F), even if $\cdot$ is a total binary operation on F (an example, which we do not give here, is given by a finite "loop" F with nontrivial "left nucleus" $U \neq F$, cf. Bruck [6, p. 57]). We will proceed by representing (4) locally with an additive function, which can then be extended throughout $U \times V \times W$.

(ii) *Local additive representation around the origin.* Let the operation $\cdot$ be defined, as above, on $(U \times VW) \cup (UV \times W)$ with image F, where $e \in U \cap V \cap W$ is given so that (5) holds, and VW and UV are defined as the sets of possible values vw, resp., uv for $u \in U$, $v \in V$, $w \in W$ (similarly for other sets below), and let $u(vw) = (uv)w$ hold. The sets U, V, W are nondegenerate intervals (since x, y and z are essential for $f(x, y, z)$ and f is continuous), and $\cdot$ is well behaved in both arguments. We will prove the following: if e is in the interior of U, V and W, then it has an open neighborhood $E \subseteq U \cap V \cap W$, so that

$$\phi(uvw) = \phi(u) + \phi(v) + \phi(w), \tag{6}$$
$$\phi(e) = 0 \quad \text{for } u \in E, v \in E, w \in E$$

hold with a well-behaved function ϕ: $D \to \mathbb{R}$, where the open interval $D \subseteq F$ is given by $D = EEE$. Instead of ϕ, we shall first construct its inverse ψ, following largely the proof by Aczél [1, pp. 54–57 and 268f] for unrestrictedly associative functions; the reader is referred to this reference for more details.

Assume that e is in the interior of $C = U \cap V \cap W$: then there are elements c, d in the interior of C with $d < e < c$. Furthermore, we can assume $cd = e$: for instance, if $cd < e < c = ce$, then $cd' = e$ holds for some $d' \in C$, where $d < d' < e$, since $\cdot$ is well behaved in its second argument, and d' can be regarded instead of d (similarly if $cd > e$). The function ψ with image $[d, c]$ shall first be defined for all rationals between -1 and 1. Let $\psi(1) = c$, $\psi(-1) = d$ and $\psi(0) = e$. For a positive integer n, the nth power u^n of $u \in C$ with respect to $\cdot$ shall be considered in the usual way (with $u^1 = u$, $u^{n+1} = u^n u$, where the latter is defined as long as u^n and u are in C). The sequence u^n for $n = 1, 2, \ldots$ is strictly increasing or strictly decreasing depending on whether u is greater or smaller than e; it furthermore exceeds c (or gets smaller than d) for sufficiently large n (cf. [1, p. 55]; note that d and c are not boundaries of C); after that, u^n is possibly undefined. As a function of u, u^n is well behaved for $u > e$ and strictly decreasing and continuous of $u < e$ (u may have to be chosen sufficiently close to e), since $\cdot$ is well behaved. So, the equation $u^n = c$ has a unique solution u in $(e, c]$, which shall be assigned to $\psi(1/n)$, and let similarly $\psi(-1/n)$ be the unique solution $v \in [d, e)$ of $v^n = d$. For positive integers n, m, $u^n u^m$ is defined whenever u^n and u^m are in C, and equal to u^{n+m} by associativity of $\cdot$ on C. For $1 \leqslant m \leqslant n$ define $\psi(m/n) = (\psi(1/n))^m \leqslant c$ and $\psi(-m/n) = (\psi(-1/n))^m \geqslant d$. Then it is easily seen (note $cd = e$) that for integers k, l with $|k|, |l|, |k + l| \leqslant n$, $\psi(k/n)$ and $\psi(l/n)$ belong to $[d, c]$ and $\psi(k/n + l/n) = \psi(k/n) \cdot \psi(l/n)$. So ψ fulfills $\psi(p + q) = \psi(p)\psi(q)$ for all rationals p, q with $|p|, |q|, |p + q| \leqslant 1$. Furthermore, ψ is strictly increasing. This still holds if ψ is extended to the reals by Dedekind cuts [1, p. 56f], and it is then a well-behaved function $[-1, 1] \to [d, c]$ since it leaves no gaps. Its inverse ϕ: $[d, c] \to [-1, 1]$ is also well behaved and fulfills $\phi(e) = 0$ and $\phi(uv) = \phi(u) + \phi(v)$ whenever $u, v, uv \in [d, c]$. Let E be any sufficiently small open neighborhood of e such that $D = EEE \subseteq$

$[d, c]$, which exists by the good behavior of $\cdot$ (the latter implies D is open; note $E \subseteq EE \subseteq D$). With ϕ restricted to D, (6) is shown.

(iii) *Arbitrary local additive representation*. Using the fact that the constants $0, 1, 2$ in (i) above are arbitrary elements of X, Y, Z, respectively, we will show next that $e = f(0, 1, 2)$ is indeed w.l.o.g. in the interior of U, V and W, and will prove the following generalization of (6): for each (u', v', w') in the interior of $U \times V \times W$, there are open neighborhoods I, J, K, L of $u'v'w', u', v', w'$ contained in F, U, V, W, respectively, where $I = JKL$, such that

$$
(7) \qquad \begin{aligned} \psi(uvw) &= \psi(uv'w') + \psi(u'vw') + \psi(u'v'w), \\ \psi(u'v'w') &= 0 \quad \text{for } u \in J, v \in K, w \in L \end{aligned}
$$

hold for a suitable well-behaved function $\psi: I \to \mathbb{R}$. To prove this, let $u' = u(x')$, $v' = v(y')$, $w' = w(z')$ be interior points of U, V, W, respectively, for suitable x', y', z'. Letting the latter take the role of $0, 1, 2$ in (i) above, we observe that there are continuous functions p, q, r (in place of u, v, w) with images P, Q, R, and a well-behaved partial associative function $\circ$ such that, as in (4),

$$
f(x, y, z) = p(x) \circ q(y) \circ r(z)
$$

holds (parentheses have been omitted). Thereby,

$$
\begin{aligned} p(x) &= f(x, y', z') = u(x)v'w', \\ q(y) &= f(x', y, z') = u'v(y)w', \\ r(z) &= f(x', y', z) = u'v'w(z). \end{aligned}
$$

The functions $(\cdot)v'w': U \to P$, $u'(\cdot)w': V \to Q$, $u'v'(\cdot): W \to R$ are well-behaved bijections, mapping, respectively, u', v' and w' to $e' = f(x', y', z') = u'v'w'$. Thus e' is in the interior of P, Q and R, and if e is not already in the interior of $U \cap V \cap W$, it can therefore be chosen this way. As proved in (ii) (cf. (6)), there are an open neighborhood $E' \subseteq P \cap Q \cap R$ of e' and a well-behaved function $\psi: I = E' \circ E' \circ E' \to \mathbb{R}$ such that $\psi(p \circ q \circ r) = \psi(p) + \psi(q) + \psi(r)$ holds for $p, q, r \in E'$, and $\psi(e') = 0$. With J, K and L as inverse images of E' under the above well-behaved bijections, (7) holds, because of $p(x) \circ q(y) \circ r(z) = f(x, y, z) = u(x)v(y)w(z)$ and, for instance, $p(x) \in E' \Leftrightarrow u(x) \in J$.

(iv) *Global extension of the additive representation*. The function uvw of (u, v, w), defined on the product $U \times V \times W$ of intervals and well behaved in each argument, has by (7) a local additive representation throughout the interior of its domain. In Debreu [9, pp. 22–25] such a representation is proved under the additional hypothesis that $X \times Z$ is PI, and it is mentioned [9, p. 25] that this follows in the large from the property in the small. Similarly, Radó [24, p. 150] states that an equivalent condition "B" (similar to the "Thomsen condition") holds globally if it holds locally. Since the argument is not spelled out at either place, we give it here in detail; in essence, the local additive representations are united to a single global one by suitable positive-affine transformations. As pointed out by a referee, this extension has to be done with some care, since adding a new three-dimensional rectangle like $J \times K \times L$ in (7) to the domain of the additive function in construction may lead to incompatibilities of the transformations, like ψ in (7), at different, disconnected places (see Wakker [30, Example 3] and [32, §2.2] for details). Here, we produce by extension only rectangles again and use the global product structure to get rid of the locality dimension by dimension, which avoids these complications; for another proof see [31].

Consider two local representations as in (7) and assume that they *agree* in two of the open intervals J, K, L, say J and K, that define their respective domain, and let the third intervals, say L and L', be intersecting, so that (7) holds with ψ, J, K, L and also with ψ', J, K, L' instead, where $L \cap L'$ is not empty. Hereby, the "origin" (u', v', w') in (7) with $\psi(u'v'w') = 0$ need initially not be the same as that for ψ', but it can be easily "shifted" since any (necessarily interior) point (a, b, c) of $J \times K \times L$ can take its place by considering $\psi - \psi(abc)$ instead of ψ, as is easily seen. So w.l.o.g. $w' \in L \cap L'$ and $\psi(u'v'w') = \psi'(u'v'w') = 0$. The two additive representations are on their common domain related by a positive-linear transformation, that is,

$$(8) \qquad \psi'(uvw) = c\psi(uvw) \quad \text{for } u \in J, v \in K, w \in L \cap L'$$

holds for a constant $c > 0$. This is well known; we give references for a standard way to prove it, which is not difficult. Let u, v, w be restricted as in (8). With $\theta(\alpha + \beta) := \psi'(\psi^{-1}(\alpha + \beta))$ for $\alpha = \psi(uvw')$, $\beta = \psi(u'v'w)$, so $\alpha + \beta = \psi(uvw)$, the "Cauchy-equation" $\theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta)$ holds (compare also [1, p. 58]) which is *restricted* to an open rectangle for the values of (α, β) that contains $(0, 0)$ since $\psi(u'v'w') = 0$. By Daróczy and Losonczi [7, pp. 239–242] (compare also [2, p. 82]), the function θ has in this case a unique extension to $\mathbb{R}$ that fulfills this equation for all reals α, β, so $\theta(\alpha) = c\alpha$ for a unique $c > 0$ since θ is well behaved on an interval [1, p. 34], which shows (8). Since ψ can be replaced by $c\psi$ without changing (7), let w.l.o.g. $c = 1$ in (8).

Fixing J and K in the preceding construction has the important consequence that the functions ψ and ψ' have no common arguments s other than those given in (8) where ψ and ψ' agree: For, let $s = abc = a'b'c'$ for $a, a' \in J$, $b, b' \in K$, $c \in L$, $c' \in L'$; the claim is that $s = uvw$ holds for some $u \in J$, $v \in K$, $w \in L \cap L'$. This is immediate if c or c' belongs to $L \cap L'$, so let this not be the case and assume w.l.o.g. $c < c'$, that is, the intervals L and L' overlap so that L' contains the larger elements. Taking any $w \in L \cap L'$, which implies $c < w < c'$ and thus $abw > s > a'b'w$ by the good behavior of $\cdot$, one notes that the interval $\{uvw | u \in J, v \in K\}$ contains s, which proves the claim. Thus, the two functions ψ and ψ' with domains $I = JKL$ and $I' = JKL'$ agree by (8) on $I \cap I' = JK(L \cap L')$ and can be extended to a single function on $I \cup I'$, call it ψ again, so that (7) holds with $L \cup L'$ instead of L. In this reasoning, the extension along the third dimension is not special, so replacing J by an interval $J \cup J'$ in (7) while K and L stay fixed is similarly possible, and correspondingly for the second dimension, enlarging K.

To extend iteratively the local additive representation in the interior of $U \times V \times W$ to a global one, take first a subcompactum $U_0 \times V_0 \times W_0$ with compact intervals U_0, V_0, W_0 and fix $u' \in U_0$, $v' \in V_0$. Consider for all $w' \in W_0$ the respective open neighborhoods $J \times K \times L$ of (u', v', w') such that (7) holds. They cover the compact set $\{(u', v')\} \times W_0$, so finitely many of them suffice. The corresponding intervals J can be replaced by their joint (finite) intersection to become independent of w', and similarly for K, since this only restricts the additive representation (7) but J and K stay neighborhoods of u', resp., v'. The corresponding intervals L unite to an open interval W' that contains W_0. For these intervals L, the extension described above can be used inductively to eventually get an additive representation on $J \times K \times W'$, that is, (7) with W' instead of L.

Next, fix only $u' \in U_0$ and consider for all $v' \in V_0$ the additive representations just shown on the open sets $J \times K \times W'$ (where $u' \in J$, $v' \in K$ and $W_0 \subseteq W'$) that cover $\{u'\} \times V_0 \times W_0$, of which again finitely many suffice. One gets rid of the dependence of J and W' on v' by taking the respective finite intersections, and this time extending along the second dimension by uniting iteratively the sets K, one gets an additive representation on $J \times V' \times W'$ where the open intervals V' and W' contain V_0, resp., W_0 and replace K, resp., L in (7). In the same way, one extends along the first dimension by considering these representations for all $u' \in U_0$. (This construction could straightforwardly be carried over to more than three dimensions.) This gives an additive representation on an open set $U' \times V' \times W'$ containing the set $U_0 \times V_0 \times W_0$ to which it can be restricted.

Consider sequences of successively larger compact intervals $U_0, U_1, U_2, \ldots$ (and similarly $V_0, V_1, \ldots$ and $W_0, W_1, \ldots$) that eventually exhaust the interior of U (resp., of V, W) and apply the preceding construction to $U_n \times V_n \times W_n$. For sufficiently large n, this set contains (e, e, e) as interior point (e is the neutral element of $\cdot$ above) which can take the place of the "origin" (u', v', w') in (7). Thus, replacing ψ by ϕ,

$$\begin{aligned} \phi(uvw) &= \phi(u) + \phi(v) + \phi(w), \\ \phi(e) &= 0 \end{aligned} \tag{9}$$

holds for $u \in U_n$, $v \in V_n$, $w \in W_n$. As n increases, this can obviously be considered as the extension of one function ϕ for which (9) eventually holds for any interior point (u, v, w) of $U \times V \times W$.

If U, V, W are not all open, equation (9) remains to be shown if some of u, v or w are corresponding boundary points of these intervals. Then $\phi(u)$, $\phi(v)$ and $\phi(w)$ are still defined because the entire intervals U, V and W are contained in the interior of $F = UVW$, since e is in the interior of U, V and W: for instance, with $e > v$ for some $v \in V$, $u = ue > uv \in F$, in particular if u is the lower endpoint of U. If u, v and w are all boundary points at the same end of U, V, W, respectively, then and only then uvw is a boundary point of F, where $\phi(uvw)$ can be defined by (9). Taking limits, the continuity of $\cdot$ and $+$ shows that (9) holds for all u, v, w, or, using (4),

$$\phi[f(x, y, z)] = \phi[u(x)] + \phi[v(y)] + \phi[w(z)],$$

which proves (1). □

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References

[1] Aczél, J. (1966). *Lectures on Functional Equations and Their Applications*. Academic Press, New York.

[2] ______ (1987). *A Short Course on Functional Equations*. D. Reidel, Dordrecht, The Netherlands.

[3] Birkhoff, G. (1967). *Lattice Theory*. (3rd Ed.), Amer. Math. Soc. Colloq. Publ. **25**. Providence, RI.

[4] Blackorby, C., Primont, D. and Russell, R. R. (1978). *Duality, Separability, and Functional Structure*. North-Holland, New York.

[5] Bliss, C. J. (1975). *Capital Theory and the Distribution of Income*. North-Holland, Amsterdam.

[6] Bruck, R. H. (1971). *A Survey of Binary Systems*. (3rd printing), Ergebnisse der Math. und ihrer Grenzgebiete **20**. Springer-Verlag, New York/Heidelberg/Berlin.

[7] Daróczy, Z. and Losonczi, L. (1967). Über die Erweiterung der auf einer Punktmenge additiven Funktionen. *Publ. Math. Debrecen* **14** 239–245.

[8] Debreu, G. (1954). Representations of a Preference Ordering by a Numerical Function. In Thrall, R., Coombs, C. and Davis, R. (Eds.), *Decision Processes*. J. Wiley, New York, 159–165.

[9] Debreu, G. (1959). Topological Methods in Cardinal Utility Theory. In Arrow, K. J., Karlin, S. and Suppes, P. (Eds.), *Mathematical Methods in the Social Sciences, 1959*. Stanford University Press, Stanford, CA, 1960, 16–26.

[10] Farquhar, P. H. (1977). A Survey of Multiattribute Utility Theory and Applications. In Starr, M. and Zeleny, M. (Eds.), *Multiple Criteria Decision Making*. North-Holland, Amsterdam, 59–90.

[11] Fishburn, P. C. (1970). *Utility Theory for Decision Making*. J. Wiley, New York.

[12] ______ and Keeney, R. L. (1974). Seven Independence Concepts and Continuous Multiattribute Utility Functions. *J. Math. Psychology* **11** 294–327.

[13] ______ and ______ (1975). Generalized Utility Independence and Some Implications. *Oper. Res.* **23** 928–940.

[14] Gorman, W. M. (1968). The Structure of Utility Functions. *Rev. Economic Studies* **35** 367–390.

[15] ______ and Vind, K. (1971). Notes and Comments on "The Structure of Utility Functions". *Rev. Economic Studies* **38** 113–116.

[16] Keeney, R. L. (1974). Multiplicative Utility Functions. *Oper. Res.* **22** 22–34.

[17] ______ and Raiffa, H. (1976). *Decisions with Multiple Objectives*. J. Wiley, New York.

[18] Kelley, J. L. (1955). *General Topology*. Van Nostrand, New York.

[19] Krantz, D. H., Luce, R. D., Suppes, P. and Tversky, A. (1971). *Foundations of Measurement Vol.* I: *Additive and Polynomial Representations*. Academic Press, New York.

[20] Mak, K. (1984). Notes on Separable Preferences. *J. Economic Theory* **33** 309–321.

[21] ______ (1986). On Separability: Functional Structure. *J. Economic Theory* **40** 250–282.

[22] ______ (1987). Coherent Continuous Systems and the Generalized Functional Equation of Associativity. *Math. Oper. Res.* **12** 597–625.

[23] Möhring, R. H. and Radermacher, F. J. (1984). Substitution Decomposition for Discrete Structures and Connections with Combinatorial Optimization. *Ann. Discrete Math.* **19** 257–356.

[24] Radó, F. (1959). Équations fonctionnelles caractérisant les nomogrammes avec trois échelles rectilignes. *Mathematica (Cluj)* **1, 24** 143–166.

[25] ______ (1959). Sur quelques équations fonctionnelles avec plusieures fonctions à deux variables. *Mathematica (Cluj)* **1, 24** 321–339.

[26] Taylor, M. A. (1973). Certain Functional Equations on Groupoids Weaker than Quasigroups. *Aequationes Math.* **9** 23–29.

[27] von Stengel, B. (1988). Decomposition of Multiattribute Expected-Utility Functions. *Ann. Oper. Res.* **16** 161–183.

[28] ______ (1991). *Eine Dekompositionstheorie für mehrstellige Funktionen mit Anwendungen in Systemtheorie und Operations Research*. Math. Systems in Economics **123**. Verlag Anton Hain, Frankfurt.

[29] Wakker, P. (1988). Continuity of Preference Relations for Separable Topologies. *Internat. Economic Rev.* **29** 105–110.

[30] ______ (1990). Counterexamples to Additive Representation on Nonfull Cartesian Products. Preprint, Fuqua School of Business, Duke University, Durham, NC.

[31] ______ (1990). From Local to Global Additive Representation. Preprint (2nd version), Fuqua School of Business, Duke University, Durham, NC.

[32] ______ (1990). Additive Representations of Preferences on Rank-Ordered Subsets of Cartesian Products. Part II: The Topological Approach. Working paper 9002, Fuqua School of Business, Duke University, Durham, NC.

[33] ______ (1991). Continuity of Transformations. *J. Math. Anal. Appl.* **162** 1–6.

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