Recursive Inspection Games

Bernhard von Stengel
Informatik 5
Armed Forces University Munich
D–8014 Neubiberg, Germany

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Abstract

Dresher (1962) described a sequential inspection game where an inspector has to distribute a given number of inspections over a larger number of inspection periods in order to detect an illegal act that an inspectee, who can count the inspector's visits, performs in at most one of these periods. This paper treats two extensions of this game. In the first, more than one illegal act is allowed. Then, under certain reasonable assumptions for the zero-sum payoffs, the optimal strategy of the inspector does not depend on the number of intended illegal acts. This allows a recursive description, which is justified formally using extensive games. The resulting recursive equation in three variables for the value of the game, which generalizes several other known equations of this kind, is solved explicitly.

In a second variation of the Dresher game, there is again only one illegal act, which is however always detected at the next inspection, with the payoff to the violator linearly increasing with the time passed in between. The solution of this game is simple and intuitive, but a conceptually sound description employs an extensive game with recursion in only one branch, and its corresponding normal form.

0. Introduction

This paper considers two generalizations of a sequential inspection game first described by Dresher [8]. The games are combinatorially interesting since their solutions are defined by non-trivial recurrences (partial difference equations) which are solved explicitly, extending previously known results [8] [10]. The main *conceptual* difficulty concerns the information of the players during the game which is here modeled conventionally by games in extensive form with information sets according to Kuhn [11]. The first considered game is described and solved recursively using a suitable manipulation of the information sets. In the second game, recursion is applied to the normal (matrix) form of the game.

Greater generality is obtained by admitting a third parameter in the game description. In Dresher's model [8], outlined in section 1, there are two parameters denoting the number of inspection periods and the number of possible controls that can be used in these periods to detect *one* illegal act. In the first generalization, a new third variable is the number of intended illegal acts. In the second game (proposed in [6] and treated here in section 5), it is the detection time that passes between illegal action and discovery. The main limitation of the models is a rather restricted uniformly linear dependence of the payoffs on the new third parameter to keep the games tractable. Practically, this seems however useful enough since the parties should be interested in direct optimizations of this parameter; for example, the inspector tries to minimize detection time.

An inspection game as considered here is a non-cooperative two-person-game whose players are called *inspector* and *inspectee*. It models a situation where the inspectee, which may be an organization or a state, is obliged to follow certain regulations but has an incentive to *violate* them. The inspector tries to minimize the impact of such violations by means of *inspections* that uncover them. A detected violation is costlier to the inspectee than legal behavior. The resources of the inspector are usually limited and complete surveillance is not possible. Then, inspections have to be randomized and the inspection game typically has a mixed equilibrium. In this paper, all games are zero-sum. A related non-zero-sum game is solved in [4], included as an appendix in this report.

A recursive treatment of the considered sequential inspection games, by induction over the number of time periods, poses some conceptual obstacles that shall be briefly outlined. A game in extensive form is represented by a tree with nodes as game states and branchings denoting the choices of the players. By looking recursively at subtrees, it seems appropriate to determine optimal strategies recursively. However, this is only possible if the players *know* they have entered the respective subtree (expressed technically: if no information set overlaps with the subtree), since only then the subtree can be properly interpreted as a subgame. Thus, recursion is possible only if "enough" subgames exist. Here, this means that both players know the actions of their opponent in all previous periods. This is usually reasonable to assume for the inspectee who can count the inspector's visits, but problematic for the inspector with respect to the uncontrolled periods since normally he obtains the information through inspections only. This is discussed in detail in section 3.

In the literature [5] [8] [13, section 5.2], comparable inspection games have been defined directly via recursion where the described subgame structure is assumed implicitly. As mentioned (and already pointed out by Kuhn [12, p.174]), this is not always legitimate or at least should be mentioned explicitly. A recursive description has the great advantage that optimal strategies for the players can be computed from recursive equations, even if no explicit formulas as given here are known (see, for example, [12]). Here, recursion is also used and justified by an analysis of the underlying extensive form.

The early sequential inspection games [8] [12] have been developed as models for inspections in the framework of a nuclear test ban treaty. For a suggested application to the arms limitations treaty on intermediate nuclear forces see [5]. The game solved in section 5 below models a particular "timeliness" problem in nuclear material safeguards and has been proposed by Canty and Avenhaus [6]; see also [7]. Inspection games in general have various applications to arms control, auditing and accounting in economics and environmental protection; some of these are surveyed in [3]. An extensive monograph is Avenhaus [2].

In practice, the goal of inspections is usually to *deter* from illegal actions altogether. This is *not* achieved in the antagonistic inspector-violator games considered here where the inspectee always acts illegally with a positive probability. However, any such game can be embedded into a simple "global" (non-zero-sum) game where the inspectee has the initial option to act legally only, and this is also his equilibrium choice provided he is sufficiently deterred from violations by the optimal inspection scheme [2, p.24]. This description is more realistic since legal behavior should be the normal situation, yet the subgame where the inspectee acts illegally is the important part since it allows optimal planning of inspection activities.

Of the five sections of this paper, section 1 describes the basic Dresher model [8]. In section 2, its first generalization is defined verbally and, via examples, in extensive form. In this game, the inspectee may intend some arbitrary number of illegal acts. An auxiliary game with additional, full knowledge of the inspector is described in section 3. It allows a recursive description and solution. The payoffs have been chosen such that optimal strategies are not altered by this manipulation of information sets and can be re-interpreted for the original game. The content of section 4 is technical, with a proof of the explicit formula for the recursively defined value of the game and a sketch of its derivation. Section 5 treats another variation of the Dresher game where the timely detection of one violation is optimized. There, a direct recursion proposed in [6] similar to Dresher's desribes the game in parts correctly, but is not fully justified. A conceptually sound approach employs a larger normal form (and not just a two-by-two game) which has an interesting solution with some intuitive appeal.

1. The Dresher model

The *Dresher model* [8] is a sequential inspection game of n *stages* or time periods. At each period, the inspector can decide to control the inspectee, using up one of m inspections allowed in total, or not to. The inspectee knows at each stage the number of past inspections. He can decide to act legally or to "violate" where he is caught if and only if (iff) he is inspected in that period. At most one violation is allowed, whose gain for the inspectee if he is undetected equals his loss if he is caught. Legal action has zero payoff. The game is zero-sum.

Dresher described and solved this game *recursively*. Its value, given as the equilibrium payoff to the inspector, for n periods with up to m inspections and a violation that may occur, is denoted by $v(n, m)$. The two-by-two game for positive n, m resulting from the possible actions of the players in the first period is shown in Figure 1.1.

Inspector \ Inspectee	legal act	violation
control	$v(n-1, m-1)$	$+1$
no control	$v(n-1, m)$	-1

Figure 1.1. The Dresher game with at most one intended violation for n periods and m inspections with value $v(n, m)$. The entries denote the payoffs to the inspector.

The table entries in Figure 1.1 are the payoffs to the inspector. If the inspectee violates in the first period, the inspector catches him if he controls and gets payoff $+1$, whereas otherwise he will not detect the violation and eventually receive payoff -1 since the inspectee will act legally in all remaining periods. If the inspectee acts legally, the game continues with $n-1$ periods and $m-1$ resp. m inspections remaining, and the respective payoffs at stage one are the values of these games. For $n = 0$, the game terminates without a violation that occurred, with

$$v(0, m) = 0. \tag{1.1}$$

If the inspector has no inspections left ($m = 0$), then the inspectee can safely violate, so

$$v(n, 0) = -1 \qquad \text{for } n > 0. \tag{1.2}$$

With (1.1) and (1.2) as initial conditions, $v(n, m)$ can be computed recursively as the value of the game in Figure 1.1. It is useful to demonstrate this with the general form of such a zero-sum game shown in Figure 1.2.

Inspector \ Inspectee	legal act	violation
control	a	b
no control	c	d

Figure 1.2. General form of an inspection game for one stage. The entries, which are restricted as in (1.3), denote the payoffs to the inspector.

The payoffs a, b, c, d to the inspector in Figure 1.2 are assumed to fulfill the restrictions

$$a \leq b, \qquad c > d, \qquad a \leq c, \qquad b > d. \tag{1.3}$$

The inequalities $a \le b$ and $c > d$ state that the inspectee (for whom these payoffs count negatively) prefers to act legally if he is controlled and to violate if he is not controlled; for greater generality, it is also admitted that $a = b$ holds, where a violation is not punished if the inspector controls, but legal action is not disadvantageous. Conversely, the inspector has an incentive not to control if the inspectee acts legally since this will usually save him an inspection he can profitably use later. An exception is the case $m = n$ in the game shown in Figure 1.1, which may occur at some later stage, where the inspector can control in every remaining period and the values $v(n-1, m-1)$ and $v(n-1, m)$ can be shown to be equal. Therefore, the corresponding inequality $a \le c$ is not strict. If the inspectee violates, control is always preferable, so $b > d$.

Excluding the cases $a = b$ and $a = c$ for the moment, the game in Figure 1.2 has a unique equilibrium in mixed strategies. If $p \in [0, 1]$ is the probability that the inspector controls, the equilibrium choice of p is to make the inspectee indifferent between legal action and violation so that he receives the same payoff $-v$, with

$$v = p \cdot a \,+\, (1-p) \cdot c = p \cdot b \,+\, (1-p) \cdot d, \tag{1.4}$$

where v is the value of the game. This is equivalent to

$$p = \frac{c-d}{c-d+b-a}, \qquad 1-p = \frac{b-a}{c-d+b-a}. \tag{1.5}$$

In other words, the probabilities p and $1-p$ of control and no control are inversely proportional to the differences $b-a$ and $c-d$ in the respective two rows in Figure 1.2, which are non-negative by (1.3). The value v of the game in Figure 1.2 is by (1.4) given by

$$v = \frac{b \cdot c - a \cdot d}{c-d+b-a}. \tag{1.6}$$

The probability q that the inspectee acts legally is similarly to (1.4) given by

$$v = q \cdot a \,+\, (1-q) \cdot b = q \cdot c \,+\, (1-q) \cdot d$$

with

$$q = \frac{b-d}{c-a+b-d}, \qquad 1-q = \frac{c-a}{c-a+b-d}, \tag{1.7}$$

which also determines v as in (1.6).

If $a = c$ holds in (1.3), the preceding equations still define a saddlepoint of the game in Figure 1.2, with $v = a$ and $q = 1$ according to (1.6) and (1.7), but p can then also be chosen larger than $(c-d)/(b-d)$ as given by (1.5). In fact, it is reasonable to assume $p = 1$, that is, the inspector controls with certainty [8, p.8]. This equilibrium in pure strategies is here also the only so-called perfect equilibrium (that is, in a certain sense robust against mistakes), since “control” weakly dominates “no control”; cf. van Damme [14, p.48]. The same applies if $a = b$ holds in (1.3), where $p = 1$ but where also the inspectee can violate with certainty. Nevertheless, the equation (1.6) for the value v of the game is in both cases still valid. The value $v(n, m)$ of the original game in Figure 1.1 is thus given by (1.6) as

$$v(n,m) = \frac{v(n-1,m) + v(n-1,m-1)}{v(n-1,m) + 2 - v(n-1,m-1)} \tag{1.8}$$

and the probability p of control in the first period according to (1.5) by

$$p = \frac{v(n-1,m)+1}{v(n-1,m)+2-v(n-1,m-1)} \qquad \text{for } m < n. \tag{1.9}$$

A similar equation holds for the probability q of legal action in the first period using (1.7).

The initial conditions (1.1) and (1.2) and the recursive equation (1.8) determine $v(n,m)$ uniquely for all non-negative integers n, m. Dresher [8] found the explicit solution

$$v(n,m) = -\binom{n-1}{m} \Big/ \sum_{i=1}^{m} \binom{n}{i}. \tag{1.10}$$

This can be verified fairly easily, and will be proved below as a special case of a more general formula; see Theorem 3.2.

This section concludes with a discussion of a conceptual problem of the Dresher model that is particularly relevant to extensions of this model. The normal form in Figure 1.1 of the Dresher game is a recursive description where the four cases resulting from the possible actions of the players in the first period are (implicitly) treated as subgames that can be replaced by their respective value. This is in fact only true for the cases where the inspector controls and no violation has taken place yet. Then, if the inspectee violates, the game terminates, corresponding to a trivial subgame without further moves of the players, or, if the inspectee acts legally, the game continues where the inspector knows that no violation has taken place, as before. The inspector does not have this knowledge if he does not control. If the inspectee violates in that period, the payoff entry in Figure 1.1 is -1 as if the game terminates, although this is not the case because the inspector will continue to control according to a certain strategy and only receive the payoff at the end. However, in that case his strategy does not influence his payoff, so he can always act as if the violation has not taken place yet since only then his behavior matters.

This is an informal justification for treating the subcase no control/legal action as a subgame and to determine the inspector's strategy whenever he has not controlled as if he were in that subgame, although he might in fact be in the subgame where the illegal act has escaped him. This informal argument can be made precise by looking at an extensive game description of the Dresher model and generalizations of it, as done in the next sections.

2. More than one violation

This section describes an extension of the Dresher model where more than one violation is allowed. This game will be described first verbally and then in extensive form, which is demonstrated by a number of examples. Determining the value of the game is the subject of the next section.

Consider the following inspection game, which depends on three non-negative integer parameters n, m, k and shall therefore be denoted by $\Gamma(n,m,k)$. As before, there are n periods of which up to m can be used for an inspection. The inspectee intends to perform up to k violations, at most one per period. He is caught iff the inspector simultaneously inspects in such a period, where the game terminates. The payoffs are zero-sum, and they depend on the number of successful violations, that is, on the number s, say, of periods where a violation but no inspection occurred, and on whether the game terminates with a caught violation, or without after all n stages are completed. For a caught violation, the inspector receives a non-negative payoff $b \geq 0$ (like the entry in Figure 1.2), otherwise payoff zero, *minus* — in both cases — the number s of violations that remained undetected. In other words, each violation that is not caught decreases the inspector's payoff by one, regardless of whether he will catch the violator at a later stage or not. This special assumption about the payoffs will be crucial for a possible recursive treatment. The information structure of the game is as in the Dresher model, that is, the inspectee knows if an inspection has taken place but the inspector does not know whether a violation occurred in a period that he did not inspect. The number k of intended violations is known to both players at the beginning of the game, as well as n, m and the payoff parameter b.

The original Dresher model is a special case of this game, with $k = 1$ intended violations and payoff $b = 1$ for a caught violation. Keeping $k = 1$ but allowing any non-negative real number b to replace the entry $+1$ in Figure 1.1, the game $\Gamma(n, m, 1)$ generalizes the Dresher model in that the reward to the inspector for a caught violation needs no longer equal his loss for an uncaught one. Since payoffs can be uniformly multiplied by any positive constant without changing their meaning, the loss to the inspector for an uncaught violation can be normalized to -1 (it is reasonable to assume that it is negative, that is, worse than legal behavior which has reference payoff zero; otherwise there would be neither an incentive to violate nor to inspect). So this is in fact the most general assumption for zero-sum payoffs in the Dresher model with at most one violation. The case $b = 0$ is also admitted, which means that the inspector does not gain by detecting a violation (and the inspectee is not punished for it), as compared to legal action; nevertheless, the inspector has still an incentive to inspect. The solution of this game, considering Figure 1.1 with $+1$ replaced by b, has been shown by Höpfinger [10] (the treatment of non-zero-sum payoffs is also very similar; see Avenhaus and von Stengel [4]). This solution will be subsumed by the solution of the more general game $\Gamma(n, m, k)$.

The number k of intended violations need not be equal to the number of violations actually performed by the inspectee when the game is played, even if he is not caught, but it is an upper limit. This leaves the inspectee the option not to violate if the inspector has too many inspections left, corresponding to the possible option of legal action (that is, no violation) in the original Dresher game. For example, the inspectee should not intend to perform more than $n - m$ violations since otherwise he will be caught with certainty, at his disadvantage. The solution below shows that this maximal number $k = n - m$ of intended violations can also be replaced by any larger number, e.g., $k = n$, to be interpreted as "violate as often as possible". This case has also been considered by Dresher in [8], cast into a recursive description (see Figure 3.3 below) that is also a special case of the one described below in Figure 3.2.

For non-negative integers n, m, k denoting the number of periods, inspections and intended violations, respectively, let the value (as the equilibrium payoff to the inspector) of the game $\Gamma(n, m, k)$ be denoted by $v(n, m, k)$. For certain boundary cases, this value can be given immediately. For $k = 0$, that is, no intended violation, the inspectee always behaves legally, so

$$v(n, m, 0) = 0. \tag{2.1}$$

If the inspector can inspect in every remaining period, for $m \geq n$, he should do so, and the inspectee will not gain by violating, so

$$v(n, m, k) = 0 \qquad \text{for } m \geq n; \tag{2.2}$$

this holds even in the case $b = 0$, where the inspectee does not lose anything by violating and might therefore do so, but will be caught immediately and only receive payoff zero (in general: $-b$). If the inspector has no inspections left ($m = 0$), then the inspectee will violate as often as he intended to, but at most once in every remaining period, each time diminishing the inspector's payoff by one:

$$v(n, 0, k) = -\min\{n, k\}. \tag{2.3}$$

As argued above, it is reasonable to assume that always $k \leq n - m$ holds; for computational convenience, this is however not assumed. It will become apparent that the value $v(n, m, k)$ of the game does not change if any number k larger than $n - m$ is replaced by $n - m$; in particular, this holds for (2.3).

For $k = 1$, that is, one intended violation, the game can be solved like the original Dresher game. Thereby, the first non-trivial case is $(n, m, k) = (2, 1, 1)$. This game is depicted in *extensive form* in Figure 2.1, following the definition by Kuhn [11] of extensive games, which shall be briefly explained. The nodes of the game tree denote states of the game, which starts at the root and terminates at a leaf which is labelled with the corresponding payoff to the inspector (the inspectee receives the corresponding negative payoff). The inner (that is, non-leaf) nodes of the tree are partitioned into *information sets* depicted by ovals, each labelled with one of the players who has to make a move by choosing an outgoing edge and thereby the successor node. The interpretation is that the player knows that he is in the information set but not at which particular node; all nodes in one information set have therefore outgoing edges marked in the same way denoting the possible actions (whose outcome might thus not be known to the player).

In Figure 2.1, the information sets labelled I_t and V_t belong to the inspector and inspectee (or "violator"), respectively, where the index t denotes a stage for convenient reference. The first information set I_1 is a singleton, so the inspector is at stage one completely informed. The two possible actions c and $\bar{c}$ are control and no control. About this choice the inspectee is not informed, so his information set V_1 at stage one contains two nodes; his actions l and $\bar{l}$ are legal and illegal action (violation). If he is at the left node of V_1 (control at the current first stage) and violates ($\bar{l}$), the game terminates with payoff b to the inspector. If the inspectee acts legally (l), he will reach the second stage, knowing that the inspector has controlled and has no inspection left, and can decide again at V_2, where he will choose $\bar{l}$ since that gives him the higher payoff. (So that leftmost branch of the game tree could be shortened to a leaf with payoff -1; a reduced form of the extensive game is shown in Figure 2.2(a).)

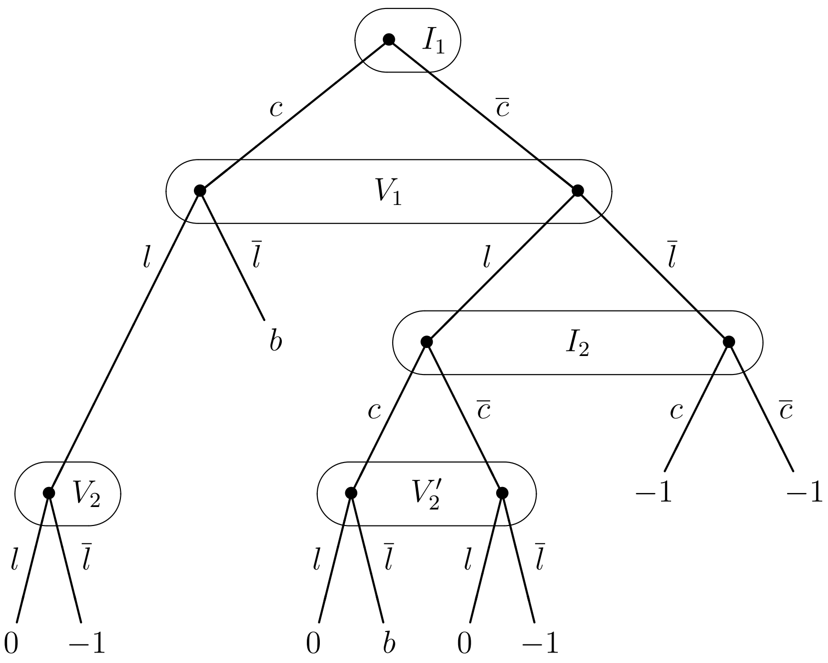
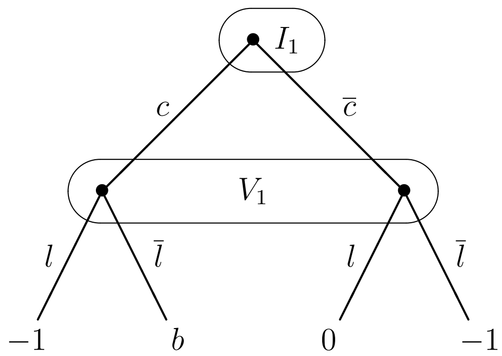


Figure 2.1. Detailed extensive form of the game $\Gamma(n, m, k)$ for $n = 2$ periods, $m = 1$ inspections and $k = 1$ intended violations between inspector (I) and inspectee (V). At each stage, their choices are c or $\overline{c}$ for control or no control, and l or $\overline{l}$ for legal or illegal action. The leaves of the game tree are labelled with the payoffs to the inspector.

If the inspectee is at the right node in V_1 (no control at the first stage), the game continues after his move, about which the inspector is not informed at stage two, depicted by the information set I_2. There, his choice between c and $\overline{c}$, control or no control, should be control since otherwise he would give up the inspection he is allowed; nevertheless, this choice is denoted here explicitly for demonstration. If the inspectee has acted illegally at stage one and not been caught (right node in I_2), he will not act illegally again, and the inspector's payoff will be -1, independent of his choice at I_2. If the inspectee has not violated (left node in I_2), the next information set V_2' after the inspector's choice gives the inspectee again a decision between l and $\overline{l}$ with corresponding payoffs. However, these reveal that the inspector can only gain by choosing control (c) at I_2, so the inspectee can assume that at V_2' he is at the left node and an optimal choice there will be legal action l.

Since the payoffs (and except for $b = 0$ even the moves) after stage one are thereby determined, the game $\Gamma(2, 1, 1)$ can be reduced to the smaller extensive game shown in Figure 2.2(a). (In fact, if the inspectee initially intended up to $k = 2$ violations, keeping $n = 2$ and $m = 1$, then, to depict $\Gamma(2, 1, 2)$, Figure 2.1 would show an extra information set V_2'' following the move after the right node in I_2 with choice between l and $\overline{l}$ for the inspectee giving payoffs -1 and $b - 1$ after control c, which the inspector will choose at I_2, and payoffs -1 and -2 after $\overline{c}$; this game would be reduced to the same game in Figure 2.2(a), equivalent to the case $k = 1 = n - m$.)

Figure 2.2(a) has a corresponding normal form, which is very simple here and shown



(a)

I_1 \ V_1	l	$\bar{l}$
c	-1	b
$\bar{c}$	0	-1

(b)

Figure 2.2. Simplified extensive form (a) and normal form (b) of the game $\Gamma(2,1,1)$.

in Figure 2.2(b). It has a mixed equilibrium with value $v(2,1,1) = -1/(b+2)$ according to (1.6) since it is of the general form shown in Figure 1.2. For $b = 1$, this also coincides with Dresher's formula (1.10).

It is useful to consider other games $\Gamma(n,m,k)$ for small numbers n,m,k in order to demonstrate the general structure of their extensive form. In the next section, it will be shown how a modified extensive game $\Gamma'(n,m,k)$ can be described recursively and how its solution can be applied to find the value $v(n,m,k)$ of the original game $\Gamma(n,m,k)$.

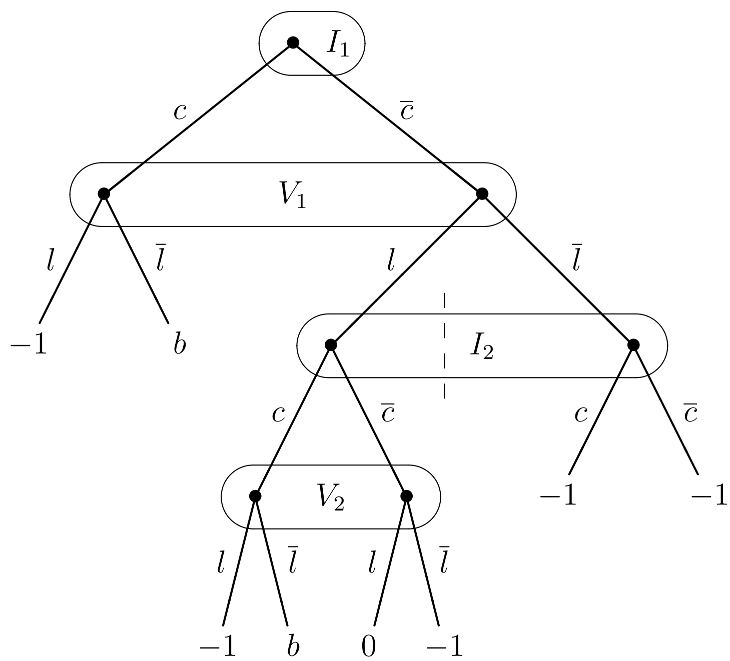


Figure 2.3. Extensive form of the game $\Gamma(3,1,1)$.

In Figure 2.3, the number of periods is $n = 3$ with $m = 1$ inspection and $k = 1$ intended violations. The beginning of the game is like that of $\Gamma(2,1,1)$ in Figure 2.1. If the inspector has used his inspection at the first stage and the inspectee acted legally, the inspectee can safely violate at a later stage with payoff -1 which is therefore given directly following that move at V_1. In the information set I_2, the left node represents the game at the second stage with no control and legal action at stage one. What follows (to the left of the dashed line) is therefore structured like the game $\Gamma(2,1,1)$ as in Figure 2.2(a). However, the inspector does not know at which node he is in I_2. If he is at the right node, his move will always result into payoff -1 independent of his actions at a later stage (if there will be still a choice), which are therefore not depicted explicitly; Figure 2.3 displays the game $\Gamma(3,1,1)$ in simplified form, which is sufficient.

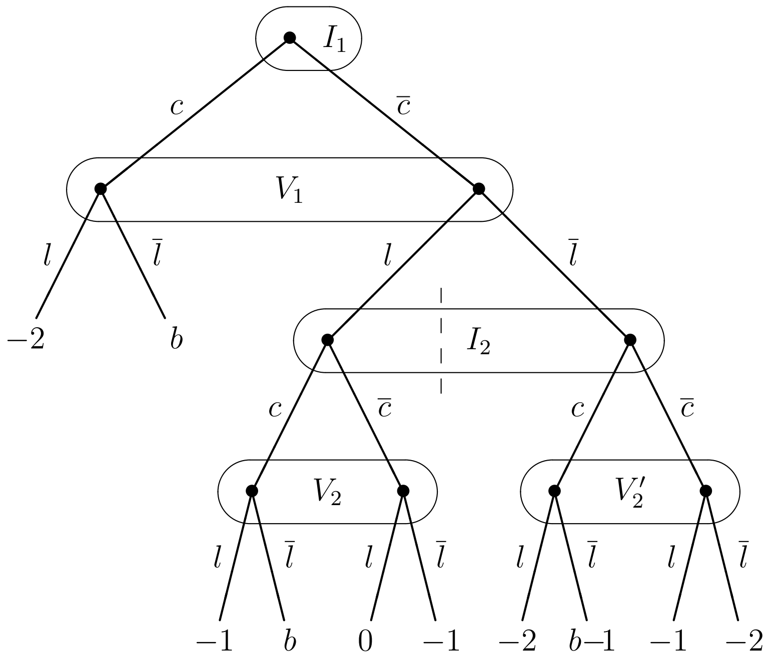


Figure 2.4. Extensive form of the game $\Gamma(3,1,2)$.

Figure 2.4 shows the game $\Gamma(3,1,2)$ with the same number of periods $n = 3$ and inspections $m = 1$ as before, but $k = 2$ intended violations. Here, if the inspector has controlled at stage one and the inspectee acted legally, the inspectee can safely violate twice at stages two and three with resulting payoff -2. The information set I_2 has two nodes distinguishing the cases where the inspectee acted legally or violated successfully at stage one. In the first case, the (left) node of I_2 has a subsequent game structure like the same node in $\Gamma(3,1,1)$ in Figure 2.3. However, if the inspectee did successfully violate at the first stage, he will still intend another violation that he might perform at stage two. Therefore, there is another information set V_2' following the move of the inspector at the right node of I_2. The resulting payoffs are like those to the left side of the dashed line except for a constant shift by adding -1. The reason is that the payoff to the inspector is diminished by one because of the successful first violation.

An even higher number $k = 3$ of intended violations is assumed in Figure 2.5 for $n = 4$ periods and $m = 1$ admitted inspections. The stages of this game $\Gamma(4,1,3)$ have a pattern similar to the previously considered games since there is only one inspection. Whenever the inspector controls, he either catches the inspectee violating at that stage, or else will have

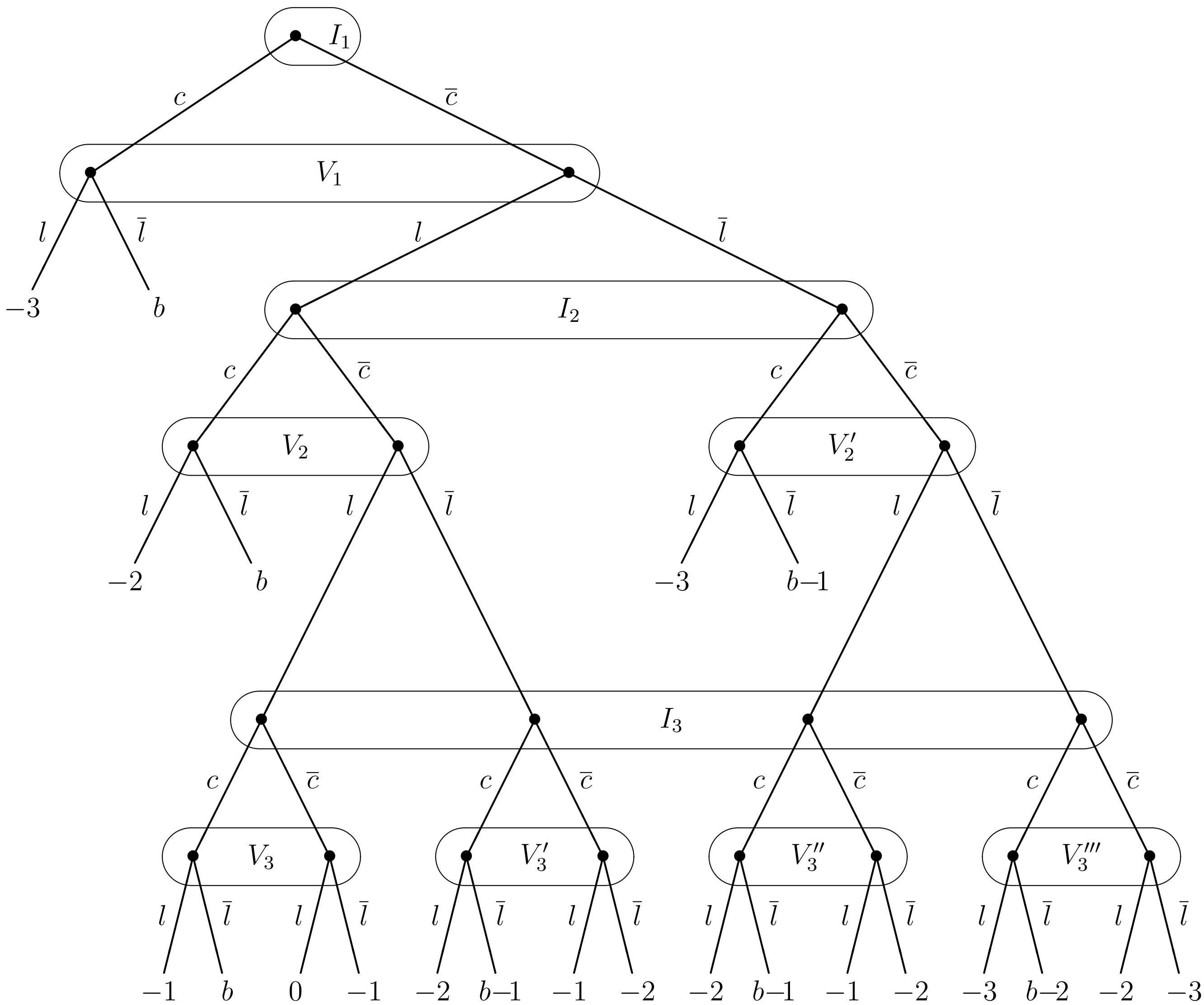


Figure 2.5. Extensive form of the game $\Gamma(4,1,3)$.

no occasion to inspect at later stages which therefore will be used for violations. Of interest is the case where the inspector did twice not control and has to decide at stage three, at the information set I_3. There are four nodes in I_3 corresponding to the possible combinations of legal or illegal action by the inspectee at stages one and two, about which the inspector is not informed. The second and third node of I_3 both denote a game state where the inspectee violated once, at period two and one, respectively. Since the payoffs only depend on the number but not on the time of successful violations, the subsequent parts of the game with the information sets V_3' and V_3'' are identical. The leftmost and the rightmost node in I_3 describe zero respectively two successful violations at the first two stages. The payoffs following the information sets V_3, V_3' and V_3'', V_3''' are therefore identical except for additive constants $0, -1, -2$.

For $n = 4$ periods, $m = 2$ inspections and up to $k = 2$ violations, the corresponding game $\Gamma(4,2,2)$ is shown in Figure 2.6. Here, if the inspector controls at the first stage

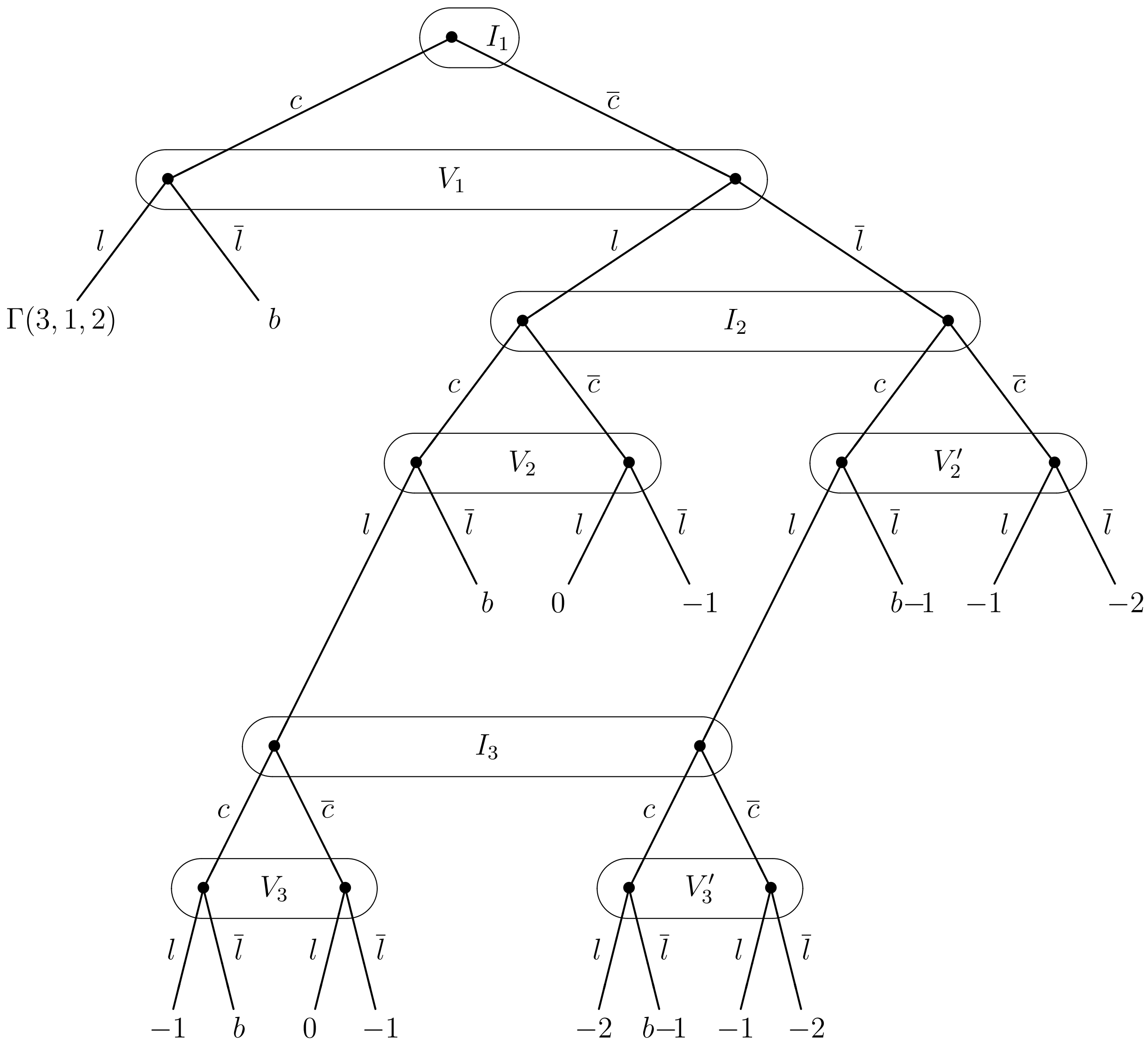


Figure 2.6. Extensive form of the game $\Gamma(4,2,2)$.

and the inspectee acts legally, the inspector has still one inspection left for the remaining three periods. Also, he will be informed that no violation has taken place at stage one since otherwise he would have caught the violator. That game part is therefore identical to $\Gamma(3,1,2)$ as shown in Figure 2.4 and, for brevity, denoted by its name as a subgame of $\Gamma(4,2,2)$ in Figure 2.6.

The remaining parts of the game $\Gamma(4,2,2)$ are given in the same manner as in the other games. There are leaves of the game whenever the game terminates because of a caught violation (with payoff b, or $b-1$ if there has been an uncaught violation before), or whenever the actions in the remaining periods are determined since either the inspector has no inspections left so the inspectee will violate as often as he intends to (here: as often as possible), or the inspector can control at every remaining period with a payoff as if the inspectee acts legally (and does so if $b>0$). Therefore, there is an information set I_3 only for the cases where the inspector has used one inspection at stage two (if he controlled at

I_1, he will be in the subgame $\Gamma(3,1,2)$ or will have caught a violation at stage one). If the inspector did not control at stages one and two, he should inspect in the remaining two periods. In a less reduced extensive description like it is Figure 2.1 for $\Gamma(2,1,1)$, subsequent to the choice $\overline{c}$ at I_2 the moves of the inspectee at the right nodes of V_2 and V_2' would all lead into another information set I_3', which is separate from I_3 since the inspector remembers his choice at I_2 (this is the condition of "perfect recall" introduced by Kuhn [11] which applies to the information sets of any game considered here). This set I_3' would be similar to I_3 in Figure 2.5 but with different payoffs, so that control will be the unique optimal choice. Such additional information sets like I_3' are explicitly necessary if the number n of periods is higher, like for $\Gamma(5,2,2)$, for example.

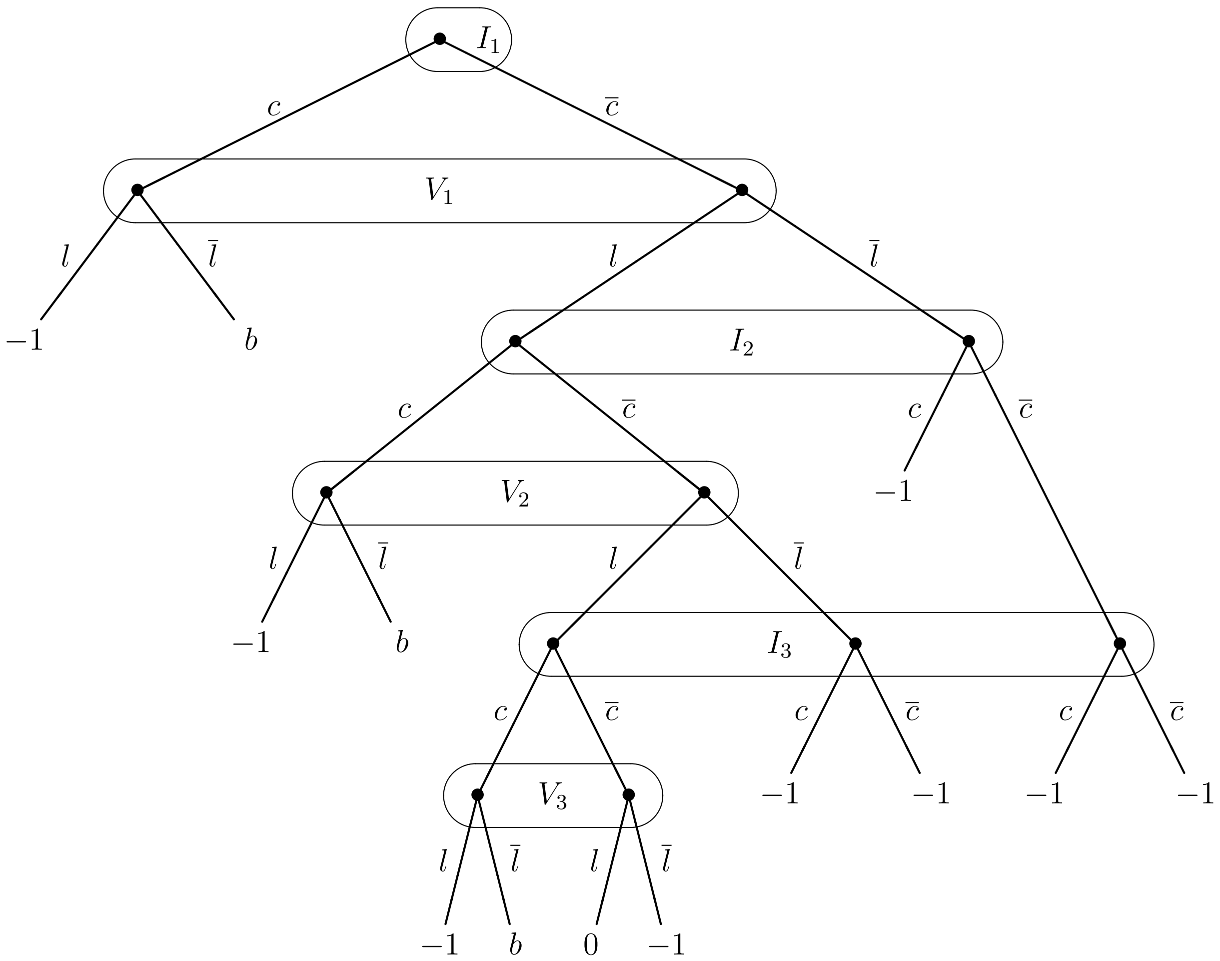


Figure 2.7. Extensive form of the game $\Gamma(4,1,1)$.

Finally, Figure 2.7 shows a situation similar to Figure 2.3 where the inspectee intends only $k = 1$ violation in $n = 4$ periods although the inspector is only permitted $m = 1$ inspection. If the inspectee has successfully violated, then the payoff to the inspector will be -1 no matter where he controls at a later stage. The inspector is, however, not informed about this so his information set I_3, for example, contains two nodes representing successful violation at stages two and one, respectively. If the move $\overline{c}$ at the right node of I_2 were not extended to I_3 but instead directly made a leaf with payoff -1, the game would not essentially differ since the middle node of I_3 serves the same purpose as the right one, denoting a game state where the inspector has already "lost". Nevertheless, the knowledge of the inspector is correctly represented by information sets like I_3 in Figure 2.7 with multiple nodes of this kind.

These examples should illustrate the extensive form of a general game $\Gamma(n,m,k)$ for any non-negative integers n,m,k. At each stage, the inspector and the inspectee make independent moves. In the game tree, these are represented sequentially, here with the inspector moving first and the inspectee second. The information sets of the inspectee contain always two elements since he is not informed about the inspector's move at the current stage, but knows the moves of all past stages. It would be equally possible to depict the inspectee's move first, whose information sets would then always be singletons. This is not done to keep the information sets of inspector, which in general are larger, as simple as possible.

The beginning of the game $\Gamma(n,m,k)$ is structured like that of $\Gamma(4,2,2)$ in Figure 2.6. For control c at the first information set I_1, following the possible actions l and $\bar{l}$ of the inspectee there is a subgame $\Gamma(n-1,m-1,k)$ respectively a leaf with payoff b. Following no control $\bar{c}$ at I_1, the next information set I_2 of the inspector contains two elements. In general, these sets grow larger whenever the inspector does not control since then the inspectee can act without the inspector's knowledge either legally or illegally at that stage. When the inspector does control, his next information set contains the same number of elements, like I_3 following I_2 in Figure 2.6, since in case of violation the game terminates. In particular, the singleton I_1 is succeeded by a singleton that is the first information set of the subgame $\Gamma(n-1,m-1,k)$.

There are leaves with corresponding payoffs whenever the actions of the inspector for the remaining periods (and thus of the inspectee) are determined as explained for $\Gamma(4,2,2)$. The payoffs for both control and no control are the same (namely, $-k$) if the inspectee has successfully performed all his k intended violations like following the right nodes in I_2 of $\Gamma(3,1,1)$ in Figure 2.3 or in I_3 of $\Gamma(4,1,1)$ in Figure 2.7. However, these information sets contain all such nodes as long as the inspector has still to decide, like I_3 in Figure 2.7.

In order to find a solution of $\Gamma(n,m,k)$, it is useful to have additional subgames in order to use a "roll-back analysis", that is, recursion, as an aid. The described game does not contain many subgames (only the leftmost branches) since, for example, the first information set of a subgame must be a singleton. A suitable modification of the information structure that does not change the optimal strategies will be described in the next section.

3. Recursive solution with an auxiliary game

This section describes a solution of the inspection game $\Gamma(n, m, k)$ defined in the previous section, where n, m and k denote the number of periods, inspections and intended violations, respectively. In a modified game $\Gamma'(n, m, k)$, the inspector is informed about all past violations even where he did not inspect. This game has additional subgames and allows a recursive description and solution. It will be shown that for the particular payoffs chosen, the equilibrium of the auxiliary game $\Gamma'(n, m, k)$ can also be interpreted as an equilibrium of the original game $\Gamma(n, m, k)$. This also justifies formally the recursive solution of the Dresher game, which is a special case.

The value of an extensive game can in principle be determined by considering its corresponding normal form. For a game in extensive form, a pure strategy of a player describes a move for each of his information sets. These combinations of moves thus define the strategies for each player, and by considering all strategy pairs and the resulting payoffs, one obtains the normal form of the game, like Figure 2.2(b) for $\Gamma(2, 1, 1)$ in Figure 2.2(a). An equilibrium of the extensive game is defined as an equilibrium of the normal form, which can be computed, for example, with the Simplex algorithm for linear programs (see, for example, Owen [13, chapter 3]).

However, it is frequently possible to find an equilibrium and to prove the saddlepoint property directly by looking at the extensive form. First, the so-called condition of *perfect recall* implies that any mixed equilibrium of the game can be found in *behavioral* strategies (Kuhn [11]). These define "locally" a probability distribution on the moves for each information set of the player and are therefore much simpler than mixed strategies which assign a probability to each pure strategy or "complete move plan". (Perfect recall means that for each information set, the player does not have additional information about where he is in that set if he remembers his past moves, which can be defined technically in terms of the game tree and the information sets, see Kuhn [11, p.213]. This condition is always fulfilled here.)

Second, it is possible to replace (recursively) *subgames* by their respective value in the game tree, thus constructing a so-called *subgame perfect* equilibrium. A subgame is a subtree of the game tree that does not overlap with any information set and therefore constitutes by itself an extensive game. For example, the game $\Gamma(4, 2, 2)$ has $\Gamma(3, 1, 2)$ as a subgame as shown in Figure 2.6; to help compute the value of $\Gamma(4, 2, 2)$, this subgame can be replaced by its value $v(3, 1, 2)$ as a direct payoff. On the other hand, the subtree of $\Gamma(3, 1, 1)$ in Figure 2.3 with the left node of I_2 as its root is not a subgame since it neither contains the information set I_2 nor is disjoint to it, but overlaps with it. That subtree would be a subgame if I_2 were cut into two new information sets along the dashed line. This is precisely what shall be done in general to construct from $\Gamma(n, m, k)$ an auxiliary game $\Gamma'(n, m, k)$ that is easier to solve.

The auxiliary game $\Gamma'(n, m, k)$ is identical to the original game $\Gamma(n, m, k)$ except that the information sets of the inspector are all singletons, so all of the original information sets of the inspector in $\Gamma(n, m, k)$ are cut apart. The interpretation of $\Gamma'(n, m, k)$ is an inspection game as before, except that *both* players are informed about all actions in past periods. In particular, the inspector knows when a violation took place even for those periods where he did not inspect. Like in $\Gamma(n,m,k)$, the information sets of the inspectee in $\Gamma'(n,m,k)$ have two elements, reflecting the fact that the inspectee is not informed about the move of the inspector at the current stage. An example for $\Gamma'(n,m,k)$ is obtained from any extensive game shown in section 2 if the information sets of the inspector are cut apart into singletons like along the dashed line in Figure 2.3 or 2.4.

In the extensive game $\Gamma'(n,m,k)$, each information set of the inspector (a singleton) defines a subgame starting at that node, in contrast to the game $\Gamma(n,m,k)$. Therefore, the game $\Gamma'(n,m,k)$ can be described recursively in terms of subgames of the same kind. For positive parameters n,m,k, the general structure of $\Gamma'(n,m,k)$ is shown in Figure 3.1.

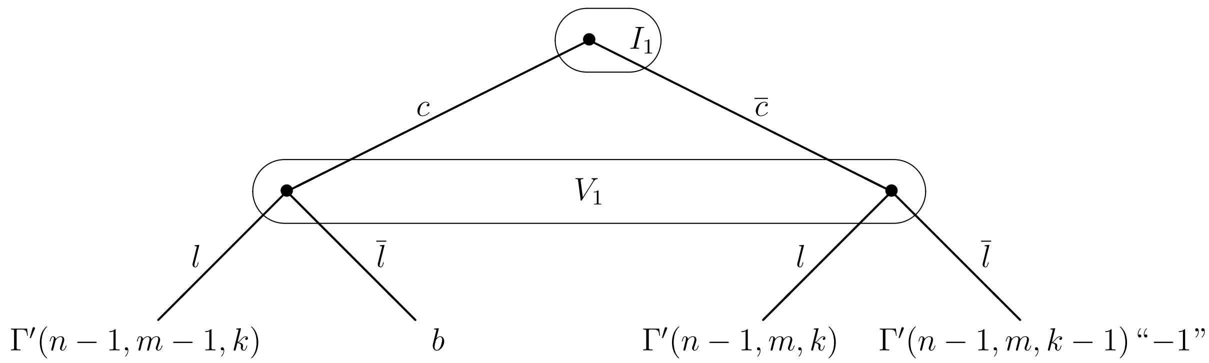


Figure 3.1. Recursive description of the auxiliary game $\Gamma'(n,m,k)$. The rightmost subgame is identical to $\Gamma'(n-1,m,k-1)$ except that all payoffs are diminished by one, indicated by the suffix "-1".

The extensive game in Figure 3.1 depicts only the moves of the players at the first stage, with suitable subgames in consequence. These subgames may be trivial, that is, be direct payoffs, like if the game terminates or if its outcome is known since one parameter has become zero. If the inspector controls (c) at I_1, the subsequent development in $\Gamma'(n,m,k)$ is essentially like in $\Gamma(n,m,k)$ as described in the previous section: for legal action (l) at V_2, the resulting subgame is $\Gamma'(n-1,m-1,k)$ with one period and one inspection less but the same number k of intended violations; if the inspectee violates $(\bar{l})$, the game terminates with payoff b. If the inspector does not control $(\bar{c})$ at I_1 and the inspectee acts legally (l), then the resulting subgame is $\Gamma'(n-1,m,k)$ where only the number n of periods is reduced by one. The last case ($\bar{c}$ and $\bar{l}$) represents a successful violation at stage one so the inspectee will intend one violation less in the remaining periods. The resulting subgame is therefore like $\Gamma'(n-1,m,k-1)$ except that all payoffs are reduced by one because of the successful violation, indicated in Figure 3.1 by the suffix "-1". So $\Gamma'(n-1,m,k-1)$ "-1" denotes the extensive game $\Gamma'(n-1,m,k-1)$ but with all payoffs shifted by -1. (When the scheme is applied recursively, some payoffs may thus be shifted several times.)

Replacing the subgames by their respective values, the value of $\Gamma'(n,m,k)$ can be computed recursively. This value shall be denoted by $v(n,m,k)$ like that for $\Gamma(n,m,k)$ for brevity; it will be shown that the values of both games are in fact equal. The resulting normal form of $\Gamma'(n,m,k)$ is shown in Figure 3.2. There, the formal suffix "-1" can be replaced by actual addition of -1 since the value of $\Gamma(n-1,m,k-1)$ "-1" is obviously $v(n-1,m,k-1)-1$.

Inspectee (V) / Inspector (I)	legal act (l)	violation ($\bar{l}$)
control (c)	$v(n-1,m-1,k)$	b
no control ($\bar{c}$)	$v(n-1,m,k)$	$v(n-1,m,k-1)-1$

Figure 3.2. Normal form, with payoffs to the inspector, of the auxiliary game $\Gamma'(n,m,k)$ with value $v(n,m,k)$.

It should be emphasized that Figure 3.2 is not a description of the original game $\Gamma(n,m,k)$ but of the auxiliary game $\Gamma'(n,m,k)$. In the original game, if the inspector does not control, he is at the next stage ignorant whether the inspectee violated, so he does not know whether he is in the lower left or lower right part of the table. A recursive description with such a table, similar to Figure 1.1 for the Dresher model, implicitly assumes that the inspector knows what happened at stage one even if he did not inspect. This has been pointed out by Kuhn [12, p.174] for a very similar model of inspections in the framework of a nuclear test ban treaty, with seismic "events" (here: periods) to be inspected that may be verified as nuclear "tests" (here: violations) for given numbers ℓ,m,n (here: n,m,k) of events, inspections and tests, respectively. In this and other applications, it may be reasonable to assume that the inspector is informed about any violations that occurred, e.g., through intelligence reports, but can only catch the inspectee and terminate the game when he actually inspects. Then the recursive model is appropriate provided the inspectee also knows that the inspector has this information, and so on, that is, if the different structure of the information sets in $\Gamma'(n,m,k)$ is common knowledge (compare Aumann [1]) to both players; this "common knowledge assumption" applies to any game model discussed here.

The two-by-two game in Figure 3.2 is of the general form shown in Figure 1.2 with a mixed equilibrium, since the restrictions (1.3) apply to its entries (as can be seen intuitively and can also be proved formally by induction). Correspondingly, the value $v(n,m,k)$ of $\Gamma'(n,m,k)$ can be computed recursively using (1.6). This recursive equation (spelled out explicitly in (3.1) below) applies only if n,m,k are all positive; if one of these parameters is zero, the appropriate initial condition (2.1), (2.2) or (2.3) stated for the game $\Gamma(n,m,k)$ holds, with the same reasoning (the case $m \geq n$ considered in (2.2) applies to $n=0$ and is otherwise subsumed by the recursive equation). These equations determine $v(n,m,k)$ for all non-negative numbers n,m,k. Using (1.5) and (1.7), the probabilities of control resp. of legal action at each stage are also determined. The recursive definition can be used for practical computations; an explicit formula will also be shown below (equation (3.3) in Theorem 3.2).

In the auxiliary game $\Gamma'(n,m,k)$, the inspector is informed about all past actions of the inspectee even if he did not inspect. At each of his information sets, he can decide with a suitable probability whether he should control or not. In the original game $\Gamma(n,m,k)$, the information sets of the inspector are in general not singletons. Whenever the inspector makes a decision at such a set, for example, at I_2 in any of the games shown in Figures 2.3 through 2.7, his move will be the same no matter at which node he is in that set. In particular, if he controls only with a certain probability, the probability of control is necessarily the same for all nodes in the set. When the information sets of Γ are cut apart, the inspector has greater freedom to choose his control probabilities, which may then be different. However, in the present case, this is *not* so.

This is the central conceptual point: With the payoffs defined for $\Gamma(n,m,k)$, the probabilities of control in the auxiliary game $\Gamma'(n,m,k)$ are the same (or can be chosen so) for all nodes within one information set of the original game $\Gamma(n,m,k)$. In other words, even if the inspector knew that a violation occurred in a period where he did not inspect, he would subsequently not control differently than without this information. Correspondingly, the optimal behavior of the inspectee would not change, either. An equilibrium for $\Gamma'(n,m,k)$ with this property can therefore be re-interpreted as an equilibrium for $\Gamma(n,m,k)$, as will be shown formally in Lemma 3.1 below.

There are two unproblematic cases where this claim can be verified immediately. Both have been considered by Dresher in the cited paper [8]. The first case is the Dresher model with $k=1$ intended violations described in section 1. Consider, for example, the auxiliary game $\Gamma'(3,1,1)$ obtained from the original game $\Gamma(3,1,1)$ shown in Figure 2.3 by separating (along the dashed line) the information set I_2 into two singletons. The subgame $\Gamma'(2,1,1)$ of $\Gamma'(3,1,1)$ starting at the left node of I_2 (the node belongs in $\Gamma'(3,1,1)$ to an information set by itself) is identical to $\Gamma(2,1,1)$ as shown in Figure 2.2. There is an optimal probability p of control (c) in this subgame, given by $p=1/(b+2)$ according to (1.5). The subgame of $\Gamma'(3,1,1)$ that starts at the right node of I_2 in Figure 2.3 gives payoff -1 for both choices c and $\bar{c}$. Therefore, the inspector can assign an arbitrary "optimal" probability, in particular p, to this choice of c. Then, at both considered nodes of $\Gamma'(3,1,1)$ the probability of control is the same and can therefore be uniquely assigned to I_2 in $\Gamma(3,1,1)$, where it remains optimal (see Lemma 3.1 below).

Similarly, the game $\Gamma'(4,1,1)$ is obtained from $\Gamma(4,1,1)$ shown in Figure 2.7 by decomposing I_2 and I_3 into singletons. All the resulting new subgames of $\Gamma'(4,1,1)$ either start with a unique optimal probability of control if the inspectee has not yet violated, or that probability is arbitrary and can be chosen equal to the determined probability assigned to the leftmost node in the respective original information set of $\Gamma(4,1,1)$; this should be done recursively, starting with the smallest subgames. Gluing the information sets back together, the constructed control probabilities can be uniquely assigned to I_2 and similarly to I_3. The same method can be applied to any auxiliary game $\Gamma'(n,m,1)$ obtained from $\Gamma(n,m,1)$. The recursive analysis by Dresher [8] described in section 1 above applies (implicitly) to the auxiliary game, but can be carried over to the original game for these reasons. Lemma 3.1 below formally justifies this cut-and-reglue construction and thereby also Dresher's approach.

The second case where the claim is rather intuitive has been mentioned briefly by Dresher in [8, p.20f]. There, the inspectee is capable of any number of violation attempts represented by $k=n$ in [8, p.20], or, equivalently, by $k=n-m$ since violation is not advantageous if the inspector can control at every remaining period. The games $\Gamma(3,1,2)$,

$\Gamma(4,1,3)$ and $\Gamma(4,2,2)$ shown in Figures 2.4, 2.5 and 2.6 are examples of this case. From the game $\Gamma(3,1,2)$ as in Figure 2.4, the auxiliary game $\Gamma'(3,1,2)$ is obtained by separating the information set I_2 into two singletons along the dashed line. The two resulting subgames of $\Gamma'(3,1,2)$ have the same payoff structure except that the payoffs of the subgame that starts at the right node of I_2 are shifted by -1. For both subgames, there is a unique optimal probability of control for the inspector. These probabilities are equal since the uniform payoff shift does not influence the optimality of strategies; this is also apparent from (1.5) and (1.7). Therefore, the probability of control can be uniquely assigned to I_2 in the original game.

The same reasoning can be used for $\Gamma(4,1,3)$ shown in Figure 2.5. There, it may be useful not to consider directly the auxiliary game $\Gamma'(4,1,3)$ but to cut apart the information set I_3 only in the middle between the second and the third node, and to separate I_2 into two singletons. Then there are two subgames starting at the nodes of I_2: for the left node, the subgame is equal to $\Gamma(3,1,2)$ as shown in Figure 2.4, whereas for the right node it is equal to $\Gamma(3,1,2)$ "-1", that is, these are again identical games except for the payoff shift, with equal optimal control probabilities that can be assigned to I_2 resp. I_3. To determine these probabilities, one can by induction consider the auxiliary games corresponding to the subgames, or directly decompose all information sets of the inspector in $\Gamma(4,1,3)$ into singletons, obtaining $\Gamma'(4,1,3)$. The subgames starting at the nodes of I_2 in this auxiliary game are apparently equal to $\Gamma'(3,1,2)$ resp. $\Gamma'(3,1,2)$ "-1".

This argument can be applied inductively to any game $\Gamma(n,m,k)$ with the maximal number $k = n - m$ of intended violations. For the auxiliary game $\Gamma'(n,m,k)$, the subgames following the inspectee's move after the inspector has not controlled ($\bar{c}$) at stage one are $\Gamma'(n-1,m,k)$ and $\Gamma'(n-1,m,k-1)$ "-1" as shown in Figure 3.1. In the first subgame, the inspectee has not violated in the first period. Then, the number $k = n - m$ of intended violations is greater than the maximal number $n-1-m$ of violations he can safely perform in the subgame, so it may as well be replaced by $k-1$. That is, the game $\Gamma'(n-1,m,n-m)$ is equal to $\Gamma'(n-1,m,n-m-1)$, as already observed in the above examples. (In a sense, the inspectee has in this subgame given up one occasion to violate in an uninspected period so he can no longer achieve the maximal number of violations; he should permit this to happen since violating with certainty is not optimal.) Thus, the two subgames following no control at stage one are equal except for the payoff shift of -1 (Theorem 3.2 below shows this formally).

If the inspectee intends to violate as often as possible, it is also intuitively clear that the inspector does not need to know whether the inspectee violated at a period where he did not inspect. If there was a successful violation, the loss of -1 is a "sunk cost" that does not change the subsequent situation. Dresher [8, p.21] represented this game as shown in Figure 3.3, that is, by the table of Figure 3.2 (with $b = 1$) where the parameter k is omitted, which for $k = n - m$ is justified; the initial conditions are

$$v(0,m) = 0, \qquad v(n,0) = -n.$$

In Figure 3.3, the inspector does not need to know whether he is in the subgame with value $v(n-1,m)$ or $v(n-1,m)-1$ (lower left resp. lower right in the table) since his behavior would not change.

Inspectee / Inspector	legal act	violation
control	$v(n-1,m-1)$	1
no control	$v(n-1,m)$	$v(n-1,m)-1$

Figure 3.3. The game $\Gamma'(n,m,n-m)$ with value $v(n,m)$ for $b=1$.

In summary, Dresher's recursive approach is justified both for the minimal number $k=1$ and maximal number $k=n-m$ of intended violations. The recursion (implicitly) defines the auxiliary games $\Gamma'(n,m,1)$ resp. $\Gamma'(n,m,n-m)$, where the inspector learns about any violation even without inspection. However, this knowledge is irrelevant since in the game $\Gamma'(n,m,1)$ the probabilities of control after a successful violation are arbitrary, whereas in the game $\Gamma'(n,m,n-m)$ they stay the same because the payoffs are uniformly shifted. Therefore, these probabilities can also be used for the original games $\Gamma(n,m,1)$ resp. $\Gamma(n,m,n-m)$.

What is rather surprising is the fact, stated in Theorem 3.2 below, that the optimal probabilities of control are also the same in both games $\Gamma(n,m,1)$ and $\Gamma(n,m,n-m)$ at each respective stage. (This would most likely have been noticed by Dresher if he had found an explicit solution for the game in Figure 3.3 with value $v(n,m)$ like it is (1.10) for the game in Figure 1.1.) This is a necessary condition if for general k the recursive solution of the auxiliary game $\Gamma'(n,m,k)$ shall be carried over to $\Gamma(n,m,k)$ in the same way as for $k=1$ and $k=n-m$. To see this, consider the case $1<k<n-m$ in Figure 3.1. If the two subgames $\Gamma'(n-1,m,k)$ and $\Gamma'(n-1,m,k-1)$ "-1" shall have the same probability of control for the first move of the inspector (as it is necessary if this probability shall be assigned to a single information set in $\Gamma(n,m,k)$), then this must obviously hold also for $\Gamma'(n-1,m,k)$ and $\Gamma'(n-1,m,k-1)$ (compare also Figure 3.2 and the remarks on Figure 3.3). Therefore, by induction, the probability of control in $\Gamma(n-1,m,k)$ must not depend on the number k of intended violations at all (and in particular be the same for $k=n-1-m$ and $k=1$). Otherwise, it would matter to the inspector whether he has moved to the lower left or lower right part of the table in Figure 3.2 and the auxiliary game $\Gamma'(n,m,k)$ could not be used as described. Specifically, for the case $(n,m,k)=(4,1,2)$, which has not yet been considered, the two subgames $\Gamma(3,1,1)$ and $\Gamma(3,1,2)$ shown in Figures 2.3 and 2.4 (or equivalently, their auxiliary games) must have the same probabilities of control at stage one.

That the probability of control in $\Gamma'(n,m,k)$ does indeed not depend on the number k of intended violations can be observed for special cases but has in fact rather little intuitive evidence. Consider, for example, the two games in Figures 1.1 and 3.3, representing $\Gamma'(n,m,k)$ for $k=1$ and $k=n-m$. By (1.5), the probabilities of control and no control are inversely proportional to the differences of the respective row entries. The difference in the first row is in both games given by $1-v(n-1,m-1)$, whereas for the second row it is $v(n-1,m)+1$ in Figure 1.1 but constant in Figure 3.3. So the ratios of these differences can only agree if the two functions $v(n,m)$ specifically fit this requirement, which seems rather unlikely (this constraint was therefore very useful in the search for the general explicit formula for $v(n,m,k)$, which will be sketched in section 4).

Since the claim requires formal proof, further discussion is deferred until Lemma 3.1 and Theorem 3.2 are stated, which have been hinted at several times. The lemma treats the cutting and regluing of information sets that allows to carry over a suitable solution of the auxiliary game $\Gamma'(n,m,k)$ to the original game $\Gamma(n,m,k)$; its proof is almost trivial. The theorem states and explicitly solves the recursive equations for $\Gamma'(n,m,k)$; its proof is deferred to section 4 because it is mainly of technical interest. This section will conclude with a discussion of Theorem 3.2 to complete the present conceptual line of reasoning.

Lemma 3.1. *Let Γ and Γ' be extensive games with perfect recall with the same game tree, the same player to move at each node and the same payoffs, but where the partition of the non-leaf nodes into information sets of Γ' is finer than that of Γ. Suppose there is an equilibrium of Γ' in behavioral strategies that, for each information set A of Γ, defines for all information sets of Γ' contained in A the same probability distribution on the moves. For these strategies, assigning this probability distribution to the information set A in Γ for each A, one obtains an equilibrium of Γ.*

Proof. Because of the condition of perfect recall, the equilibria of Γ and Γ' can be expressed in terms of behavioral strategies (Kuhn [11]). Let there be two players (the argument for more than two players is entirely analogous but not of interest here), so an equilibrium of Γ' is given by a pair of behavioral strategies, one for each player. By assumption, each information set of Γ' is a subset of an information set A of Γ (belonging to the same player who is to move). This set inclusion may be proper, so that some sets A of Γ are cut apart to yield information sets of Γ'. By induction, it suffices to assume that just one information set A of Γ is cut into two information sets B and C of Γ' and that all remaining information sets of Γ' and Γ are equal. (If there are several information sets of Γ to be cut apart, with possibly several cuts, Γ' is obtained in finitely many such steps with just one cut at a time, starting with the information sets closest to the leaves to maintain the condition of perfect recall.)

W.l.o.g., let A be an information set of the first player. Any behavioral strategy for this player in Γ is given by (a,p) where a is a probability distribution on the moves at A and p is a tuple of probability distributions, one for each of his other information sets. Similarly, a behavioral strategy for the first player in Γ' is given by (b,c,p) with probability distributions b and c on the moves at B resp. C and a tuple p of distributions for the remaining information sets, where p is a tuple as in Γ. A behavioral strategy for the second player, in Γ and Γ' alike, is a tuple q of distributions on the moves at each of his information sets.

Since all nodes in A, and thus in B and C, permit by definition the same moves, the probability distributions a, b and c are of the same kind, so that the distribution a defined for A can also be applied to B or C. Thus, a behavioral strategy (a,p) in Γ of the first player defines also a behavioral strategy (a,a,p) in Γ'. For these strategies, given some strategy q of the other player, the behavior and the outcomes in Γ and Γ' are identical. In fact, the game Γ can be considered as a restricted form of the game Γ' where the first player is only allowed to use behavioral strategies of the form (a,a,p) in Γ' (that is, strategies (b,c,p) with $b=c$), which, conversely, define corresponding strategies (a,p) in Γ.

With this observation, the proof is almost done. By assumption, there is an equilibrium of Γ' given by a pair of behavioral strategies $\big((a^*,a^*,p^*),q^*\big)$ for the first and second player, respectively, where a^* represents the two identical probability distributions on the moves at B and C. With G and H denoting the expected payoffs to the two players (where $G=-H$ for zero-sum games), the equilibrium conditions are

$$G\big((a^*,a^*,p^*),q^*\big) \ge G\big((b,c,p),q^*\big)$$

for all behavioral strategies (b,c,p) of the first player, and

$$H\big((a^*,a^*,p^*),q^*\big) \ge H\big((a^*,a^*,p^*),q\big)$$

for all behavioral strategies q of the second player. The first inequality implies with the added restriction $b=c$ the inequality

$$G\big((a^*,a^*,p^*),q^*\big) \ge G\big((b,b,p),q^*\big)$$

for all b,p, which together with the inequality for H states the equilibrium property of the pair of behavioral strategies $\big((a^*,p^*),q^*\big)$ in Γ, as claimed. (Remark: With extra notational effort, this argument can also be used if more than one information set is cut apart, so induction is not essential.) □

Lemma 3.1 shall be applied to the games $\Gamma(n,m,k)$ and $\Gamma'(n,m,k)$ in place of Γ and Γ'. The information sets of the inspectee are the same in both games. Each information set of the inspector in $\Gamma'(n,m,k)$ contains only a single node. As explained before Lemma 3.1, the optimal probabilities of control in the auxiliary game $\Gamma'(n,m,k)$, which can be determined by looking recursively at its subgames, are the same for all nodes contained in an information set of the original game $\Gamma(n,m,k)$ if and only if they are independent of the number of intended violations at that stage. The following theorem states that this is the case, and describes the solution of both games.

Theorem 3.2. *Let $v(n,m,k)$ denote the value of the auxiliary game $\Gamma'(n,m,k)$ for non-negative integers n,m,k. For positive n,m,k, this value is according to Figure 3.2 (with $b\ge 0$) recursively defined by*

$$v(n,m,k) = \frac{b\cdot v(n-1,m,k) - v(n-1,m-1,k)\cdot\big(v(n-1,m,k-1)-1\big)}{v(n-1,m,k)-v(n-1,m,k-1)+1+b-v(n-1,m-1,k)}, \tag{3.1}$$

and otherwise by the initial conditions

$$v(0,m,k)=0, \qquad v(n,0,k)=-\min\{n,k\}, \qquad v(n,m,0)=0. \tag{3.2}$$

The unique solution to these equations is explicitly given by $v(n,m,k)=0$ if $n<m$, otherwise by

$$v(n,m,k) = \frac{\binom{n-k}{m+1}-\binom{n}{m+1}}{s(n,m)} \tag{3.3}$$

where

$$s(n,m) := \sum_{i=0}^{m} \binom{n}{i} b^{m-i}. \tag{3.4}$$

Let $p(n,m,k)$ for $n,m>0$ denote the optimal probability of control (c) for the first move of the inspector in $\Gamma'(n,m,k)$ (that is, at I_1 in Figure 3.1), which can be chosen equal to one for $n \le m$. For $n > m$, this probability is for $k > 0$ uniquely given by

$$p(n,m,k) = \frac{s(n-1,m-1)}{s(n,m)}, \qquad 1-p(n,m,k) = \frac{s(n-1,m)}{s(n,m)}, \tag{3.5}$$

so it is independent of k, and it is arbitrary for $k=0$. Therefore, this solution applies also to the original game $\Gamma(n,m,k)$, which has value $v(n,m,k)$.

The proof of Theorem 3.2 is deferred to section 4. The recursive equation (3.1) is a partial difference equation in three variables that is not easy to solve, in particular since the function $v(n,m,k)$ is not simple enough to be easily guessed. Section 4 therefore also sketches several approaches, including the one actually used, that may help to arrive at the explicit solution (3.3). Even a straightforward verification that (3.3) fulfills (3.1) and (3.2) is rather laborious, but it can be shortened by considering the original equation (1.4) for the value of the more general game in Figure 1.2. Figure 3.2 is a special case of such a game since it has a mixed equilibrium. This will also be proved formally; in particular, $v(n,m,k)$ is for $m < n$ a strictly increasing function of m, that is, the inspector gains if he is admitted more inspections, which is intuitively obvious but not immediate from (3.1) or (3.3).

The explicit form (3.3) is replaced by the equation $v(n,m,k) = 0$ for $n < m$ since then $s(n,m) = 0$ if $b = 0$; for $b > 0$, this case distinction is not necessary. For $n \le m$, the boundary condition (2.2) holds which can be weakened to $v(0,m,k) = 0$ as in (3.2).

Equation (3.3) shows that if the number k of intended violations is greater than the number $n-m$ of uninspected periods, the game $\Gamma(n,m,k)$ has the same value $v(n,m,k)$ as for $k = n-m$ since $\binom{n-k}{m+1}$ is zero iff $k \ge n-m$. Also, the initial condition $v(n,0,m) = -\min\{n,k\}$ is thereby represented in closed form. Furthermore, for $k > 0$ (3.3) shows

$$v(n,m,k) - v(n,m,k-1) = -\binom{n-k}{m} \Big/ s(n,m)$$

(compare also (4.4) in the proof of Theorem 3.2 below), so as the number k of intended violations increases, its effect on the value of $\Gamma(n,m,k)$ diminishes. Except for $n \le m$ or $k = 0$, this value $v(n,m,k)$ is negative so the inspectee violates with a positive probability $1 - q(n,m,k)$, say, which is not explicitly stated but can be found by applying (1.7) to the game in Figure 3.2 with $q = q(n,m,k)$.

The solution (3.3) also generalizes Dresher's formula (1.10). The Dresher model is the game $\Gamma(n,m,1)$ where the reward b to the inspector for a caught violation equals his loss for an uncaught one. This assumption $b = 1$ is generalized to any $b \ge 0$ by the change in the dominator $s(n,m)$ shown in (3.4); this solution (for $k = 1$) has been found by Höpfinger [10], as mentioned in section 2. (Also, the proof of (3.3) for general k shown in section 4 is shorter than the proof by Dresher [8, pp.11–16] of the special case.)

The probability of control for the case $k = 1$ considered by Dresher applies to any game $\Gamma(n, m, k)$ independently of k. The equation for $p = p(n, m, k)$ in (3.5) is simpler than (1.9) using (1.10). Equation (3.5) also reveals some numerical properties of $p(n, m, k)$. For example, let $b = 1$, so $s(n-1, m-1) < s(n-1, m)$ for $m < n$ (compare also (4.8) below). This shows that the probability $p(n, m, k)$ of control in the first period is (for $b = 1$) always less than 1/2 as long as the inspector cannot inspect at every remaining period, even for $m = n-1$. If he can inspect only once ($m = 1$), it is easily seen that this inspection is optimally used with equal probability in one of the first $n-1$ periods and with a probability $(b+1)$ times as high (e.g., twice as likely for $b = 1$) in the last period. Further numerical properties are discussed in Dresher [8] and by Brams, Davis and Kilgour [5], who consider the recursive equation for the inspection game $\Gamma'(n, m, 1)$ but with a somewhat different interpretation (however, they do not give the explicit solution found by Höpfinger [10] for $b \neq 1$, nor do they discuss that the recursive description implies full knowledge of the inspector as pointed out by Kuhn [12, p.174] and laid out above).

The game $\Gamma(n, m, k)$ is solved recursively via the auxiliary game $\Gamma'(n, m, k)$ in fairly the same way as for the case $k = n - m$ briefly considered by Dresher [8, p.20f] (see Figure 3.3 above), since as there, each successful violation, of which for $k > 1$ there may be several, adds uniformly -1 to the payoff. As already pointed out before Lemma 3.1, the generalization to the case $1 < k < n - m$ is, however, deeper than may seem at first sight. (Also, there may exist applications where an inspectee would indeed attempt only a limited number of violations, but more than one.) Since the value $v(n, m, k)$ does depend on k, the matrix in Figure 3.2 is for different k "twisted" and not merely "shifted", so it is not immediate that the control probability $p(n, m, k)$ in $\Gamma'(n, m, k)$ is independent of k, which is necessary to apply it to $\Gamma(n, m, k)$.

Besides the formal proof, the author has not found a more intuitive reason why $p(n, m, k)$ is independent of k. It is not sufficient that an uncaught violation reduces the payoff by some amount that is "out of the pocket" of the inspector and leaves his future actions unaffected. For, this "sunk cost" must also be the same for each violation, that is, the payoff function must be *linear* in the number of successful violations. Otherwise, simple examples show that the control probabilities in $\Gamma'(n, m, k)$ generally depend on past violations so the recursive solution can no longer be carried over to $\Gamma(n, m, k)$.

The recursive approach fails similarly if a caught violation terminates the game with a constant payoff b to the inspector that is independent of previous successful violations (for instance, one may think of a thief who has to return the loot of past thefts when he is caught). In this case, the inspector would in general gain if he knew that the inspectee violated (who, in turn, would then also behave differently), since further violations become more risky to the inspectee. A solution of this non-recursive inspection game, whose payoffs are fairly naturally defined, would be interesting. The value of this game is bounded from below by $v(n, m, k)$ as given by (3.3), since its payoffs are greater than or equal to the payoffs of $\Gamma(n, m, k)$.

In the game $\Gamma(n, m, k)$, the inspector does not even need to know the number k of intended violations in order to behave optimally (which may be useful operationally, although the game-theoretic interpretation of the game and its value is then no longer clear). As shown, this is necessary to solve the game $\Gamma(n,m,k)$ recursively with payoffs that are linear in the number s, say, of successful violations when the game terminates, as assumed here. There may be a recursive solution without this assumption. Namely, adding s as a fourth parameter (initially zero) besides n, m and k, the inspector is in the auxiliary game $\Gamma'(n,m,k,s)$ after no control at the first stage either in the subgame $\Gamma'(n-1,m,k,s)$ or $\Gamma'(n-1,m,k-1,s+1)$ depending on whether the inspectee acted legally or not. In the original game $\Gamma(n,m,k,s)$, the inspector is not informed about this but at least knows $k+s$, that is, the sum of intended and successful violations stays the same. With suitably generalized payoffs (both for caught and uncaught violations), his control probabilities may, unlike here, depend on the initial value of k, but still may stay invariant with respect to $k+s$, that is, be the same in the original information sets of $\Gamma(n,m,k,0)$ so that Lemma 3.1 can be applied. It might be interesting to find and characterize payoffs with this property, although they would probably look contrived just to allow for the recursive approach. The game $\Gamma(n,m,k)$ considered here, however, seems to be a fairly canonical extension of the inspection games in Figures 1.1 and 3.3 introduced by Dresher.

4. Proof and derivation of the explicit formula

This section gives a proof of Theorem 3.2. Since it is much harder to find an explicit formula for a recurrence like (3.1) than to verify it, a possible derivation of the solution is also sketched. The author found it by elementary computations. The most useful computational simplifications are obtained from the original game-theoretic equation (1.4), both for the proof and the derivation. A more systematic approach, for example with the help of generating functions, would be interesting, which might perhaps also allow a combinatorial or probabilistic interpretation of the result.

Proof of Theorem 3.2. The real-valued function $v(n,m,k)$ of non-negative integers n,m,k is uniquely defined by the initial conditions (3.2) and the recursive equation (3.1). The equation (3.3) defines $v(n,m,k)$ explicitly; since it is undefined for the case $b=0$ and $n<m$ where the denominator $s(n,m)$ given by (3.4) is zero, it is for $n<m$ replaced by $v(n,m,k)=0$. By induction on n, the following statements will be shown for all $n,m,k\geq 0$:

(a) The explicit definition of $v(n,m,k)$ agrees with (3.2) and (3.1).

(b) For positive n,m,k, the auxiliary game $\Gamma'(n,m,k)$ is represented in Figure 3.2 which is a special case of Figure 1.2; its value is given by (3.1). If n, m or k is zero, the value of $\Gamma'(n,m,k)$ is given by (3.2).

(c) For $n,m>0$, the optimal probability $p(n,m,k)$ of control at the first stage of the game $\Gamma'(n,m,k)$ has the properties stated in the theorem.

First, note that the explicit definition of $v(n,m,k)$ implies the equations (3.2) since then obviously $v(0,m,k)=0$ and $v(n,m,0)=0$ hold, and

$$\begin{aligned} v(n,0,k) &= \binom{n-k}{1} - \binom{n}{1} \\ &= \begin{cases} -k & \text{if } k<n \\ -n & \text{if } k \geq n \end{cases} \\ &= -\min\{n,k\}. \end{aligned}$$

These equations represent the value $v(n,m,k)$ for the boundary situations of the game $\Gamma(n,m,k)$ as described at the beginning of section 2, and in the same way of $\Gamma'(n,m,k)$. Furthermore, (a), (b) and (c) hold for $n=0$.

Let n be positive and assume, as inductive hypothesis, that (a), (b) and (c) are true for $n-1$ instead of n; these statements shall be proved for n. They are true for $m=0$, so let $m>0$. For $k=0$, either choice of the inspector will result in the payoff $v(n-1,m-1,0)=v(n-1,m,0)=0$ by (3.2), so $p(n,m,0)$ can be chosen arbitrarily and (c) holds, as well as (a) and (b). Thus, let also $k>0$.

Since n, m and k are positive, the figures

$$\begin{aligned} a &:= v(n-1,m-1,k), \\ c &:= v(n-1,m,k), \\ d &:= v(n-1,m,k-1)-1 \end{aligned}$$

are defined and, by (a), given according to the explicit definition. Therefore, if $n \leq m$, obviously $a=c=0$ and $d=-1$, and if $n>m$, (3.3) yields

$$a = \frac{\binom{n-1-k}{m} - \binom{n-1}{m}}{s(n-1,m-1)}, \tag{4.1}$$

$$c = \frac{\binom{n-1-k}{m+1} - \binom{n-1}{m+1}}{s(n-1,m)}, \tag{4.2}$$

$$d = \frac{\binom{n-k}{m+1} - \binom{n-1}{m+1}}{s(n-1,m)} - 1\,. \tag{4.3}$$

With these abbreviations, the game in Figure 3.2 has the table of the game shown in Figure 1.2, whose solution can be used if the inequalities (1.3) hold. These will be shown next.

The inequality $a \leq b$ holds for $n \leq m$ where $a = 0 \leq b$, otherwise $\binom{n-1-k}{m} < \binom{n-1}{m}$, so $a < 0 \leq b$ by (4.1) since $s(n-1,m-1)>0$.

The inequality $c>d$ holds for $n \leq m$ and is otherwise by (4.2) and (4.3) equivalent to

$$\binom{n-1-k}{m+1} - \binom{n-k}{m+1} > -s(n-1,m)$$

or

$$\binom{n-1-k}{m} < s(n-1,m) = \sum_{i=0}^{m-1} \binom{n-1}{i} b^{m-i} + \binom{n-1}{m},$$

which is true since $n > m$ and $k > 0$.

The inequality $a \le c$ holds for $n \le m$ where $a = 0 = c$. Let $n > m$, which implies the strict inequality $a < c$, as follows. The numerator $r(n, m, k)$ of the term in (3.3),

$$r(n,m,k) := \binom{n-k}{m+1} - \binom{n}{m+1},$$

fulfills

$$r(n,m,k) - r(n,m,k-1) = \binom{n-k}{m+1} - \binom{n-k+1}{m+1} = -\binom{n-k}{m},$$

so $r(n,m,k) = r(n,m,k-1) - \binom{n-k}{m}$, which by iteration and with $r(n,m,0) = 0$ yields

$$r(n,m,k) = -\sum_{j=1}^{k} \binom{n-j}{m} \tag{4.4}$$

(see also Feller [9, p.3, equation (12.6)]). Then $a < c$ is equivalent to

$$\frac{r(n-1,m-1,k)}{s(n-1,m-1)} < \frac{r(n-1,m,k)}{s(n-1,m)}$$

which by (4.4) holds if

$$\frac{-\binom{n-1-j}{m-1}}{s(n-1,m-1)} \le \frac{-\binom{n-1-j}{m}}{s(n-1,m)} \tag{4.5}$$

is true for $j = 1, \ldots, k$ and if at least one of these inequalities (4.5) is strict. For $j > n - m$, (4.5) holds non-strictly since both sides are zero. For $j \le n - m$ (which exists), (4.5) is as a strict inequality equivalent to

$$\begin{aligned}
s(n-1,m) &> \frac{n-j-m}{m} \cdot s(n-1,m-1),\\
b^m + \sum_{i=1}^{m} \binom{n-1}{i} b^{m-i} &> \frac{n-j-m}{m} \cdot \sum_{i=1}^{m} \binom{n-1}{i-1} b^{m-i},\\
b^m + \sum_{i=1}^{m} \binom{n-1}{i-1} \Big(\frac{n-i}{i} - \frac{n-j-m}{m}\Big) b^{m-i} &> 0,\\
b^m + \sum_{i=1}^{m-1} \binom{n-1}{i-1} \Big(\frac{n(m-i)}{im} + \frac{j}{m}\Big) b^{m-i} + \binom{n-1}{m-1} \frac{j}{m} &> 0
\end{aligned}$$

which is true. This proves $a < c$ for $n > m$ as claimed.

Finally, the last inequality $b > d$ of (1.3) holds since $b \ge 0 > -1 \ge d$.

The game $\Gamma'(n, m, k)$ with the normal form in Figure 3.2 is therefore a special case of the game in Figure 1.2 whose value $v = v(n, m, k)$ is given by (1.6), that is, by the recursive equation (3.1). This shows (b).

To prove (a), it can be shown directly that (3.1) is fulfilled by (3.3). However, this computation is rather complicated but can be shortened if the representation of $p(n, m, k)$ in (3.5) is proved first. The game in Figure 1.2, which is equal to Figure 3.2, has the optimal control probability $p = p(n, m, k)$. Let $n \le m$. Then $a = 0 = c$, where p can be chosen equal to one as argued following equation (1.7) in section 1. Also, both (3.1) and the explicit definition yield $v(n,m,k)=0$. This shows (c) and (a) for $n\le m$.

Therefore, let $n>m$, which implies $a<c$ as shown above, so p is uniquely determined by (1.4), that is, by

$$p\cdot a\,+\,(1-p)\cdot c=p\cdot b\,+\,(1-p)\cdot d\,. \tag{4.6}$$

In order to show the equations (3.5), it suffices that with $p=p(n,m,k)$ they fulfill (4.6). For that purpose, note the following properties of $s(n,m)$. First,

$$\begin{aligned} s(n,m) &= \sum_{i=0}^{m}\binom{n}{i}b^{m-i} \\ &= \binom{n}{0}b^{m}+\sum_{i=1}^{m}\Bigl(\binom{n-1}{i}+\binom{n-1}{i-1}\Bigr)b^{m-i} \\ &= \binom{n-1}{0}b^{m}+\sum_{i=1}^{m}\binom{n-1}{i}b^{m-i}+\sum_{i=0}^{m-1}\binom{n-1}{i}b^{m-1-i} \\ &= s(n-1,m)+s(n-1,m-1)\,. \end{aligned} \tag{4.7}$$

Second,

$$\begin{aligned} s(n-1,m) &= \sum_{i=0}^{m}\binom{n-1}{i}b^{m-i} \\ &= \sum_{i=0}^{m-1}\binom{n-1}{i}b^{m-1-i}\cdot b+\binom{n-1}{m} \\ &= b\cdot s(n-1,m-1)+\binom{n-1}{m}\,. \end{aligned} \tag{4.8}$$

Let $p=s(n-1,m-1)/s(n,m)$ as in (3.5); with (4.7), this implies the second equation in (3.5), namely $1-p=s(n-1,m)/s(n,m)$. Thus, by (4.1) and (4.2),

$$\begin{aligned} p\cdot a+(1-p)\cdot c &= \frac{s(n-1,m-1)}{s(n,m)}\cdot\frac{\binom{n-1-k}{m}-\binom{n-1}{m}}{s(n-1,m-1)}+\frac{s(n-1,m)}{s(n,m)}\cdot\frac{\binom{n-1-k}{m+1}-\binom{n-1}{m+1}}{s(n-1,m)} \\ &= \frac{\binom{n-k}{m+1}-\binom{n}{m+1}}{s(n,m)}\,. \end{aligned}$$

Furthermore, by (4.3) and (4.8),

$$\begin{aligned} p\cdot b+(1-p)\cdot d &= \frac{s(n-1,m-1)}{s(n,m)}\cdot b+\frac{s(n-1,m)}{s(n,m)}\cdot\left(\frac{\binom{n-k}{m+1}-\binom{n-1}{m+1}}{s(n-1,m)}-1\right) \\ &= \frac{1}{s(n,m)}\cdot\Bigl(b\cdot s(n-1,m-1)+\binom{n-k}{m+1}-\binom{n-1}{m+1}-s(n-1,m)\Bigr) \\ &= \frac{1}{s(n,m)}\cdot\Bigl(\binom{n-k}{m+1}-\binom{n-1}{m+1}-\binom{n-1}{m}\Bigr) \\ &= \frac{\binom{n-k}{m+1}-\binom{n}{m+1}}{s(n,m)}\,. \end{aligned}$$

This shows (4.6) and thereby (3.5) and (c).

Furthermore, both terms in (4.6) define the value $v = v(n, m, k)$ of the game as in (1.4), which implies the equation (1.6) that here has the complicated form (3.1). Since the term in question is precisely the explicit formula in (3.3) for $v(n, m, k)$, also (a) is true, which completes the induction.

By Lemma 3.1, this solution of the auxiliary game $\Gamma'(n, m, k)$ also applies to the original game $\Gamma(n, m, k)$. □

The following is a short account of the derivation of the explicit solution (3.3) for the equations (3.1) and (3.2) which may be of interest to some readers. The problem is substantially simplified by the equation (4.10) below which the author found with a computation that, in retrospect, is unnecessarily long, but which will be sketched briefly for illustration.

The game $\Gamma'(n, m, k)$ in Figure 3.2 was initially considered for $b = 1$ like the Dresher games with $k = 1$ and $k = n - m$ shown in Figures 1.1 and 3.3. Assume that the game has a mixed equilibrium, with $0 < k \le n - m$ and $m < n$ to avoid degenerate cases. The value $v = v(n, m, k)$ of the game is given by (1.6) with

$$\begin{aligned} a &= v(n-1, m-1, k), \\ c &= v(n-1, m, k), \\ d &= v(n-1, m, k-1) - 1, \end{aligned}$$

resulting in (3.1), with $b = 1$.

Computing the value $v(n, m, k)$ for small numbers n, m, k gives rise to the conjecture that the inspection probability $p = p(n, m, k)$ as defined in Theorem 3.2 is independent of k. Obviously, p as given by (1.5) is independent of k iff $(b - a)/(c - d)$ is, that is, iff

$$\frac{1 - v(n-1, m-1, k)}{v(n-1, m, k) - v(n-1, m, k-1) + 1} =: f(n, m) = \frac{1 - v(n-1, m-1, 1)}{v(n-1, m, 1) + 1}. \tag{4.9}$$

The right term in (4.9) is obtained by letting $k = 1$. There is an explicit representation of $f(n, m)$ since $v(n, m, 1)$ is known by (1.10) but this is very inconvenient for computation; this applies also to p given by (1.9) using (1.10). Rather, (4.9) could be proved inductively using (3.1) or be of help to obtain an explicit solution to (3.1).

A convenient special case to start with is $m = 1$ where $a = -k$ which simplifies (3.1). It is possible to first guess and then prove closed formulas for $v(n, 1, 1)$, then for $v(n, 1, 2)$, $v(n, 1, 3)$ etc., which indicate

$$v(n, 1, k) = \frac{-\, k \cdot n + \binom{k+1}{2}}{n + 1}$$

which can also be verified by (3.1). For $m = 1$, (4.9) is rather simple and can be proved by induction on n using (3.1) and the explicit representation of $v(n, 1, k)$. This proof can be redone with very little reference to the explicit representation. Considering in this way $v(n, m, k)$ instead of $v(n, m, 1)$ and using (4.9) as inductive hypothesis, some nice cancellations show that a necessary condition for (4.9) is

$$\frac{1 - v(n, m, k)}{2 - v(n-1, m, k-1)} = g(n, m), \tag{4.10}$$

that is, the term on the left must also be independent of k.

These considerations can be cut short by considering, as in the above proof of Theorem 3.2, the original equation (1.4) for the value $v = v(n,m,k)$ of the game rather than taking the recursive equation (3.1) as a starting point. Namely, by (1.4), $v = p + (1-p)d$ which implies $1 - v = (1-p)\cdot(1-d)$ which shows (4.10) with $g(n,m) = 1-p$; nevertheless, (4.10) alone is very useful.

With $k = 1$ in (4.10), one easily obtains with (1.10)

$$g(n,m) = \frac{1 - v(n,m,1)}{2} = \frac{s(n-1,m)}{s(n,m)}. \tag{4.11}$$

Note that (4.11) implies (3.5). With $w(n,m,k) := 1 - v(n,m,k)$, (4.10) yields

$$w(n,m,k) = g(n,m)\cdot\Big(1 + w(n-1,m,k-1)\Big)$$

which is a rather simple recurrence in essentially a single variable since n and k are decreased simultaneously. Drop the parameter m which remains unchanged both in this equation and in (4.11), so

$$w(n,k) = \frac{s(n-1)}{s(n)}\cdot\Big(1 + w(n-1,k-1)\Big).$$

In particular, since $v(n,m,0) = 0$,

$$\begin{aligned}
w(n,0) &= 1,\\
w(n+1,1) &= \frac{s(n)}{s(n+1)}\cdot\Big(1 + w(n,0)\Big) = \frac{s(n)}{s(n+1)}\cdot 2,\\
w(n+2,2) &= \frac{s(n+1)}{s(n+2)}\cdot\Big(1 + w(n+1,1)\Big)\\
&= \Big(s(n+1) + s(n)\cdot 2\Big)\Big/ s(n+2),\\
w(n+3,3) &= \frac{s(n+2)}{s(n+3)}\cdot\Big(1 + w(n+2,2)\Big)\\
&= \Big(s(n+2) + s(n+1) + s(n)\cdot 2\Big)\Big/ s(n+3),\\
&\vdots\\
w(n+k,k) &= \left(\sum_{i=0}^{k-1} s(n+i) + s(n)\right)\Big/ s(n+k).
\end{aligned}$$

This gives the explicit formula

$$v(n+k,m,k) = 1 - w(n+k,k) = \frac{s(n+k) - \sum_{i=0}^{k-1} s(n+i) - s(n)}{s(n+k)}.$$

In the right term, m is still omitted for simplicity; its numerator $r(n+k,m,k)$ can be simplified further:

$$
\begin{aligned}
r(n,m,0) &= 0,\\
r(n+1,m,1) &= s(n+1) - 2\cdot s(n)\\
&= \sum_{i=0}^{m}\binom{n+1}{i} - 2\cdot\sum_{i=0}^{m}\binom{n}{i}\\
&= \binom{n+1}{0} + \sum_{i=1}^{m}\Bigl(\binom{n}{i} + \binom{n}{i-1}\Bigr) - 2\cdot\sum_{i=0}^{m}\binom{n}{i}\\
&= -\binom{n}{m},\\
r(n+2,m,2) &= s(n+2) - s(n+1) - 2\cdot s(n)\\
&= s(n+2) - 2\cdot s(n+1) + s(n+1) - 2\cdot s(n)\\
&= -\binom{n+1}{m} - \binom{n}{m},\\
&\vdots\\
r(n+k,m,k) &= -\sum_{j=1}^{k}\binom{n+k-j}{m}.
\end{aligned}
$$

This is the representation (4.4) for $r(n,m,k)$ which nicely generalizes Dresher's formula (1.10) for $k=1$ and also takes account of the degenerate cases $k>n-m$ or if n, m or k is zero as in (3.2). It can be shown that this fulfills (3.1) using the properties (4.7) and (4.8) with $b=1$ of $s(n,m)$ that are already helpful to verify the recursive equation (1.8) for the explicit formula (1.10). A necessary equation is thereby

$$
r(n,m,k) = \binom{n-k}{m+1} - \binom{n}{m+1}
$$

which is easily proved by induction and yields the explicit representation (3.3) with $b=1$.

Höpfinger [10, p.5] considered the game in Figure 1.1 where the payoffs $+1$ and -1 to the inspector for caught and uncaught violation are replaced by b and $-e$ with two positive constants b and e. One of these parameters can be normalized to one, and for $e=1$ Höpfinger's solution differs from (1.10) only by the more general denominator $s(n,m)$ as shown in (3.4). Since (1.10) and (3.3) with $b=1$ also have the same denominator, one can try the more general denominator for the game in Figure 3.2 for any $b>0$, and the proof still works since only (4.8) is needed in its more general form.

As shown in the above proof of Theorem 3.2, the verification of the explicit formula (3.3) can be streamlined further by considering the original equation (1.4) instead of (3.1), and one can also admit the case $b=0$.

5. One violation, optimizing timeliness of detection

The following game is another variation of the Dresher model which is different from the game discussed in the previous sections. Again, there are n periods $1, \ldots, n$, where the inspector can allocate m inspections to these periods, $0 \le m \le n$. The inspectee will commit at most one illegal act, but unlike the situation considered by Dresher, this will be discovered not only at an inspection at the current period but (with certainty) at the next inspection that takes place, where the *payoff* to the inspectee is given by the number of time periods the violation remains undiscovered. Thus, this payoff is the difference $d \ge 0$ between the period t of violation and the period $t + d$ of the next inspection, where there are no inspections at periods $t, t+1, \ldots, t+d-1$. If the violation occurs at a period t after the inspector has used up all m inspections, the payoff to the inspectee is $n+1-t$, so one can think of an additional inspection at an additional period with number $n+1$. Legal action (no violation) has payoff zero, like a violation that is instantly discovered. The game is zero-sum. The inspectee is informed about all past inspections.

This game has been described by Canty and Avenhaus in [6, section 3.3]. It models an actual safeguards problem in the surveillance of nuclear reactor sites, where m so-called interim inspection activities have to be distributed over a number n of interim inspection periods besides a major inspection that regularly takes place at the end of these periods (which is here the not counted inspection at period $n+1$). In this application, an optimal inspection scheme should show how to minimize the detection time of an optimized violation (deviation of nuclear material), so no particular reward for legal action is assumed for the inspectee.

Canty and Avenhaus gave a recursive description of the game that is not fully justified (as noted by them in [6, p.26]), not even via an auxiliary game like the one constructed in section 3, but which nevertheless yields the correct value of the game. It can, however, be partly justified by detailed considerations of an extensive game description and its corresponding normal form, which will be developed here, but the ad-hoc recursive description shall be given first.

Denote the game by $\Gamma(n, m)$ with n periods and m allocable inspections as described. It is useful to consider in this zero-sum game the payoffs to the *inspectee* since they are always non-negative (the same could have been done for the games in the previous sections, but Dresher described the game in [8] with payoffs to the inspector, and this convention was kept since thereby the payoff parameter b in Figure 3.2 is non-negative). Let $v(n, m)$ be the value of $\Gamma(n, m)$ as the equilibrium payoff to the inspectee. The parameters n and m are non-negative integers with $m \le n$ (the last restriction can be omitted by stipulating that for $m > n$, the game $\Gamma(n, m)$ is equivalent to $\Gamma(n, n)$). For $m = 0$, the inspectee will optimally violate at the first period and receive the maximal payoff n, and for $m = n$, the inspector will inspect at every remaining period, where the inspectee receives payoff 0 no matter when or if he violates, so

$$v(n, 0) = n, \qquad v(n, n) = 0. \tag{5.1}$$

For the interesting case $0 < m < n$, Canty and Avenhaus [6, p.27] described the game

Inspector \ Inspectee	legal act	violation
control	$v(n-1,m-1)$	0
no control	$v(n-1,m)$	$v(n-1,m)+1$

Figure 5.1. Ad-hoc recursive description of the inspection game $\Gamma(n,m)$ of "timeliness of detecting one violation" for $0 < m < n$ with value $v(n,m)$ and initial conditions (5.1). The entries denote here the payoffs to the inspectee.

recursively by the two-by-two game in Figure 5.1 displaying the options of the players at the first period. For control at period one, the payoffs to the inspectee are $v(n-1,m-1)$ respectively 0 for legal action and violation. For no control and legal action at the first period, the inspectee receives the payoff $v(n-1,m)$. These table entries have obvious explanations. However, the payoff for no control and violation is $v(n-1,m)+1$, which can only be justified temptatively, for example as follows: If the players behave optimally, the inspectee should be indifferent between legal action and violation, receiving in both cases the value of the game as payoff. If he violates at stage two and was not controlled at stage one, his payoff should therefore be $v(n-1,m)$. If he violates instead already at stage one, he gains one payoff unit so his payoff is $v(n-1,m)+1$.

As said, this reasoning is not quite convincing. It was argued in section 3 that the recursive description implies that the four cases resulting from the actions of the players at the first period constitute proper subgames of the game. Here, this would mean that the inspector knows whether the inspectee violated if he did not control, or at least that this information would not change the inspector's behavior. This is however not so since if the inspector knew that the violation occurred he would control with certainty. In this respect, one could however justify the recursive description as follows: The inspector has to act as if the violation has not taken place yet, that is, like at the beginning of $\Gamma(n,m)$, since otherwise he would have to control immediately, using up his inspections in the first m periods, which is certainly not optimal since his guaranteed payoff would be much lower than if he spread out his inspections evenly, for example. So if the inspector did not control at the first period, he would then act like in the game $\Gamma(n-1,m)$, only with his expected payoff diminished by one if the inspectee violated at the first period. Again, this is not yet a complete argument.

Before describing the game $\Gamma(n,m)$ properly, it is useful to have the solution of the game of Figure 5.1 at hand. The value of this game is given by a recursive equation whose explicit solution can be guessed after computing a few values.

Lemma 5.1. *The value $v(n,m)$ of the game $\Gamma(n,m)$ shown in Figure 5.1 is for $0 < m < n$ given by*

$$v(n,m) = \frac{v(n-1,m-1)\cdot(v(n-1,m)+1)}{v(n-1,m-1)+1}$$

and for $m=0$ by $v(n,0)=n$ and for $m=n$ by $v(n,n)=0$. The explicit solution to these *equations is*

$$v(n,m) = \frac{n-m}{m+1}.$$

The optimal probabilities p of control and q of legal action at the first period are given by

$$p = \frac{m}{n}, \qquad q = \frac{m}{m+1}.$$

Proof. The initial conditions for $v(n,m)$ are as in (5.1). Assume, as inductive hypothesis, that the game $\Gamma(n,m)$ has for $0 < m < n$ a mixed equilibrium, which is true for $n = 2$. Then it is of the general form shown in Figure 1.2 except for the inequalities (1.3) which are reverted since Figure 5.1 shows the payoffs to the inspectee. Nevertheless, the equation (1.4) for the value $v = v(n,m)$ is valid, as well as (1.6) and the equations (1.5) and (1.7) for p and q; in the fractions shown there, one may multiply denominator and numerator by -1 to make them positive. This yields the given recurrence for $v(n,m)$, whose explicit solution is verified at once. It confirms that there is a mixed equilibrium for all n if $0 < m < n$, and the given formulas for p and q using (1.5) and (1.7). $\square$

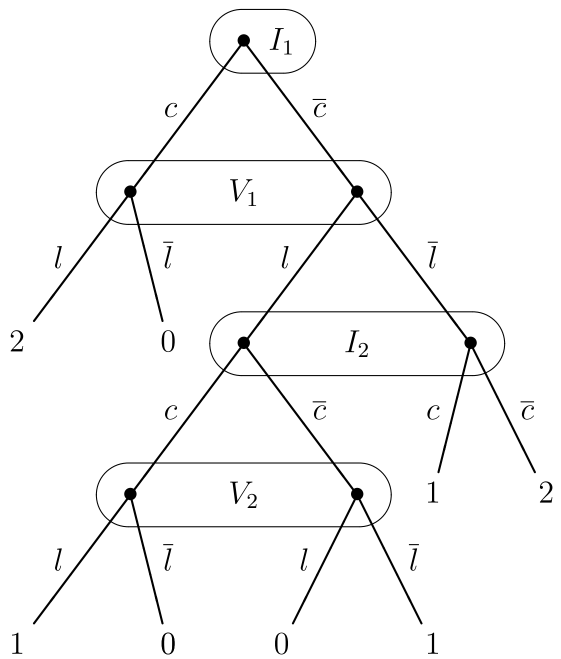


Figure 5.2. Extensive description of the game $\Gamma(3,1)$. The notation is as in Figure 2.1 except that payoffs are shown for the inspectee.

In the following, the game $\Gamma(n,m)$ will be described in a conceptually sound fashion using extensive games. Figure 5.2 shows a reduced extensive form of $\Gamma(3,1)$; the notational conventions are as in section 2. Since the inspector has in $\Gamma(3,1)$ only one inspection, the game is determined after the inspector used his control, where he either detects a violation or the inspectee violates at the next period to receive the maximal payoff. Also, the inspector should control at the last period if he hasn't done so before, so only two periods need be displayed in Figure 5.2.

$I \backslash V$	$l\,l$	$l\,\bar{l}$	$\bar{l}$
c	2	2	0
$\bar{c}\,c$	1	0	1
$\bar{c}\,\bar{c}$	0	1	2

Figure 5.3. Normal form of the game $\Gamma(3,1)$. The strategies of the inspector I and of the inspectee V denote sequences of conditional moves according to the extensive form shown in Figure 5.2.

This game has no proper subgames, and it is also not possible to employ Lemma 3.1 by splitting I_2 into two singletons, for example, since this would change the optimal strategies. Therefore, the game is converted to its normal form shown in Figure 5.3. The strategies c, $\bar{c}\,c$ and $\bar{c}\,\bar{c}$ of the inspector mean that he uses his inspection at the first, second respectively third period. For the inspectee, $l\,l$ denotes legal action at the first period, and also legal action at the second period *provided* the inspector did not control at stage one. If the inspector controls, the inspectee should violate immediately afterwards, resulting in the payoff 2 for the strategy pair $(c, l\,l)$. That is, $l\,l$ is a conditional strategy to be read according to the sequence of moves in the extensive form of Figure 5.2. Similarly, $l\,\bar{l}$ and $\bar{l}$ are the other strategies of the inspectee.

It is easy to verify that the game $\Gamma(3,1)$ in Figure 5.3 has value $v(3,1) = 1$ with optimal strategies $(1/2, 0, 1/2)$ for the inspectee and $(1/3, 1/3, 1/3)$ or $(0, 1, 0)$ for the inspector or, as usual, any convex combination of these two strategies.

The value of this game and the choices of the players at the first period are in agreement with the recursive solution of Lemma 5.1, considering only the strategy $(1/3, 1/3, 1/3)$ of the inspector, which is also correctly described by Lemma 5.1 for stage two. However, the optimal behavior of the inspectee at V_2 in Figure 5.2 is to choose legal action l with certainty, since otherwise the strategy $l\,\bar{l}$ in Figure 5.3 would get a positive probability which is not optimal against the strategy $\bar{c}\,c$ of the inspector. This is not in agreement with the recursive description of Figure 5.1 which suggests choosing l and $\bar{l}$ at V_2 with equal probability, if any recommendation is obtained from Lemma 5.1 at all, since for stage two, $\Gamma(2,1)$ is not a subgame of $\Gamma(3,1)$ in the extensive form. This shows that a full analysis of the game $\Gamma(n,m)$ requires the extensive form, although it will be seen that its value and an optimal strategy of the inspector are correctly described by Lemma 5.1.

The extensive form of the game $\Gamma(n,m)$ for general n,m is indicated in Figure 5.4. For

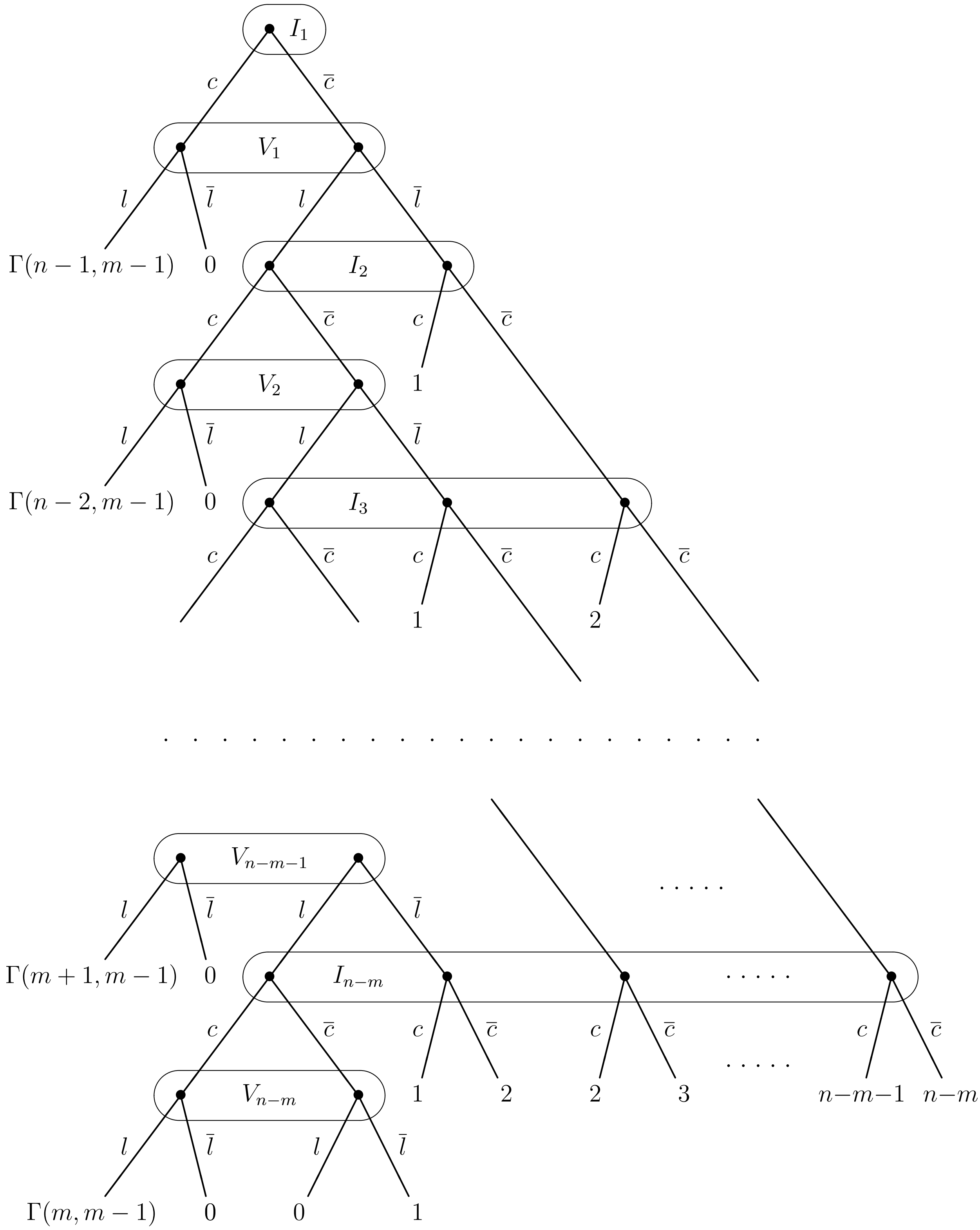


Figure 5.4. General form of the extensive game $\Gamma(n, m)$ for $0 < m < n$, with initial conditions (5.1) for its value $v(n, m)$ and payoffs to the inspectee.

each stage t where the inspector has not yet controlled, there is an information set I_t of the inspector, which needs only be specified for $t = 1, \ldots, n-m$ since after stage $n-m$, the inspector will inspect in every period. These information sets indicate that the inspector is not informed about the past moves of his opponent, whereas the inspectee is only uncertain about the choice of the inspector at the current period. Whenever the inspector controls at stage t and catches the inspectee, who violated at period $t-d$ with $d \geq 0$ (for $d = 0$ at the current period), the game terminates with payoff d to the inspectee. Otherwise, the inspector knows that an illegal action is yet to occur, and the game continues with the subgame $\Gamma(n-t, m-1)$.

$I \backslash V$	$l^{n-m}\bar{l}$	$\cdots$	$l\,l\,\bar{l}$	$l\,\bar{l}$	$\bar{l}$
c	$v(n-1, m-1)$	$\cdots$	$v(n-1, m-1)$	$v(n-1, m-1)$	0
$\bar{c}\,c$	$v(n-2, m-1)$	$\cdots$	$v(n-2, m-1)$	0	1
$\bar{c}\,\bar{c}\,c$	$v(n-3, m-1)$	$\cdots$	0	1	2
$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$
$\bar{c}^{\,n-m}c$	0	$\cdots$	$n-m-2$	$n-m-1$	$n-m$

Figure 5.5. Normal form of the game $\Gamma(n,m)$ shown in Figure 5.4, where the subgames $\Gamma(n-t, m-1)$ for $1 \leq t \leq n-m$ are replaced by their values $v(n-t, m-1)$. The dashed lines indicate the four subcases resulting form the moves of the players at stage one.

In the corresponding normal form of the game $\Gamma(n,m)$ shown in Figure 5.5, these subgames $\Gamma(n-t, m-1)$ are replaced by their values $v(n-t, m-1)$ as payoff entries, so this is a recursive description which is justified by the recursion one can use in the extensive form. The strategies of the inspector are denoted by $\bar{c}^{\,t}c$ for $t = 0, \ldots, n-m$, where $t+1$ is the first period where the inspector controls. Similarly, $l^t\bar{l}$ denotes a strategy of the inspectee indicating that he violates at period $t+1$ *provided* the inspector has not controlled in between, in which case the players have entered the corresponding subgame, represented by its value in the normal form in Figure 5.5.

As in Figure 5.3, the strategies in Figure 5.5 denote sequences of conditional moves in the extensive form in Figure 5.4, except for the last row and the first column, which should have been labelled $\bar{c}^{\,n-m}$ and l^{n-m}, but this would have complicated the normal form unnecessarily. Note that the last row with the strategy $\bar{c}^{\,n-m}c$ of the inspector implies control with certainty at period $n-m+1$ after his choice $\overline{c}$ at the information set I_{n-m} in Figure 5.4, where the payoffs with possible values between 0 and $n-m$ are indicated directly in the extensive form. Similarly, the first column is given by the strategy $l^{n-m}\overline{l}$ where the inspectee waits until the inspector controls at period t (using the strategy $\overline{c}^{\,t-1}c$) where the players enter the game $\Gamma(n-t,m-1)$ with value $v(n-t,m-1)$, except for the latest possible time $t=n-m+1$, after the inspector chose $\overline{c}$ at I_{n-m} and the inspectee chose l at V_{n-m} in Figure 5.4. Then, the inspectee receives payoff zero even if he does not actually violate at period $n-m+1$ but any time later or even not at all; all these behaviors are equivalent to and therefore represented by the strategy $l^{n-m}\overline{l}$.

The payoff matrix in Figure 5.5 is divided into four parts, indicated by the dashed lines, that result from the actions of the players at the first period. If the inspector controls and the inspectee acts legally, the players enter the subgame $\Gamma(n-1,m-1)$ with value $v(n-1,m-1)$. Control c and violation $\overline{l}$ terminate the game with payoff 0. For no control $\overline{c}$ at the first period and violation $\overline{l}$, there is a column with payoffs increasing with the time where the inspector uses his first inspection, which terminates the game. The most interesting situation results from no control $\overline{c}$ and legal action l at period one. Then, the strategies $\overline{c}^{\,t}c$ and $l^s\overline{l}$ for $1\le t,s\le n-m$ define a game that has the same normal form as $\Gamma(n-1,m)$ according to Figure 5.5 except that from the move sequences $\overline{c}^{\,t}c$ and $l^s\overline{l}$ that define the strategies, the first moves $\overline{c}$ respectively l should be omitted, diminishing t and s by one.

This indicates that the recursive normal form of $\Gamma(n,m)$ in Figure 5.5 can be reduced to the shorter ad-hoc recursive description in Figure 5.1, except for the lower right part resulting from the choices $\overline{c}$ and l at stage one. The following paragraphs give a heuristic argument that also this lower right part can be replaced by $v(n-1,m)+1$, as the lower left part is replaced by $v(n-1,m)$.

First, note that an equilibrium strategy for the inspectee in the game $\Gamma(n,m)$ assigns a positive probability to illegal action $\overline{l}$ at the first period, since otherwise, the inspector would with certainty not control at period one, which reduces the game in Figure 5.5 to the part at the left of and below the dashed lines, that is, to $\Gamma(n-1,m)$; by repetition of the argument, one would obtain a saddlepoint of $\Gamma(n,m)$ given by the pure strategies $\overline{c}^{\,n-m}c$ and $l^{n-m}\overline{l}$, which is not the case.

For a strategy of the inspector given by a probability distribution on the rows, each column of the game matrix has an expected value which will be called its *column value.* Thus, for an optimal strategy of the inspector, the column value for $\overline{l}$ is the value of the game, that is, a maximal column value, since otherwise the inspectee would not choose $\overline{l}$ with positive probability.

Second, one may reasonably assume that an optimal strategy of the inspector is also conditionally optimal in the sense that the conditional probability distribution on his moves given that he has not chosen c is optimal for the lower left part in Figure 5.5, that is, in the game matrix of $\Gamma(n-1,m)$. Then, the last column of that matrix has under the conditional distribution column value $v(n-1,m)$, and the column to the right of it with entries $1,2,\ldots,n-m$, which is the same column shifted by $+1$, has expected value $v(n-1,m)+1$ just as in the original recursive description in Figure 5.1.

This is strong enough evidence that the formula for $v(n,m)$ given in Lemma 5.1 is the correct value of the game $\Gamma(n,m)$ in Figure 5.5. The following theorem describes the solution of $\Gamma(n,m)$ precisely.

Theorem 5.2. *The value $v(n,m)$ of the game $\Gamma(n,m)$ in Figure 5.5 is*

$$v(n,m) = \frac{n-m}{m+1}. \tag{5.2}$$

An optimal strategy of the inspector with the probability p_t assigned to $\bar{c}^t c$ for $0 \le t \le n-m$ is

$$p_0 = \frac{m}{n}, \qquad p_t = (1 - p_{t-1}) \cdot \frac{m}{n-t} \qquad \text{for } 1 \le t \le n-m. \tag{5.3}$$

Applied to the entire game $\Gamma(n,m)$ in Figure 5.4 with all subgames spelled out, this strategy of the inspector is equivalent to selecting each subset A of $\{1,\ldots,n\}$ with $|A| = m$ with equal probability as the set of inspected periods. The unique optimal strategy of the inspectee with the probability q_s assigned to $l^s\bar{l}$ for $n-m \ge s \ge 0$ is

$$q_{n-m} = \frac{m}{m+1}, \qquad q_s = 0 \quad \text{for } 0 < s < n-m, \qquad q_0 = \frac{1}{m+1}. \tag{5.4}$$

Applied to the entire game, this means that the inspectee selects with equal probability $1/(m+1)$ a number $r \in \{0,1,\ldots,m\}$, waits until the inspector has used r inspections and violates immediately afterwards (if $r = 0$, at the first period).

Proof. The assertion will be proved by induction on m, starting with $n = 2$ since Figures 5.4 and 5.5 are recursive definitions of the game $\Gamma(n,m)$ for $0 < m < n$, where the values of the games $\Gamma(n,0)$ and $\Gamma(n,n)$ are defined by (5.1) (note that for these base cases of the recursive definition of $\Gamma(n,m)$ which are not considered in the theorem, (5.2) is also valid). For the entire game, the strategies will be interpreted after the induction is completed. The induction will show the statements about (5.2), (5.3) and (5.4), and in addition that with the inspector strategy defined in (5.3), *all* columns of the matrix have column value $v(n,m)$.

These assertions are true for $n = 2$, where $m = 1$, with the game matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and unique optimal strategies given by $p_0 = p_1 = q_1 = q_0 = 1/2$ and value $v(2,1) = 1/2$, which is the column value for both columns.

Let $n > 2$ and assume, as inductive hypothesis, that the assertions hold for all games $\Gamma(n-1,m)$ with $0 < m < n-1$. Let $0 < m < n$. If $m = n-1 > 1$, then $v(n-1,m-1) = 1/m$ by (5.2), using the inductive hypothesis. Then, the game matrix is $\begin{pmatrix} 1/m & 0 \\ 0 & 1 \end{pmatrix}$ with unique optimal strategies specified by $p_0 = q_0 = m/(m+1)$ and $p_1 = q_1 = 1/(m+1) = v(n,m)$, which shows (5.3), (5.4) and (5.2), and both columns have column value $v(n,m)$. Therefore, let $m < n-1$.

Assume that the inspector selects the row $\bar{c}^t c$ for $0 \le t \le n-m$ with probability p_t as defined in (5.3). The iterative definition of the probabilities $p_1,\ldots,p_{n-m}$ shows that they have the first factor $1 - m/n$, so

$$p_t = (1 - m/n) \cdot r_{t-1} \qquad \text{for } 1 \le t \le n-m \tag{5.5}$$

with

$$r_0 = \frac{m}{n-1}, \qquad r_t = (1 - r_{t-1}) \cdot \frac{m}{n-1-t} \qquad \text{for } 1 \le t \le n-1-m. \tag{5.6}$$

Note that $p_0 + p_1 + \cdots + p_{n-m} = 1$ iff $r_0 + \cdots + r_{n-m-1} = 1$ which, by iteration, is easily seen to be true. According to (5.6), $r_0, \ldots, r_{n-1-m}$ are the probabilities of the inspector strategy for $\Gamma(n-1, m)$ prescribed by (5.3), which, by the induction hypothesis, is optimal for $\Gamma(n-1, m)$ and, furthermore, yields the column value

$$v(n-1, m) = \frac{n-1-m}{m+1} = \frac{n}{m+1} - 1$$

for all columns of the game matrix of $\Gamma(n-1, m)$. This game matrix is the part of the matrix of $\Gamma(n, m)$ in Figure 5.5 to the left of and below the dashed lines. Therefore, the first $n - m$ columns of the game matrix of $\Gamma(n, m)$ have by (5.5) column value

$$\begin{aligned} &p_0 \cdot v(n-1, m-1) + (1 - m/n) \cdot v(n-1, m) \\ &= \frac{m}{n} \cdot \frac{n-m}{m} + \frac{n-m}{n} \cdot \Big(\frac{n}{m+1} - 1\Big) = \frac{n-m}{m+1}, \end{aligned}$$

where $v(n-1, m-1)$ is determined by (5.2) using the inductive hypothesis or, if $m = 1$, equation (5.1). The column value of the last column $\bar{l}$ is, comparing it with the last but one column,

$$(1 - m/n) \cdot (v(n-1, m) + 1) = \frac{n-m}{n} \cdot \frac{n}{m+1} = \frac{n-m}{m+1}.$$

This shows that the inspector strategy given by (5.3) yields for all columns of $\Gamma(n, m)$ the same column value $(n-m)/(m+1)$.

Finally, it will be shown that this is also the value of the game since the inspectee can use a strategy so that all rows have row value less than or equal to $(n-m)/(m+1)$, where these inequalities in fact have to be equalities since each row is selected with a positive probability by the inspector. These $n - m + 1$ equations uniquely determine the optimal probabilities $q_0, \ldots, q_{n-m}$ of the inspectee, with q_s for the column $l^s\bar{l}$, $0 \le s \le n-m$, as follows. The first row yields the equation

$$\begin{aligned} \frac{n-m}{m+1} &= (1 - q_0) \cdot v(n-1, m-1) + q_0 \cdot 0 \\ &= (1 - q_0) \cdot \frac{n-m}{m} \end{aligned}$$

or

$$1 - q_0 = \frac{m}{m+1}, \qquad q_0 = \frac{1}{m+1}.$$

The equation for the second row is

$$\begin{aligned} \frac{n-m}{m+1} &= (1 - q_0 - q_1) \cdot v(n-2, m-1) + q_1 \cdot 0 + q_0 \cdot 1 \\ &= \Big(\frac{m}{m+1} - q_1\Big) \cdot \frac{n-1-m}{m} + \frac{1}{m+1} \\ &= \frac{n-1-m+1}{m+1} - q_1 \cdot \frac{n-1-m}{m} \end{aligned}$$

or $q_1 = 0$. For the third row, one obtains

$$\begin{aligned}\frac{n-m}{m+1} &= (1-q_0-q_2)\cdot v(n-3,m-1)+q_0\cdot 2\\ &= \Big(\frac{m}{m+1}-q_2\Big)\cdot\frac{n-2-m}{m}+\frac{2}{m+1}\\ &= \frac{n-2-m+2}{m+1} \;-\; q_2\cdot\frac{n-2-m}{m}\end{aligned}$$

or $q_2 = 0$. In the same fashion, the equations for the rows $\bar{c}^{\,s}c$ up to $s = n-m-1$ yield $q_s = 0$ for $0 < s < n-m$. Then, the equation for the last row holds for any real number q_{n-m}, which in turn is determined by $1 = \sum_{s=0}^{n-m} q_s = q_0 + q_{n-m}$, which proves (5.4) (note that these equations determine $q_0, \ldots, q_{n-m}$ as probabilities; negative solutions would have invalidated the result). This completes the induction.

The strategy of the inspector given by (5.3) is equivalent to the "global" strategy ϕ, say, of selecting with equal probability $1/\binom{n}{m}$ an m-subset A of $\{1,\ldots,n\}$ as the set of inspected periods (an m-set is a set of cardinality m) and then controlling at the stages i with $i \in A$ until the game terminates. This will be shown by induction on n; it is true for $m = 0$ and $m = 1$. With the strategy ϕ, the probability of control c at the first period is $\binom{n-1}{m-1}/\binom{n}{m} = m/n$ since the m-subsets A of $\{1,\ldots,n\}$ with $1 \in A$ are in obvious one-to-one correspondence with the $(m-1)$-subsets of $\{2,\ldots,n\}$. By this argument, the strategy ϕ after control c at period one agrees with (5.3) applied to $\Gamma(n-1,m-1)$, using the inductive hypothesis. If $1 \notin A$, that is, no control $\bar{c}$ at stage one, ϕ selects an m-subset of the $(n-1)$-set $\{2,\ldots,n\}$ like in $\Gamma(n-1,m)$ (by the inductive hypothesis), which is precisely the conditional strategy of (5.3) given $\bar{c}$ at stage one as remarked for (5.6).

Similarly, if the inspectee uses the described "global" strategy ψ, say, of selecting with equal probability any number r from $\{0,1,\ldots,m\}$ and violating immediately after the inspector has used r inspections, then ψ results in (5.4), as follows. At the first period, the inspectee violates according to (5.4) with probability $q_0 = 1/(m+1)$, which is equal to the probability of $r = 0$ using ψ. Assume that he does not violate at stage one. As long as the inspector does not control, the inspectee will choose legal action l at any information set V_t in Figure 5.4 for $0 < t \le n-m$ both according to (5.4) and to ψ. If the inspector controls at stage t, $0 \le t \le n-m$, then at the first stage of the ensuing subgame $\Gamma(n-t,m-1)$, (5.4) prescribes violation with probability $1/m$, which under ψ is the conditional probability of $r = 1$ given $r = 0$. In the same manner, ψ is seen to agree with (5.4) throughout all stages of the game. There is one case where ψ is more specific than (5.4), namely if the inspectee has waited too long so the inspector can inspect at every remaining period, where the extensive game just specifies payoff zero, shown in the normal form in the lower left corner. In that case, ψ prescribes to let some inspections pass and then to violate, where the immediate detection with payoff zero yields the same result. □

The inspector strategy described in Theorem 5.2 is the same as in Lemma 5.1, but here, for the game $\Gamma(n,m)$ shown in Figures 5.4 and 5.5, it is not necessarily unique, as the game $\Gamma(3,1)$ in Figure 5.3 demonstrates. With the inspector strategy (5.3), the inspectee is indifferent between all his strategies, as shown in the proof of Theorem 5.2 and in the preceding informal remarks. Nevertheless, for small examples of $\Gamma(n,m)$, it is easy to construct inspector strategies where the only optimal answers of the inspectee are the two strategies in the first and last column in Figure 5.5 chosen by him with positive probability.

The optimal strategy (5.4) of the inspectee is unique. It results in the same row value for all inspector strategies, which can by itself be verified at once. At the first stage, the randomized choice of the inspectee between legal and illegal action agrees also with Lemma 5.1. However, after no control and no violation at period one his strategy is not apparent from the ad-hoc description of Figure 5.1, since then he acts *legally* with certainty until the inspector uses his next inspection after which the inspectee randomizes again at the following stage.

An intuitive explanation for this behavior of the inspectee is that at stage one, he should be indifferent between legal and illegal action, but if he did not violate and was not controlled, his situation has worsened since then the inspector can distribute the same number of inspections over fewer periods, so then legal action is strictly better. Of course, this situation will continue to worsen for the inspectee if the inspector continues not to control since in the end the inspector can control at every remaining period. In a sense, a violator needs some "nerves" to execute this strategy. Also, this shows what a difference it would make to the inspector to know what actually happened at a period where he did not control: not only would he inspect at once if he learned about a violation, but if he knew of legal action he would save up his inspections to "corner" the inspectee. However, without this knowledge, postponing control bears the risk of high losses, at the worst the highest payoff $n - m$ to the inspectee shown in the lower right corner in Figure 5.4 or 5.5. This interesting violation strategy of the game $\Gamma(n, m)$ is revealed by its extensive form.

To conclude this paper, a formal equivalence is noted between the game $\Gamma(n, m)$ of this section and $\Gamma(n, m, k)$ as defined and discussed in sections 2 and 3 for the special case $b = 0$ and $k = n - m$. Namely, as pointed out in section 3, the maximal number k of intended violations in $\Gamma(n, m, k)$ is $n - m$ since any larger k makes no difference to $k = n - m$. In that case, the parameter k can be dropped, resulting in the game table of Figure 3.3 (with payoffs to the inspector) where the parameter $-b$ describing the punishment of the inspectee for a caught violation is set to $b = 1$. In general, b can be nonnegative, so in particular $b = 0$ is allowed (which slightly complicates the proof of Theorem 3.2 as compared to $b > 0$ but has been admitted for the present comparison). With $b = 0$, the payoff entry 1 in Figure 3.3 is replaced by 0, resulting in Figure 5.1 if payoffs are considered for the inspectee (with $v(n-1, m-1)+1$ instead of $v(n-1, m-1)-1$). Note that also the initial conditions for Figure 3.3 agree with (5.1). Thus, the explicit solution (3.3) in Theorem 3.2 for $b = 0$ can be used, which is given by

$$\binom{n}{m+1} \Big/ s(n, m) = \binom{n}{m+1} \Big/ \binom{n}{m} = \frac{n-m}{m+1}$$

(note that the sign of the value of the game is reversed, and $m \leq n$ is always assumed). This agrees with Lemma 5.1, which, however, is much easier proved by itself.

A game-theoretic explanation of this formal equivalence would justify Figure 5.1 as a description of the "timeliness" game $\Gamma(n, m)$, but does not seem to exist since the violation strategies are different. The two games have in common that a caught violation is not worse to the inspectee than legal action, and both have a common optimal inspector strategy. The game $\Gamma(n,m,n-m)$ with $b=0$ very nearly results in the game $\Gamma(n,m)$ of this section if the inspectee is *forced* to act illegally at every period following his first violation, since then he is caught at the next inspection and receives the payoff equal to the number of periods passed in between. The only thing missing is the certain additional control at the additional period $n+1$. Note that the unique optimal violation strategy in $\Gamma(n,m,n-m)$ is a gamble with a positive probability q of legal action at each period described in Lemma 5.1, so violations with certainty following the first violation are not optimal. It is interesting to observe that one obtains the same value and optimal inspector strategy for $\Gamma(n,m,n-m)$ with $b=0$ by forcing these follow-up violations and putting an additional period with a (forced) additional inspection at the end of the game.

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