Equilibrium Algorithms for Two-Player Games

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Nash equilibria of bimatrix games

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ \boxed{3} & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ \boxed{4} & 3 \end{bmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \boxed{\begin{matrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{matrix}} \qquad B = \boxed{\begin{matrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{matrix}}$$

$$x = \boxed{\begin{matrix} 2/3 \\ 1/3 \\ 0 \end{matrix}} \qquad x^T B = \boxed{\begin{matrix} 5/3 & 5/3 \end{matrix}}$$

$$Ay = \boxed{\begin{matrix} 4 \\ 4 \\ 3 \end{matrix}} \qquad y^T = \boxed{\begin{matrix} 1/3 & 2/3 \end{matrix}}$$

only **pure best responses** can have probability > 0

Best response condition

Let $\mathbf{x}$ and $\mathbf{y}$ be mixed strategies of player I and II, respectively. Then $\mathbf{x}$ is a best response to $\mathbf{y}$

$\iff$ for all pure strategies i of player I:

$$x_i > 0 \implies (\mathbf{Ay})_i = u = \max\{(\mathbf{Ay})_k \mid 1 \le k \le m\}.$$

Here, $(\mathbf{Ay})_i$ is the ith component of $\mathbf{Ay}$, which is the expected payoff to player I when playing row i.

Proof.

$$\begin{aligned}\mathbf{xAy} &= \sum_{i=1}^{m} \mathbf{x}_i (\mathbf{Ay})_i = \sum_{i=1}^{m} \mathbf{x}_i (u - (u - (\mathbf{Ay})_i)) \\ &= \sum_{i=1}^{m} \mathbf{x}_i u - \sum_{i=1}^{m} \mathbf{x}_i (u - (\mathbf{Ay})_i) = u - \sum_{i=1}^{m} \mathbf{x}_i (u - (\mathbf{Ay})_i) \le u,\end{aligned}$$

because $\mathbf{x}_i \ge 0$ and $u - (\mathbf{Ay})_i \ge 0$ for all i. Furthermore, $\mathbf{xAy} = u \iff \mathbf{x}_i > 0$ implies $(\mathbf{Ay})_i = u$, as claimed.

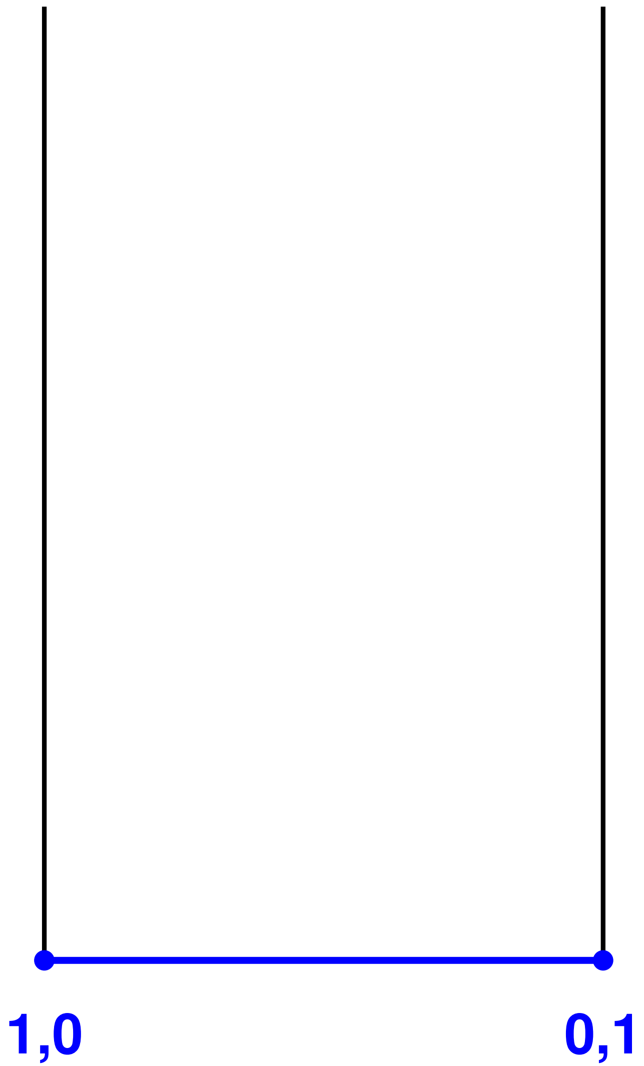
Best responses to mixed strategy of player 2

$$\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 0 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & 3 \end{array} = A$$

payoffs to
player I

1,0 0,1

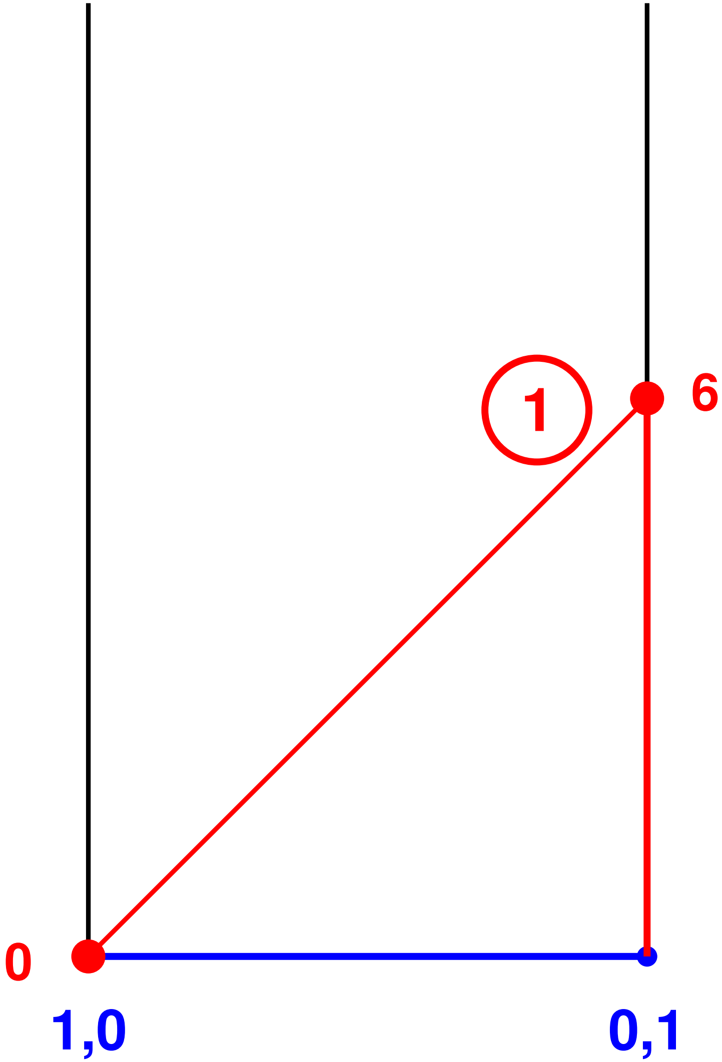
Best responses to mixed strategy of player 2



$$\begin{array}{c c} & \begin{array}{cc} 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cc} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{array}\right] \end{array} = A$$

payoffs to player I

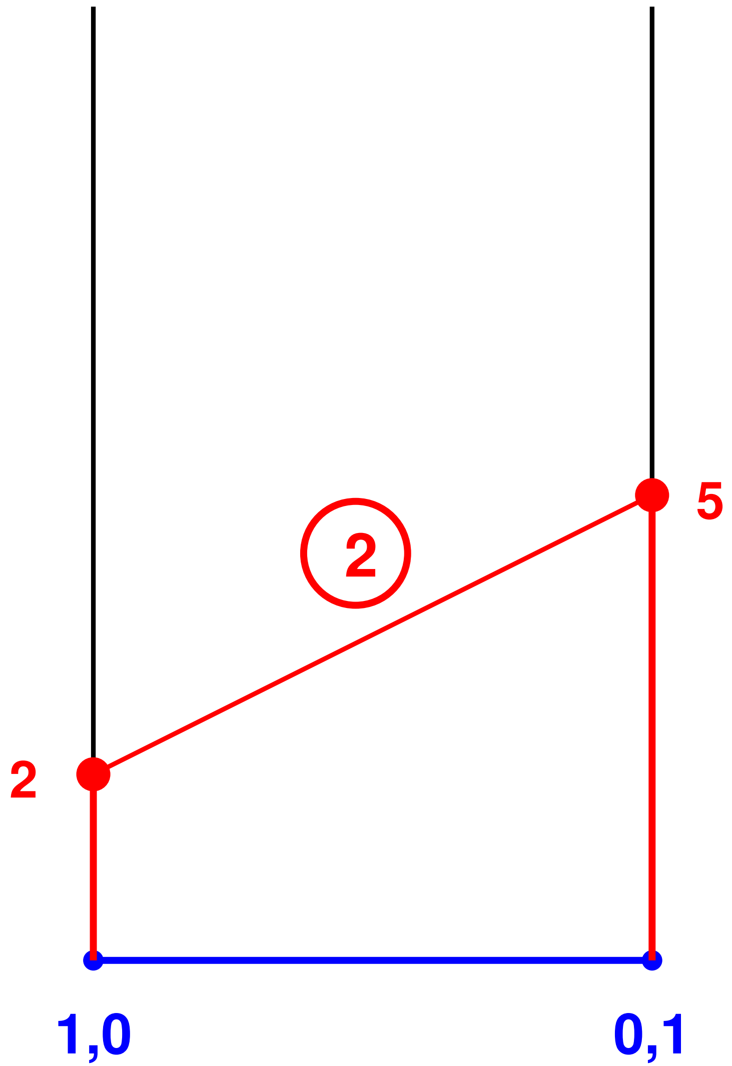
Best responses to mixed strategy of player 2



$$\begin{array}{c|cc}
 & 4 & 5 \\
\hline
1 & 0 & 6 \\
2 & 2 & 5 \\
3 & 3 & 3
\end{array} = A$$

payoffs to
player I

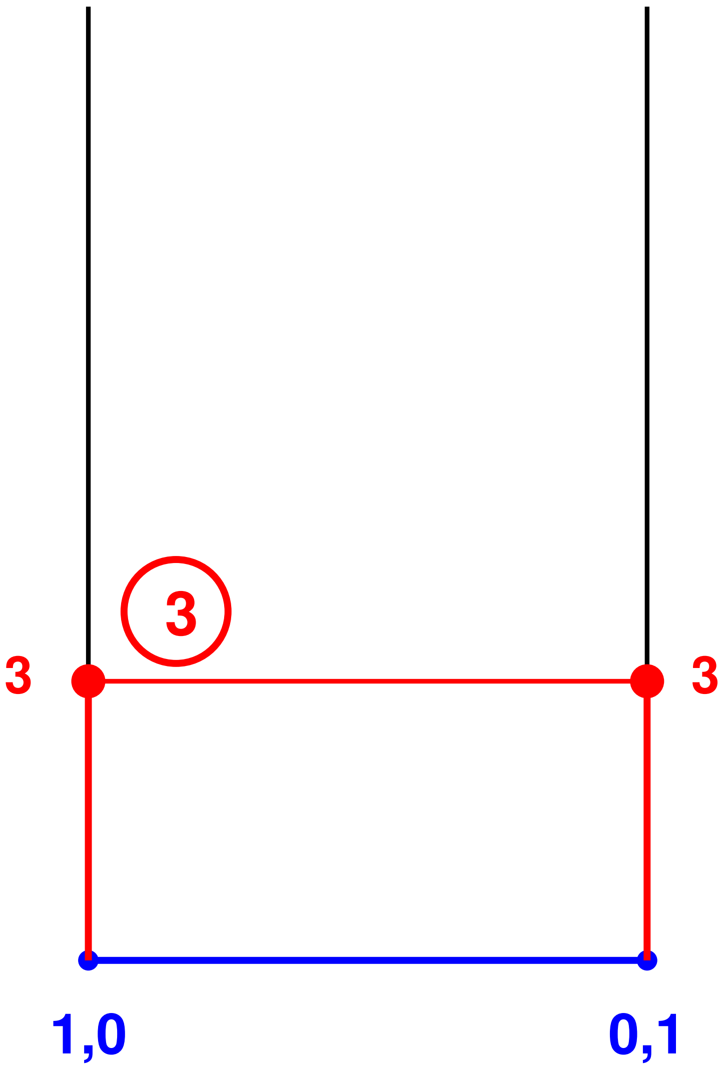
Best responses to mixed strategy of player 2



$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 1 & 0 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & 3 \\ \hline \end{array} = A$$

payoffs to
player I

Best responses to mixed strategy of player 2



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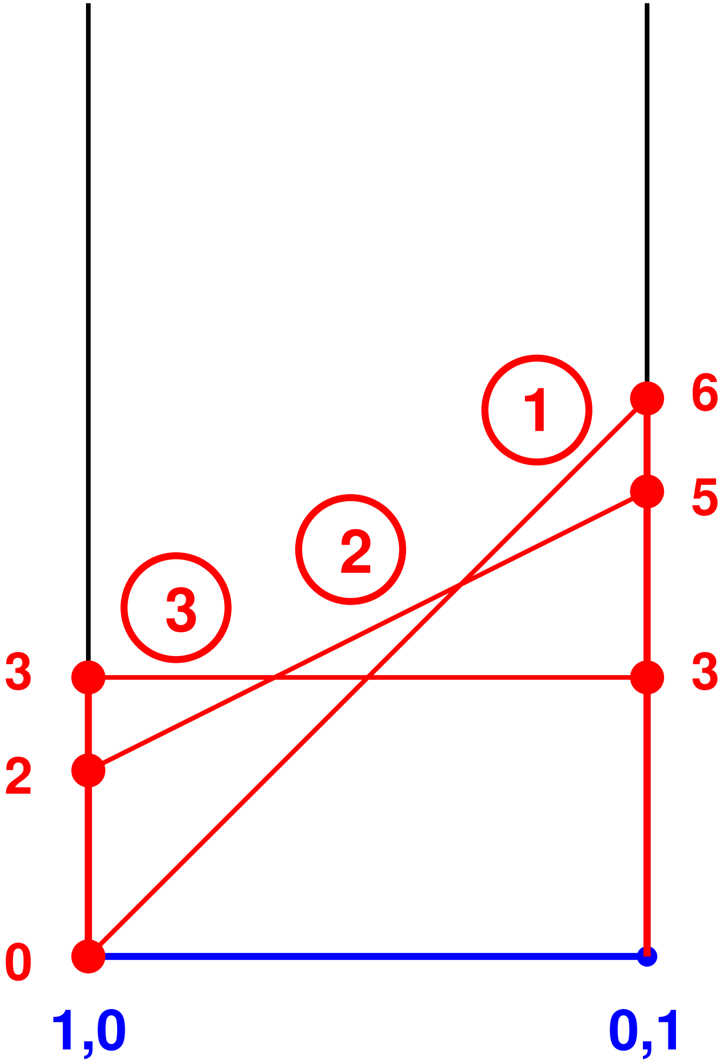
payoffs to
player I

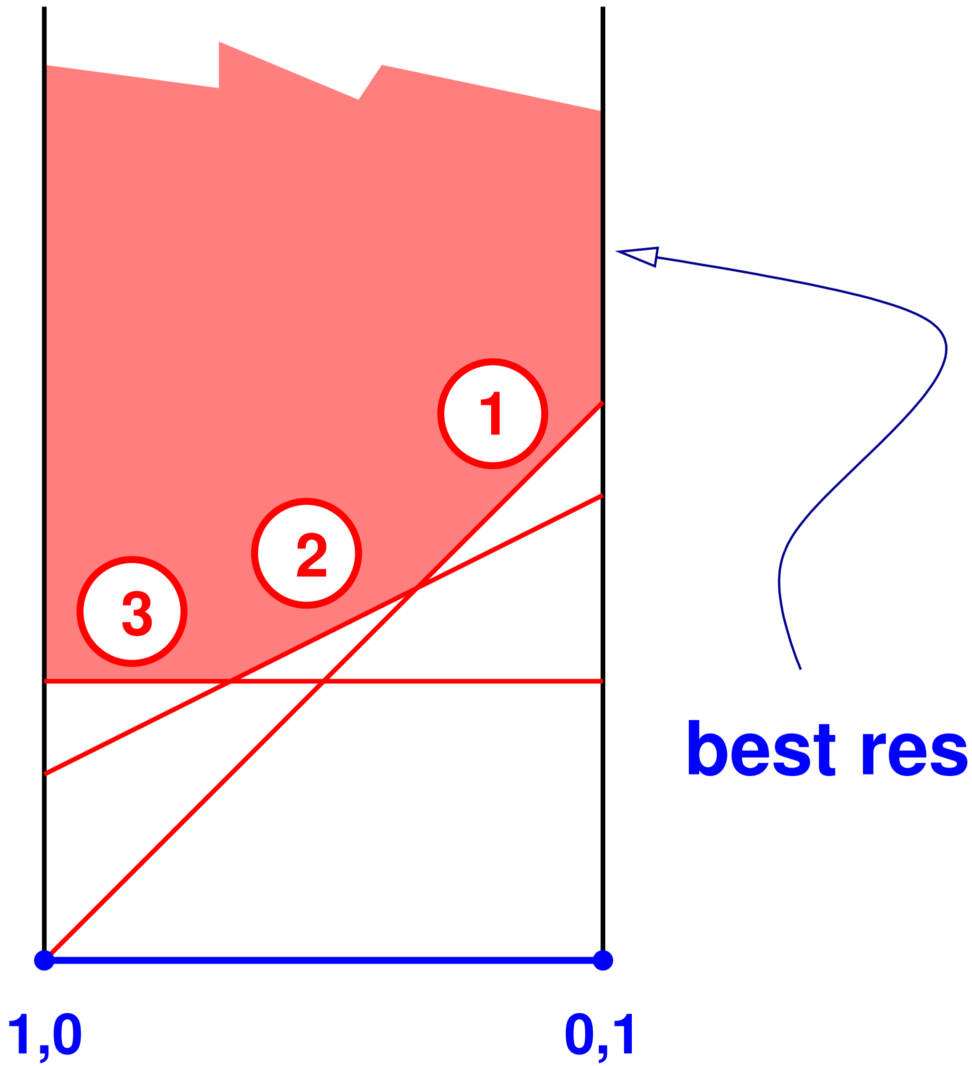
Best responses to mixed strategy of player 2



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{cc} 4 & 5 \\ \hline 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{array} = A$$

payoffs to
player I

Best responses to mixed strategy of player 2

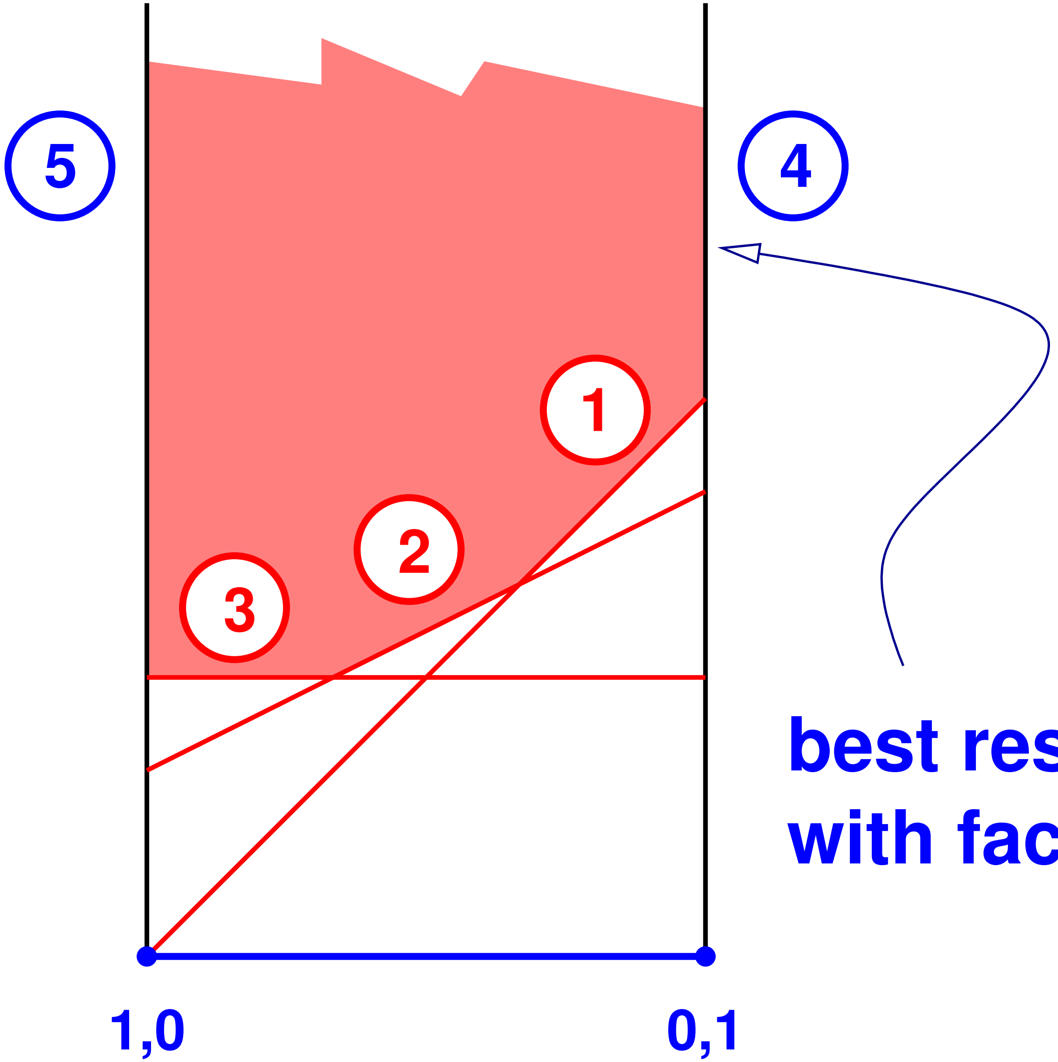


best response polyhedron

$$\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 0 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & 3 \end{array} = A$$

payoffs to
player I

Best responses to mixed strategy of player 2

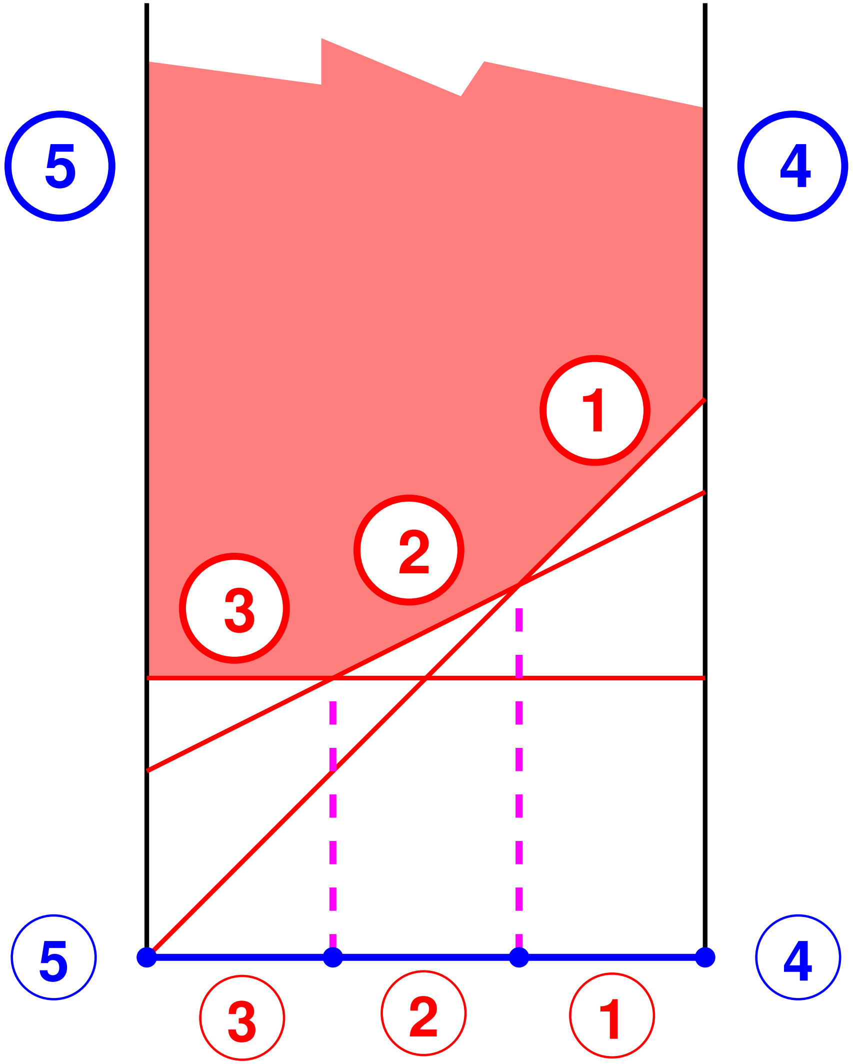


$$\begin{array}{cc} & \begin{array}{cc} 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \boxed{\begin{array}{cc} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{array}} \end{array} = A$$

payoffs to
player I

best response polyhedron with facet labels

Best responses to mixed strategy of player 2



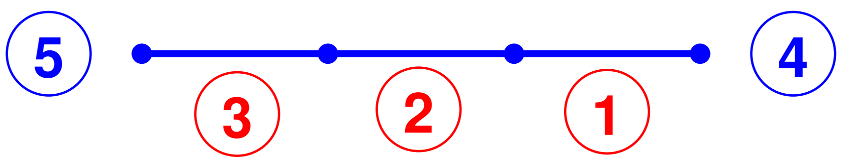
$$\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 0 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & 3 \end{array} = A$$

payoffs to
player I

Best responses to mixed strategy of player 2

$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 1 & 0 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & 3 \\ \hline \end{array} = A$$

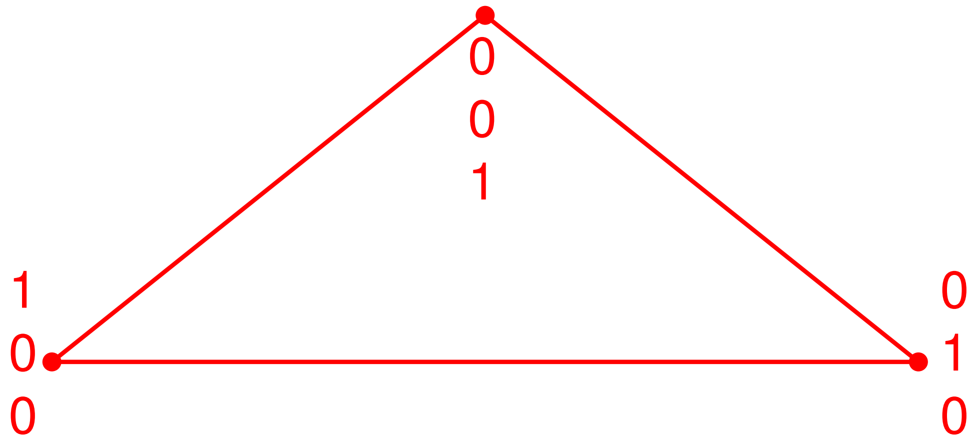
payoffs to
player I



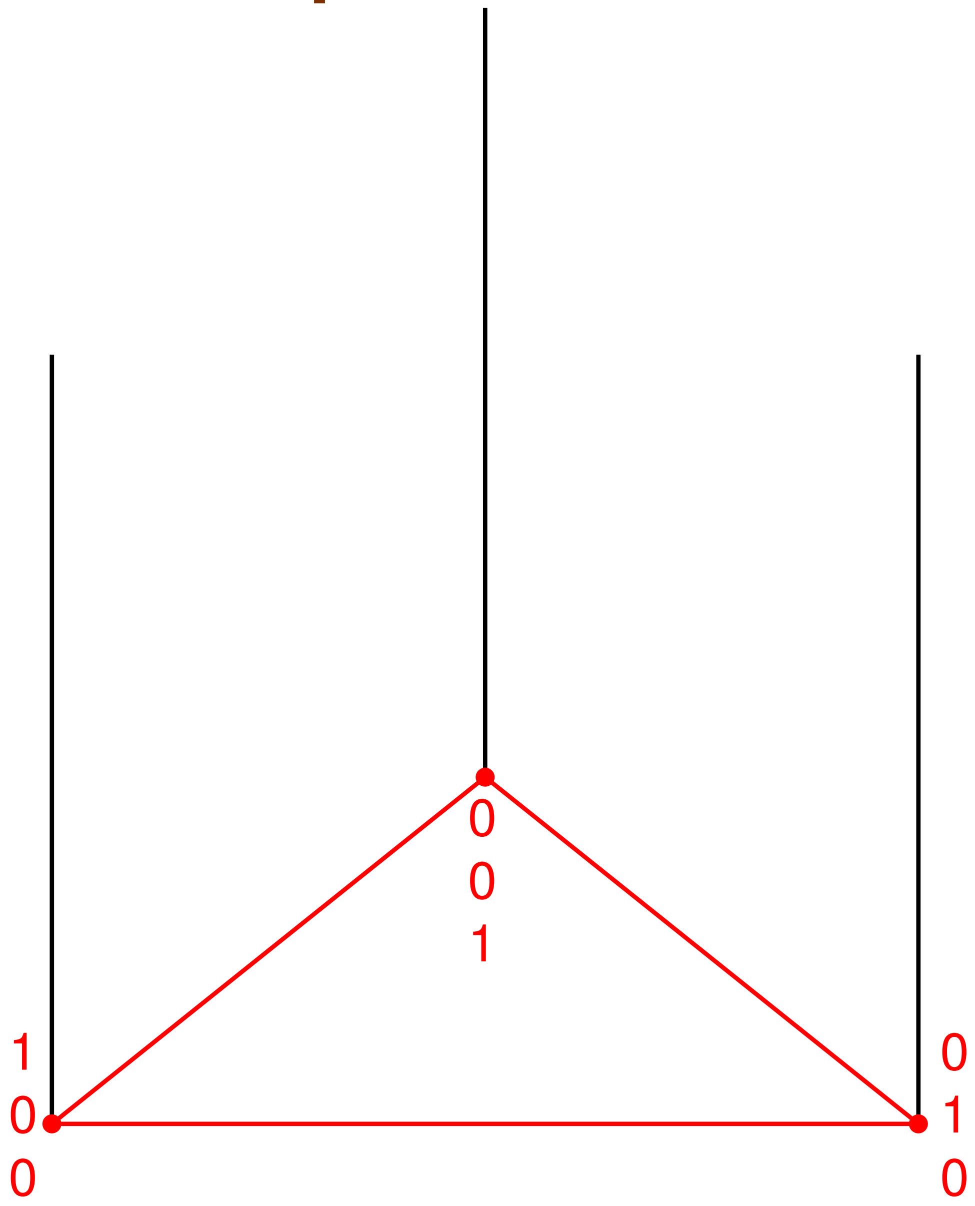
Best responses to mixed strategy of player 1

$$\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & 3 \end{array} = B$$

payoffs to
player II



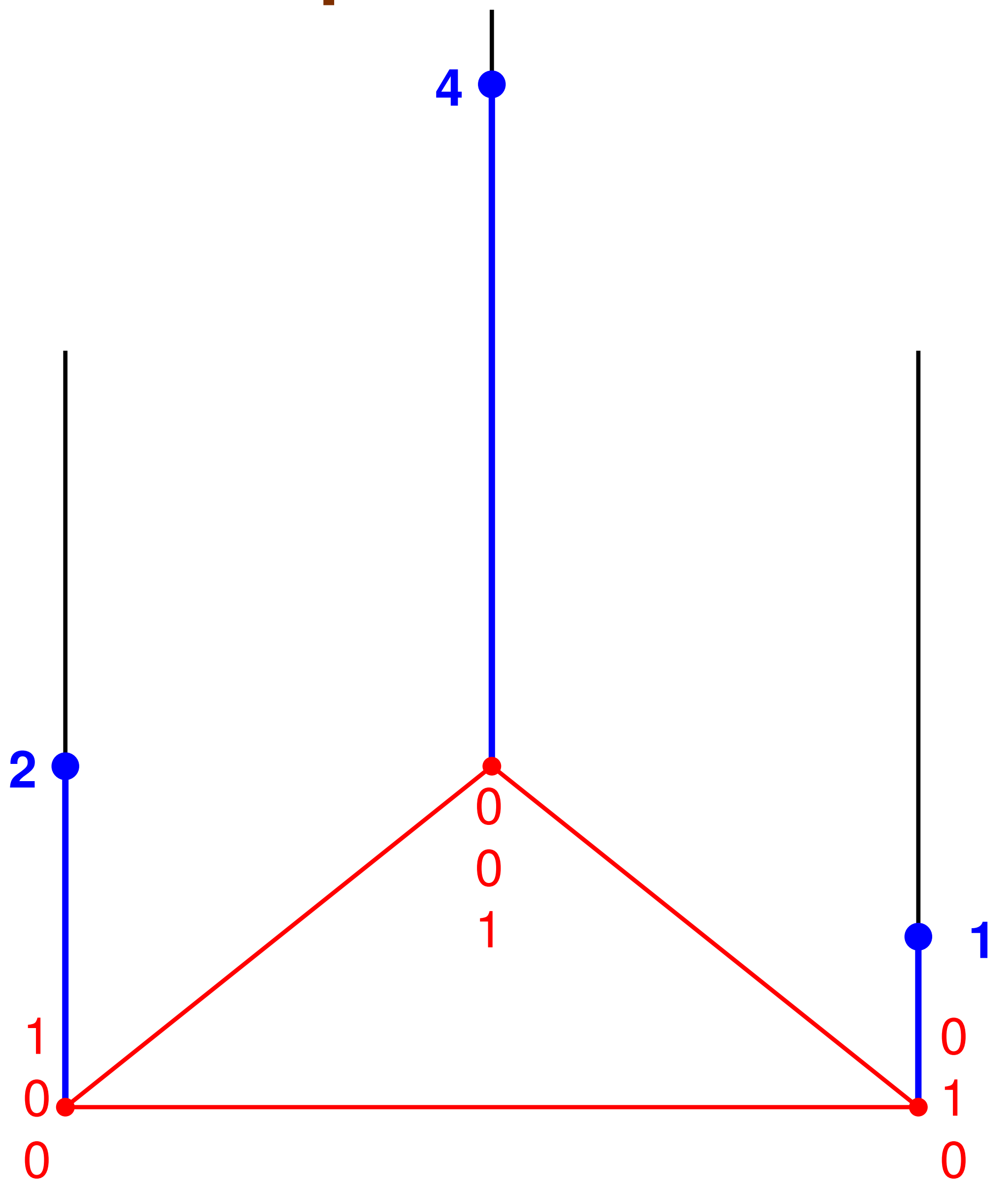
Best responses to mixed strategy of player 1



$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & 3 \\ \hline \end{array} = B$$

payoffs to
player II

Best responses to mixed strategy of player 1

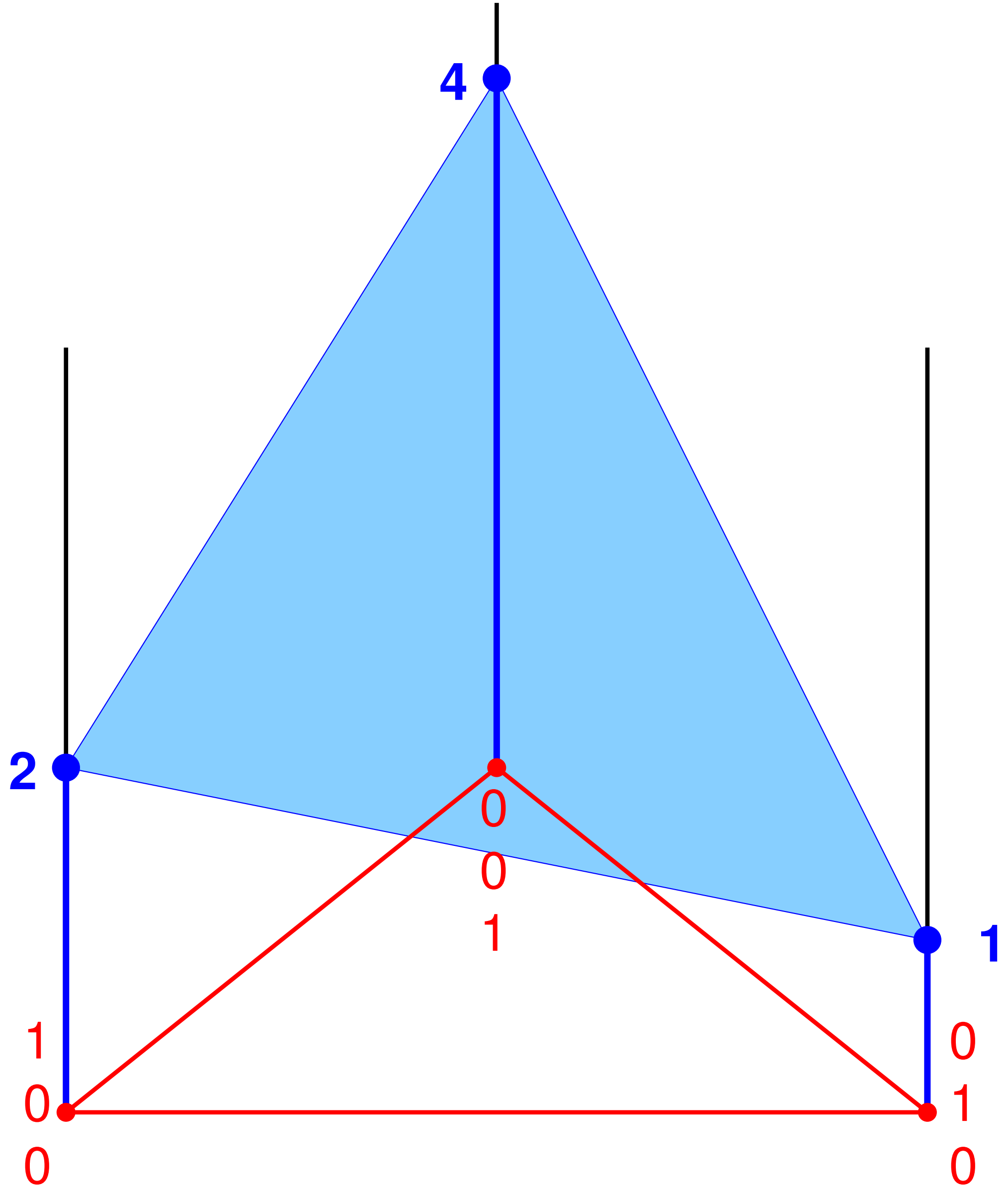


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

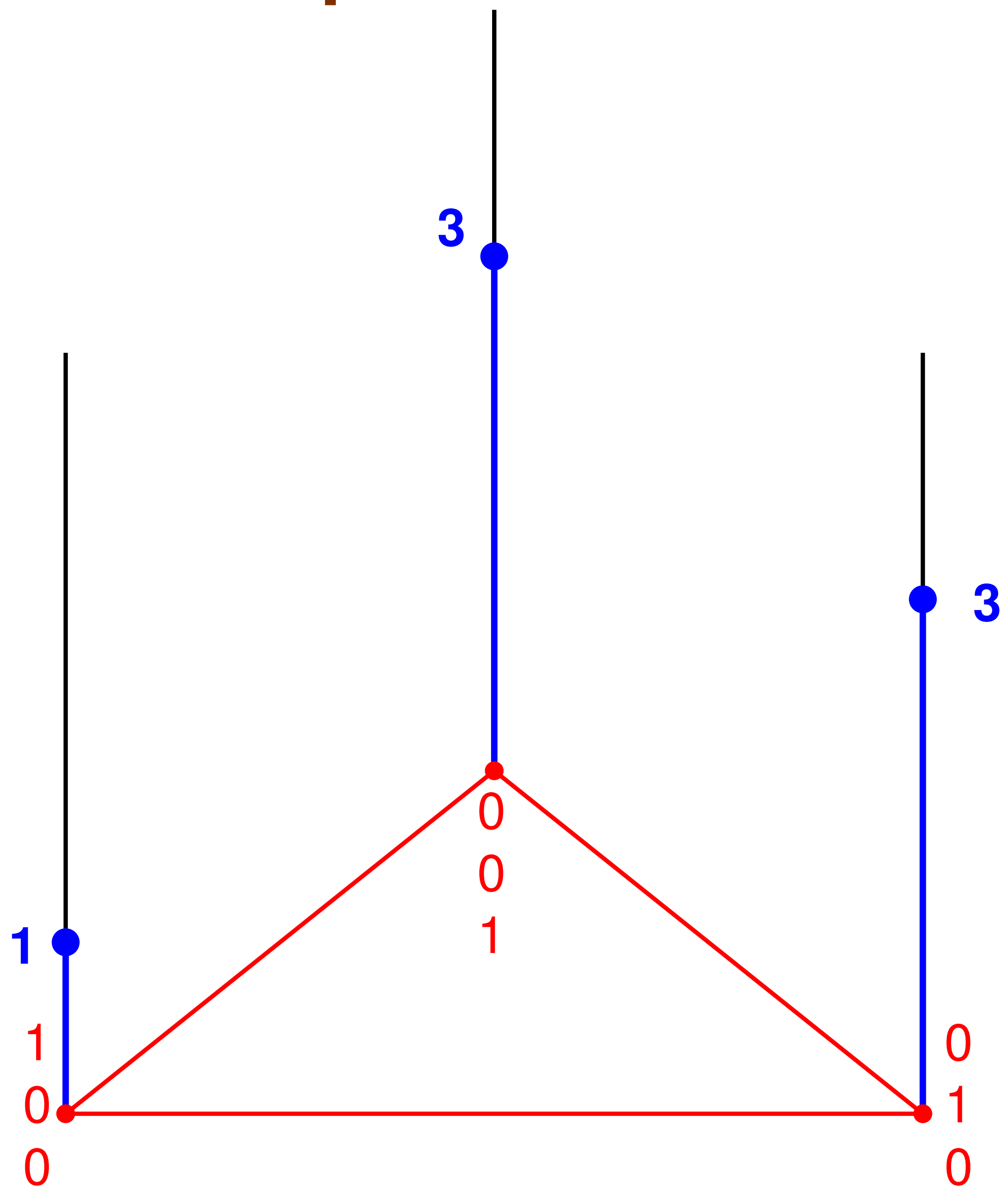


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to player II

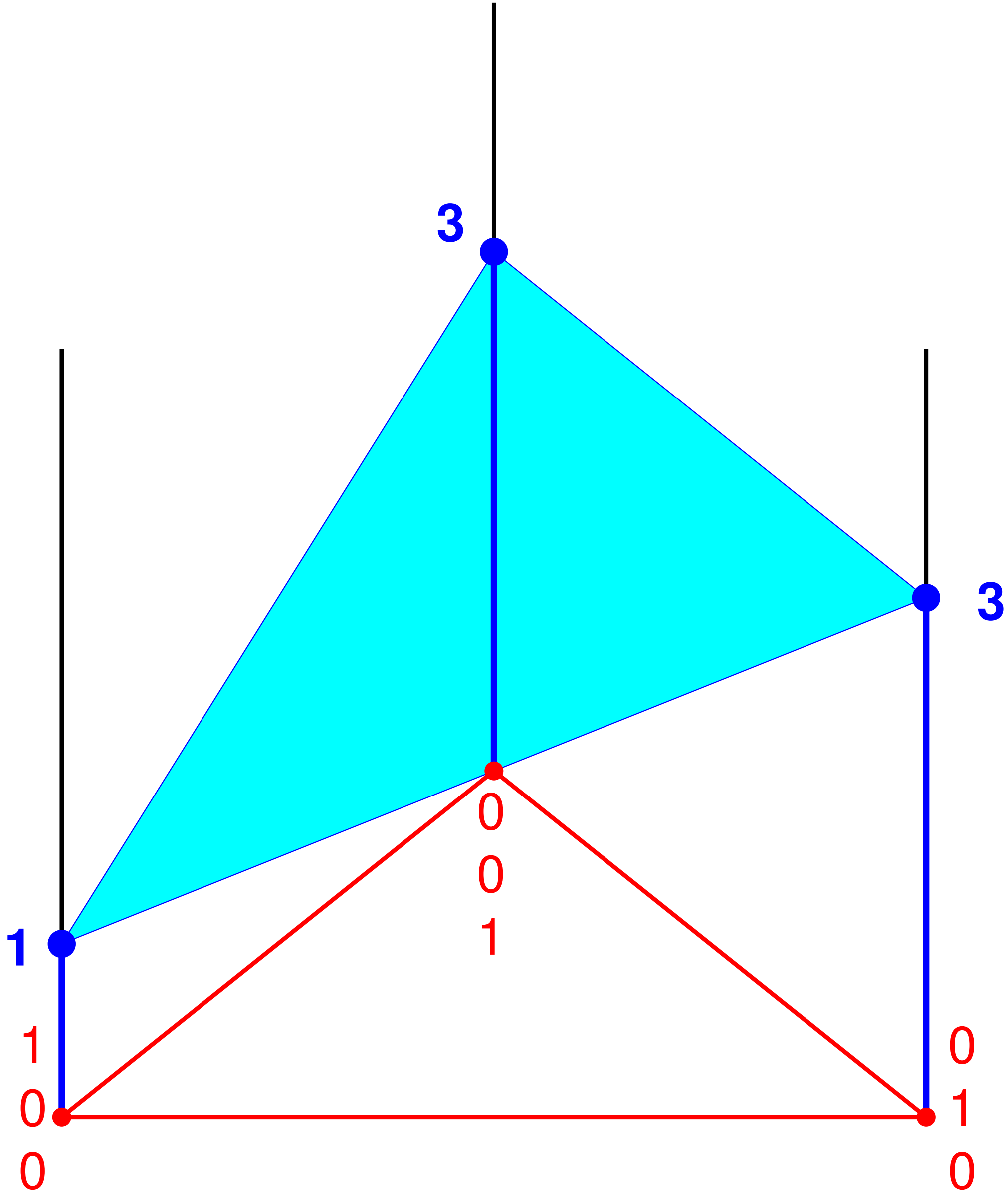
Best responses to mixed strategy of player 1



$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & 3 \\ \hline \end{array} = B$$

payoffs to
player II

Best responses to mixed strategy of player 1



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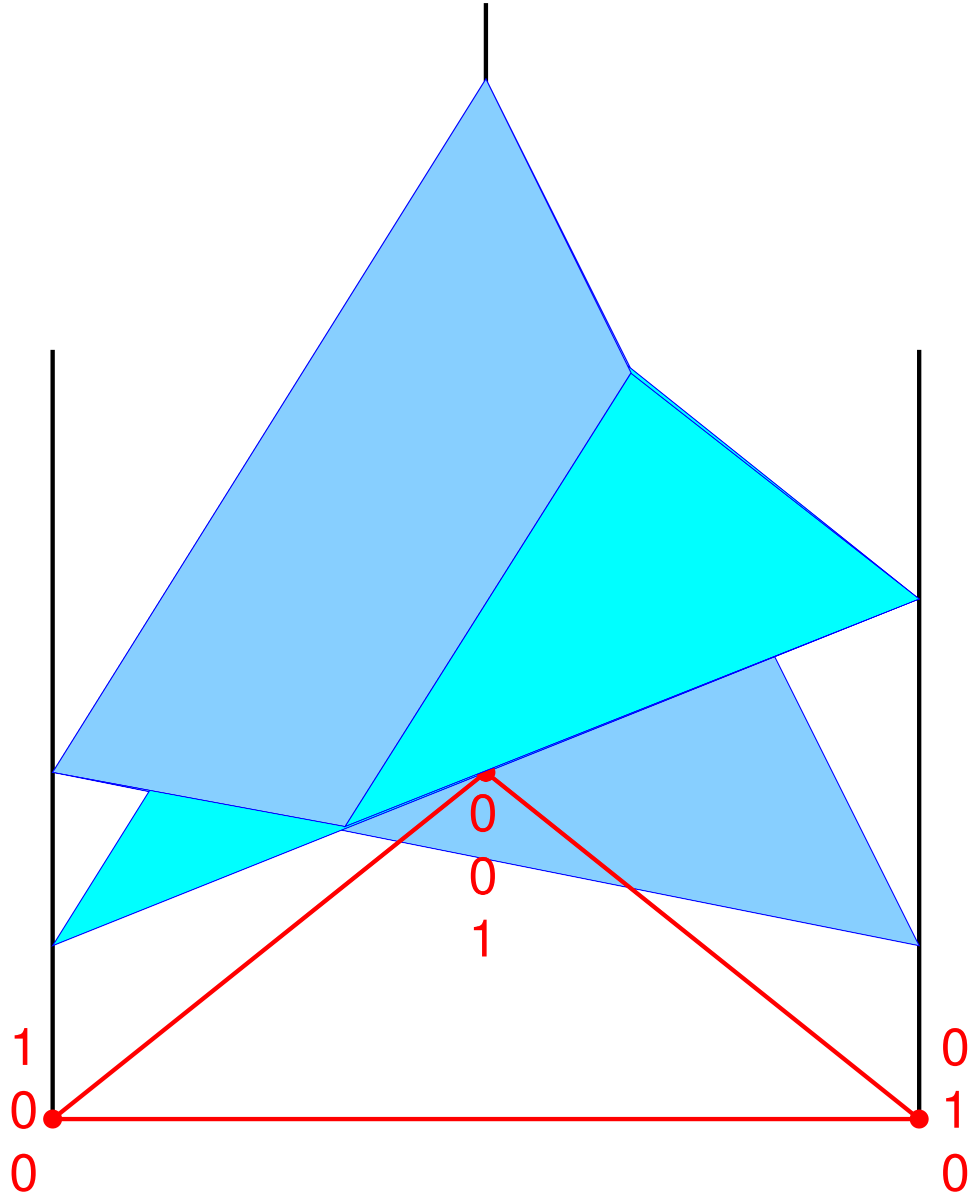
payoffs to
player II

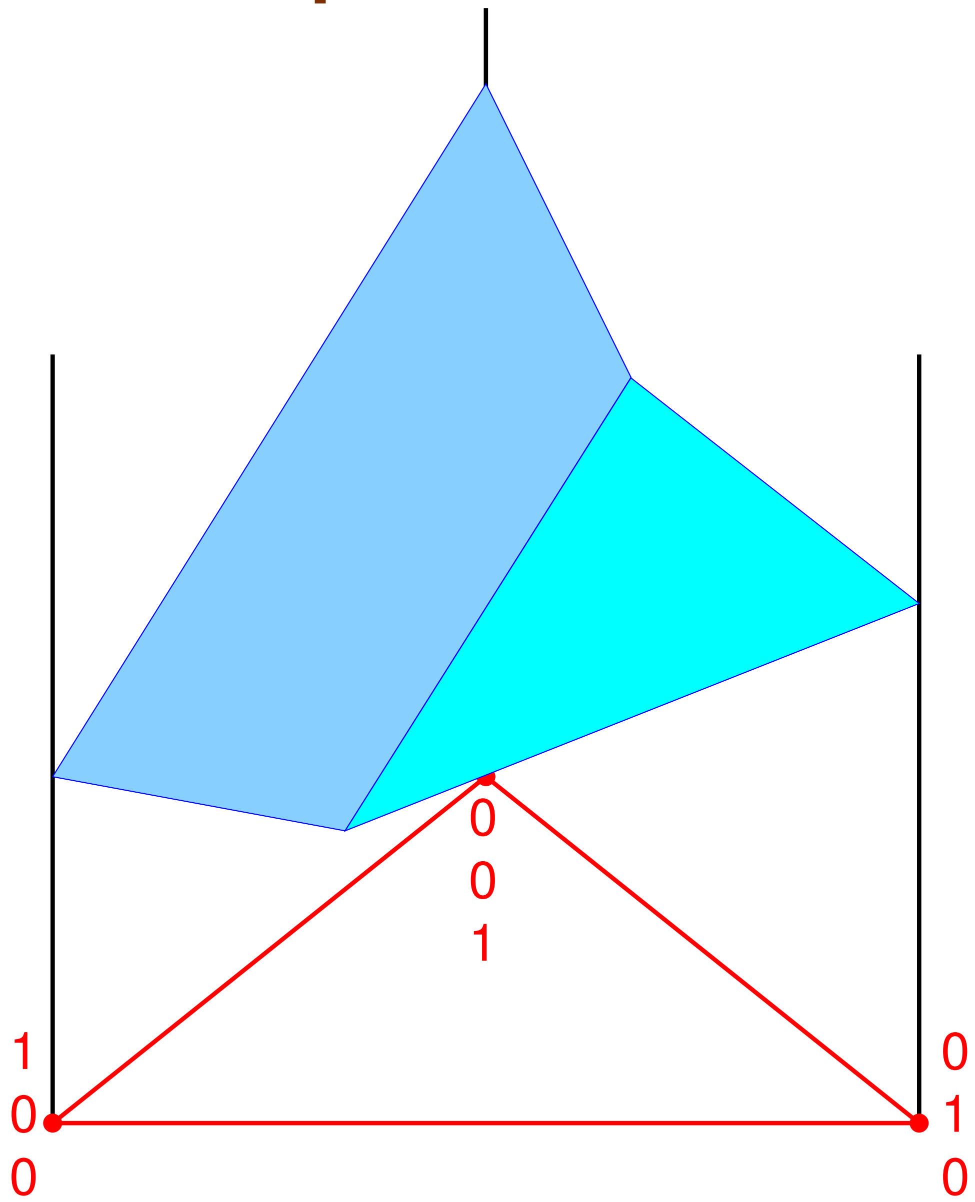
Best responses to mixed strategy of player 1



$$\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & 3 \end{array} = B$$

payoffs to
player II

Best responses to mixed strategy of player 1

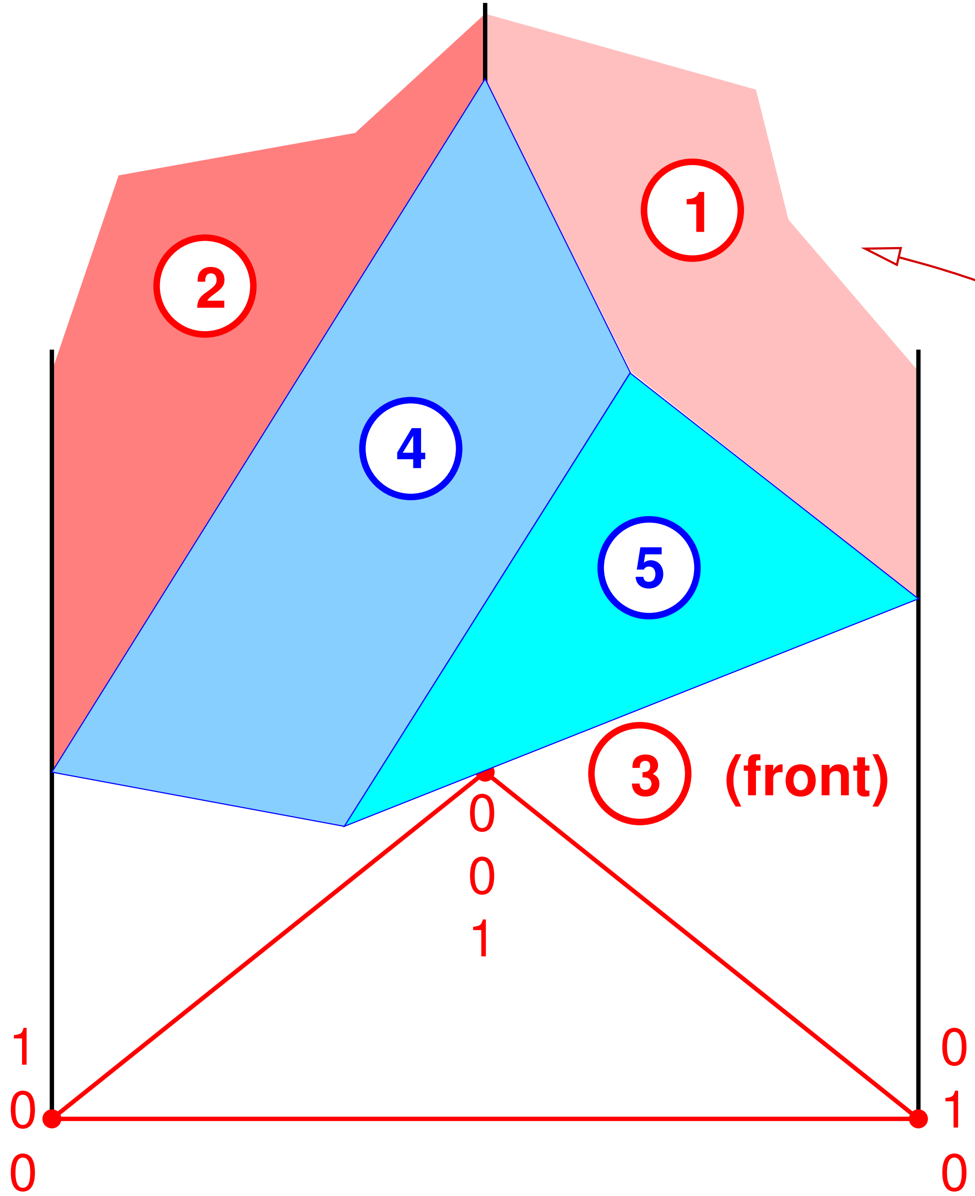


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to player II

Best responses to mixed strategy of player 1



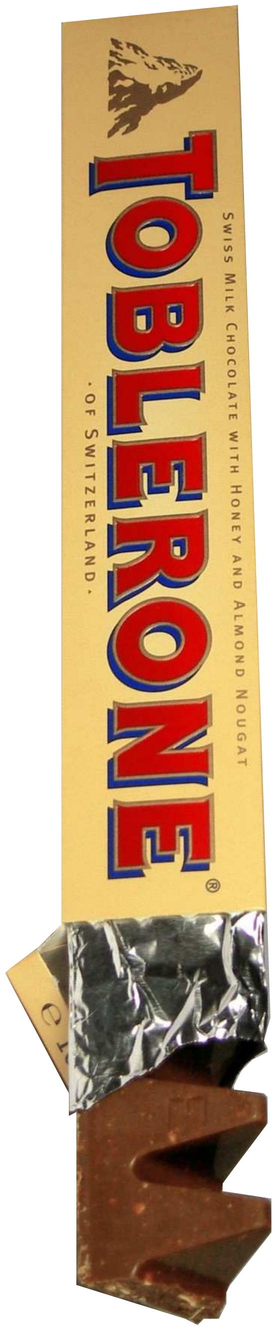
$$\begin{array}{c c} & \begin{array}{cc} 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix} \end{array} = B$$

payoffs to
player II

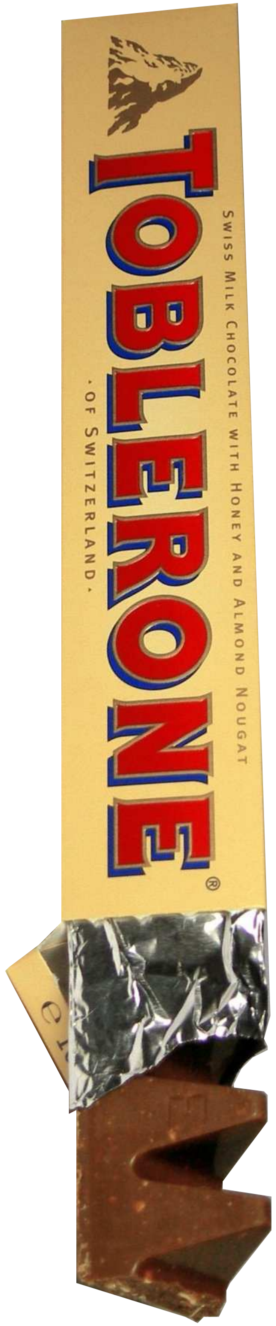
**best response
polyhedron
with facet labels**

Alternative view

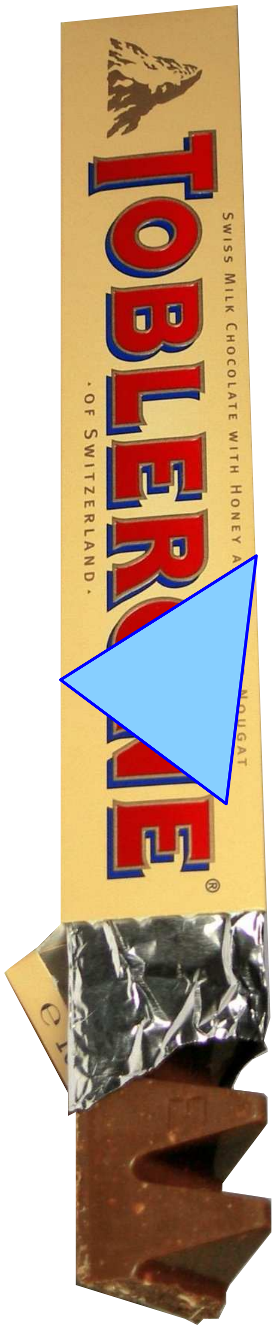
Chop off Toblerone prism



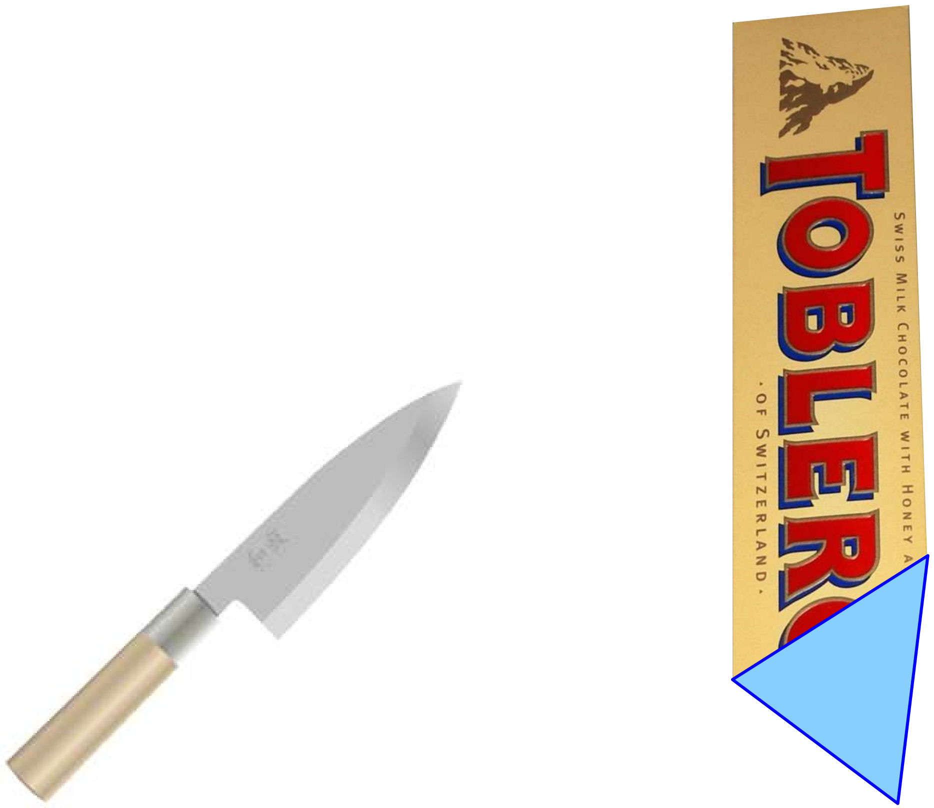
Chop off Toblerone prism



Chop off Toblerone prism

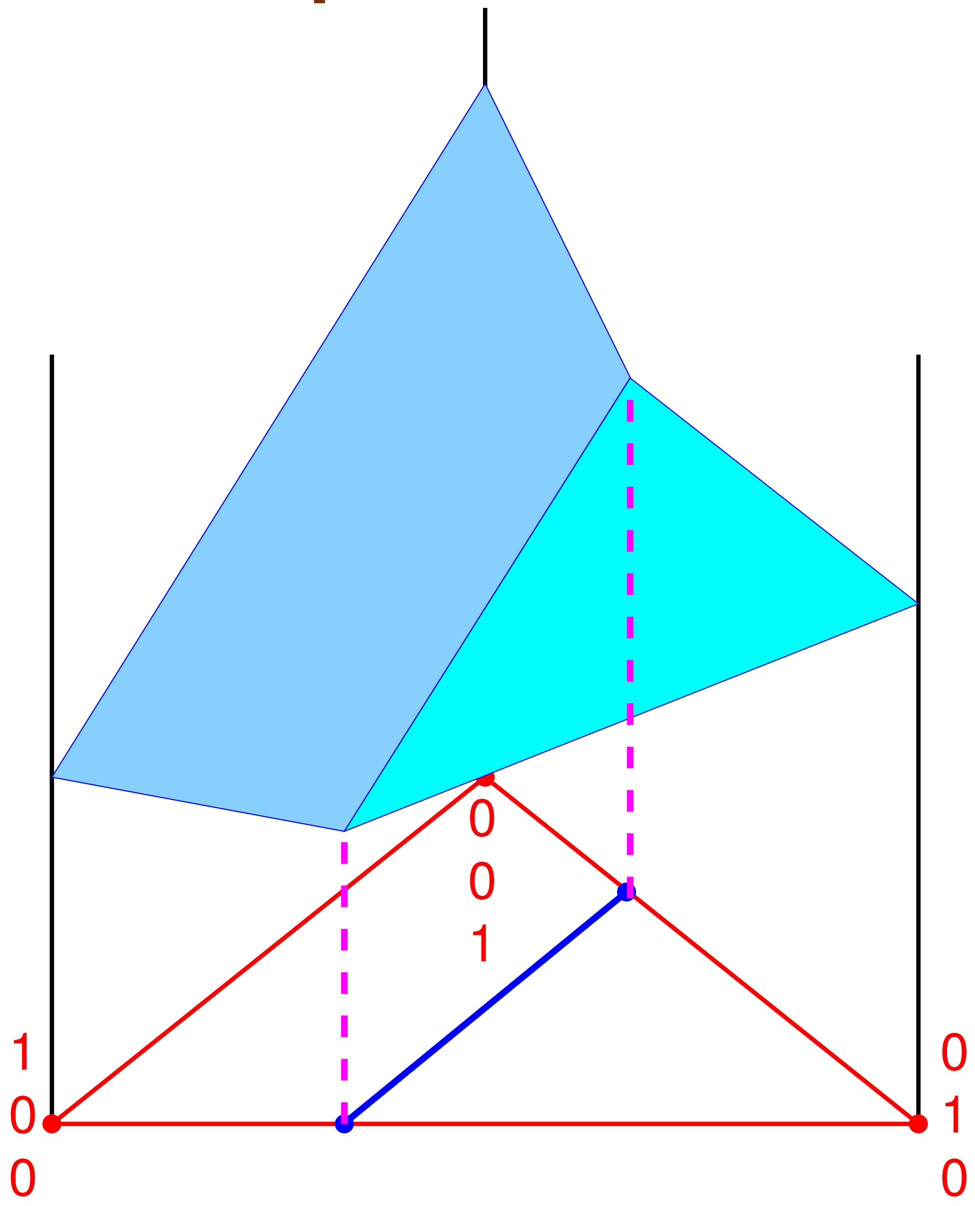


Chop off Toblerone prism



Chop off Toblerone prism

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

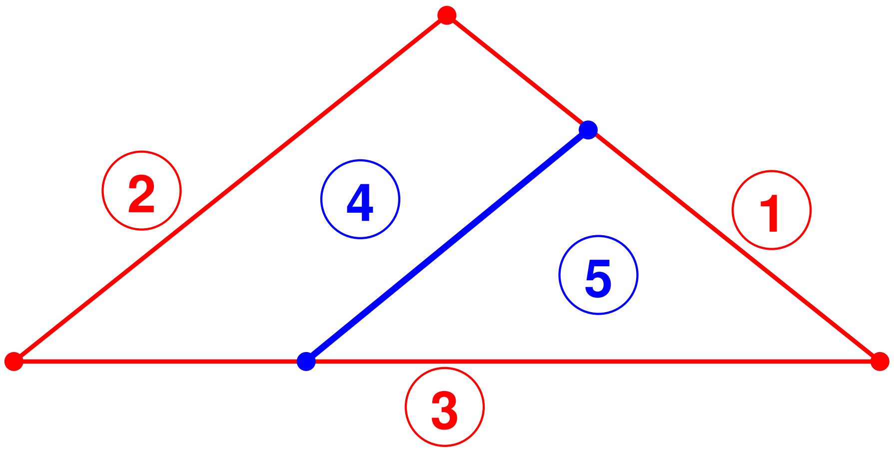
= B

payoffs to
player II

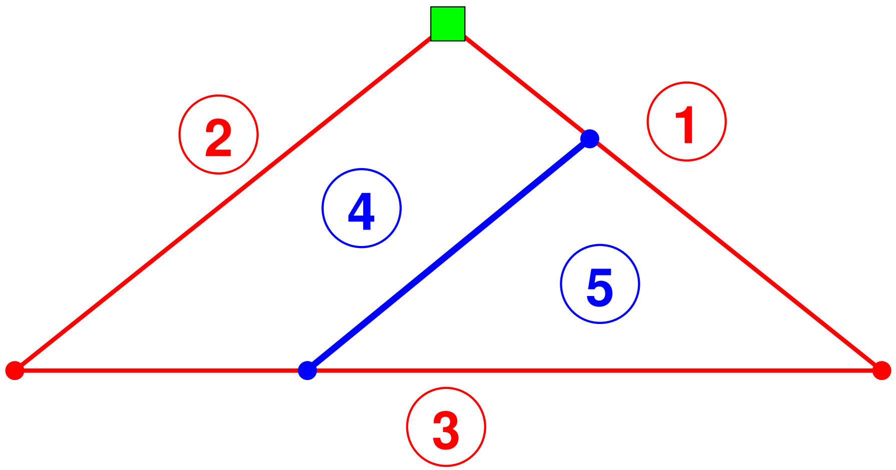
Best responses to mixed strategy of player 1

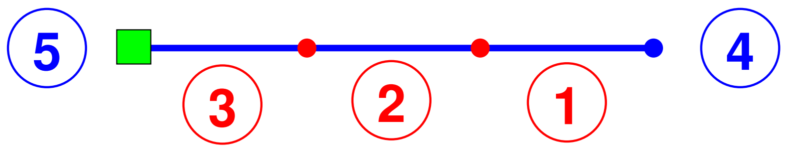
$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & 3 \\ \hline \end{array} = \mathbf{B}$$

payoffs to
player II

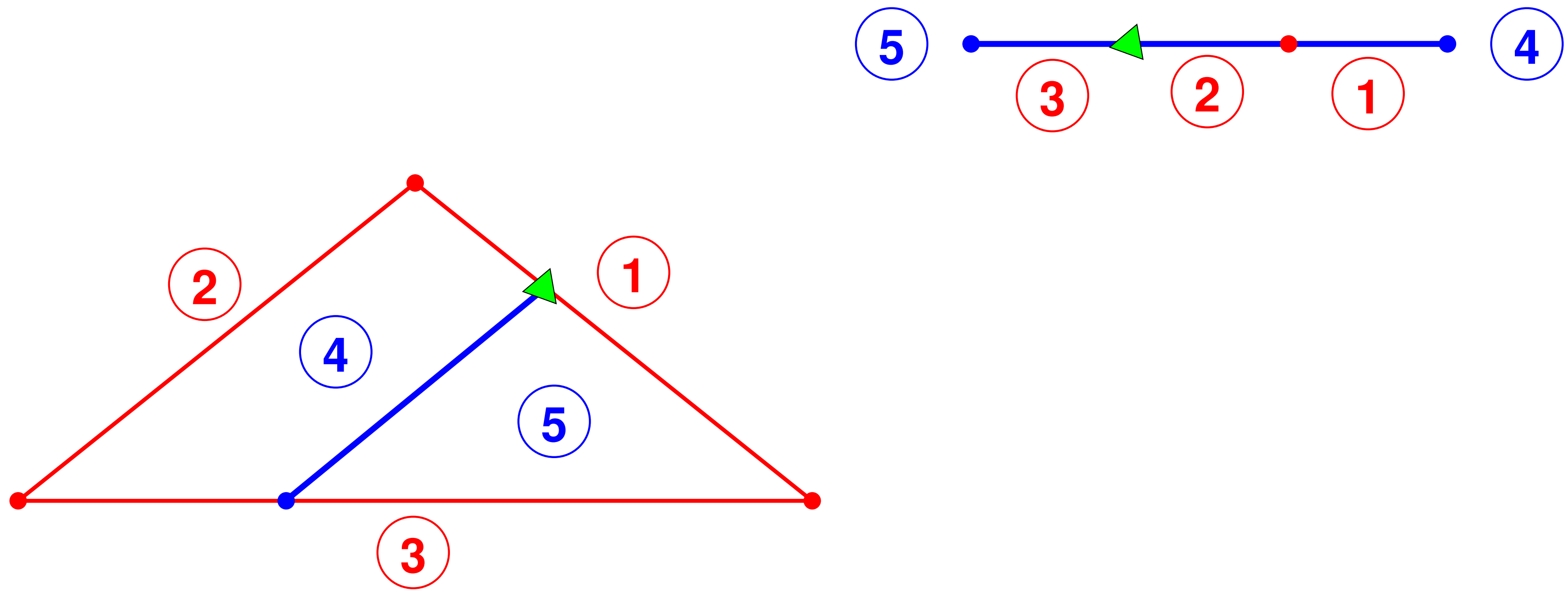


Equilibrium = completely labeled strategy pair

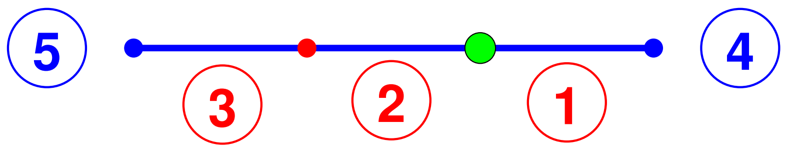


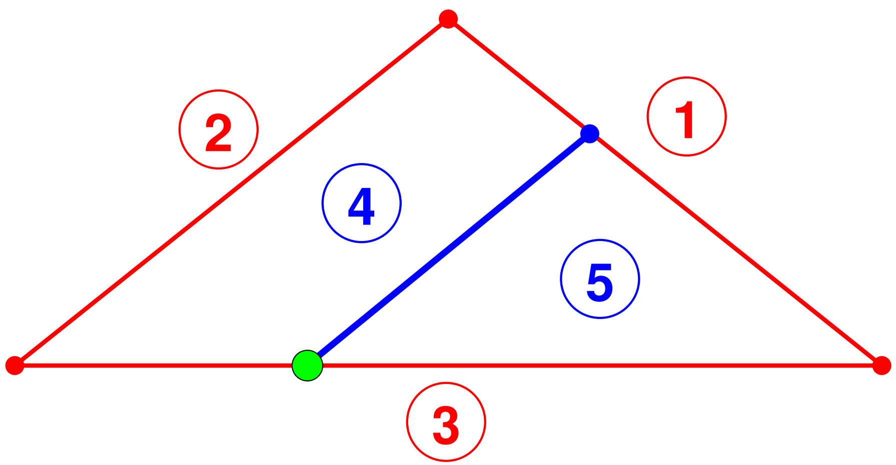


Equilibrium = completely labeled strategy pair



Equilibrium = completely labeled strategy pair





Constructing games using geometry

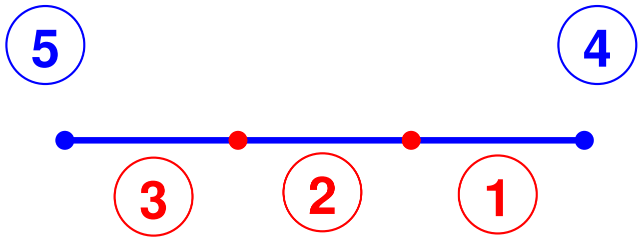
low dimension: 2, 3, (4) pure strategies:

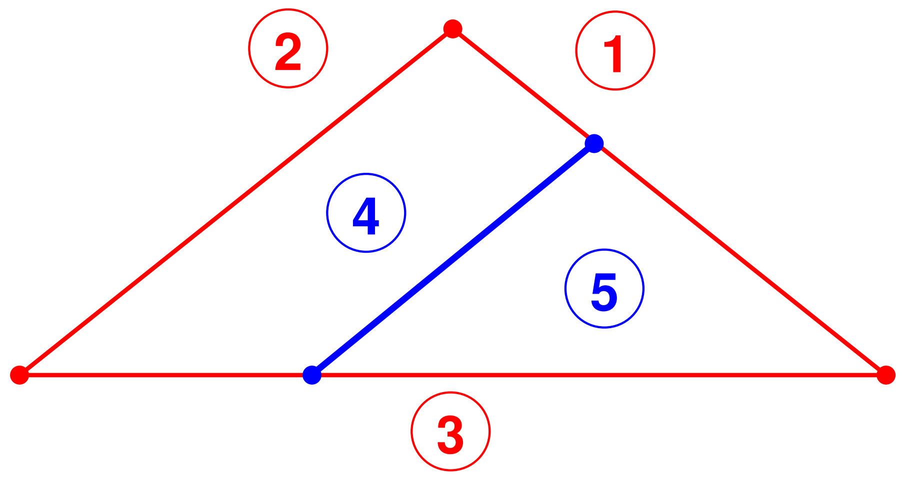
subdivide mixed strategy simplex into
response regions, label suitably

high dimension:

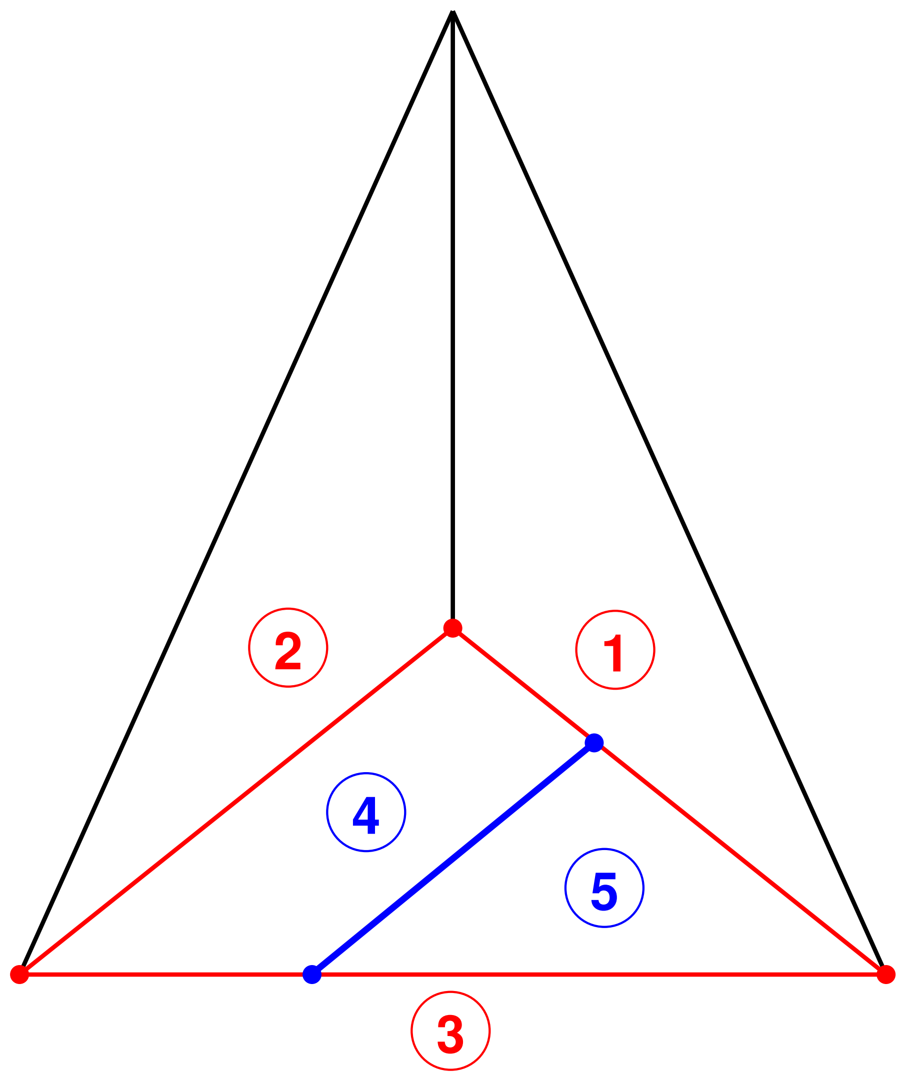
use polytopes with known combinatorial structure
e.g. for constructing games with many equilibria,
or long Lemke-Howson computations
[Savani & von Stengel, *FOCS 2004*,
Econometrica 2006]

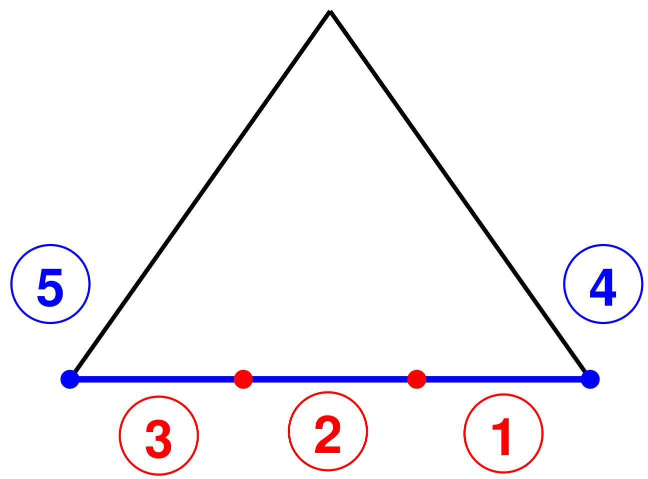
The Lemke–Howson algorithm





The Lemke–Howson algorithm





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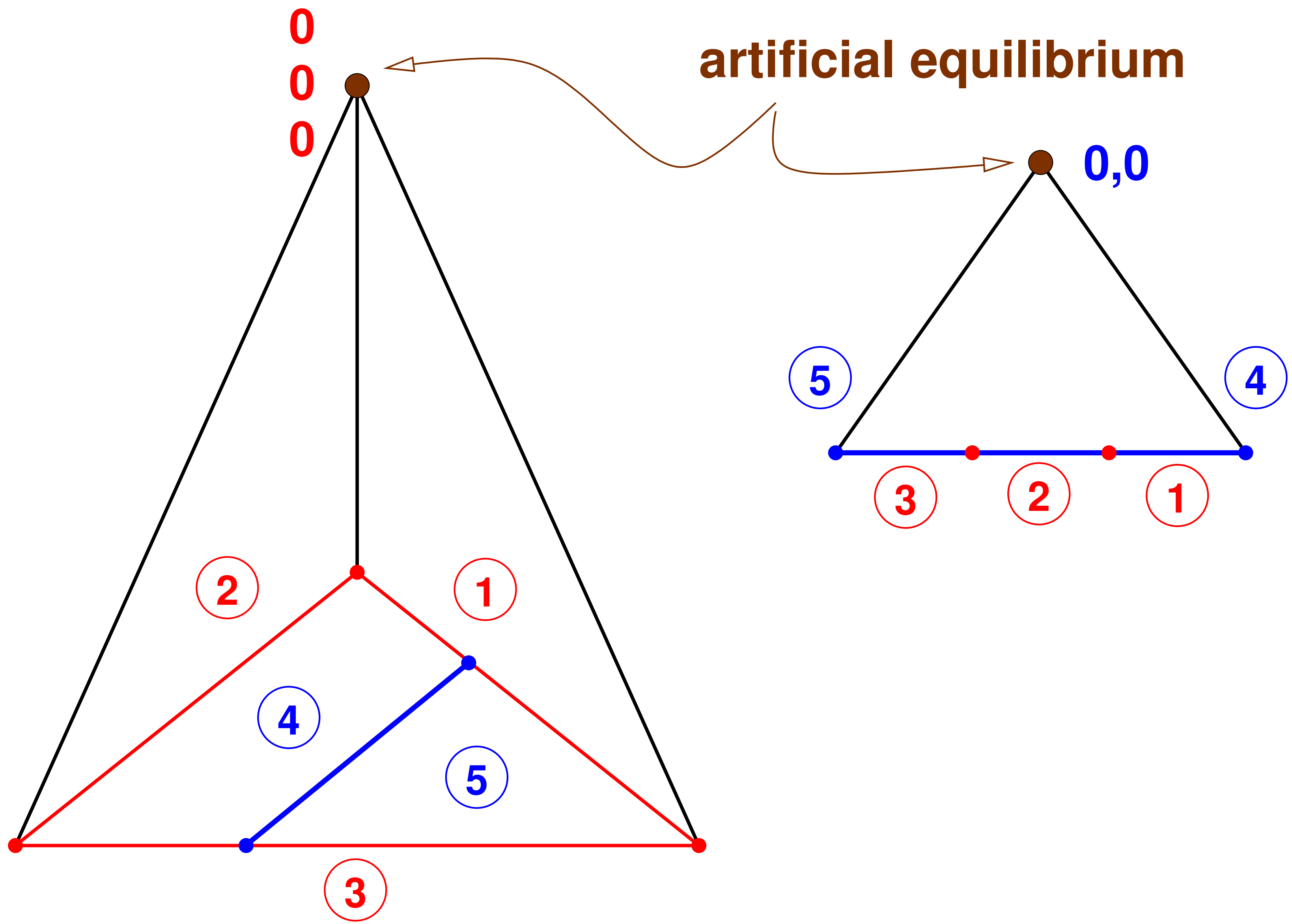

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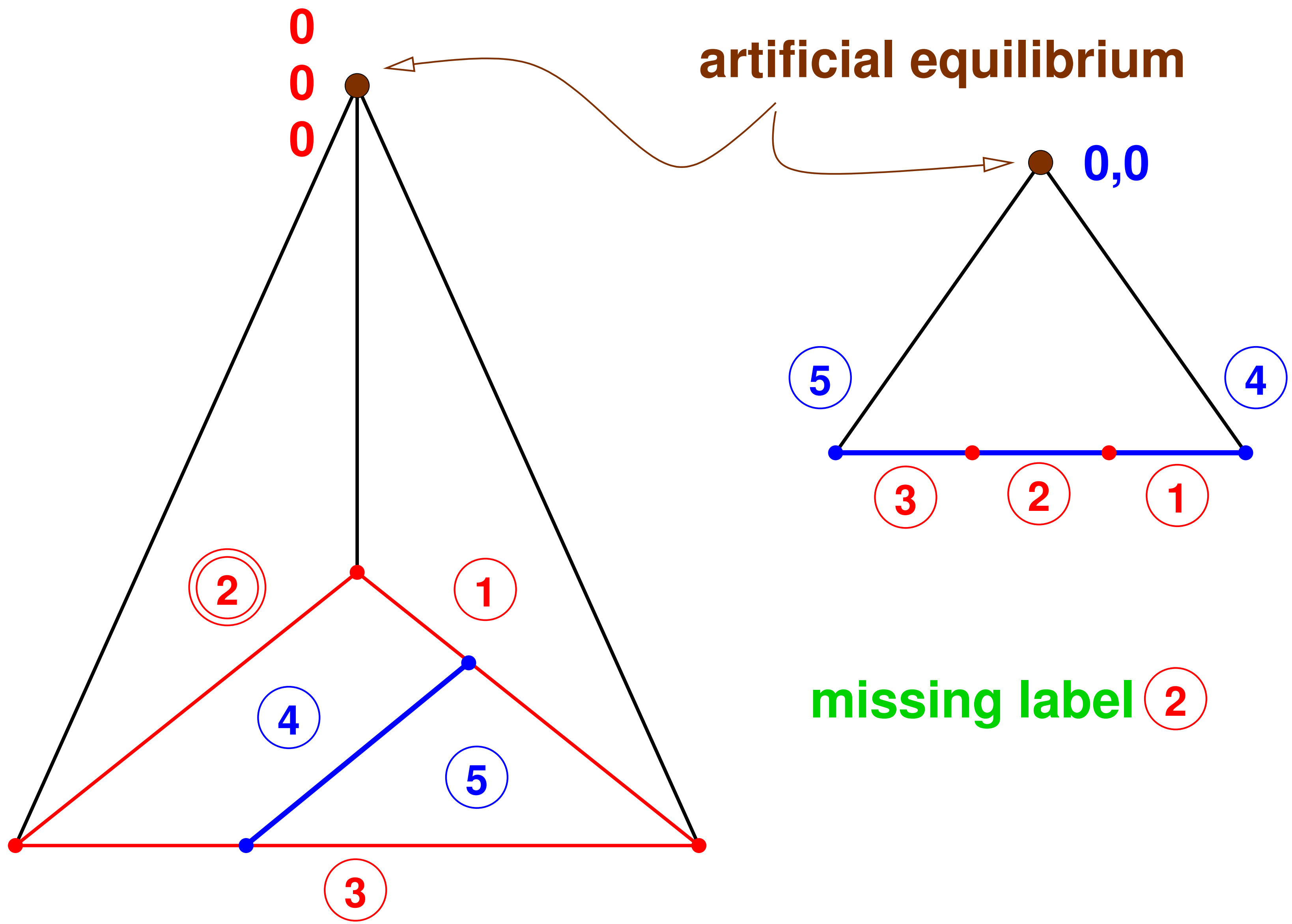

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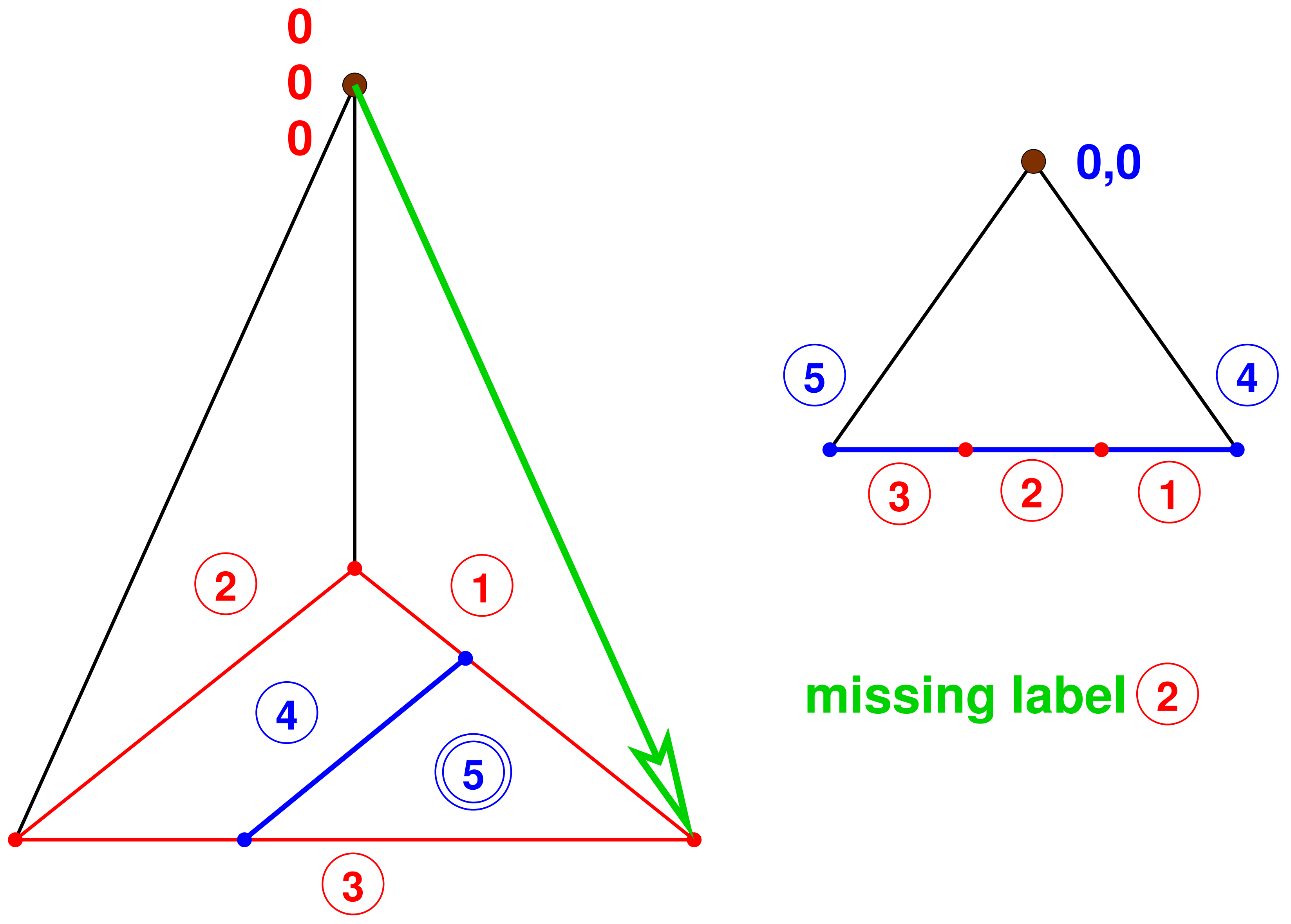

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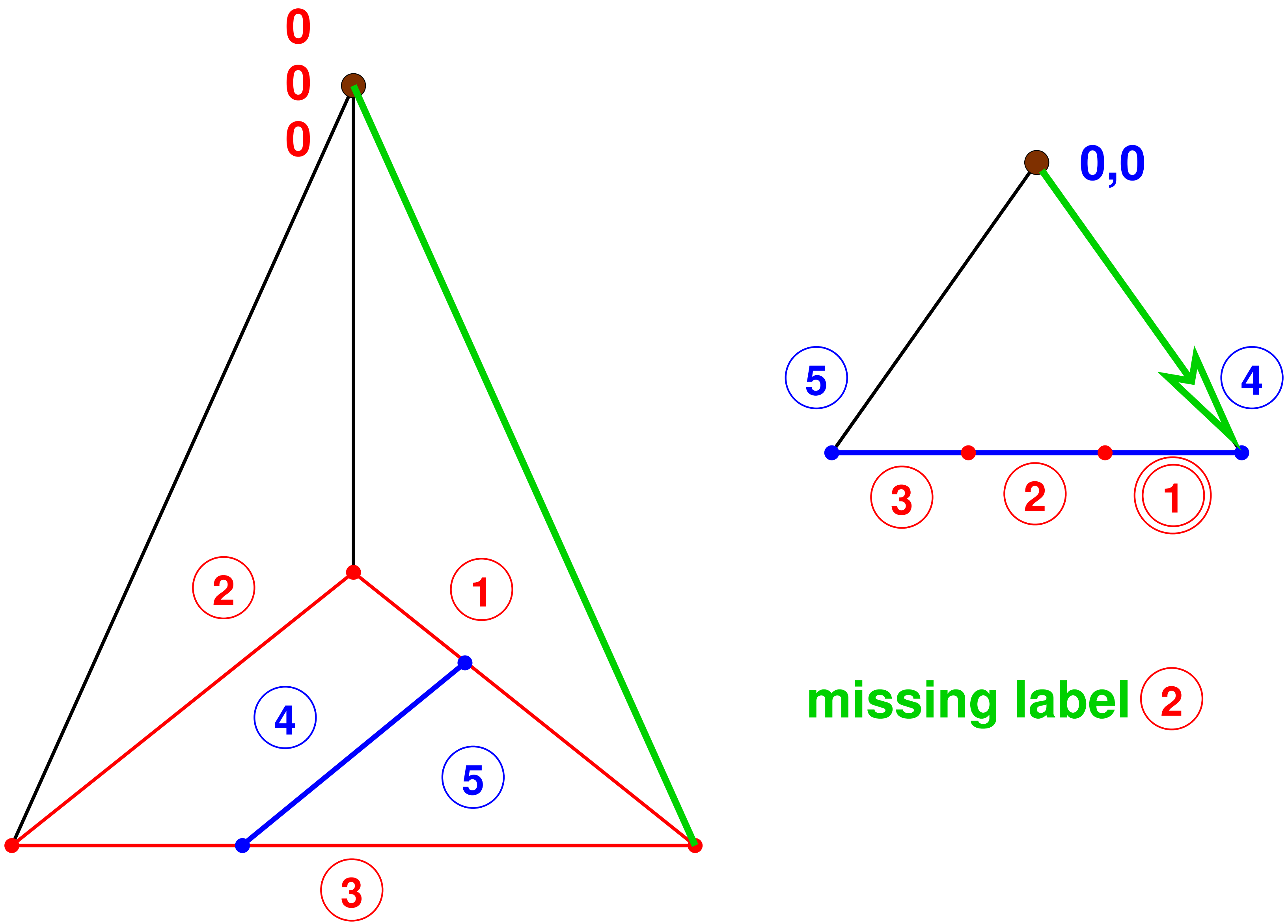

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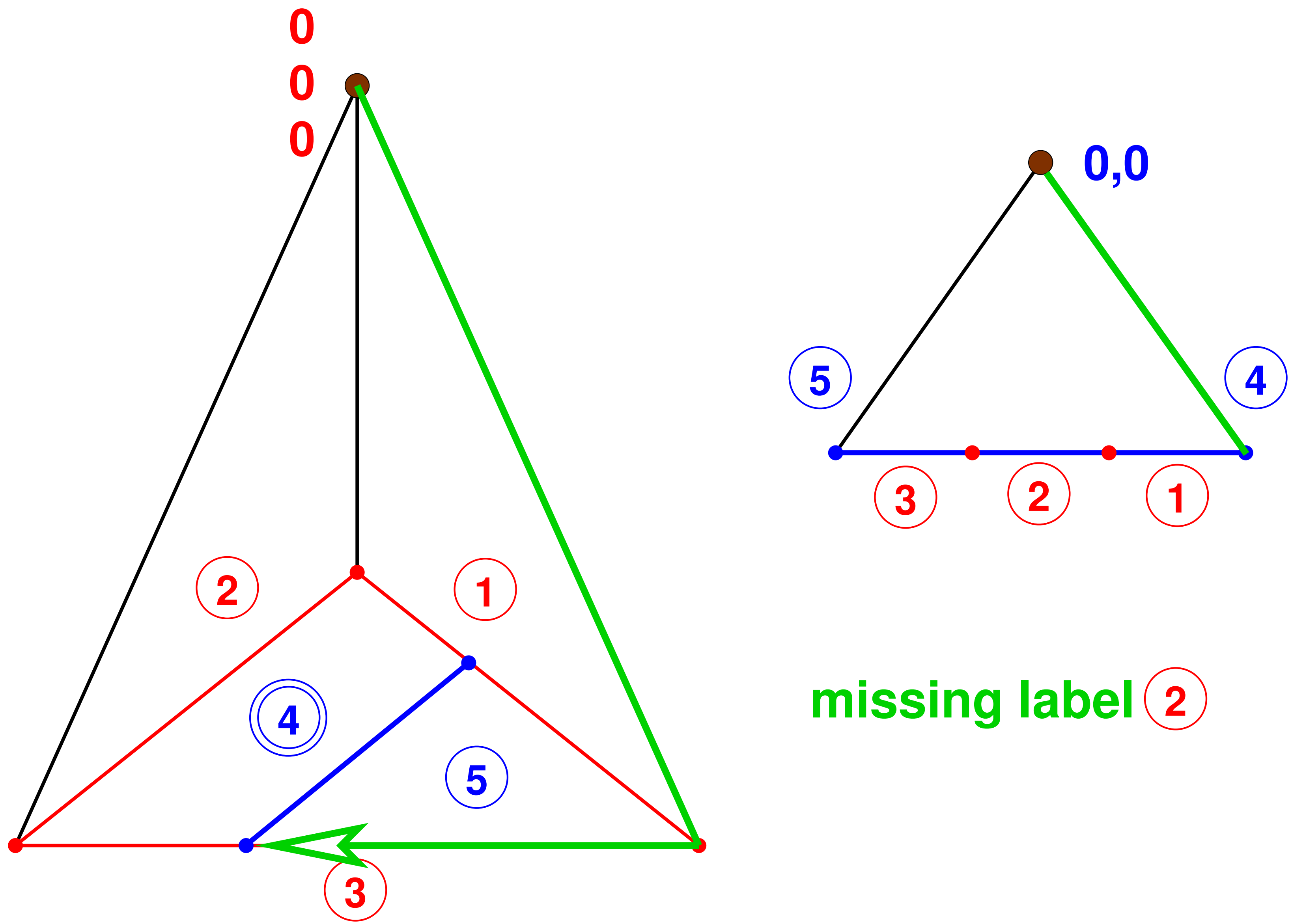

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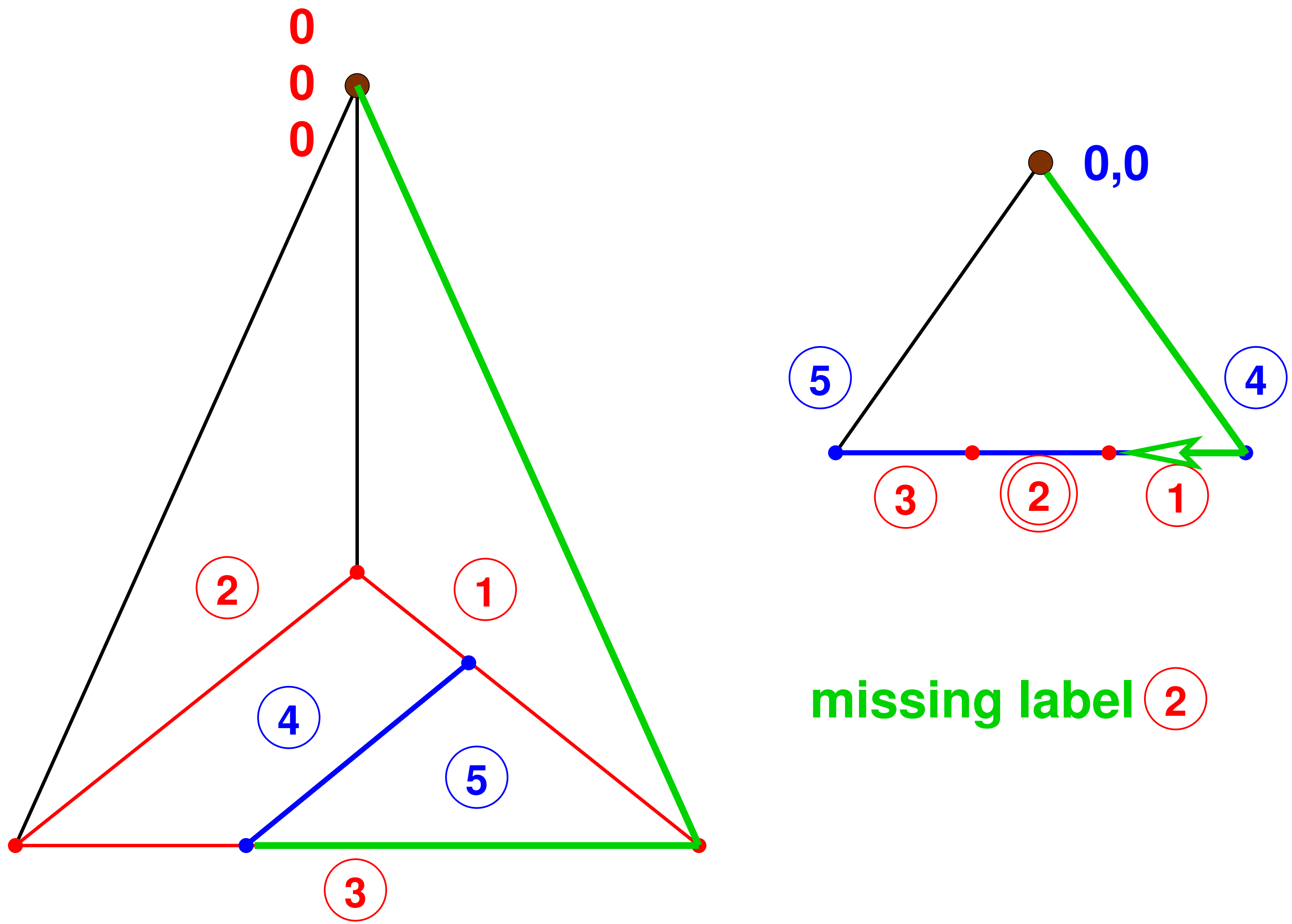

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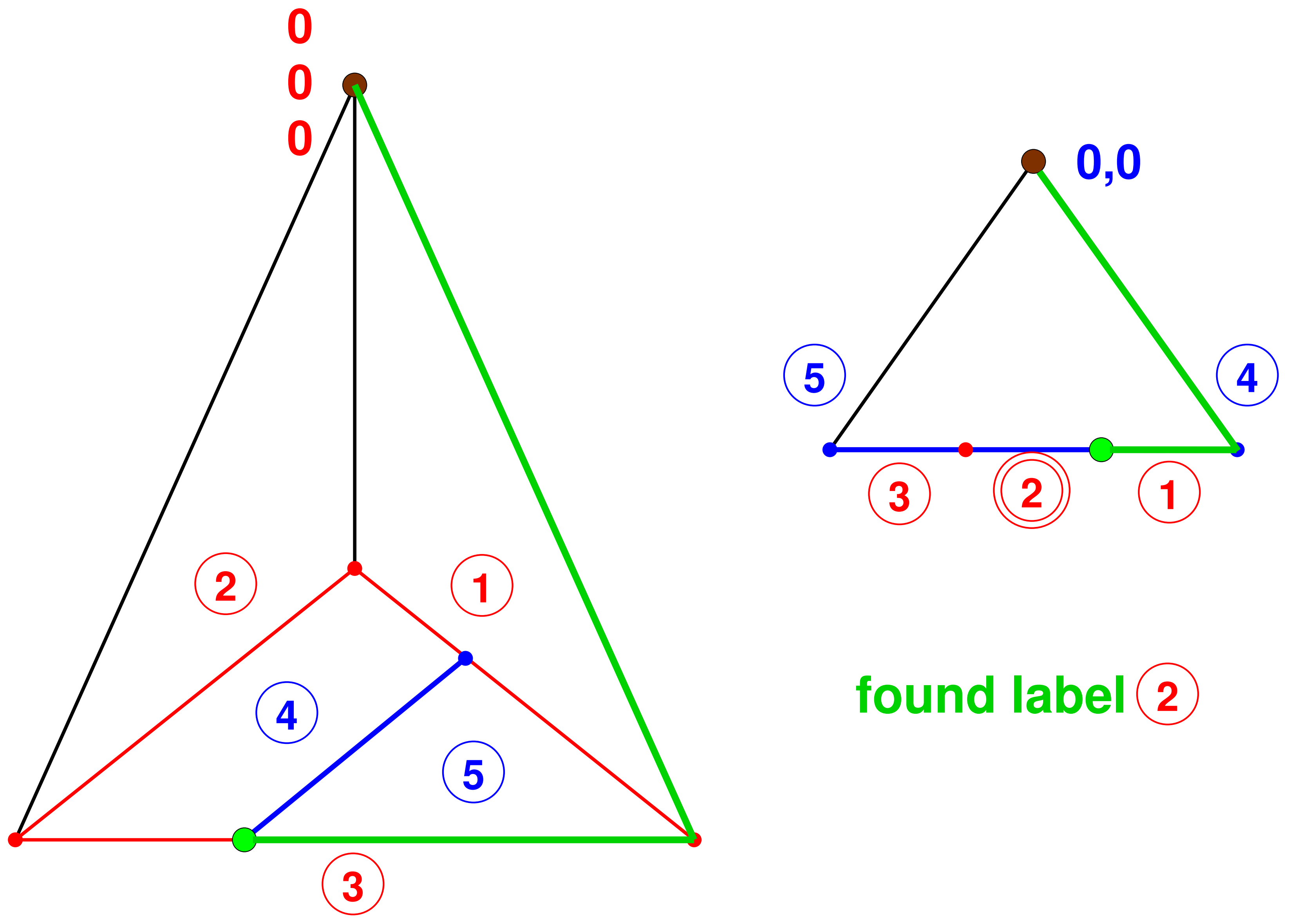


Why Lemke-Howson works

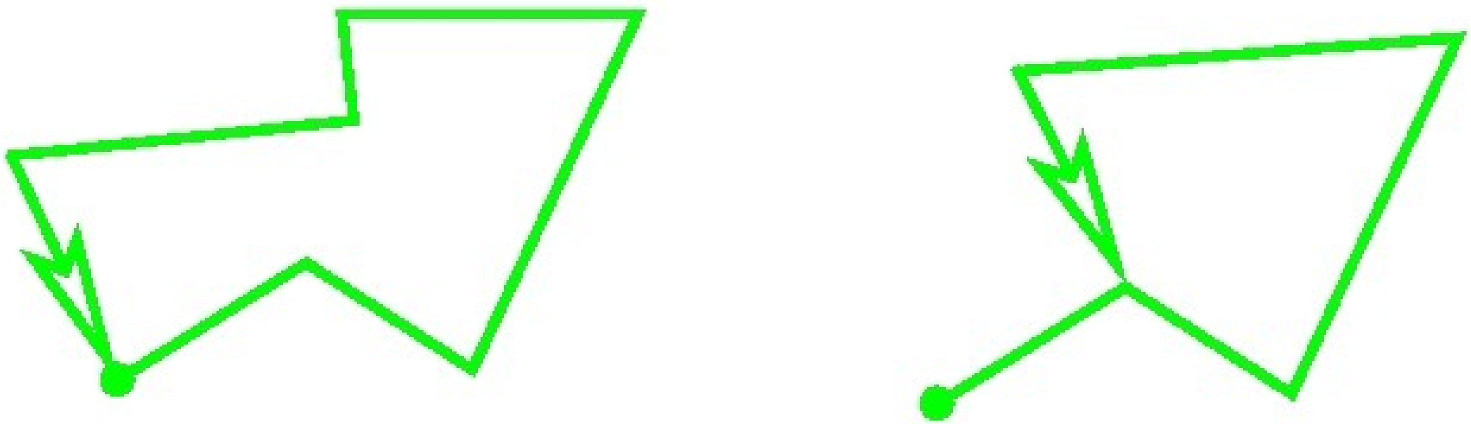
LH finds at least one Nash equilibrium because

- finitely many "vertices"

for nondegenerate (generic) games:

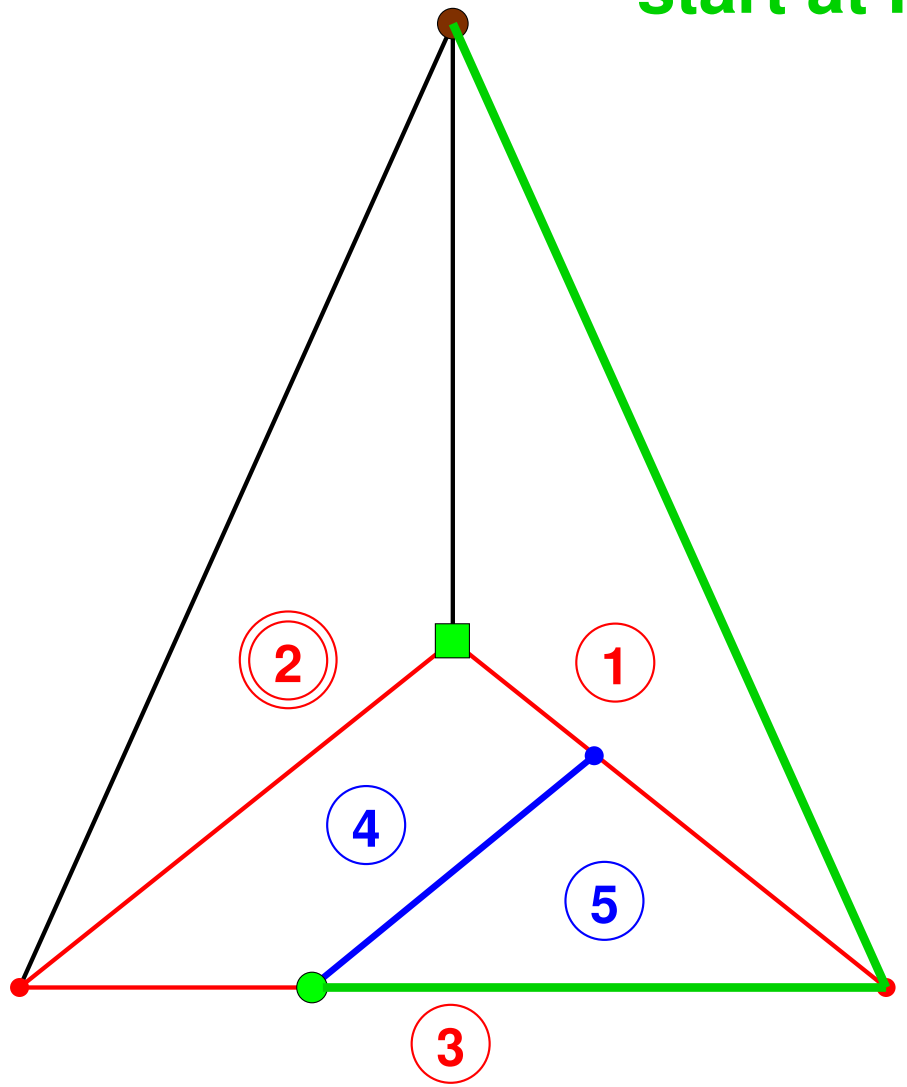
- **unique** starting edge given missing label
- **unique** continuation

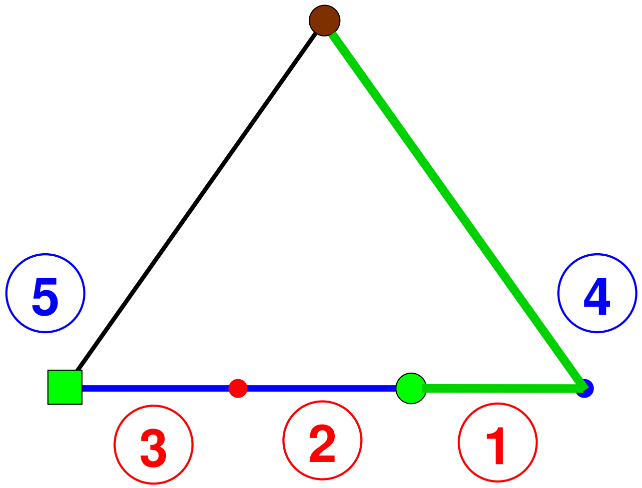
$\Rightarrow$ precludes "coming back" like here:



The Lemke–Howson algorithm

start at Nash equilibrium

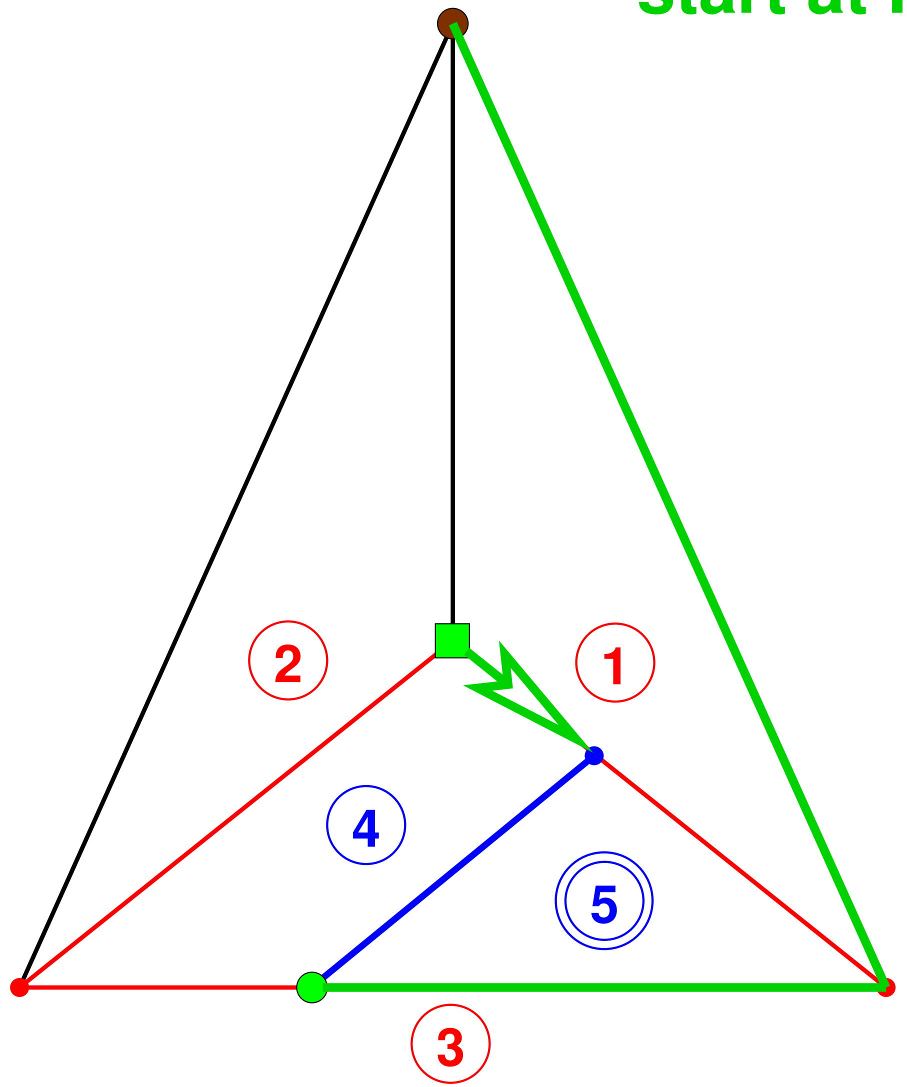


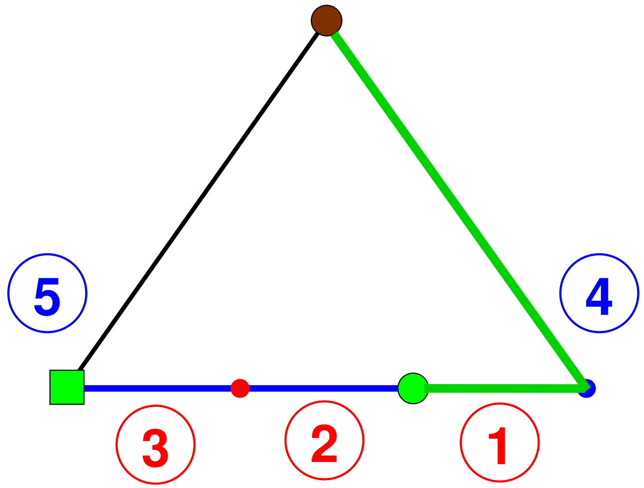


missing label 2

The Lemke–Howson algorithm

start at Nash equilibrium

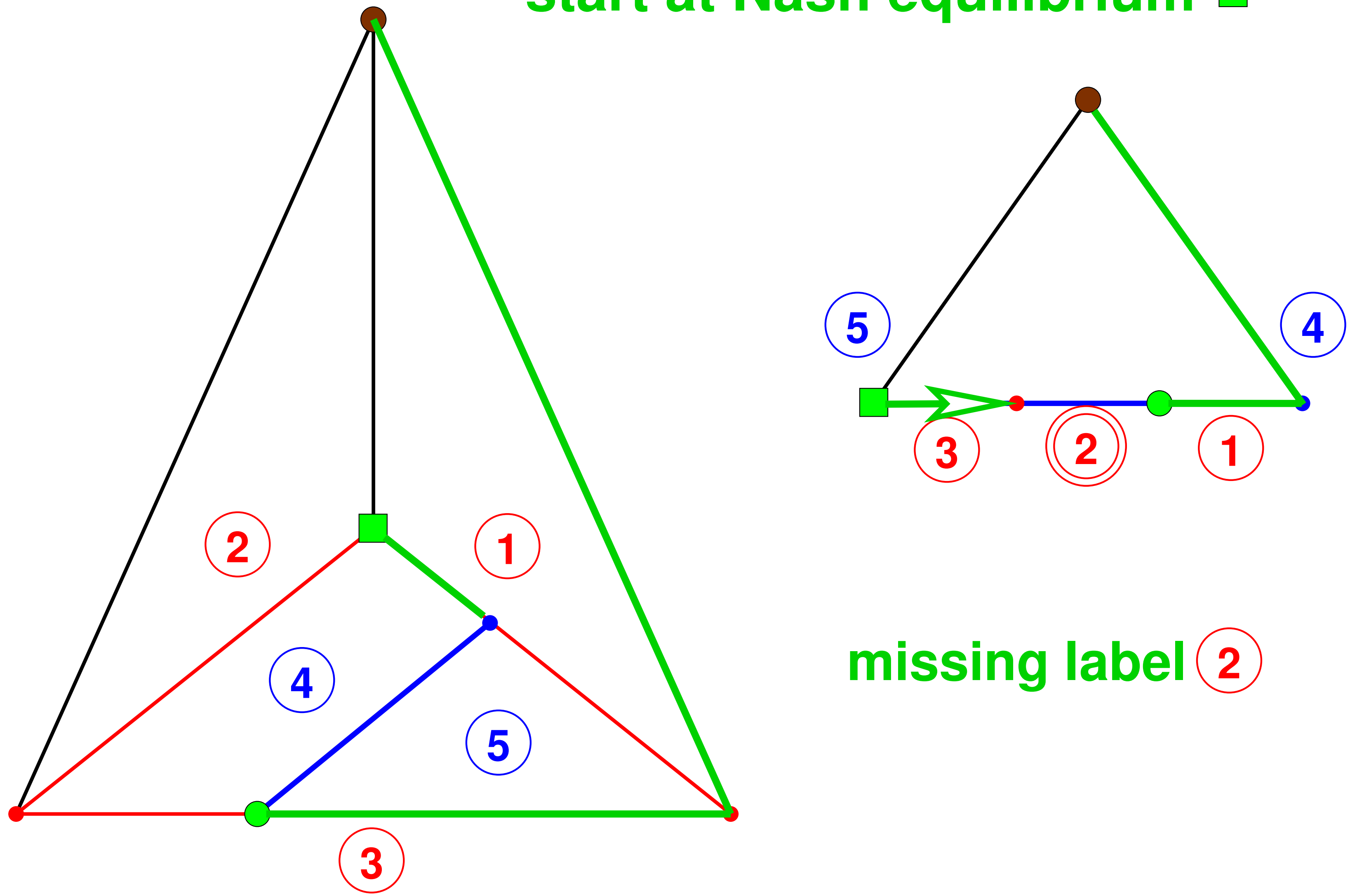




missing label 2

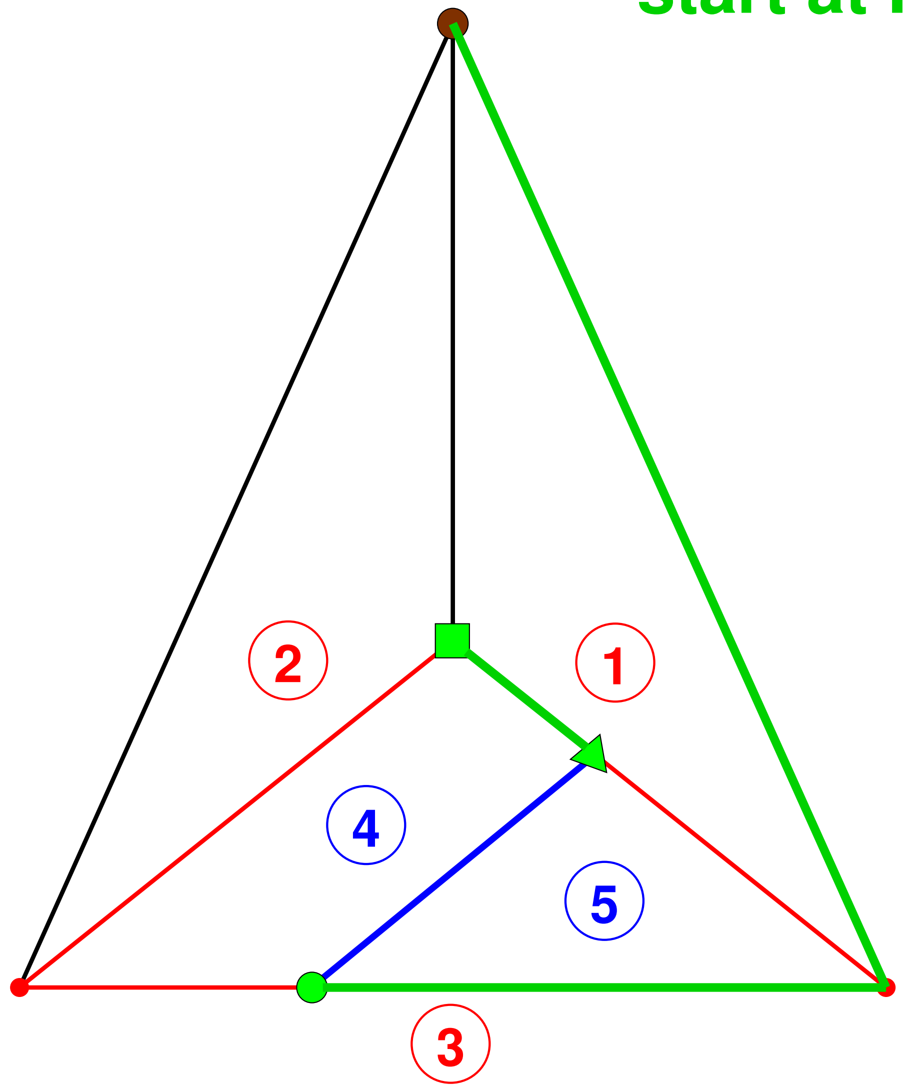
The Lemke–Howson algorithm

start at Nash equilibrium

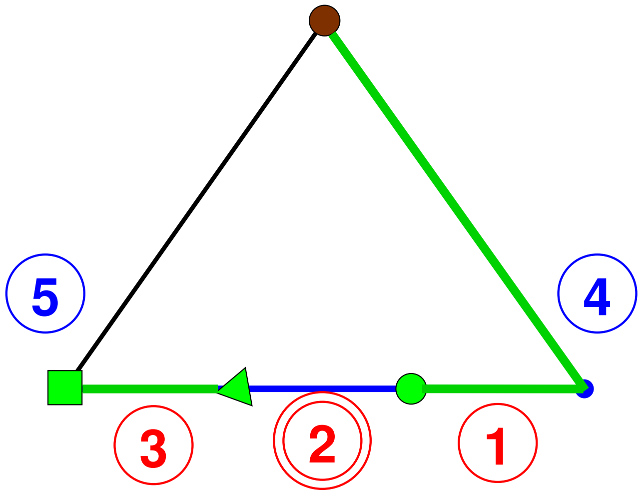


missing label 2

Odd number of Nash equilibria!



start at Nash equilibrium ■



found label 2

Nondegenerate bimatrix games

Given: $m \times n$ bimatrix game (A, B)

$X = \{ x \in \mathbf{R}^m \mid x \geq \mathbf{0},\ x_1 + \ldots + x_m = 1 \}$
$Y = \{ y \in \mathbf{R}^n \mid y \geq \mathbf{0},\ y_1 + \ldots + y_n = 1 \}$

$\text{supp}(x) = \{ i \mid x_i > 0 \}$
$\text{supp}(y) = \{ j \mid y_j > 0 \}$

(A, B) **nondegenerate** $\Leftrightarrow$ $\forall\, x \in X,\ y \in Y$:

$| \{ j \mid j \text{ best response to } x \} | \leq | \text{supp}(x) |,$
$| \{ i \mid i \text{ best response to } y \} | \leq | \text{supp}(y) |.$

Nondegeneracy via labels

	$m \times n$ bimatrix game (A,B) **nondegenerate**
$\Leftrightarrow$	no $x \in X$ has more than m labels, no $y \in Y$ has more than n labels.

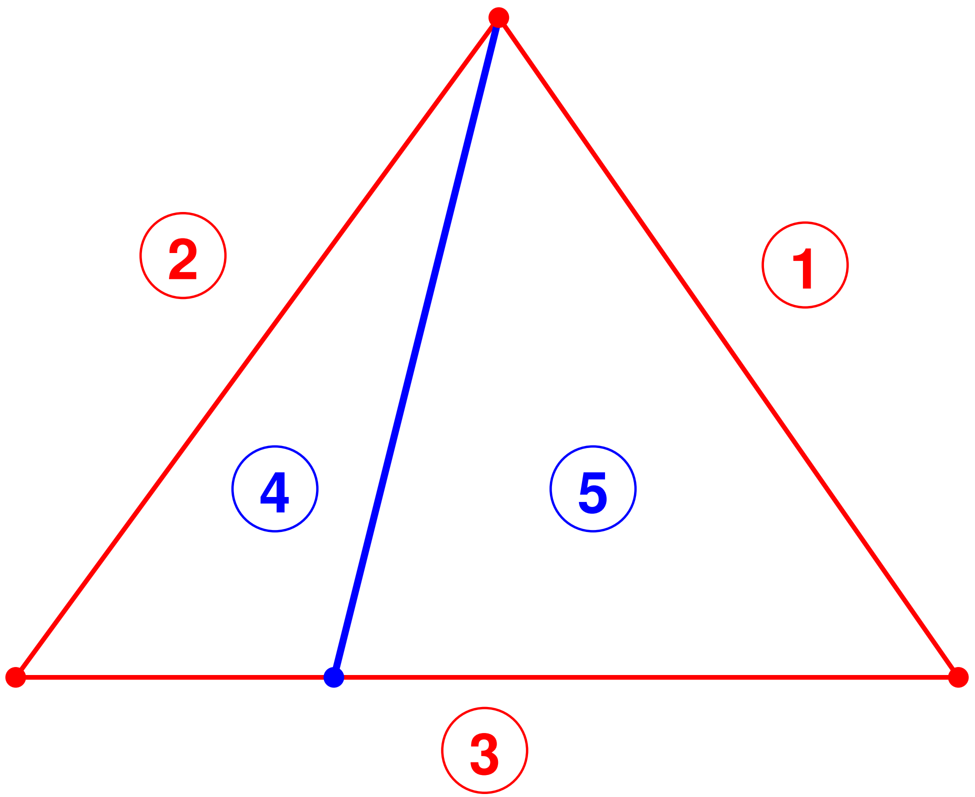
E.g. x with $> m$ labels,
s labels from $\{ 1 , . . . , m \}$,

$\Rightarrow$ $> m-s$ labels from $\{ m+1 , . . . , m+n \}$

$\Leftrightarrow$ $> |\mathrm{supp}(x)|$ best responses to x.

$\Rightarrow$ degenerate.

Example of a degenerate game



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \left[\begin{array}{cc} 2 & 1 \\ 1 & 3 \\ 4 & 4 \end{array}\right] \end{array} = B$$

payoffs to
player II

Handling degenerate games

Lemke–Howson implemented by pivoting, i.e., changing from one *basic feasible solution* of a linear system to another by choosing an entering and a leaving variable.

Choice of entering variable via complementarity (only difference to simplex algorithm for linear programming).

Leaving variable is *unique* in nondegenerate games.

In degenerate games: *perturb* system by adding $(\varepsilon, \ldots, \varepsilon^n)^\top$, creates nondegenerate system.
Implemented *symbolically* by lexicographic rule.