SUBSTITUTION DECOMPOSITION OF MULTILINEAR FUNCTIONS WITH APPLICATIONS TO UTILITY AND GAME THEORY

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Abstract: A theory of decomposition "by substitution" for multi-linear (i.e. multi-affine) functions is presented. A representation theorem for such functions is shown to be given by a Moebius inversion formula. The concept of autonomous sets of variables (a "linear separability" of some kind, also known as "generalized utility independence") captures the decomposition possibilities of a multi-linear function. Their entirety can be hierarchically represented by a so-called composition tree. Distinguished, strong forms of decompositions are shown to be given by multiplicative or additive functions. Important applications to the theories of multi-attribute expected-utility functions, switching circuits and cooperative n-person games are outlined.

1. Multi-Linear Functions

The substitution of a Boolean function of several variables into a variable of another Boolean function yields a new Boolean function. Inverting this process is called substitution decomposition of a Boolean function, and is important for applications since it reduces the complexity of the switching circuit the function represents. We will show that multi-linear functions can be decomposed in a similar way. The presented scheme will comprise decomposition methods known for expected-utility functions, Boolean functions and cooperative games, as presented in the final sections. The results are of the kind that certain decomposition possibilities imply others, and that some of them lead to specific, e.g. additive representations of the decomposed function. To the author's knowledge, the results are original where not attributed to others, in particular the proof of (1.) and the obtained unification of the decomposition theory for utility functions and games.

A linear (also called affine) function is here understood as a real function $G:\mathbf{R}\to\mathbf{R}$ such that $G(t)=a\cdot t+b$ for suitable real a,b. Obviously, if G is invertible, i.e. if $a\neq 0$, then G^{-1} is linear, and so is the functional composition of two linear functions G,H. A function of several variables is called multi-linear if it is linear in each variable (i.e. if the other variables are fixed), and n-linear if there are n variables. Without explicit notice, all variables of a function

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are assumed to be essential. This can be done without loss of generality, since for a given function, a variable that is not essential can be dropped, and in the cases below where a function of several variables is defined in terms of others, all its variables must be essential, too.

Further below, we will define and investigate possible decompositions of a function $f:\mathbf{R}^n \to \mathbf{R}$. The argument of this function is a vector, and in order to identify its components, we let throughout

$M = \{1,\dots,n\}$,

with the understanding that the elements of M refer to the coordinates of the space $\mathbf{R}^n$. For $x \in \mathbf{R}^n$, $i \in M$ and $A \subseteq M$, let x_i be the i-th component of the vector x and x_A be the projection of x onto $\mathbf{R}^{|A|}$, more precisely the subvector of x consisting of the components x_k with $k \in A$. $x_\emptyset$ is the empty vector, which can be considered as the identity element of pairing, i.e. $(y, x_\emptyset) = y$. We refer to the special vectors of $\mathbf{R}^n$ only consisting of 0's and 1's (the corners of the n-dimensional unit cube) by 1_A, where A is the set of coordinates that have the value 1; that is, $(1_A)_i = 1$ if $i \in A$ and $(1_A)_i = 0$ if $i \in M\text{-}A$, for $A \subseteq M$.

The following theorem asserts that a multilinear function is a polynomial in its variables (that is, a sum of products of non-negative powers of these variables), where each variable appears in at most its first power. The coefficients of this polynomial are unique, and can be computed from the function evaluated for arguments that are either 0 or 1. Applied to utility functions, this theorem asserts a so-called "quasi-additive" representation of a utility function, which is distinguished by the fact that it requires the estimation of 2^n "scaling parameters" (cf. FISHBURN / KEENEY [4, p.938] and (10.a) below).

(1.) Theorem. Let $n \geq 1$. Then $f:\mathbf{R}^n \to \mathbf{R}$ is multi-linear iff

(a) $f(x) = \Sigma_{A \subseteq M}\, c_A \, \Pi_{i \in A}\, x_i$,

where the c_A's are unique real numbers given by

(b) $c_A = \Sigma_{B \subseteq A}\, (-1)^{|A-B|} \cdot f(1_B)$, for $A \subseteq M$.

Proof: The proof will be given in three parts. First, it will be shown that with (b), (a) is true if $x = 1_S$, for any $S \subseteq M$. Second, that there is at most one choice for each c_A, $A \subseteq M$, such that (a) holds. Third, that (a) holds for any $x \in \mathbf{R}^n$ given f is n-linear; the converse is obvious.

Let (b) hold. Then for $S \subseteq M$,

$$\begin{aligned} &\Sigma_{A \subseteq M}\, c_A \, \Pi_{i \in A}\, (1_S)_i \\ = \;&\Sigma_{A \subseteq S}\, c_A \\ = \;&\Sigma_{A \subseteq S}\, \Sigma_{B \subseteq A}\, (-1)^{|A-B|} \cdot f(1_B) \end{aligned}$$

$= \Sigma_{B\subseteq S} \Sigma_{A:\, B\subseteq A\subseteq S} (-1)^{|A-B|} \cdot f(1_B)$

$= \Sigma_{B\subseteq S} f(1_B) \cdot \Sigma_{T\subseteq S-B} (-1)^{|T|}$

$= f(1_S),$

since the second sum in the last but one line is 1 for S-B = ∅, otherwise $(1-1)^{|S-B|}$, i.e. 0 for B≠S, by the binomial theorem. (Remark: This was a special case of the usual proof of a so-called Moebius inversion formula, cf. ROTA [10].)

To prove the second part, the uniqueness of the c_A's, read (a) as proved as a system of 2^n equations for 2^n unknowns c_S, S⊆M (after OWEN [9, p. P-79]):

$$f(1_S) = \Sigma_{A\subseteq S}\, c_A \qquad (S\subseteq M).$$

It is sufficient to show that the corresponding set of homogeneous equations

$$0 = \Sigma_{A\subseteq S}\, c_A \qquad (S\subseteq M)$$

has only the trivial solution. Indeed, the assumption $c_S\neq 0$ for some minimally chosen S would yield the contradiction $0 = \Sigma_{A\subseteq S}\, c_A = c_S \neq 0$.

For the third part, let (b) hold, f be n-linear, and k be a natural number. <k shall denote the set of all the members of M less than k, similarly ≥k = {k,...,n}, etc. Thus, ≥1 = M and <1 = ∅. As before, $x_{>k}$ shall denote the vector x projected on its last n-k coordinates, e.g. for $x=1_C$, C⊆M. We prove for 1≤k≤n+1 by induction on k:

$$(*) \qquad f(y, (1_C)_{\geq k}) = \Sigma_{A\subseteq <k} (\Pi_{i\in A}\, y_i) \cdot \Sigma_{B\subseteq C}\, c_{A\cup B} \qquad \text{for all } y\in \mathbf{R}^{k-1},\ C\subseteq \geq k.$$

For k=1, this equation has been proved in the first part above. If k=n+1, (*) represents (a) as to be proved for all $x\in\mathbf{R}^n$. Assume (*) holds for some k, 1≤k≤n. Proving it for k+1 amounts to showing

$$f(y, z, (1_C)_{>k}) = \Sigma_{A\subseteq <k} (\Pi_{i\in A}\, y_i) \cdot \Sigma_{B\subseteq C}\, c_{A\cup B} + \Sigma_{A\subseteq <k} (\Pi_{i\in A}\, y_i) \cdot z \cdot \Sigma_{B\subseteq C}\, c_{A\cup\{k\}\cup B}$$

for all $y\in\mathbf{R}^{k-1}$, $z\in\mathbf{R}$, C⊆>k.

A similar equation is known from the fact that f is linear in its k-th variable z, viz.

$$f(y, z, w) = a(y,w)\cdot z + b(y,w) \qquad \text{for all } y\in\mathbf{R}^{k-1},\ z\in\mathbf{R},\ w\in\mathbf{R}^{n-k},$$

for some suitable functions $a,b:\mathbf{R}^{n-1}\to\mathbf{R}$. Letting z=0 and z=1 in this equation, one can conclude

$$b(y,w) = f(y,0,w),$$
$$a(y,w) = f(y,1,w) - f(y,0,w).$$

But for $w = (1_C)_{>k}$, for any C⊆>k, f(y,0,w) and f(y,1,w) are known from the induction hypothesis. It is easy to verify that in this case (*) yields indeed

$$b(y,w) = \Sigma_{A\subseteq <k} (\Pi_{i\in A}\, y_i) \cdot \Sigma_{B\subseteq C}\, c_{A\cup B},$$
$$a(y,w) = \Sigma_{A\subseteq <k} (\Pi_{i\in A}\, y_i) \cdot \Sigma_{B\subseteq C}\, c_{A\cup\{k\}\cup B},$$

which remained to be shown. (End of Proof.)

2. Autonomous Sets of Coordinates

We introduce the concept of decomposing a function of several variables by "linearly separating" specific sets of variables, which will be called "autonomous".

(2.) Definition. Let $n \geq 1$, $f: \mathbf{R}^n \to \mathbf{R}$. Then A is called autonomous (with respect to f) with corresponding divisor h (of f), if $\emptyset \neq A \subseteq M$, $B=M-A$, $h: \mathbf{R}^{|A|} \to \mathbf{R}$, and there is some function $g: \mathbf{R} \times \mathbf{R}^{|B|} \to \mathbf{R}$ that is linear in its first variable, such that

$$f(x) = g(h(x_A), x_B)$$

holds for all $x \in \mathbf{R}^n$.

The preceding definition includes the decomposition of multi-linear functions "by substitution": it is easy to verify that if f is n-linear, then A is autonomous with respect to f iff there exists an $|A|$-linear function h and a $1+|B|$-linear function g (with $B=M-A$) such that $f(x) = g(h(x_A), x_B)$. The important restriction is indeed that g is linear in the variable for which the divisor h is substituted. M is always an autonomous subset of M with respect to $f: \mathbf{R}^n \to \mathbf{R}$, and if f is n-linear, any singleton $\{i\}$, $i \in M$, is autonomous. A multi-linear function is called indecomposable if it has no other autonomous sets. For any n, there are indecomposable n-linear functions [8, p.271]. We will show that indecomposable multi-linear functions form one type of "building block" which occurs in a unique hierarchical decomposition of any given n-linear function.

If $e,f: \mathbf{R}^n \to \mathbf{R}$, let e and f be called isomorphic if $e(x)=G(f(x))$ for some invertible linear $G: \mathbf{R} \to \mathbf{R}$. According to the introductory remarks on linear functions, "is isomorphic to" is an equivalence relation. The following lemma states that this equivalence "preserves" in some sense the decompositions regarded here.

(3.) Lemma. Let A be autonomous with respect to $f: \mathbf{R}^n \to \mathbf{R}$ with a corresponding divisor h, and $B=M-A$. Then

(a) H is a divisor of f corresponding to A iff h and H are isomorphic.

(b) If F and f are isomorphic, then A is autonomous with respect to F, and h is a divisor of F.

(c) If $f(x) = g(h(x_A), x_B)$, and g is linear in its first variable, then g is unique.

Proof: If h and H are isomorphic, H is obviously a divisor of f. The converse follows from the observation that f itself with the variables corresponding to B fixed at suitable values (such that the resulting function is not constant) is isomorphic to any divisor that corresponds to A. (b) is obvious, and (c) holds since a linear function is determined if only two different values for its argument are given. (End of Proof.)

The preceding lemma shows that the system of autonomous subsets of M with respect to a given function $f: \mathbf{R}^n \to \mathbf{R}$ characterizes the decomposition possibilities of f, or of any function isomorphic to f; the possible corresponding divisors or functional representations as in (3.c) are then determined. We will show in section 3 that this system can be represented by a so-called "composition tree", using the fact that specific relationships hold between different autonomous sets. The following lemma treats the case of two comparable autonomous sets, stating a certain "transitivity" of the divisor property.

(4.) Lemma. Let $\emptyset \neq B \subseteq A \subseteq M$, and A be autonomous with respect to $f:\mathbf{R}^n \to \mathbf{R}$ with corresponding divisor h. Then B is autonomous with respect to f iff B is autonomous with respect to h.

Proof: The consideration is similar to that for (3.a) above. Note that by (3.b), the autonomy of B with respect to h does not depend on the choice of h. (End of Proof.)

By (4.), it suffices to examine autonomous proper subsets of M that are maximal, since one can then recursively look at the corresponding divisors and their decompositions. Two maximal autonomous sets can either be disjoint or overlap. The latter is the more interesting case. It applies for instance to the following theorem, which states that the system of autonomous sets is closed under non-disjoint unions and intersections.

(5.) Theorem. Let A and B be non-disjoint autonomous sets with respect to $f:\mathbf{R}^n \to \mathbf{R}$. Then $A \cup B$ and $A \cap B$ are autonomous.

Proof: Cf. FISHBURN / KEENEY [3, p.931, La.1] or VON STENGEL [12, p.36].

In view of the observation in the proof of (3.a), the preceeding assertion for the intersection is very obvious, but not so for the union. If in (5.), $A \cup B = M$, the result is uninteresting, too. But (4.) and (5.) allow us to confine ourselves to just this case for further investigations on overlapping autonomous sets A,B, since otherwise $A \cup B$ is an autonomous proper subset of M and then A and B are not maximal. In what follows, a function $\bullet:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, denoted "infix", is associative iff $u\bullet(v\bullet w) = (u\bullet v)\bullet w$ for all real u,v,w; the parentheses can then consequently be omitted.

(6.) Theorem. Let M be the disjoint union of the non-empty sets A, B, C, and let $A \cup B$ and $B \cup C$ be autonomous with respect to $f:\mathbf{R}^n \to \mathbf{R}$. Then

$$f(x) = a(x_A) \bullet b(x_B) \bullet c(x_C)$$

for suitable functions a, b, c, and a bilinear (i.e. 2-linear) associative function $\bullet$

Proof: Cf. (8.) below.

The possible bilinear associative functions in (6.) are characterized as follows:

(7.) Theorem. $\bullet$ is a bilinear associative function iff either

(a) $G(x \bullet y) = G(x) \cdot G(y)$,

where G is a unique invertible linear function, or

(b) $G(x \bullet y) = G(x) + G(y)$,
where G is linear and invertible, and unique up to a nonzero multiplicative constant. [Remark: in (b), it can be assumed w.l.o.g. that $G(t) = t + r$, for some unique real r; then $x \bullet y = x + y + r$.]

Proof: A function $\bullet$ defined by $x \bullet y = G^{-1}(G(x) \cdot G(y))$, or with + instead of $\cdot$, is associative and bilinear because $\cdot$ and + are. The converse follows in a straightforward way from the associativity equation for a bilinear function $\bullet$, using that by (1.),
$x \bullet y = pxy + qx + ry + s$ holds for some real numbers p,q,r,s. (End of Proof.)

We remark that the equation in (6.) implies that A and C are autonomous with corresponding divisors a and c, respectively. With (4.) and (5.), this shows that union, intersection and differences of overlapping autonomous sets are autonomous (in fact, this also holds for the symmetrical difference, since (7.) implies that all bilinear associative functions are symmetric). We will however use a generalization of (6.), given next. A partition of a set S is thereby understood as a set of non-empty, pairwise disjoint sets whose union is S.

(8.) Theorem. Let $\{A_j | 1 \le j \le k\}$ be a partition of M, $k \ge 3$, and $M\text{-}A_j$ be autonomous with respect to $f{:}\mathbf{R}^n \to \mathbf{R}$ for $1 \le j < k$ (note these are k-1 conditions). Then

(a) there exist: an invertible linear function G, and suitable functions a_j, $1 \le j \le k$, such that $G(f(x)) = a_1(x_{A1}) \bullet \ldots \bullet a_k(x_{Ak})$,
where $\bullet$ is either multiplication or addition.

(b) Given (a), $\bigcup_{j \in L} A_j$ is autonomous with respect to f, for any non-empty subset L of {1,..,k}.

Proof: For (a), cf. FISHBURN / KEENEY [4, La. 2]. (b) holds by (3.b) and because both multiplication and addition are bilinear, symmetric and associative. (End of Proof.)

Under the assumptions of (8.), A_j is autonomous for $1 \le j \le k$ according to (8.b), with corresponding divisor a_j as in (8.a). If h_j is a given divisor that corresponds to A_j, a_j is isomorphic to h_j by (3.a). The linear transformations to obtain a_j from h_j for $1 \le j \le k$, and the transformation of the given function f to represent it as a product or sum (i.e. G in (8.a)) can be determined by evaluating f for a number of suitably chosen arguments, somewhat similarly to (1.b). Under additional monotonicity assumptions, O(k) many (e.g. k+2) arguments suffice. These assumptions of so-called "utility independence", explained in section 4 below, usually hold for the particular application to utility functions. The well-known "multiplicative / additive representation" of a utility function, which is asserted by theorem (8.), is therefore distinguished by the fact that it requires the estimation of only proportionally many (and not exponentially many as in (1.)) "scaling parameters" as compared to the number of variables (cf. KEENEY / RAIFFA [6, p.289]).

3. The Unique Hierarchical Decomposition

The system of autonomous sets with respect to a given function f is ordered by inclusion. The following theorem, if applied iteratively in a "top-down" manner, gives a complete description of this system for n-linear f.

(9.) Theorem. Let f be n-linear, $n \geq 2$. Then there exists a unique partition P(f) of M, $P(f) = \{A_j \mid 1 \leq j \leq k\}$, such that either

(a) $k \geq 2$, and A_j is autonomous (with respect to f) for $1 \leq j \leq k$, or

(b) $k \geq 3$, and $\bigcup_{j \in L} A_j$ is autonomous with respect to f, for any non-empty subset L of $\{1,..,k\}$,

and in either case,

(c) any other set that is autonomous with respect to f, except M, is a proper subset of an element of P(f).

Proof: Cf. GORMAN [5, p.375, Th.2] or [12, p.39].

If an n-linear function f is given, P(f) as described in (9.) is a partition of M into autonomous sets. It then suffices to consider the corresponding divisors in order to obtain the autonomous sets with respect to f other than the elements of P(f) or their unions (in case (9.b) holds), because of (9.c) and (4.). f can be functionally expressed in terms of these divisors and an additional multi-linear function (which may be called "quotient" [8, p.269]) that corresponds to P(f), as follows.

(10.) Theorem. Let f be n-linear, $n \geq 1$. If n=1, f is itself an invertible linear function. If $n \geq 2$, let $P(f) = \{A_j \mid 1 \leq j \leq k\}$ as in (9.). For the cases (a) and (b) as in (9.), the following assertions hold:

(a) If h_j is any divisor corresponding to A_j, for $1 \leq j \leq k$, then

$$f(x) = g(h_1(x_{A1}), \ldots, h_k(x_{Ak}))$$

for a suitable k-linear function g, which is indecomposable.

(b) There are divisors h_j corresponding to A_j, for $1 \leq j \leq k$, and an invertible linear function G, such that

$$G(f(x)) = h_1(x_{A1}) \bullet \ldots \bullet h_k(x_{Ak}),$$

where $\bullet$ is either multiplication or addition.

Proof: Because of (8.), only (a) needs to be shown. From the k conditions that A_j is autonomous with corresponding divisor h_j, for $1 \leq j \leq k$, it follows (in steps) that the equation in (a) is an admissible definition of $g:\mathbf{R}^k \rightarrow \mathbf{R}$, and that g is linear in each variable. (End of Proof.)

We will refer to the cases (a) and (b) in (9.) and (10.) by saying that the top quotient of f is prime or degenerate, respectively (after [8, p.328]). Applying the previous two theorems iteratively, a given n-linear function f can be represented as a closed term T of multi-linear functions of fewer variables (if f is not indecomposable) using all the decomposition possibilities of f. There are some simplifications possible because of the freedom to choose the divisors of f corresponding to the elements of P(f) in case of (10.a), which is useful if some of these divisors in turn have degenerate top quotients and need to be linearly transformed to be representable as products or sums, or are trivial 1-linear functions. The syntactical structure of the substitutions in T can be represented by a tree in an obvious way. This tree, properly labeled, carries all the information about the complete decomposition of f. It is called the composition tree of f, in detail described in MÖHRING / RADERMACHER [8, p.328]. It is worth noting that in particular the system of autonomous sets, which is possibly exponential in size, can be described by this data structure of linear complexity. (Of course, the representation of an involved k-linear function g requires exponential space ($O(2^k)$) according to (1.), if g is indecomposable.)

The material about the substitution decomposition of multi-linear functions presented here focused on existence- and uniqueness properties of certain decompositions and their representations. There are of course other, in particular algorithmic aspects of the theory; for some considerations concerning the special case of Boolean functions, cf. [1], [8, pp.332ff]. We will conclude this paper by sketching briefly some applications of the above results to utility functions, Boolean functions, and cooperative n-person games.

4. Application to Expected-Utility Functions

Most of the results above originated with the decomposition theory for expected-utility functions (also called von Neumann-Morgenstern utility functions, here utitlity functions for short). A utility function is real-valued and defined on the set of possible outcomes of a decision and represents the decision maker's preferences and risk behavior. Each considered action (decision alternative) is represented by a probability distribution on outcomes, and the most preferred one is that with the highest expected value of the utility function. The outcomes can usually be represented by vectors of real-valued attribute levels. In a particular decision analysis, a utility function $f:\mathbf{R}^n \rightarrow \mathbf{R}$ needs to be constructed in order to identify the decision maker's behavior, a task that is substantially alleviated using decomposition methods (cf. KEENEY / RAIFFA [6], FARQUHAR [2]), in particular the one presented above. The concept of autonomy as in definition (2.) is thereby strengthened to that of so-called "utility independence".

A set of coordinates (i.e., attributes) is called utility independent if it is autonomous with respect to the utility function f, where the corresponding divisor is substituted into a function that is linear and increasing in the respective variable. This concept can be behaviorally interpreted, using the fact that a utility function is unique only up to increasing linear transformations. A utility independent set of attributes is one for which it makes sense to speak of a "sub-utility" function (given by the corresponding divisor), that is, of a unique preference and risk behavior for these attributes independent of the levels of the other attributes. This condition can be tested and is frequently observed (cf. [6]). In the theorems above, the condition of autonomy of a set can of course in assumptions be strengthened to that of utility independence, but not generally in conclusions, e.g. not in (8.b) (cf. VON STENGEL [12, p.60]). However, "utility independent" can be uniformly substituted for "autonomous" in (3.) (if only increasing "isomorphisms" are admitted), (4.) and (5.) [12, pp.34,36].

The multiplicative or additive representation of a given utility function as in (8.) is particularly useful, if it applies, because it allows the decision analyst to establish a utility function with a minimal amount of information obtained from the decision maker for each attribute [6, p.292]. For the multiplicative representation, the linear transformation G in (8.a) can be interpreted as asserting a "substitutivity" of the attributes if it is negative, and a "complementarity" if it is positive [6, pp.240f]. The hierarchical decomposition of a utility function is of practical relevance since it usually corresponds to a natural hierarchy of attributes, and thus enhances an overview of the preference structure. In most applications, a utility function can <u>not</u> be assumed to be n-linear. This can however frequently be achieved by replacing each attribute by a suitable real-valued function of the attribute values, i.e. by postulating that each singleton $\{i\}$, $i \in M$, is autonomous. Where this cannot be done, for example if an attribute does not belong to any autonomous set except M, the respective variable can be trivially added to express an additional dependency, e.g. as a further argument to g in (10.a) (in the polynomial representation, the coefficients c_A as in (1.b) then become functions of this variable). The utility function may still be expressed as described as an arithmetic expression in several functions of the variables, each of which is of considerably smaller dimension than n. The concept of the composition tree, slightly generalized, is thereby still applicable (cf. GORMAN [5, pp.375f]).

<u>5. Application to Switching Circuit Decomposition</u>

The results presented in sections 2 and 3 can be directly applied to the theory of the decomposition of switching circuits (cf. e.g. CURTIS [1], or [8]), because of the following observation. Theorem (1.) implies that an n-linear function corresponds uniquely to, and can thus be identified with, a function $f:\{0,1\}^n \to \mathbf{R}$. If the range of f can be restricted to be $\{0,1\}$, f can be considered as a Boolean function of n variables. In this case, a set A is autonomous with respect to f iff (with B=M-A)

(*) $\quad f(x) = g(h(x_A), x_B)$

holds for all $x \in \{0,1\}^n$ with <u>Boolean</u> functions g,h. Namely, a Boolean function G(t) of one variable is either given by 0, 1, t or 1-t, that is, linear. Conversely, a divisor h of f corresponding to the autonomous set A can be assumed to be given as f with some variables properly fixed, as observed in the proof of (3.a); thus without loss of generality, g and h in (*) are Boolean if f is. Equation (*) states a so-called "simple disjoint decomposition" of the Boolean function f [8, p.270]. With this concept replaced for that of "autonomy" of definition (2.), the results above can be found in the relevant literature (e.g. [1]). It is worth noting the counterparts to (6.), (8.) and (10.b): degenerate top quotients of a Boolean function are given (up to complementation of Boolean values) by products or sums modulo 2 [8, p.244]. This can be derived from theorem (7.), as follows.

(11.) Theorem. $\bullet$ is a Boolean function of two variables that is associative iff either

(a) $G(x \bullet y) = G(x) \cdot G(y)$ or

(b) $G(x \bullet y) = G(x) \oplus G(y)$,

for all $x,y \in \{0,1\}$, where G is a unique invertible function $\{0,1\} \to \{0,1\}$, and $\cdot$ and $\oplus$ are multiplication and addition modulo 2, respectively.

Proof: Application of (7.), where $\bullet$ is considered as bilinear and associative, yields

$$x \bullet y = G^{-1}(G(x) \cdot G(y)), \qquad \text{for all } x,y \in \{0,1\}$$

for some invertible linear G; the case (7.b) can be excluded since otherwise $\bullet$ would be three- and not two-valued. G maps $\{0,1\}$, in two possible ways, on a two-element set S of reals, which must be closed under multiplication in order that in the above equation, G^{-1} yields values in $\{0,1\}$. It is quickly seen that there are only two choices for S, (a) $S=\{0,1\}$ or (b) $S=\{-1,1\}$, where the latter can be represented by $\{0,1\}$ if multiplication is replaced by $\oplus$. (End of Proof.)

6. Application to Game Theory

Finally, the results of sections 1 to 3 can also be directly applied to so-called cooperative n-person games, for the particular instances treated in SHAPLEY [11], MEGIDDO [7] and OWEN [9]. A cooperative n-person game is thereby understood as a function $f:\{0,1\}^n \to \mathbf{R}$ whose values f(x) denote a "payment" to the "coalition" B if $x=1_B$, for a subset B of the set M of "players". Usually f is assumed to be monotonic in each variable. f corresponds uniquely to an n-linear function (which in turn can also be interpreted as a game, cf. [9]), as mentioned above. A so-called simple game [11] is a monotonic Boolean function, with only "winning" or "losing" as possible payments, e.g. for modelling systems of voting. An autonomous set A defines a so-called committee, that is, a set of players that behaves as if "playing" a game (with unique rules) by itself whenever it contributes to a coalition. Many of the concepts introduced in Sections 2 and 3 have suitable interpretations (e.g. (4.) corresponds to [11, Th.4], [7, La.3.3]); in comparison to [7], our proofs could be simplified because of the observed invariance of autonomy under invertible linear transformations (cf. (3.)).

7. References

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