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DECOMPOSITION OF MULTIATTRIBUTE EXPECTED-UTILITY FUNCTIONS

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Abstract

This paper integrates and extends the theory of the decomposition of multiattribute expected-utility functions based on "utility independence". In a preliminary section, the standard decision model of expected utility is briefly discussed, including the fact that the decision maker's preference for *lotteries with two outcomes* determines the utility function uniquely. The decomposition possibilities of a utility function are captured by the concept of *autonomous sets* of attributes, an "affine separability" of some kind known as "generalized utility independence". *Overlapping* autonomous sets lead to biaffine-associative, i.e. *multiplicative* or *additive* decompositions. The multiplicative representation shows that autonomy has stronger *closure properties* than utility independence, for instance with respect to set-theoretic difference. Autonomy is also a concept with a wider scope since it applies to the decomposition of Boolean functions, games and a number of other topics in combinatorial optimization. This relationship to the well-known theory of *substitution decomposition* in discrete mathematics also reveals a kind of "discrete core" behind the decomposition of utility functions. The entirety of autonomous sets can be represented by a compact data structure, the so-called *composition tree*, which frequently corresponds to a natural hierarchy of attributes. Multiplicative/additive or *multi-affine* functions correspond to the hierarchy steps. The known representation of multi-affine functions is shown to be given by a *Moebius inversion formula*. The entire approach has the advantage that it allows the application of more sophisticated representation methods on a detailed level, whereas it employs only *finite set theory* and *arithmetic* on the main levels in the hierarchy.

Keywords: Decomposition, multiattribute, utility function, utility independence

1. Motivation

This paper presents the decomposition of multiattribute expected-utility functions based on the concept of (generalized) utility independence. This approach has been known in the literature for more than a decade (cf. Keeney and Raiffa [12] and references). The presentation given here is comprehensive and largely self-contained, and provides a number of results and simplifications in addition to those already known, as indicated in the abstract. The main methodological improvement is the "*composition tree*" (introduced by Gorman [10] for value functions), a compact data structure describing a *unique* "top-down" iterative

decomposition of the given function (which is more systematic than the "utility independent chains" suggested in [12, sect. 6.9]). The underlying independence condition asserts an "affine separability" of a set of coordinates (i.e. attributes) of the "outcome" space that is the domain of the utility function. This condition is invariant under invertible affine transformations, which in turn can therefore be regarded as a kind of "isomorphisms" [24, p. 468]; intuitively, this means that (conditional) utility functions can always be suitably "scaled", which simplifies a number of representation theorems. The use of the conditions essentially involves operations on finite sets (of coordinates) and these isomorphisms, as indicated in sect. 4. The composition tree can be regarded as a suitable representation for the resulting set-theoretic structure.

Since the decomposition theorems of this approach rely only on conditions involving finite sets and arithmetical expressions, *discrete representations* of parameters (like attribute levels) can be freely used. This is in contrast to theorems about value functions that typically rely on topological conditions (like connectedness of domains, requiring the use of real intervals); an example is the additive decomposition of a value function that holds if the attributes are "mutually preferentially independent" (cf. [12, p. 105] or Gorman [10, p. 382]). The approach is therefore more robust as far as possible misinterpretations of its assumptions are concerned. Also, the involved (positive-) affine transformations are trivial in comparison to the (usually necessary) monotonical transformations for value functions. The conceptually easier theory presented here may therefore even be used for modelling decisions under certainty (with the interpretation of "measurable value functions", cf. Dyer and Sarin [5]). This point is further promoted by Miyamoto [14], who argues that functions that are affine in a suitably chosen real "scale" for each attribute are likely to be found in quite a number of measurement settings (like that of differences as in [5]). The theory is also an alternative to the widely used additive models since the possible introduction of a hierarchy and the use of multiplicative and (small-dimensional) multi-affine functions allows the decision maker to model a greater variety of applied decision problems and/or to attain (with reasonable effort) a higher level of precision in his modelling.

Since the approach is compatible with discrete representations (and also related to decomposition theories for discrete structures, cf. the beginning of sect. 4), it seems particularly suitable for the use of utility functions in automated domains, like *decision support systems*. This holds in particular for the data structure given by the composition tree. This data structure can serve as a model to systematically store and aggregate all the information gathered during an interactive construction of a utility function, as specified in sect. 5. A paper with details on algorithmic aspects of the decomposition of utility functions is in preparation.

2. The decision model of expected utility

In this section, we will briefly recapture the decision model in which a utility function reflects the decision maker's preferences and risk behavior. The decision situation is thereby understood to be given by a set of decision *alternatives* (i.e. possible actions, including that of not doing anything), one of which is to be selected. It is assumed that the choice is determined by the decision maker's *preference* for *outcomes* of the decision. This preference can be regarded as a binary relation given by the (hypothetical) choice between any two actions or outcomes. If the consequences of the decision are *certain*, actions and outcomes can be identified. The concept of a *value function* applies to this situation. This is a real-valued function u defined on outcomes such that $u(x) \geqslant u(y)$ iff outcome x is (not necessarily strictly) preferred to outcome y; consequently, a most preferred action is one with highest value of the value function. Obviously, a value function u can be replaced by a function $G(u(x))$ of x, where G is strictly increasing on the range of u. Apart from such strictly increasing transformations, a value function is unique.

The choice might however be given among actions with *uncertain* consequences. Then a frequently used approach is that of identifying each decision alternative with a *probability distribution* on the set of outcomes; one-point measures thereby represent safe choices. For this model, a *utility* function (also called "von Neumann-Morgenstern utility function") is a real-valued function defined on outcomes that is "weighted" with the occurring probabilities to represent the preference for alternatives. In other words, the *expected value* of the utility function expresses the decision maker's preference for a probability distribution on outcomes. This implies that an (implicit) preference for gambles, i.e. a *risk behavior*, will be specified explicitly via the utility function.

Of course, a utility function can only represent a preference relation satisfying certain constraints, which however do *not* involve any preconceptions about "utility" as a psychological mechanism. Certain weak *consistency* requirements about the decision maker's preference and risk behavior suffice to guarantee the existence of a real-valued function that is a utility function as described above, whose values may permit some *interpretation* as "personal utility". These requirements are known as the so-called "axioms of rationality" by von Neumann and Morgenstern [21], which constitute one of the fundamental contributions to decision theory. A major condition of these is that the preference is a so-called total pre-order, that is, a binary relation that is reflexive, transitive and total; any real-valued (utility) function necessarily induces such a preference relation. Practically, any preference that is not a total pre-order is inconsistent in the sense that in the relevant cases, e.g. for incomparable alternatives (if the preference relation is not total), a choice can not be uttered and is thus arbitrary, which results in factual indifference. Nevertheless, questionaires directed to the decision maker may possibly reveal inconsistencies in his preference structure. It is assumed here

that the decision maker is interested in correcting such inconsistencies, and perhaps clarifying his preferences with the aid of a decision analysis. In practice, decision makers calling for decision support *generally strive for consistency* (cf. [11, p. 157], [12, pp. 7–19], [19, pp. 81ff]). A second important consistency requirement involves the decision maker's preference for *lotteries* (or gambles) of outcomes. If x and y are the two possible outcomes of a lottery, where x will occur with probability p, $0 \leqslant p \leqslant 1$, let this lottery be denoted by xpy. (By finite iteration, all lotteries (with finitely many outcomes) can thus be obtained if y is thereby permitted to be a lottery itself.) The so-called "independence of irrelevant alternatives" then states that if z is any lottery and p a nonzero probability, xpz is preferred to ypz if outcome x is preferred to outcome y. It is reasonable to assume this axiom as well (cf. Raiffa [17, sect. 4.9]). In conjunction with the requirement that the preference relation is a total pre-order, this condition essentially guarantees the existence of a utility function. There are other, not mentioned conditions of rather technical interest (cf. [21, p. 26]), in particular if probability distributions besides *finite-discrete* ones are admitted (e.g. for asserting integrability of the utility function; cf. Fishburn [7, th. 10.3]).

The "axioms of rationality" assume that the decision maker's preference relation is defined for all possible lotteries of outcomes. As concerns testing these axioms, it might be somewhat hard for the decision maker to grasp the meaning of such complex prospects. Nevertheless, test choices between some kinds of gambles are clearly necessary in order to get information on a risk behavior (cf. Wilhelm [24, p. 481]). If the preference is already assumed to be consistent, however, lotteries with only *two* possible outcomes suffice for establishing a utility function uniquely, except a trivial "*scaling*", i.e. a choice of scale and origin, as the following lemma asserts. (Therefore, any behavioral concept given by a property of the utility function, e.g. that of "utility independence" below, can also be expressed in terms of choices among two-outcomes lotteries.) In the literature, the uniqueness of the utility function up to positive-affine transformations is usually asserted under the assumption of (full) convexity of the regarded set of probability distributions (e.g. [21], [6, p. 61], [24, prop. 3]). Debreu [4, p. 17f] shows that a *continuous* utility function on a connected domain is already uniquely determined by the decision maker's preference for *even-chance* lotteries with two sure outcomes. The next lemma, however, employs two-outcome lotteries with *arbitrary* probabilities in order to avoid topological conditions.

(1.) LEMMA

Let the preference relation be defined for all outcomes x, y and gambles xpy with $0 \leqslant p \leqslant 1$, and let it be represented by a utility function u, in the sense that

(*) $\int u \, dP \geqslant \int u \, dQ$ iff P is (not necessarily strictly) preferred to Q

holds for any P, Q taken from a set of probability distributions on outcomes that includes these one- and two-point measures. Then v is another such utility function iff $v = a \cdot u + b$ for some suitable reals a, b, $a > 0$.

Proof

(A similar proof is given in Wilhelm [24, p. 489f]). Clearly, if u has the property (*), so does $a \cdot u + b$ if $a > 0$, since the integral is a linear functional. On the other hand let (*) hold also with v instead of u. This and the assumptions imply for $0 \leqslant p \leqslant 1$ and any outcomes x, y, z:

$u(x) > u(y)$ iff $v(x) > v(y)$,

$u(x) = u(y)$ iff $v(x) = v(y)$,

$p \cdot u(x) + (1-p) \cdot u(y) = u(z)$ iff $p \cdot v(x) + (1-p) \cdot v(y) = v(z)$

(note that $p \cdot u(x) + (1-p) \cdot u(y)$ is the expected value $\int u \, dP$ of u with respect to $P = xpy$).

Let i and o be outcomes such that $u(i) > u(o)$ (if u is constant, so is v; in this case let $a = 1$, $b = v - u$). Then the system of equations

$a \cdot u(i) + b = v(i)$

$a \cdot u(o) + b = v(o)$

has a unique solution in a, b, where $a > 0$ by the above. For another outcome z, for instance, let

$u(i) \geqslant u(z) \geqslant u(o)$

hold. Then $u(z) = p \cdot u(i) + (1-p) \cdot u(o)$ for some p, $0 \leqslant p \leqslant 1$, and therefore (s.a.)

$v(z) = p \cdot v(i) + (1-p) \cdot v(o)$,

thus $v(z) = p \cdot [a \cdot u(i) + b] + (1-p) \cdot [a \cdot u(o) + b] = a \cdot u(z) + b$. Similarly, if $u(z) > u(i)$, let $u(i) = p \cdot u(z) + (1-p) \cdot u(o)$ for some p, $0 \leqslant p \leqslant 1$; since $p \neq 0$ holds because of $u(i) \neq u(o)$, one can conclude $v(z) = a \cdot u(z) + b$ analogously, and in the same fashion for $u(o) > u(z)$. Thus for all z, $v(z) = a \cdot u(z) + b$. (End of proof.)

If the presented model is accepted for a decision analysis, there is the *practical need* to determine and to represent the set of decision alternatives, the set of outcomes, the probability distributions on outcomes associated with the alternatives, and the utility function. We will focus on aspects of constructing the utility function. The outcomes can usually be represented by vectors of *attribute* levels, which for each attribute can, for example, be real values or elements of a given finite set, e.g. to indicate how a particular sub-goal of the decision is met. (The attributes must usually be chosen judiciously to model the decision situation successfully, cf. Keeney and Raiffa [12, pp. 50ff]. In fact, this is about the most essential part in any practical work on decision support. In trying to organize this task, focusing on *value aspects* seems to be advantageous, cf. Keeney [11].) The utility function then takes vectors as arguments. The task of constructing it is substantially alleviated using "decomposition" methods, in particular the one discussed in the following (for others, cf. the overview in Farquhar [6]).

3. The concept of utility independence

Constructing a utility function is a hard task for the decision maker, if the dependency on the different attribute levels has to be observed for all attributes at the same time. This task is conceptually even more difficult if lotteries are involved [12, p. 311]. The *decomposition* approach therefore aims at testing whether a utility function can be expressed in a simple manner as a kind of aggregation in terms of functions of fewer variables. Each of these functions should only involve a fraction of the attributes and thereby express a restricted and conceptually easier dependency of the decision maker's preference on the achievement of attribute levels. The concept of "utility independence" applies to sets of attributes and asks for a unique preference and risk behavior concerning these attributes, independent of the levels of the other ones. In order to define this concept technically, we introduce some conventions, as follows.

Let M be a set of names for the attributes, or more generally just a finite set; an element of M shall be called *coordinate*. S_i shall be a given set representing the possible levels for attribute $i \in M$ (e.g. a real interval for the possible amounts of money if i represents "costs"). For a set A of coordinates, let $S_A = \prod_{i \in A} S_i$ be the space spanned by the "coordinate axes" S_i corresponding to the coordinates i in A. For a vector $x \in S_M$, let x_A be the subvector consisting of the components corresponding to $A \subseteq M$, i.e. the projection of x onto S_A. By x_i we denote, for the sake of simplicity, the projection $x_{\{i\}}$ of x on $S_{\{i\}}$. (Note that $x_\emptyset$ is the empty vector, which can be considered as the identity element of pairing, i.e. $(y, x_\emptyset) = y$.) In summary, let the following assertions generally hold.

(2.) CONVENTIONS

Let M be a non-empty finite set, and S_i be a non-empty set for each $i \in M$. For $A \subseteq M$, let $S_A = \prod_{i \in A} S_i$, and for $x \in S_M$, let $x_A = (x_i)_{i \in A}$, where $x = x_M$.

We assume that a utility function exists, given (according to the preceding conventions) as a function u: $S_M \to \boldsymbol{R}$. "Utility independence" is then defined as follows.

(3.) DEFINITION

Let u: $S_M \to \boldsymbol{R}$ be a utility function. Then A is called *utility independent* (with respect to u), with corresponding *sub-utility function* h, if A is a non-empty set of coordinates (i.e. $\emptyset \neq A \subseteq M$), $B = M - A$, h: $S_A \to \boldsymbol{R}$, and there are some suitable functions a, b: $S_B \to \boldsymbol{R}$, such that the following assertions hold:

(a) $a(y) > 0$ for all $y \in S_B$,

(b) $u(x) = a(x_B) \cdot h(x_A) + b(x_B)$ for all $x \in S_M$.

Utility independence of a set A of attributes can be interpreted as follows (cf. also Keeney and Raiffa [12, sect. 5.2]; for a formal characterization in terms of the preference relation the utility function represents, cf., for instance, [6, p. 66], or [14, p. 166]). Let B be the set of the remaining attributes, i.e. $B = M - A$. A vector x of attribute levels ($x \in S_M$) splits into the two subvectors x_A, x_B; for simplicity of notation, we assume x_A is the "first half" of x, i.e. $x = (x_A, x_B)$. If x is varied in such a way that x_B is always the same subvector $b \in S_B$, the corresponding values $u(x_A, b)$ of the utility function express the decision maker's choice in dependence on the levels of the attributes in A, i.e. of x_A. In view of lemma (1.), the assertions (3.a) and (3.b) then state that this restricted preference and risk behavior (which can be tested with two-outcome lotteries) is the same regardless of the choice b for x_B, and can be described by the sub-utility function h. This sub-utility function is therefore constructed as a by-product if A is successfully tested as utility independent (but not necessarily the functions a, b in (3.b)). Utility independence is frequently observed, or in practice is at least a good approximation of the decision maker's behavior (cf. Keeney and Raiffa [12, pp. 264ff, p. 370]).

The notion of utility independence is in (3.) defined as a property of the utility function. Its practical importance is given by the fact that it expresses a certain behavior of the decision maker which can be tested before the utility function is completely known. The information that a set of attributes is utility independent can then in turn be used to conclude that the utility function has a specific representation, like (3.b), whose constituent parts (in (3.b), the functions a, h, b) can be more easily constructed than the utility function directly. Particularly useful is the information about several utility independent sets of attributes, above all, if these sets overlap. Using these conditions jointly, so-called multiplicative/additive or multi-affine representations of the utility function hold, as discussed in the next sections.

Subsequently, we will mathematically investigate the implications of utility independence with respect to a given utility function u. The "positive-affine separability" of u given by this concept can be weakened to an "*affine separability*" if the requirement (3.a) is dropped, without impairing the possible investigations; in fact, the resulting theory is more coherent, as will be indicated. As concerns the behavioral interpretation, the notion defined by (3.) without condition (3.a) is known as "generalized utility independence" (cf. Fishburn and Keeney [8], [9]). It includes the modified assertion of (3.a) given by "$a(y) \geqslant 0$ for $y \in S_B$", which defines a unique preference and risk behavior for the attributes in A, except a possible complete indifference. It might be practically relevant to consider this weaker condition. Also allowing for $a(y) < 0$ for some $y \in S_B$ means a complete reversal of preferences. However, this case as a third (and only) additional possibility is probably rather rare; nevertheless, it might apply to some situations [9, p. 929].

4. Autonomous sets of coordinates

In this and the next section, we will regard "affine decompositions" of some given function of several variables, such as a utility function with several attributes. The underlying concept of "generalized utility independence" will thereby be called "autonomy" (cf. Möhring and Radermacher [15] for a discussion of analogous concepts in a variety of applications concerning combinatorial optimization, relational systems, set systems and Boolean functions). Whenever appropriate, additional remarks and interpretations concerning the more special notion of utility independence will be made.

An *affine* (sometimes also called linear) function is here understood as a real-valued function G defined on a subset T of $\boldsymbol{R}$ such that $G(t) = a \cdot t + b$ holds for all $t \in T$, for suitable reals a, b. If G is affine, it can be directly extended to an affine function $\boldsymbol{R} \to \boldsymbol{R}$; this extension is unique if the original domain T contains at least two elements. Obviously, if G is invertible, i.e. if $a \neq 0$, then G^{-1} is affine, and so is the functional composition of two affine functions. A function of several variables is called affine *in a variable* if that variable takes real values and the function is affine whenever the other variables are fixed. It is called *multi-affine* (or *n-affine*) if it is affine in each of its (n) variables. A variable of a function of one or more variables is called *essential* if the function actually depends on that variable. Without explicit notice, the variables of a function are always assumed to be essential. This can be done without loss of generality, since for a given function, a variable that is not essential can be dropped, and in the cases below where a function of several variables is defined in terms of others, all its variables must be essential, too.

(4.) DEFINITION

Given (2.), let f: $S_M \to \boldsymbol{R}$. Then A is called *autonomous* (with respect to f) with corresponding *divisor* h (of f), if $\emptyset \neq A \subseteq M$, $B = M - A$, h: $S_A \to \boldsymbol{R}$, and there is some function: g: $\boldsymbol{R} \times S_B \to \boldsymbol{R}$ that is affine in its first variable, such that

$$f(x) = g(h(x_A),\ x_B)$$

holds for all $x \in S_M$.

M is always an autonomous set of coordinates with respect to f: $S_M \to \boldsymbol{R}$, with f itself as a corresponding divisor. However, there are generally no other trivial autonomous subsets of M, with the following exception: if a "coordinate axis" S_i has only two elements for some $i \in M$, the singleton $\{i\}$ is autonomous (S_i must contain at least two elements, since otherwise x_i would be an inessential variable of f). This holds because in that case, the two elements of S_i can be mapped by a suitable function h to two reals, e.g. 0 and 1, which in turn can be mapped to any given real values by an affine (but not necessarily strictly increasing) function. By this observation, parts of well-known decomposition theories for *Boolean* func-

tions (cf. e.g. Curtis [3], or [15]), or for more general functions of two-valued variables (like characteristic functions of *cooperative n-person games*, as in Shapley [20] and Megiddo [13]), turn out to be special cases of the "affine" decomposition theory based on definition (4.) presented here. These applications (for details cf. von Stengel [22], [23]) indicate that it can be useful in certain respects to study the concept of "autonomy" rather that of "utility independence", since it has a wider scope.

If $e, f: S_A \to \boldsymbol{R}$, for some $A \subseteq M$, let e and f be called *isomorphic* if $e(y) = G(f(y))$ for $y \in S_A$, for some invertible affine function $G: \boldsymbol{R} \to \boldsymbol{R}$. According to the introductory remarks on affine functions, "is isomorphic to" is an equivalence relation. The following lemma states that this equivalence "preserves" in some sense the decompositions regarded here; it simplifies a number of proofs (in particular of lemmas 2, 3 in Miyamoto [14] about proper choices of "sequences of scales").

(5.) LEMMA

Given (2.), let A be autonomous with respect to $f: S_M \to \boldsymbol{R}$ with a corresponding divisor h, and $B = M - A$. Then

(a) H is a divisor of f corresponding to A iff h and H are isomorphic.
(b) If F and f are isomorphic, then A is autonomous with respect to f, and h is a divisor of F.
(c) If $f(x) = g(h(x_A), x_B)$, and g is affine in its first variable, then g is unique.

Proof

If h and H are isomorphic, H is obviously a divisor of f. The converse follows from the observation that f itself, with the variables corresponding to B *fixed* at suitable values (such that the resulting function is not constant), is isomorphic to any divisor that corresponds to A. Assertion (b) is obvious, and (c) holds since an affine function is determined if only two different values for its argument are given. (End of proof.)

The preceding lemma shows that the *system of autonomous subsets* of M with respect to a given function $f: S_M \to \boldsymbol{R}$ *characterizes the decomposition possibilities* of f, or of any function isomorphic to f; the possible corresponding divisors or functional representations as in (5.c) are then determined. We will demonstrate in sect. 5 that this system can be represented very succintly by a so-called "composition tree", using the fact that specific relationships hold between different autonomous sets. In doing so, we try to embed the present problem into the general framework discussed in [15]. Seen from this general point of view, certain properties of autonomy are needed in the present context. One such property is a kind of *transitivity* of autonomy, which we will indeed obtain in the following lemma that treats the case of two comparable autonomous sets.

(6.) LEMMA

Let $\emptyset \neq B \subseteq A \subseteq M$, and A be autonomous with respect to f: $S_M \to \boldsymbol{R}$ with corresponding divisor h. Then B is autonomous with respect to f iff B is autonomous with respect to h.

Proof

The consideration is similar to that for (5.a) above. Note that by (5.b), the autonomy of B with respect to h does not depend on the choice of h. (End of proof.)

Lemmas (5.) and (6.) hold correspondingly–by the same arguments–for "utility independent" instead of "autonomous", if "isomorphic" is re-defined as "equal up to a (not only invertible, but) strictly increasing affine transformation", and if "divisor" is replaced by "sub-utility function". The so modified assertion (5.b) states the desirable property that utility independence does not depend on the particular choice of the utility function, which is unique only up to positive-affine transformations. A sub-utility function can also be arbitrarily scaled by (5.a). The statement corresponding to (6.) asserts that utility independence can be regarded within a natural hierarchy of attributes. In an application of that statement, A may be a large set of attributes (e.g. comprising the "financial aspects" of the decision) that defines a main objective of the decision and is utility independent, and a subset of it (e.g. "maintenance costs") may correspond to a particular sub-objective that is "locally independent" and therefore utility independent, too ("via transitivity"; cf. also [12, pp. 340ff]).

With respect to some given function f: $S_M \to \boldsymbol{R}$, it suffices by (6.) to examine autonomous proper subsets of M that are *maximal*, since one can then recursively look at the corresponding divisors and their decompositions. Two maximal autonomous sets can either be disjoint or overlap. The latter is the more interesting case. It applies for instance to the following theorem, which states that the system of autonomous sets is closed under non-disjoint unions and intersections.

(7.) THEOREM

Let A and B be non-disjoint autonomous sets with respect to f: $S_M \to \boldsymbol{R}$. Then $A \cup B$ and $A \cap B$ are autonomous.

Proof

Cf. Fishburn and Keeney [9, p. 931, La. 1] or von Stengel [22, p. 36].

In view of the observation in the proof of (5.a), the preceding assertion for the intersection is very obvious, but not so for the union. If in (7.), $A \cup B = M$, the result is uninteresting, too. But (6.) and (7.) allow us to confine ourselves to just this case for further investigations on overlapping autonomous sets A, B, since

otherwise $A \cup B$ is an autonomous proper subset of M and then A and B are not maximal. In what follows, a function $\cdot\colon \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$, written in infix notation, is *associative* iff $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ for all real u, v, w; the parentheses can then consequently be omitted.

(8.) THEOREM

Let M be the disjoint union of the non-empty sets A, B, C, and let $A \cup B$ and $B \cup C$ be autonomous with respect to $f\colon S_M \to \boldsymbol{R}$. Then

$f(x) = a(x_A) \cdot b(x_B) \cdot c(x_C)$

for suitable functions a, b, c, and a biaffine (i.e. 2-affine) associative function $\cdot$.

Proof

In view of (9.), the assertion is a special case of (10.) below.

The equation in (8.) implies that A and C are autonomous with corresponding divisors a and c, respectively. With (6.) and (7.), this shows that *union, intersection* and *differences* of *overlapping autonomous sets* are *autonomous*. In fact, this also holds for the *symmetrical difference*, since the following theorem implies that all biaffine associative functions are symmetric. This theorem characterizes the possible biaffine associative functions in (8.). Interestingly, it can also be used to characterize the associative *Boolean* functions known for switching circuit decomposition, given (up to complementation) by addition and multiplication modulo 2; cf. von Stengel [23, p. 526].

(9.) THEOREM

$\cdot$ is a biaffine associative function iff for all real x, y either

(a) $G(x \cdot y) = G(x) \cdot G(y)$,
where G is a unique invertible affine function, or

(b) $G(x \cdot y) = G(x) + G(y)$,
where G is affine and invertible, and unique up to a nonzero multiplicative constant. [Remark: in (b), it can be assumed w.l.o.g. that $G(t) = t + r$, for some unique real r; then $x \cdot y = x + y + r$.]

Proof

A function $\cdot$ defined by $x \cdot y = G^{-1}(G(x) \cdot G(y))$, or with $+$ instead of $\cdot$, is associative and biaffine because $\cdot$ and $+$ are, and because G is affine. The converse follows in a straightforward way from the associativity equation for a biaffine function $\cdot$ (cf. Aczél [1, p. 59f]), using the fact that such a function is of the form $x \cdot y = pxy + qx + ry + s$ for some real numbers p, q, r, s. The latter is actually asserted by theorem (14.) below. (End of proof.)

Is it interesting to consider theorems 7 and 8 with respect to the concept of utility independence instead of autonomy. Theorem 7 then also holds and is

proved analogously (cf. Keeney and Raiffa [12, p. 317]). The replacement of "autonomous" by "utility independent" trivially leaves theorem 8 valid, since it only strengthens the assumptions. However, the conclusion has different implications if the involved biaffine associative function is "multiplicatively represented" as in (9.a). Given (9.a), application of G on both sides of the equation in (8.) yields

$$G(f(x)) = G\big((a(x_A) \cdot b(x_B)) \cdot c(x_C)\big) = G(a(x_A) \cdot b(x_B)) \cdot G(c(x_C))$$
$$(*) = G(a(x_A)) \cdot G(b(x_B)) \cdot G(c(x_C)).$$

It is possible that the first and the third factor on the right never change sign, e.g. are always positive. In that case, $B \cup C$ and $A \cup B$ are utility independent (with respect to $G(f(x))$, and also with respect to f; it is thereby not relevant whether G is a positive- or negative-affine function, nor which sign the factors have, since one can always choose the sign of the sub-utility function appropriately). Nevertheless, it is not possible to conclude that A (or C) is utility independent, since in $(*)$, the second factor may change sign. This holds, for instance, for the function $f\colon [0, 1]^3 \to \boldsymbol{R}$ defined by $f(x_1, x_2, x_3) = (x_1 + 1) \cdot (x_2 - 0.5) \cdot (x_3 + 1)$. Here, the sets $\{2,3\}$ and $\{1,2\}$ are utility independent subsets of M $(= \{1, 2, 3\})$, but their differences $\{3\}$ and $\{1\}$ are not, nor is in fact their symmetrical difference $\{1, 3\}$. The collection of *utility independent* sets of coordinates with respect to a given function is therefore *only* closed with respect to *union* and *intersection* of overlapping sets. In comparison with the remark following (8.), this shows that "autonomy" has nicer mathematical properties than "utility independence", as concerns the deduction of new decomposition possibilities from known ones. Furthermore, invertible affine transformations, which may be negative, are the natural isomorphisms with respect to the concept of autonomy (cf. (5.) above), whereas utility independence is only invariant under the positive-affine transformations admitted for utility functions. But for multiplicative representations of utility functions f, as $(*)$ above, negative transformations G may be necessary. In fact, given certain scaling conventions for sub-utility functions (cf. (11.) below), a negative transformation can be interpreted as asserting a frequently observed "*substitutivity*" of the attributes, whereas a positive transformation describes a "*complementarity*" (cf. Keeney and Raiffa [12, p. 240f]).

The following theorem is an extension of (8.), with the generalized assumption of several overlapping autonomous sets of coordinates. A *partition* of a set S is thereby understood as a set of non-empty, pairwise disjoint sets whose union is S.

10.) THEOREM

Let $\{A_j \mid 1 \leqslant j \leqslant k\}$ be a partition of M, $k \geqslant 3$, and $M - A_j$ be autonomous with respect to $f\colon S_M \to \boldsymbol{R}$ for $1 \leqslant j < k$ (note these are $k - 1$ conditions). Then

a) there exist: an invertible affine function G, and suitable functions a_j, $1 \leqslant j \leqslant k$, such that $G(f(x)) = a_1(x_{A_1}) \bullet \cdots \bullet a_k(x_{A_k})$,
where $\bullet$ is either multiplication or addition.

(b) Given (a), $\bigcup_{j \in L} A_j$ is autonomous with respect to f, for any non-empty subset L of $\{1, \ldots, k\}$.

Proof

For (a), cf. Fishburn and Keeney [9, La. 2]; (b) holds by (5.b) and because both multiplication and addition are biaffine, symmetric and associative. (End of proof.)

The preceding theorem asserts that a given function can be multiplicatively or additively represented if certain strong assumptions about autonomous sets of coordinates hold. However, it would be desirable to have additionally an explicit criterion to decide which of the representations hold, and to determine the necessary affine transformation (i.e. G in (10.a)) of the given function. Furthermore, under the assumptions of (10.), A_j is autonomous for $1 \leqslant j \leqslant k$ according to (10.b), with corresponding divisor a_j as in (10.a). In practice, there is usually some *given* divisor h_j that corresponds to A_j, for $1 \leqslant j \leqslant k$, for instance in form of a sub-utility function. Then a_j is isomorphic to h_j by (5.a), and the affine transformations to obtain a_j from h_j also need be determined (for $1 \leqslant j \leqslant k$). This information can indeed be obtained by evaluating the given function for a number of suitably chosen arguments, essentially to obtain "equations" out of which the "unknowns" (i.e. the pairs of reals describing the affine transformations in question) can be extricated. Under additional assumptions of utility independence, $O(k)$ many (e.g. $k+2$) arguments suffice, as the following theorem demonstrates.

(11.) THEOREM

Let (2.) hold, $M = \{1, \ldots, k\}$, $f\colon S_M \to \boldsymbol{R}$ and L be autonomous with respect to f for all $L \subseteq M$, $L \neq \emptyset$. Let $\{j\}$ be utility independent with respect to f for $1 \leqslant j \leqslant k$, with corresponding sub-utility function $h_j\colon S_j \to \boldsymbol{R}$, and for suitable o_j, $i_j \in S_j$:

(a) $h_j(o_j) = 0$, $h_j(i_j) = 1$ for $j \in M$, and furthermore $f(o_1, \ldots, o_k) = 0$, $f(i_1, \ldots, i_k) = 1$.

Then f is uniquely determined by the positive real numbers c_j given by

(b) $c_j = f(o_1, \ldots, o_{j-1}, i_j, o_{j+1}, \ldots, o_k)$ for $j \in M$.

If $\sum_{j \in M} c_j = 1$, then

(c) $f(x) = \sum_{j \in M} c_j \cdot h_j(x_j)$ for $x \in S_M$,

otherwise

(d) $G(f(x)) = \prod_{j \in M} G(c_j \cdot h_j(x_j))$ for $x \in S_M$,

where $G(t) = d \cdot t + 1$ for $t \in \boldsymbol{R}$; d is thereby the only root greater than -1 besides 0 of the polynomial $p\colon \boldsymbol{R} \to \boldsymbol{R}$ given by $p(t) = \prod_{j \in M}(c_j \cdot t + 1) - t - 1$.

Proof

Cf. Keeney and Raiffa [12, p. 238] (for $k = 2$), [12, p. 289, p. 347f] (for $k \geqslant 3$).

The preceding theorem asserts the well-known "multiplicative/additive representation" of a utility function under the assumption of so-called "mutual utility independence" of the given attributes [12, p. 289]. (As a mere notational simplification, the underlying partition of the set M of coordinates in (10.) is in (11.) assumed to be given by the set of singletons; this can always be achieved by re-defining the underlying "coordinate axes" S_i in (2.) as the subspaces S_A for the elements A of the original partition, or, in terms of utility theory, by regarding "vector attributes" as new single attributes.) In (11.a), it is assumed that each sub-utility function h_j, for the attribute $j \in M$, is scaled by suitable "reference" attribute levels o_j, i_j. Since $\{j\}$ is utility independent, o_j can be interpreted as a less (possibly least) preferred level of attribute $j \in M$ compared to i_j. Correspondingly, both the individual sub-utility functions and the entire utility function f can be assumed to be scaled by 0 and 1, given these less- and more-preferred attribute levels as respective arguments. (11.b) says that each "scaling constant" c_j, $1 \leqslant j \leqslant k$, is obtained by arising the "reference level" of the corresponding attribute from low (o_j) to high (i_j) and evaluating the utility function. Of course, these constants make sense only with respect to the given scale (i.e., the reference levels), and can therefore *not* be interpreted as indicators of *absolute importance* of the attributes (cf. Keeney and Raiffa [12, sect. 5.9]). For the multiplicative representation (11.d), the sign of the affine transformation G indicates, as mentioned, whether the attributes interact as substitutes ($d < 0$) or complements ($d > 0$). The root d of the given polynomial that determines this transformation can be found using a suitable *numerical procedure*, e.g. Newton's method.

A representation of the utility function as a product or sum is in practice very desirable since the amount of information that is to be obtained from the decision maker is then only *proportional* to the number of attributes, viz. one "one-dimensional" sub-utility function and one "scaling constant" in the form of an absolute utility value per attribute. Furthermore, the required independence conditions can then also be tested with proportional effort, e.g. by establishing (with the notation used in (11.)) $\{i, i+1\}$ as autonomous for $1 \leqslant i < k$ and using the properties of overlapping autonomous sets as remarked after (8.). (Using the notion of preferential independence and further topological conditions (connectedness of the domain and continuity), even further simplifications are possible, which may in practice be quite relevant; cf. Keeney and Raiffa [12, p. 311], or Fishburn and Keeney [8, th. 2].). The additive representation allows additional independence tests, since it is characterized by a certain "lack of interaction" between the attributes (cf. Fishburn [7, ch. 11], [12, sect. 6.5], [6, p. 64f]). In the next section, it will be shown that the multiplicative/additive representation applies to one of two basic types of "quotient" that occurs as a "building block" in a unique hierarchical decomposition of a given function.

5. The unique hierarchical decomposition

The system of autonomous sets of coordinates with respect to a given function f is ordered by inclusion. The following theorem, if applied iteratively in a "top-down" manner, gives a complete description of this system. The theorem actually ties in with the general theory of substitution decomposition [15], since only the set-theoretic closure properties of overlapping autonomous sets stated above are used, as well as the fact that smaller autonomous sets can be regarded with respect to suitable divisors (lemma (6.)); these properties are also characteristic for the general setting [15, pp. 313–315].

(12.) THEOREM

Given (2.), let f: $S_M \to \boldsymbol{R}$, and let A be the set of those coordinates that are contained in at least one proper subset of M that is autonomous with respect to f. Then there exists a unique partition $P(f)$ of A, $P(f) = \{A_j | 1 \leqslant j \leqslant k\}$, such that either

(a) $k \geqslant 0$ ($k \geqslant 2$ if $A = M$), and A_j is autonomous with respect to f for $1 \leqslant j \leqslant k$, or

(b) $k \geqslant 3$, $A = M$, and $\bigcup_{j \in L} A_j$ is autonomous with respect to f, for any non-empty subset L of $\{1, \ldots, k\}$,

and in either case

(c) any other set that is autonomous with respect to f, except M, is a proper subset of an element of $P(f)$.

Proof

Cf. Gorman [10, p. 375, th. 2].

With respect to a given function f: $S_M \to \boldsymbol{R}$, the autonomous proper subsets of M can be *stepwise described* using the preceding theorem (12.). The union of these sets is in (12.) called A. If there are no such sets, i.e. if $A = \emptyset$, the given function is not "affine decomposable"; this is the termination condition for the iterative application of (12.). Otherwise, $P(f)$ as described in (12.) is a partition of A into autonomous sets. It suffices to consider the corresponding divisors in order to obtain the autonomous sets with respect to f other than those which are the elements of $P(f)$ or their unions (in case (12.b) holds), because of (12.c) and (6.). These divisors are "proper" in that they depend on fewer variables than f, which is asserted by the conditions on k in (12.a) and (12.b). f can be functionally expressed in terms of these divisors and an additional function (which may be called "quotient" [15, p. 269]) that corresponds to $P(f)$, as asserted by the next theorem. In case of (12.b), this function is either a product or a sum as in (10.a). (12.a) represents the (so far not considered) case that the maximal autonomous sets with respect to f are disjoint, and that they possibly do not include some coordinates, viz. those of $M - A$. Except with respect to an only

trivially expressible dependency on these coordinates, the mentioned "top quotient" of f is a multi-affine function; this single functional representation captures the "affine separabilities" of f given by several disjoint autonomous sets.

(13.) THEOREM

Given (2.), let f: $S_M \to \boldsymbol{R}$, and let A and $P(f)$ be given as in (12.), $P(f) = \{A_j | 1 \leqslant j \leqslant k\}$, and $B = M - A$. (B shall be called the *free* set [10, p. 313].) For the cases (a) and (b) as in (12.), the following assertions respectively hold:

(a) If h_j is any divisor corresponding to A_j, for $1 \leqslant j \leqslant k$, then

$$f(x) = g\big(h_1(x_{A_1}), \ldots, h_k(x_{A_k}),\ x_B\big)$$

for a suitable function g: $\boldsymbol{R}^k \times S_B \to \boldsymbol{R}$, which is affine in each of its first k variables.

(b) There are divisors h_j corresponding to A_j, for $1 \leqslant j \leqslant k$, and an invertible affine function G, such that

$$G(f(x)) = h_1(x_{A_1}) \cdot \cdots \cdot h_k(x_{A_k}),$$

where $\cdot$ is either multiplication or addition.

Proof

Because of (10.), only (a) needs to be shown. Let h_j be a divisor corresponding to A_j, $1 \leqslant j \leqslant k$. For $0 \leqslant i \leqslant k$, we define a function

g_i: $\boldsymbol{R}^i \times S_{B_i} \to \boldsymbol{R}$, where $B_i = M - \bigcup_{j=1}^{i} A_j$, by

(*) $g_i\big(h_1(x_{A_1}), \ldots, h_i(x_{A_i}),\ x_{B_i}\big) = f(x)$, for $x \in S_M$.

We show by induction on i that (*) is an *admissible definition* of g_i, that is, independent of the particular choices y or z from S_{A_j}, if $h_j(y) = h_j(z)$, for $1 \leqslant j \leqslant i$, and that g_i is *affine* in its j-th variable for $1 \leqslant j \leqslant i$. (For real values of its j-th argument not in the range of h_j, $1 \leqslant j \leqslant i$, the functional value of g_i is not of interest.) The claim is trivially true for $i = 0$. Assume, as induction hypothesis, that for some i, $1 \leqslant i \leqslant k$,

$$g_{i-1}\big(h_1(x_{A_1}), \ldots, h_i(x_{A_{i-1}}),\ x_{A_i \cup B_i}\big) = f(x)$$

holds for $x \in S_M$, where g_{i-1} is affine in each of its first $i-1$ variables. Furthermore, by assumption (autonomy of A_i),

$$g'\big(h_i(x_{A_i}),\ x_{M-A_i}\big) = f(x)$$

holds for some g': $R \times S_{M-A_i} \to \boldsymbol{R}$ that is affine in its first variable. Regarding (*) as a definition, the preceding two equations show that it is admissible, and that if the arguments other than $h_j(x_{A_j})$ are fixed, the resulting function is affine, for $1 \leqslant j \leqslant i$. This completes the induction step. Finally, we let $g = g_k$, with $B = B_k$, which then fulfills the assertion of (13.a) as it was to be shown. (End of proof.)

As shown, the decomposition of a given function f, based on the concept of autonomy, can be hierarchically structured. This structure can be depicted by a tree, which is the same as the so-called "utility tree" introduced by Gorman [10] for value functions. It will be called here the *composition tree* for f (after [15, p. 328]); it also applies to a unique hierarchical decomposition known for certain discrete structures, like Boolean functions, graphs and more general relational systems and set systems (in a slightly specialized form, since "free sets" are absent because of the remarks following (4.); cf. Möhring and Radermacher [15, pp. 324ff]). The composition tree is a data structure that, properly labeled, carries all the information about the decomposition possibilities of f. It can be described as follows. Each node of the tree is a non-empty set of coordinates (with an optional label); the successors of an inner (non-leaf) node form a partition of their predecessor. The root of the tree is the set M of all coordinates, which define the space S_M that is the domain of the underlying function f. The node is labeled by "(a)" or "(b)", say, according to whether case (a) or (b) applies in theorem (12.). Each element of $P(f)$ becomes a successor node of the root and serves as the root of a subtree that is the composition tree for the corresponding divisor; these nodes are autonomous sets. For the case (12.a), it is possible that the free set B as defined in (12.) is not empty. It then becomes an additional successor node (possibly the only one), and a leaf of the tree. Any leaf is such a free set, since in an iterative application of (12.), eventually $A = \emptyset$ must hold, e.g. if M is a singleton. Any set that is autonomous with respect to f can be obtained from the composition tree for f, either as an inner node of the tree, or as the union of some successors of a node labeled "(b)". Note that thereby the system of autonomous sets, which is possibly exponential in size (as compared to the number of coordinates), can be described by a data structure of *linear size* [15, p. 330]. A forthcoming paper will elaborate more precisely on the computational complexity of algorithms based on the presented decomposition approach.

The composition tree for f: $S_M \to \boldsymbol{R}$ describes (as presented so far) the autonomous sets. It can, in a canonical fashion, carry further information about the functional representation of the divisors of f corresponding to the inner nodes, based on theorem (13.); this information is associated with the "quotient". For a node of type "(b)", it can be given by a pair of real numbers to define the affine function G (or G^{-1}) in (13.b), and an information "bit" to indicate whether the multiplicative or the additive representation applies. For "(a)" type nodes, the quotient g as in (13.a) can be represented by 2^k real-valued functions defined on the space S_B, where B is the free set (if B is empty, these functions are constants), according to the next theorem. This theorem is given in [9, p. 938], but is here proved more formally. It is thereby shown that the special form of the involved parameters is naturally proved using the so-called "Moebius inversion formula"; furthermore, the uniqueness of these parameters is provided, which is missing in [12, p. 293], as Miyamoto [14, p. 172] remarks; finally, a properly defined induction on the number of arguments allowed to take arbitrary real values

completes the reasoning in [9, p. 938], and is simpler than the proof by Miyamoto [14, pp. 158ff] of this result.

Essentially, the following theorem asserts that a k-affine function is a polynomial in its variables (that is, a sum of products of non-negative powers of these variables), where each variable appears in at most its first power (in [14] called "first order polynomial"). The coefficients of this polynomial are unique, and can be computed from the function evaluated for arguments that are either 0 or 1. The employed notation is to be understood as follows: for $x \in \boldsymbol{R}^k$, $1 \leqslant i \leqslant k$, x_i is the i-th component of x, i.e. $x = (x_1, \ldots, x_k)$. The special vectors of $\boldsymbol{R}^k$ only consisting of 0's and 1's (the corners of the k-dimensional unit cube) are called 1_A, where A is the set of coordinates that have the value 1; that is, $(1_A)_i = 1$ if $i \in A$ and $(1_A)_i = 0$ if $i \in K - A$, for $A \subseteq K = \{1, \ldots, k\}$.

(14.) THEOREM

Let $k \geqslant 0$, $K = \{1, \ldots, k\}$, S be a non-empty set, g: $\boldsymbol{R}^k \times S \to \boldsymbol{R}$, and let g be affine in each of its first k variables. Then

(a) $g(x, s) = \sum_{A \subseteq K} c_A(s) \cdot \prod_{i \in A} x_i$ for $x \in \boldsymbol{R}^k$, $s \in S$,

where, for each $A \subseteq K$, c_A: $S \to \boldsymbol{R}$ is a unique function given by

(b) $c_A(s) = \sum_{B \subseteq A} (-1)^{|A-B|} \cdot g(1_B, s)$, for $s \in S$.

Proof

Let $s \in S$ be given. With f: $\boldsymbol{R}^k \to \boldsymbol{R}$ defined by $f(x) = g(x, s)$ for $x \in \boldsymbol{R}^k$ and $d_A = c_A(s)$ for $A \subseteq K$, the equations (a) and (b) are then equivalent to

(a′) $f(x) = \sum_{A \subseteq K} d_A \cdot \prod_{i \in A} x_i$

(for $x \in \boldsymbol{R}^k$), and

(b′) $d_A = \sum_{B \subseteq A} (-1)^{|A-B|} \cdot f(1_B)$,

where f is k-affine by assumption.

The proof will now be given in three parts. First, it will be shown that with (b′), (a′) is true if $x = 1_C$, for any $C \subseteq K$. Second, that there is at most one choice for each d_A, $A \subseteq K$, such that (a′) holds. Third, that (a′) holds for any $x \in \boldsymbol{R}^k$ given f is k-affine. Let (b′) hold. Then for $C \subseteq K$,

$$\sum_{A \subseteq K} d_A \prod_{i \in A} (1_C)_i = \sum_{A \subseteq C} d_A = \sum_{A \subseteq C} \sum_{B \subseteq A} (-1)^{|A-B|} \cdot f(1_B)$$

$$= \sum_{B \subseteq C} \sum_{A : B \subseteq A \subseteq C} (-1)^{|A-B|} \cdot f(1_B)$$

$$= \sum_{B \subseteq C} f(1_B) \cdot \sum_{T \subseteq C-B} (-1)^{|T|} = f(1_C),$$

since the second sum in the last line is 1 for $C - B = \emptyset$, otherwise $(1-1)^{|C-B|}$, i.e. 0 for $B \neq C$, by the binomial theorem. (Remark: This is a special case of the usual proof of a so-called *Moebius inversion formula*, cf. Rota [18].)

To prove the second part, the uniqueness of the d_A's, read (a') as proved as a system of 2^k equations for 2^k unknowns d_C, $C \subseteq K$ (after Owen [16, p. P79]):

$$f(1_C) = \sum_{A \subseteq C} d_A \quad (C \subseteq K).$$

It is sufficient to show that the corresponding set of *homogeneous* equations

$$0 = \sum_{A \subseteq C} d_A \quad (C \subseteq K)$$

has only the trivial solution. Indeed, the assumption $d_C \neq 0$ for some minimally chosen C would yield the contradiction $0 = \sum_{A \subseteq C} d_A = d_C \neq 0$.

For the third part, let (b') hold, f be k-affine, and n be a natural number. $< n$ shall denote the set of all the members of K less than n, similarly $\geqslant n = \{n, \ldots, k\}$, etc. Thus, $\geqslant 1 = K$ and $< 1 = \emptyset$. Analogously to (2.), $x_{>n}$ shall denote the vector x projected on its last $k-n$ coordinates, e.g. for $x = 1_C$, $C \subseteq K$. We prove for $1 \leqslant n \leqslant k+1$ by induction on n:

$$(*)\quad f(y, (1_C)_{\geqslant n}) = \sum_{A \subseteq < n} \left(\prod_{i \in A} y_i \right) \cdot \sum_{B \subseteq C} d_{A \cup B} \quad \text{for all } y \in \boldsymbol{R}^{n-1},\ C \subseteq \geqslant n.$$

For $n = 1$, this equation has been proved in the first part above. If $n = k+1$, (*) represents (a') as to be proved for all $x \in \boldsymbol{R}^k$. Assume (*) holds for some n, $1 \leqslant n \leqslant k$. Proving it for $n+1$ amounts to showing

$$f(y, z, (1_C)_{>n}) = \sum_{A \subseteq < n} \left(\prod_{i \in A} y_i \right) \cdot \sum_{B \subseteq C} d_{A \cup B} + \sum_{A \subseteq < n} \left(\prod_{i \in A} y_i \right) \cdot z \cdot \sum_{B \subseteq C} d_{A \cup \{n\} \cup B}$$

for all $y \in \boldsymbol{R}^{n-1}$, $z \in \boldsymbol{R}$, $C \subseteq > n$.

A similar equation is known from the fact that f is affine in its n-th variable z, viz.

$$f(y, z, w) = a(y, w) \cdot z + b(y, w) \quad \text{for all } y \in \boldsymbol{R}^{n-1},\ z \in \boldsymbol{R},\ w \in \boldsymbol{R}^{k-n}$$

for some suitable functions a, b: $\boldsymbol{R}^{k-1} \to \boldsymbol{R}$. Letting $z = 0$ and $z = 1$ in this equation, one can conclude

$$b(y, w) = f(y, 0, w),$$

$$a(y, w) = f(y, 1, w) - f(y, 0, w).$$

But for $w = (1_C)_{>n}$, for any $C \subseteq > n$, $f(y, 0, w)$ and $f(y, 1, w)$ are known from the induction hypothesis. It is easy to verify that in this case (*) yields indeed

$$b(y, w) = \sum_{A \subseteq < n} \left(\prod_{i \in A} y_i \right) \cdot \sum_{B \subseteq C} d_{A \cup B},$$

$$a(y, w) = \sum_{A \subseteq < n} \left(\prod_{i \in A} y_i \right) \cdot \sum_{B \subseteq C} d_{A \cup \{n\} \cup B},$$

which remained to be shown. (End of proof.)

Theorem (13.) shows that a given function f, say a utility function, can be iteratively constructed using two types of "quotient" as in (13.a) and (13.b). These assertions have their explicit counterparts in theorems (14.) and (11.) (the latter with additional assumptions). This analogy is particularly strong if the free set B in (13.a) is empty (i.e. $S = S_\emptyset$ in (14.); subsequently, a variable ranging in $S_\emptyset$ is inessential and will thus be omitted as an argument). In that case, $f(x) = g(h_1(x_1), \ldots, h_k(x_k))$ holds (with the simplifying assumption as in (11.) that K is the set of attributes, i.e. $M = K = \{1, \ldots, k\}$). If the divisors h_j are scaled by suitable "reference" attribute levels $o_j, i_j \in S_j$ by $h_j(o_j) = 0$ and $h_j(i_j) = 1$ for $1 \leqslant j \leqslant k$ as in (11.a), there are 2^k "reference vectors" $I_A \in S_M$ that can be formed from these, where $(I_A)_j = i_j$ for $j \in A$ and $(I_A)_j = o_j$ for $j \in K - A$, for all $A \subseteq K (= M)$. Since $f(I_A) = g(1_A)$ holds (for $A \subseteq K$), the utility values corresponding to these reference vectors determine the "scaling parameters" c_A as in (14.b). For instance, for $j \in K$, $c_{\{j\}} = g(1_{\{j\}}) - g(1_\emptyset) = f(I_{\{j\}}) - f(I_\emptyset)$. With the scaling convention $f(I_\emptyset)$ $(= f(o_1, \ldots, o_k)) = 0$ for f, this is actually equation (11.b). In fact, (11.c) and (11.d) can be transformed into a single closed form (by distributing the factors in (11.d)) that is a special case of (14.a) substituted into (13.a) (with $S = S_\emptyset$), sometimes also called the "quasi-additive representation" (cf. Keeney and Raiffa [12, p. 289]). The uniqueness assertion in (14.) shows that this form is unique, which could be added as a further claim in (11.). The utility values $f(I_A)$ for $A \subseteq K$ can in practice be determined using suitable lotteries (cf. Keeney and Raiffa [12, p. 303]).

If an inner node of the composition tree for f is labeled "(a)", and has k successors, but no free set (leaf) among them, 2^k reals c_A for $A \subseteq K = \{1, \ldots, k\}$ can be permanently associated with this node to represent the corresponding divisor (in terms of the divisors that correspond to the successors) according to (14.a). If it has an additional leaf B as a successor, real-valued functions defined on S_B take the place of these reals. In some sense, the presented decomposition approach ends and gives no further information as far as the representation of these functions is concerned. This representation depends on the domain of the functions, that is, on the "coordinate axes" S_i, $i \in B$, about which no assumption is made in definition (4.). The coordinate axes may, for example, be finite sets or real intervals to represent possible attribute levels, and corresponding function

values may be represented by tables, or by some ad-hoc means like polynomials for interpolation.

For a utility function f, any leaf B of the composition tree for f is ideally a singleton, such that only functions depending on the possible levels of a single attribute have to be estimated, which can be done using methods of "*one-dimensional* utility theory" (cf. Keeney and Raiffa [12, ch. 4]). However, some care has to be taken concerning a *proper scaling* of these functions. If (with the notation of the preceding paragraph) $k = 0$, then B is a minimal autonomous set, and there is only 1 (i.e. 2^0) function to be determined, which is the divisor (h, say) corresponding to B. (Note that B appears twice in the composition tree, as a leaf and its predecessor; this is systematic since in regarding the tree as a "bottom up" evaluation scheme for $f(x)$, $x \in S_M$, one can associate the respective subvector x_B of the given vector x with the leaf, and the real value $h(x_B)$ with the inner node.) The scaling of this divisor can be done arbitrarily. For $k \geqslant 1$, the functions c_A for $A \subseteq K = \{1, \ldots, k\}$ defined on S_B as in (14.b) are "conditional utility functions" that need to be scaled in a mutually consistent fashion. For instance, the special case $k = 1$ in (13.a) and (14.) describes the familiar situation $P(f) = \{A\}$, i.e. A is autonomous with respect to f, and $f(x) = g(h(x_A), x_B)$ for a function g that is affine in its first argument. If for suitable "reference" subvectors o, $i \in S_A$, $h(o) = 0$ and $h(i) = 1$, then, according to (14.a),

$g(x_1, s) = c_\emptyset(s) + c_{\{1\}}(s) \cdot x_1$, where by (14.b)

$c_\emptyset(s) = g(0, s) = g(h(o), s) = f(o, s)$ and

$c_{\{1\}}(s) = g(1, s) - g(0, s) = g(h(i), s) - g(h(o), s) = f(i, s) - f(o, s)$

(assuming a suitable arrangement of the coordinates such that those in A "come first" in a vector $x \in S_M$). The "conditional utility functions" of s $f(i, s)$ and $f(o, s)$ thereby need to be estimated with lotteries subject to an arbitrary scale to determine their "relative shape", and then be scaled according to the scaling of f with specific choices of arguments s, that is, again with the aid of "scaling constants" given by absolute utility values. This holds also in general, if 2^k ($k \geqslant 1$) conditional utility functions need to be determined (cf. also [12, p. 245, p. 326]). For the case of free sets with two or more attributes, *other approaches*, like direct estimation methods using interpolation (cf. Bell [2]) or more general decomposition approaches (cf. [12, sec. 5.7], and Farquhar [6] and references), *can apply*.

The presented approach should provide a helpful, systematic tool for decision analysis. Many of its features could and should be automated in an interactive computer program. This program could perform automatic scaling and re-translation of utility values, and the all-important consistency checks and sensitivity analyses (cf. e.g. [12, p. 310, pp. 349ff]). Furthermore, it could observe that qualitative properties like risk aversion (for single attributes, cf. [12, pp. 159ff];

for multiple attributes ("substitutivity"), cf. [12, p. 240]) are maintained during changes, and compute and display the identified independence conditions with the composition tree. Of course, this area of research could only be touched upon here.

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