

# Equilibrium Algorithms for Two-Player Games

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# Nash equilibria of bimatrix games

$$A = \begin{array}{|c|c|} \hline 0 & 6 \\ \hline 2 & 5 \\ \hline 3 & 3 \\ \hline \end{array} \quad B = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 3 \\ \hline 4 & 3 \\ \hline \end{array}$$

Nash equilibrium =

pair of strategies  $x$ ,  $y$  with

$x$  best response to  $y$  and

$y$  best response to  $x$ .

# Mixed equilibria

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}$$

$$x^T B = \begin{bmatrix} 5/3 & 5/3 \end{bmatrix}$$

$$A y = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

$$y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

only **pure best responses** can have probability  $> 0$

## Best response condition

Let  $\mathbf{x}$  and  $\mathbf{y}$  be mixed strategies of player I and II, respectively. Then  $\mathbf{x}$  is a best response to  $\mathbf{y}$

$\iff$  for all pure strategies  $i$  of player I:

$$x_i > 0 \implies (\mathbf{A}\mathbf{y})_i = u = \max\{(\mathbf{A}\mathbf{y})_k \mid 1 \leq k \leq m\}.$$

Here,  $(\mathbf{A}\mathbf{y})_i$  is the  $i$ th component of  $\mathbf{A}\mathbf{y}$ , which is the expected payoff to player I when playing row  $i$ .

*Proof.*

$$\begin{aligned} \mathbf{x}\mathbf{A}\mathbf{y} &= \sum_{i=1}^m \mathbf{x}_i (\mathbf{A}\mathbf{y})_i = \sum_{i=1}^m \mathbf{x}_i (u - (u - (\mathbf{A}\mathbf{y})_i)) \\ &= \sum_{i=1}^m \mathbf{x}_i u - \sum_{i=1}^m \mathbf{x}_i (u - (\mathbf{A}\mathbf{y})_i) = u - \sum_{i=1}^m \mathbf{x}_i (u - (\mathbf{A}\mathbf{y})_i) \leq u, \end{aligned}$$

because  $\mathbf{x}_i \geq 0$  and  $u - (\mathbf{A}\mathbf{y})_i \geq 0$  for all  $i$ . Furthermore,

$\mathbf{x}\mathbf{A}\mathbf{y} = u \iff \mathbf{x}_i > 0$  implies  $(\mathbf{A}\mathbf{y})_i = u$ , as claimed.

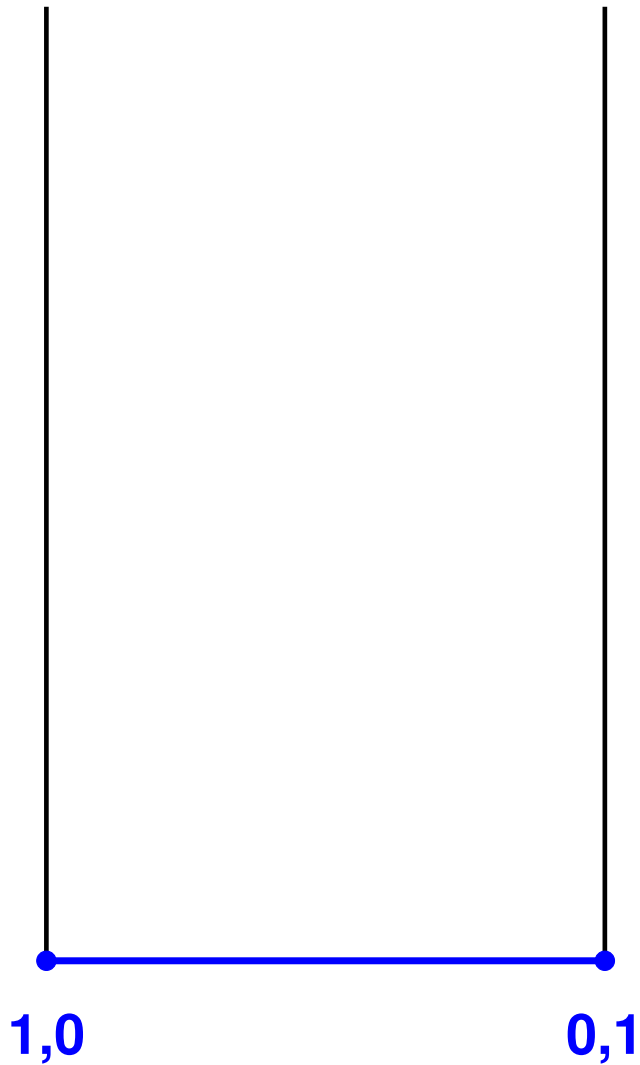
# Best responses to mixed strategy of player 2

	<b>4</b>	<b>5</b>	
<b>1</b>	<b>0</b>	<b>6</b>	<b>= A</b>
<b>2</b>	<b>2</b>	<b>5</b>	
<b>3</b>	<b>3</b>	<b>3</b>	

payoffs to  
player I



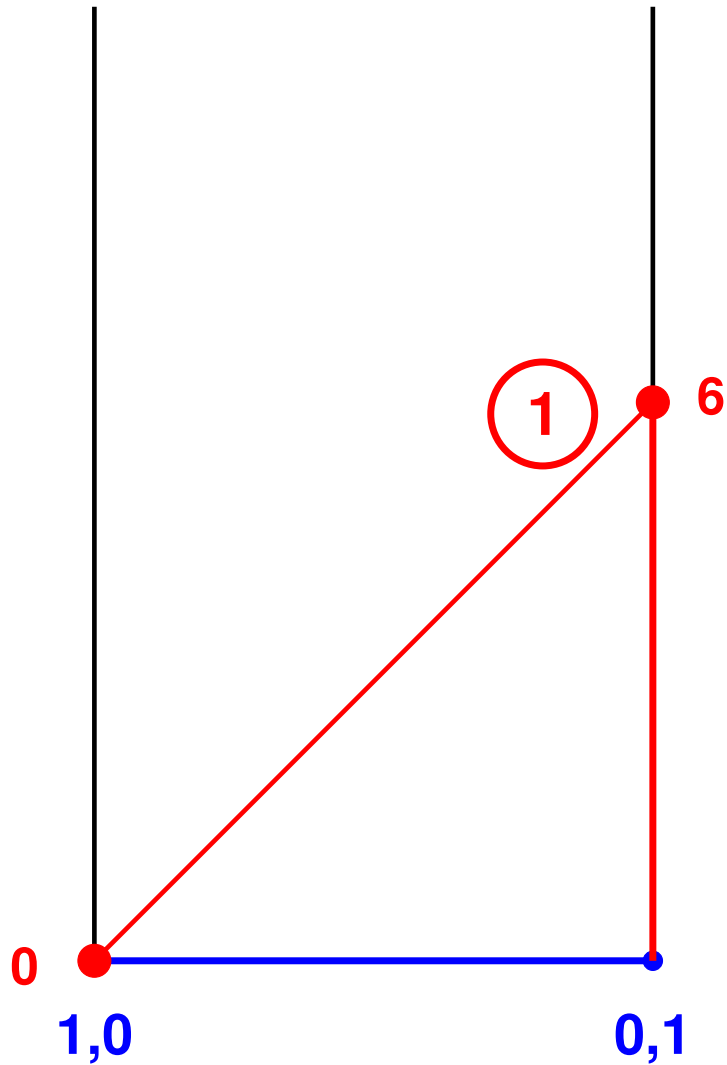
# Best responses to mixed strategy of player 2



	<b>4</b>	<b>5</b>	
<b>1</b>	<b>0</b>	<b>6</b>	<b>= A</b>
<b>2</b>	<b>2</b>	<b>5</b>	
<b>3</b>	<b>3</b>	<b>3</b>	

payoffs to  
player 1

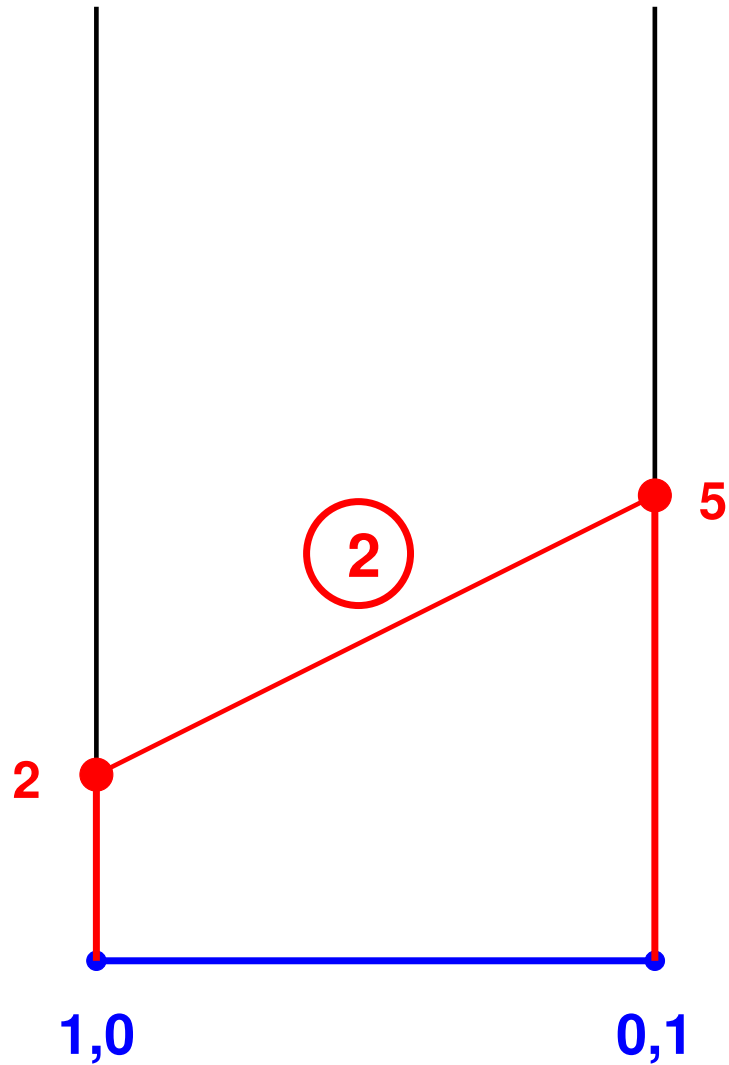
# Best responses to mixed strategy of player 2



	<b>4</b>	<b>5</b>	
<b>1</b>	0	6	
<b>2</b>	2	5	= A
<b>3</b>	3	3	

payoffs to  
player 1

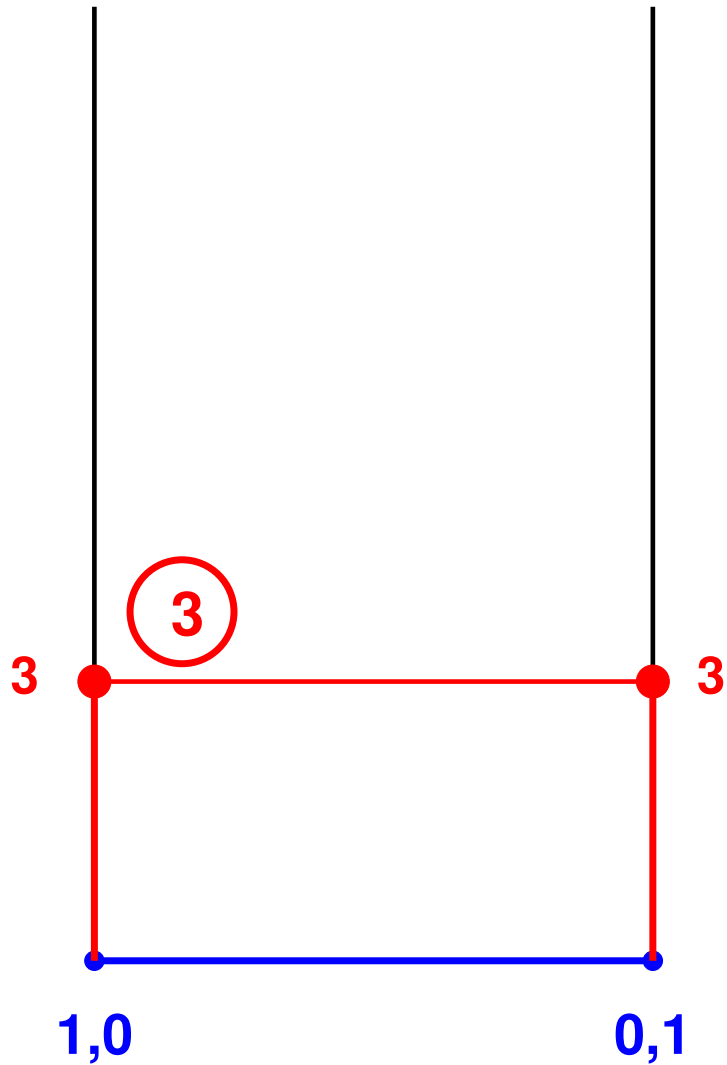
# Best responses to mixed strategy of player 2



	<b>4</b>	<b>5</b>	
<b>1</b>	0	6	
<b>2</b>	2	5	= A
<b>3</b>	3	3	

payoffs to  
player I

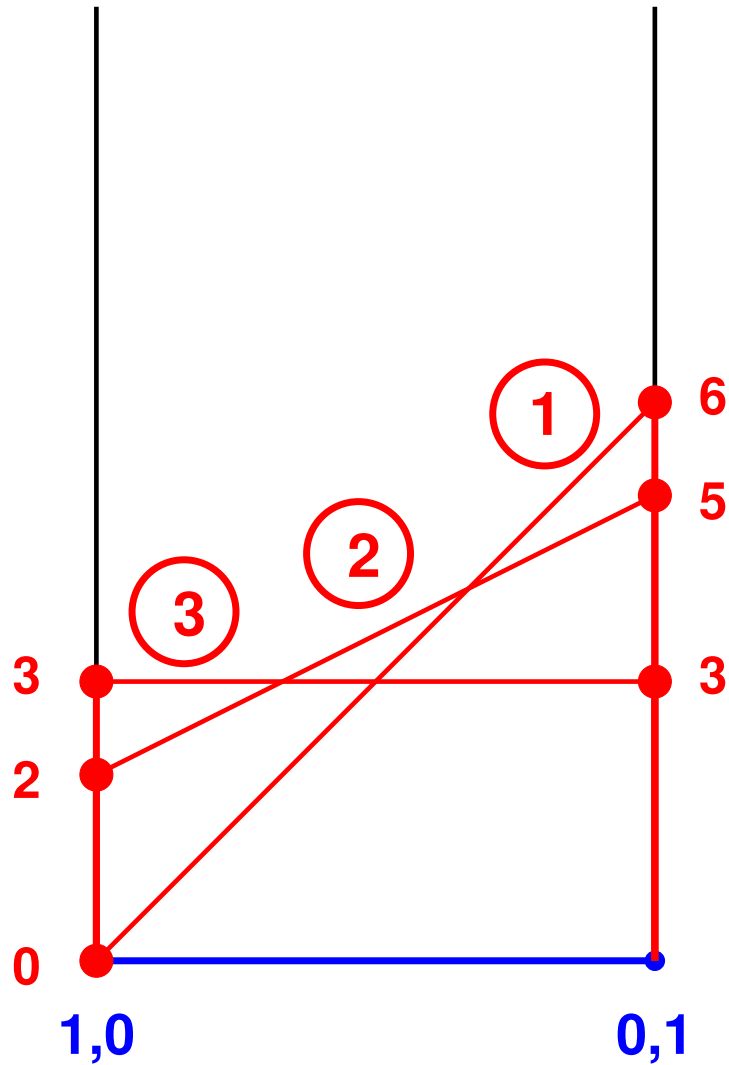
# Best responses to mixed strategy of player 2



	<b>4</b>	<b>5</b>	
<b>1</b>	<b>0</b>	<b>6</b>	
<b>2</b>	<b>2</b>	<b>5</b>	<b>= A</b>
<b>3</b>	<b>3</b>	<b>3</b>	

payoffs to  
player 1

# Best responses to mixed strategy of player 2

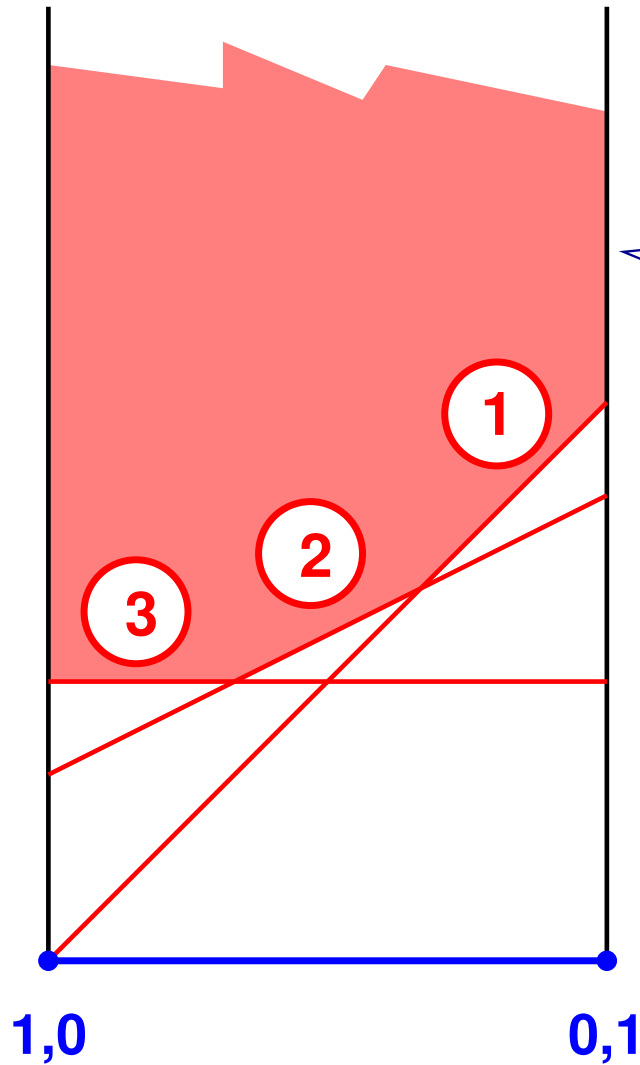


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to  
player I

# Best responses to mixed strategy of player 2



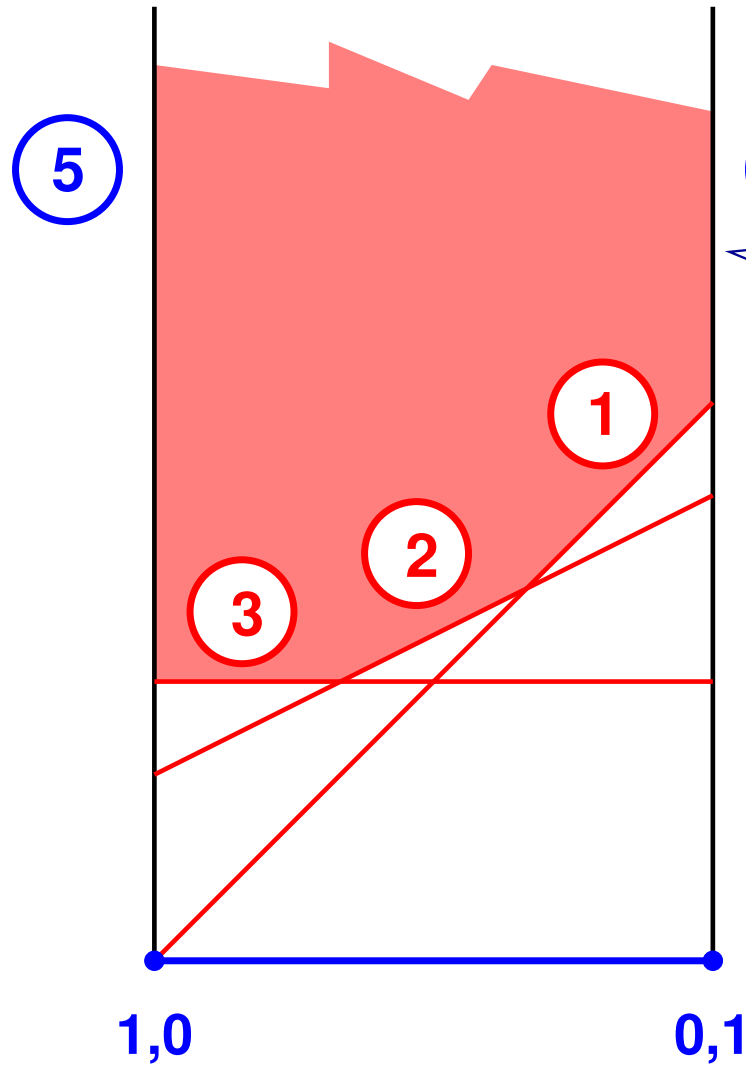
	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to  
player I

best response polyhedron

# Best responses to mixed strategy of player 2

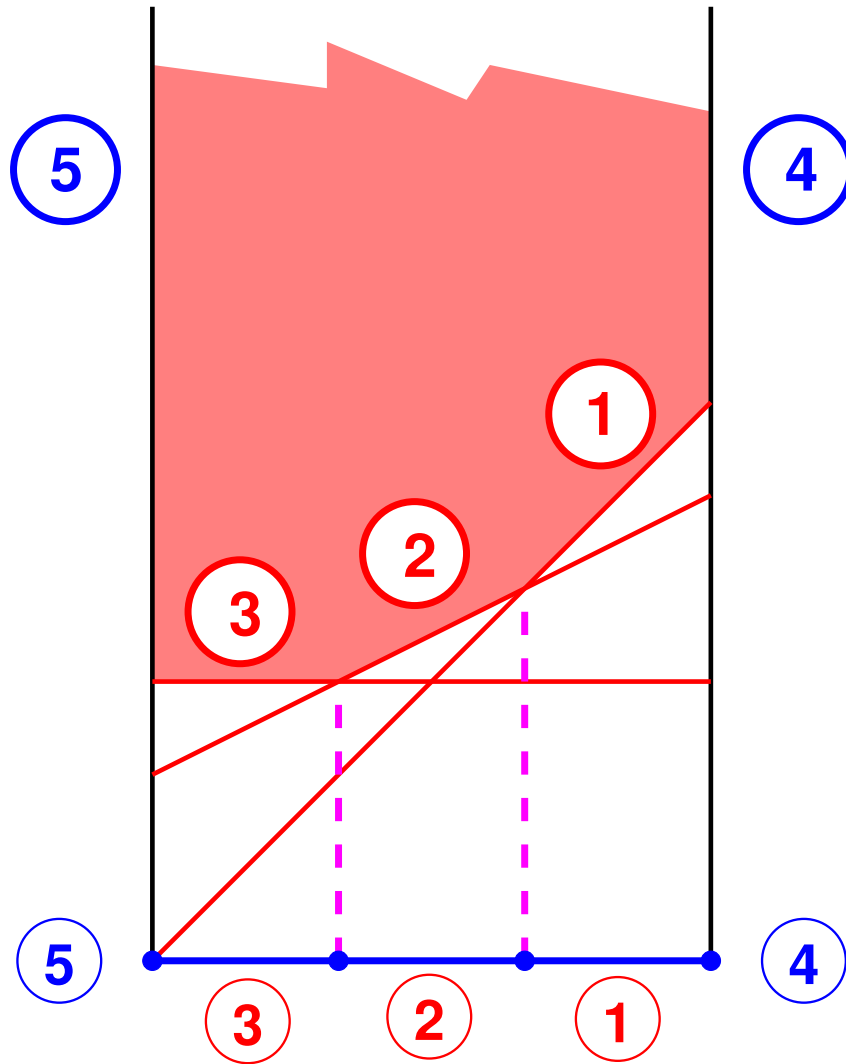


	<b>4</b>	<b>5</b>	
<b>1</b>	<b>0</b>	<b>6</b>	<b>= A</b>
<b>2</b>	<b>2</b>	<b>5</b>	
<b>3</b>	<b>3</b>	<b>3</b>	

payoffs to  
player I

**best response polyhedron  
with facet labels**

# Best responses to mixed strategy of player 2



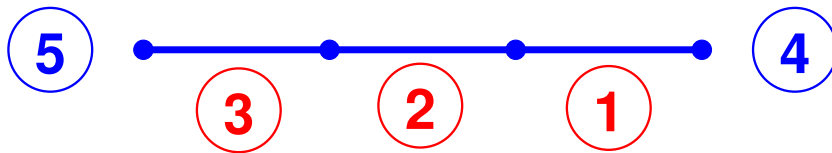
	(4)	(5)	
(1)	0	6	= A
(2)	2	5	
(3)	3	3	

payoffs to  
player 1

# Best responses to mixed strategy of player 2

	<b>4</b>	<b>5</b>	
<b>1</b>	<b>0</b>	<b>6</b>	<b>= A</b>
<b>2</b>	<b>2</b>	<b>5</b>	
<b>3</b>	<b>3</b>	<b>3</b>	

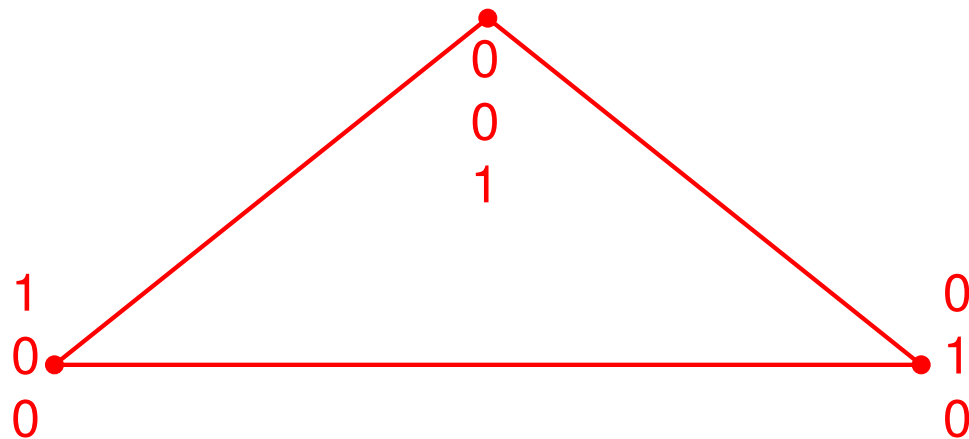
payoffs to  
player I



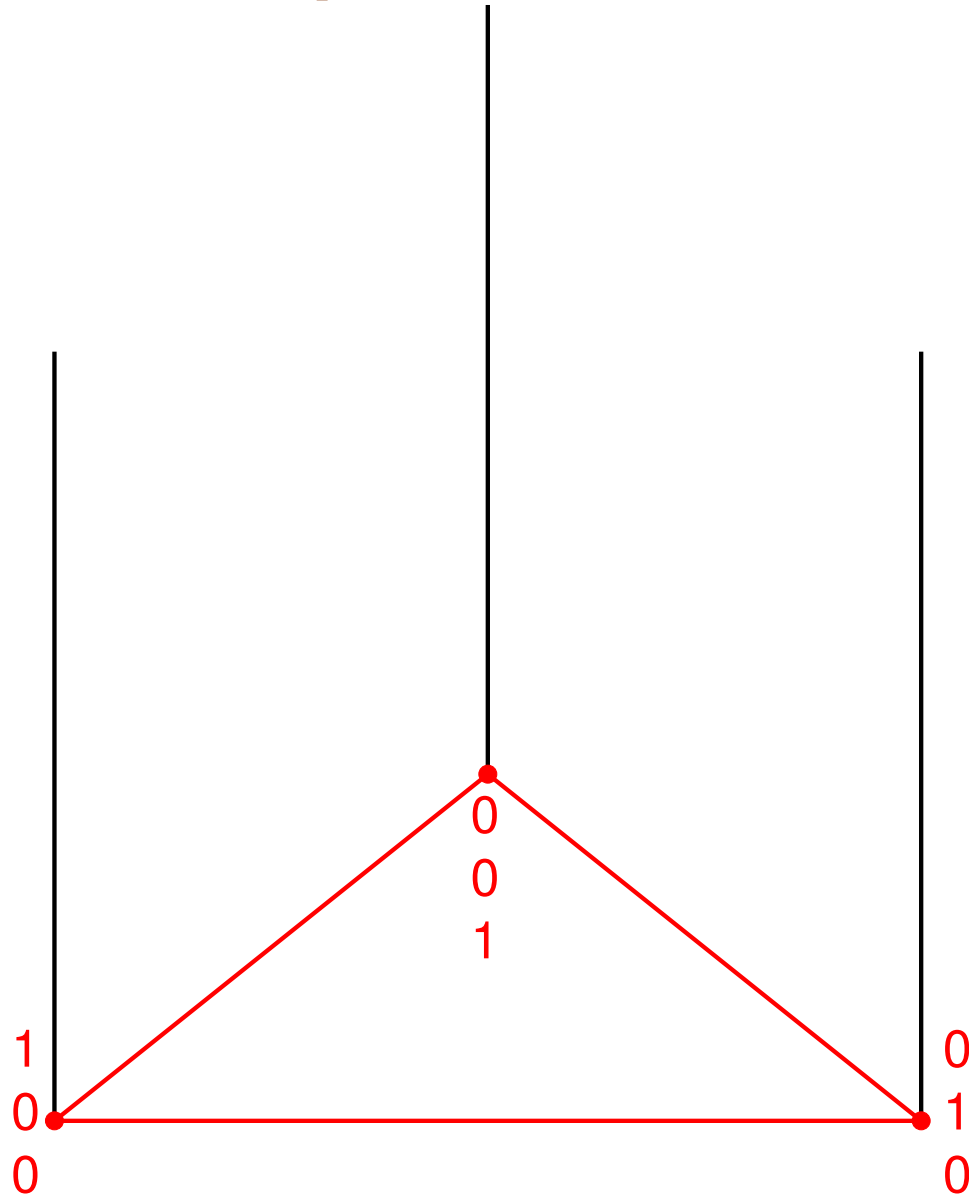
# Best responses to mixed strategy of player 1

	<b>4</b>	<b>5</b>	
<b>1</b>	<b>2</b>	<b>1</b>	= B
<b>2</b>	<b>1</b>	<b>3</b>	
<b>3</b>	<b>4</b>	<b>3</b>	

payoffs to  
player II



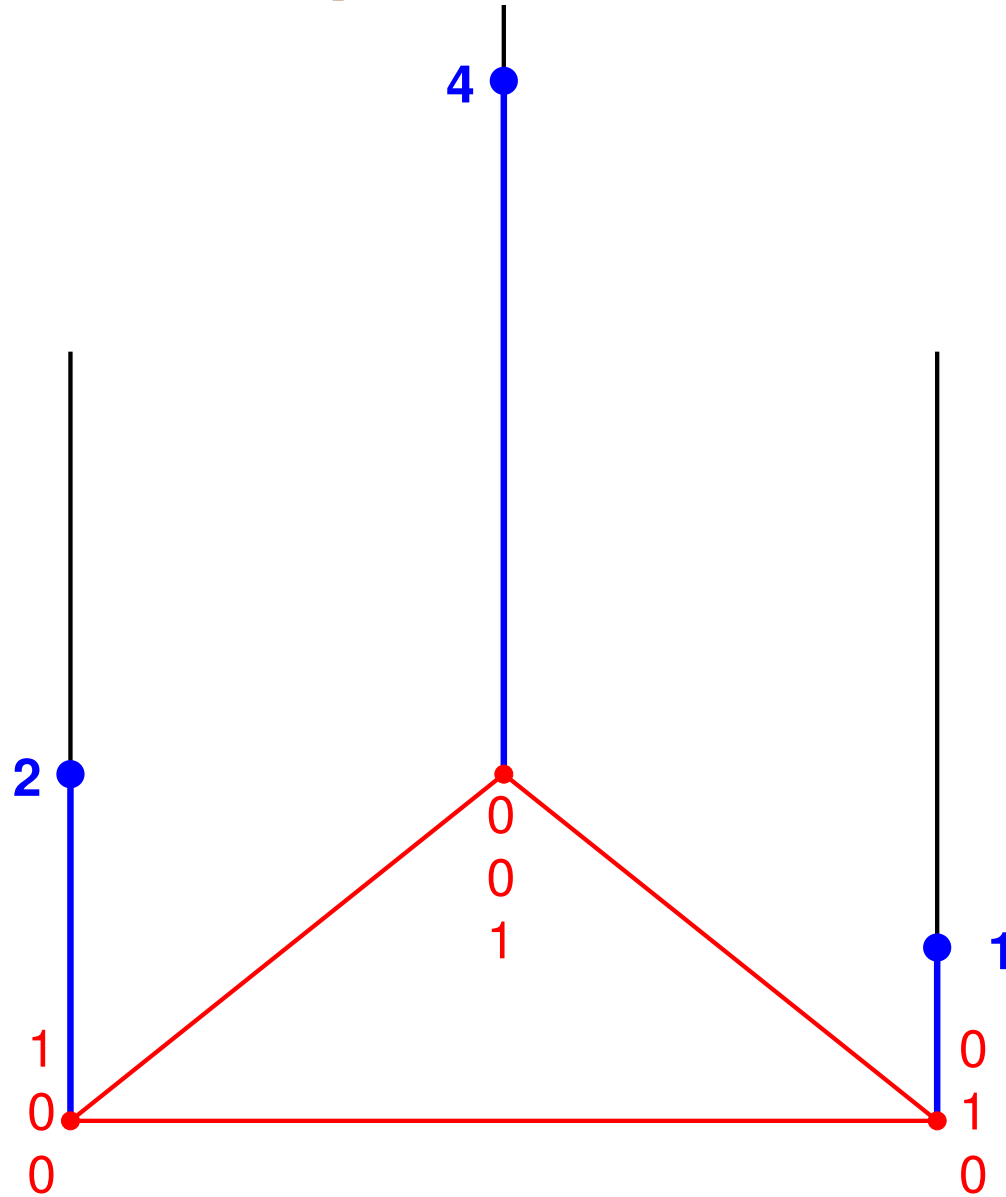
# Best responses to mixed strategy of player 1



	(4)	(5)	
(1)	2	1	= B
(2)	1	3	
(3)	4	3	

payoffs to  
player II

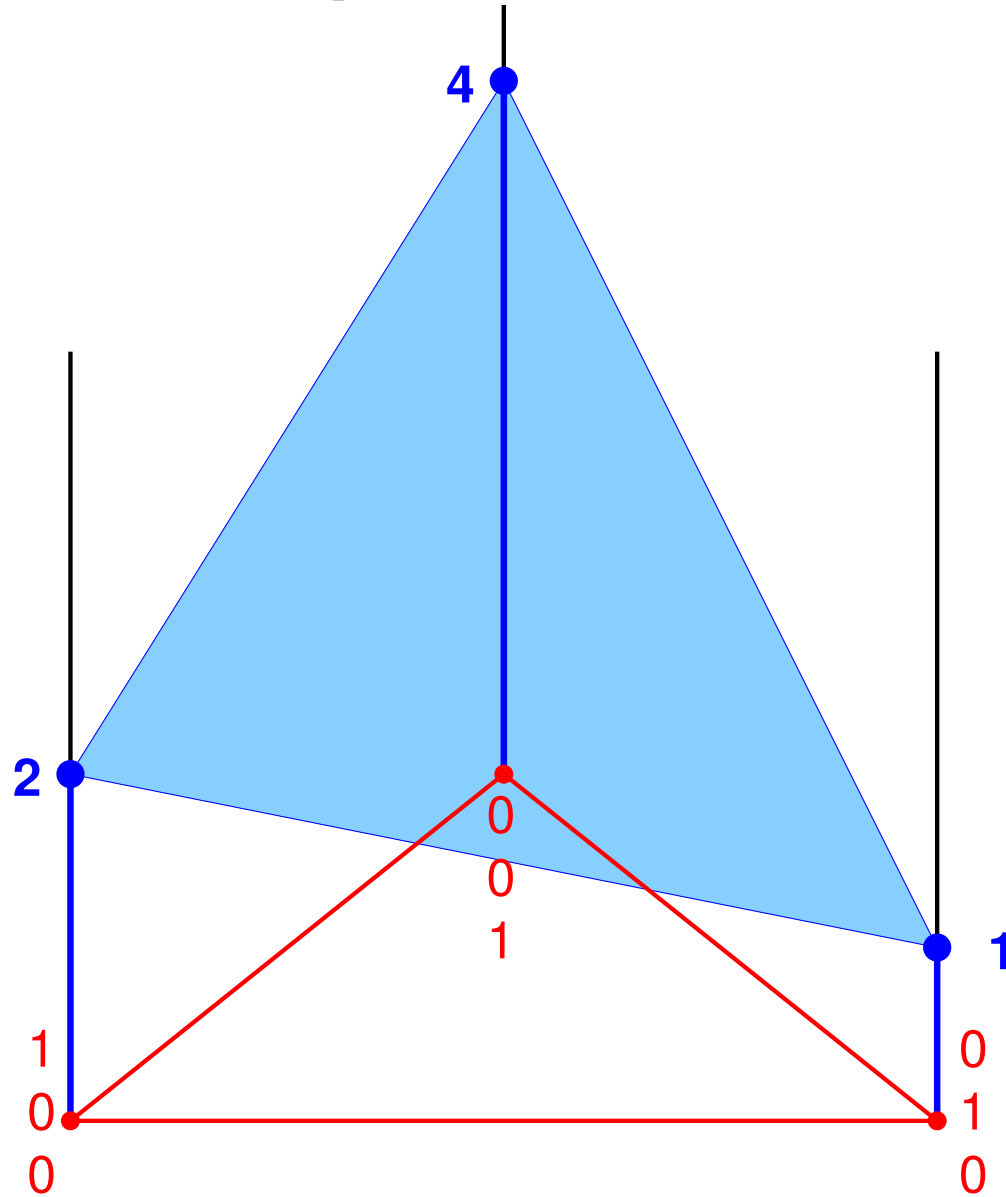
# Best responses to mixed strategy of player 1



	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to  
player II

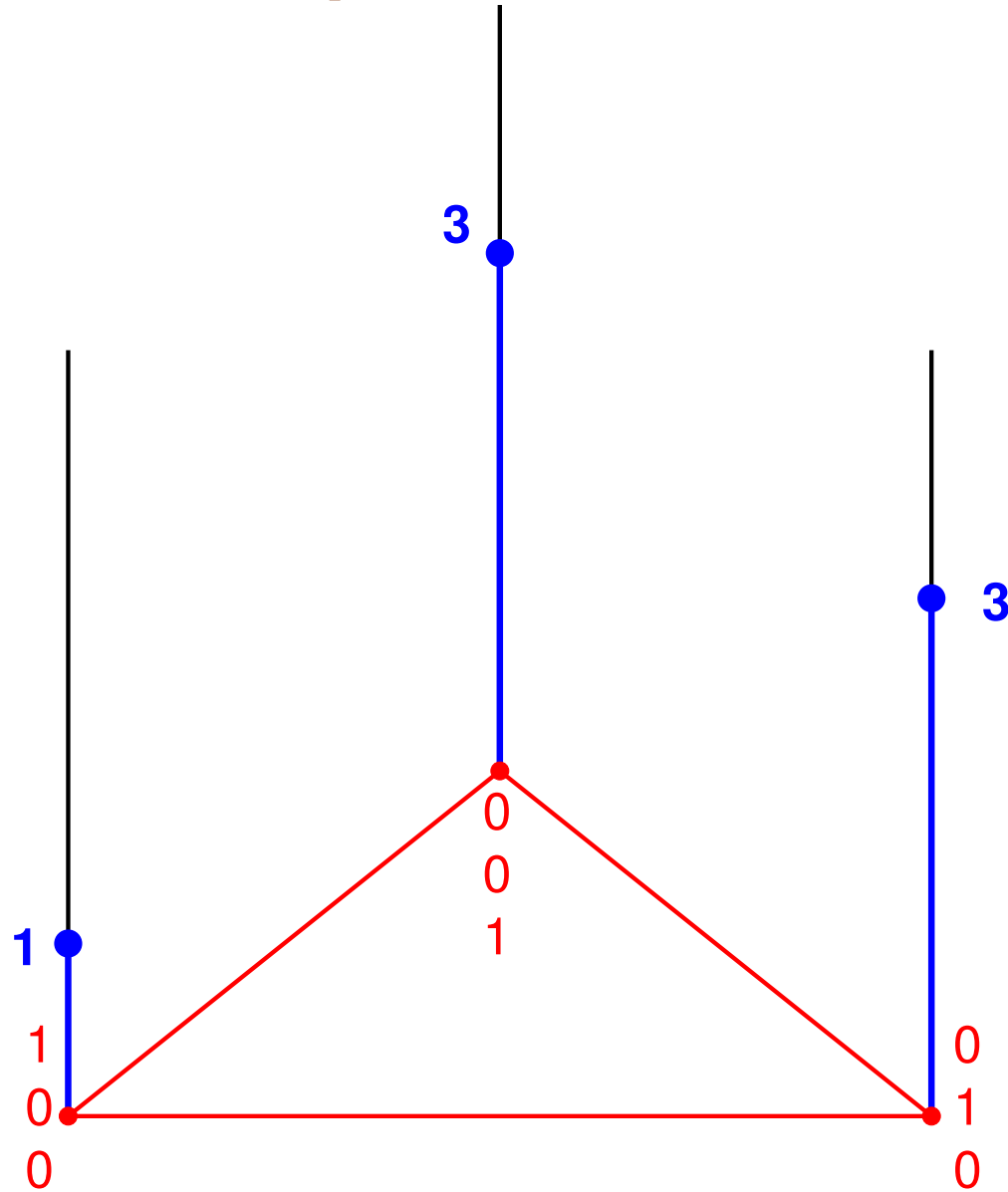
# Best responses to mixed strategy of player 1



	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to  
player II

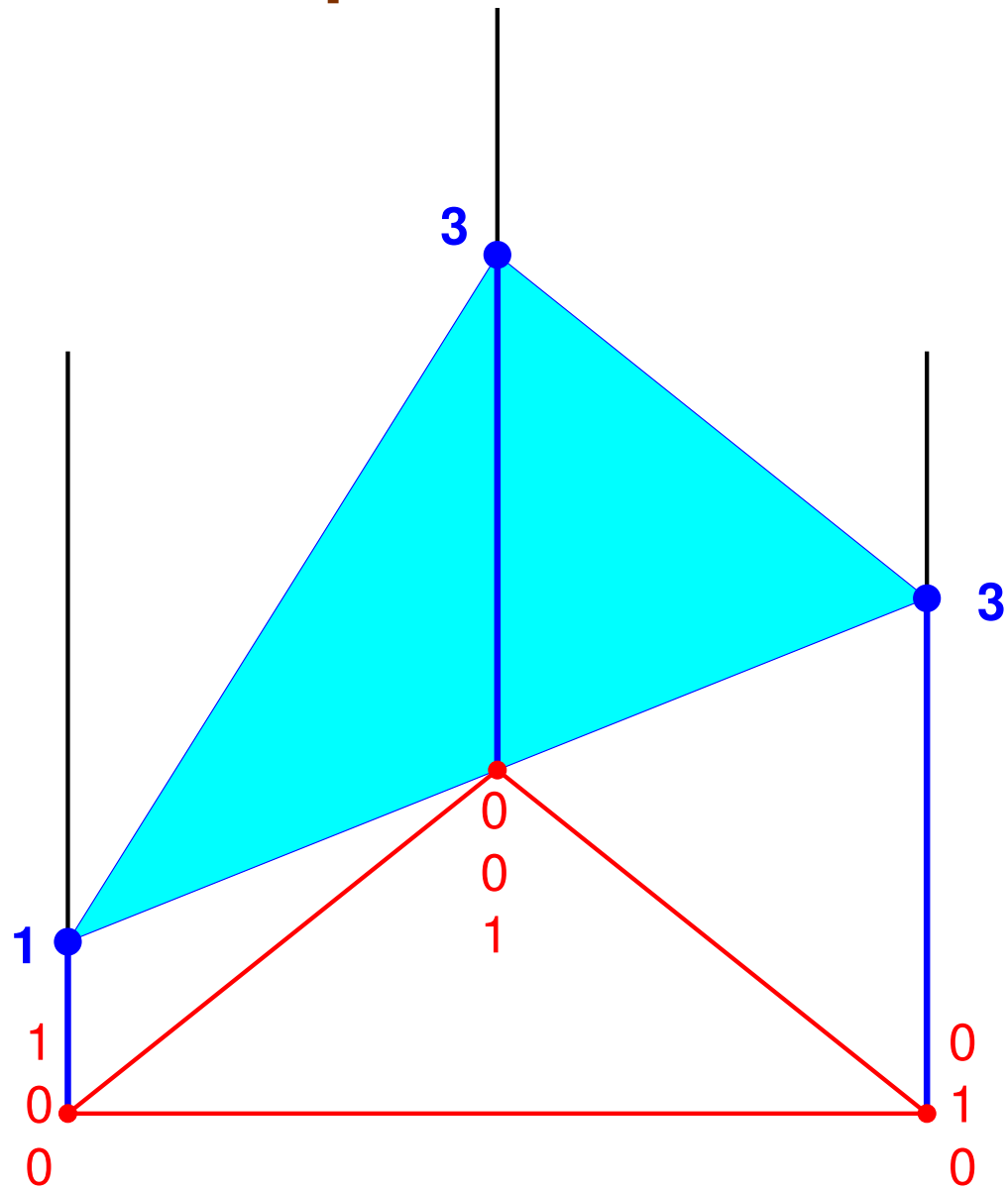
# Best responses to mixed strategy of player 1



	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to  
player II

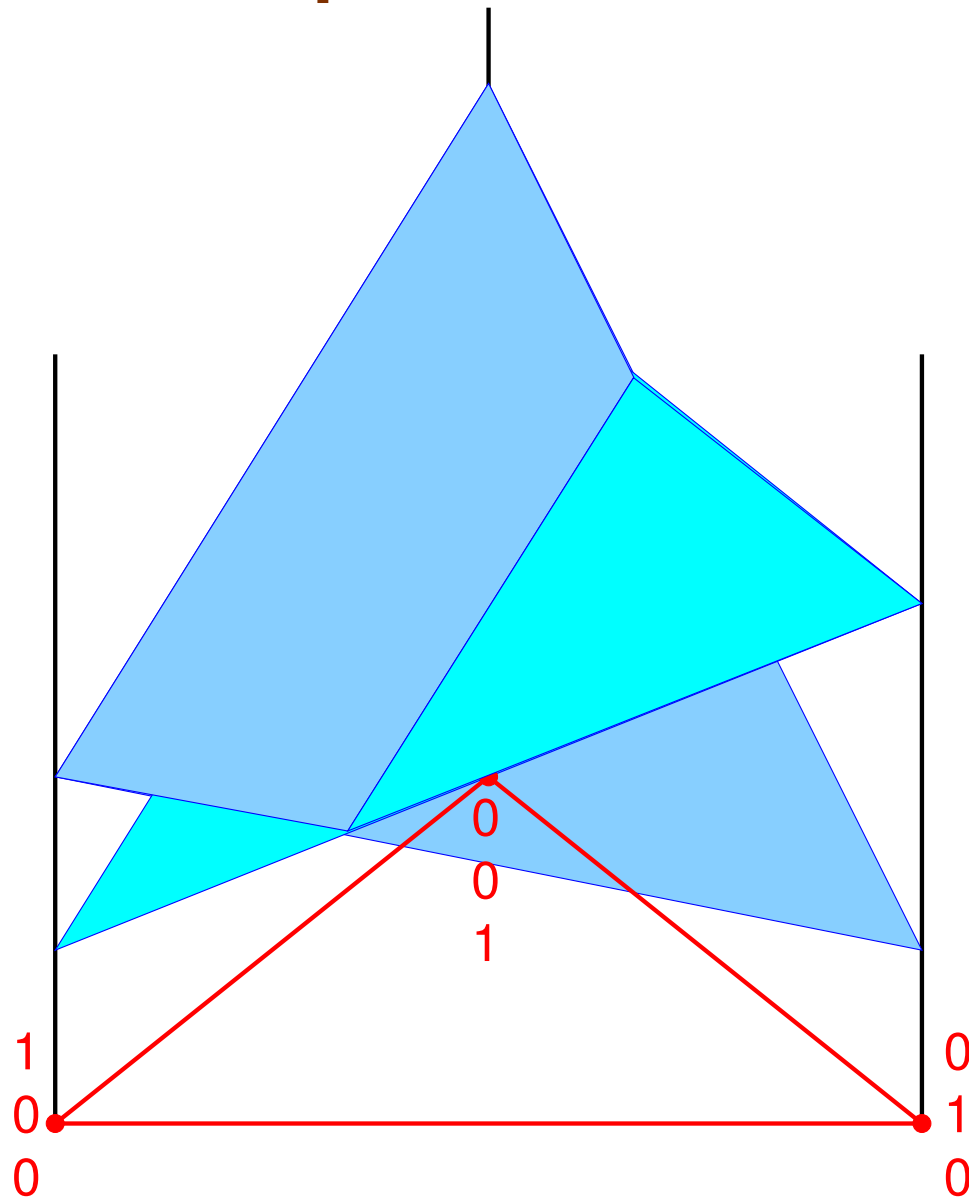
# Best responses to mixed strategy of player 1



	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to  
player II

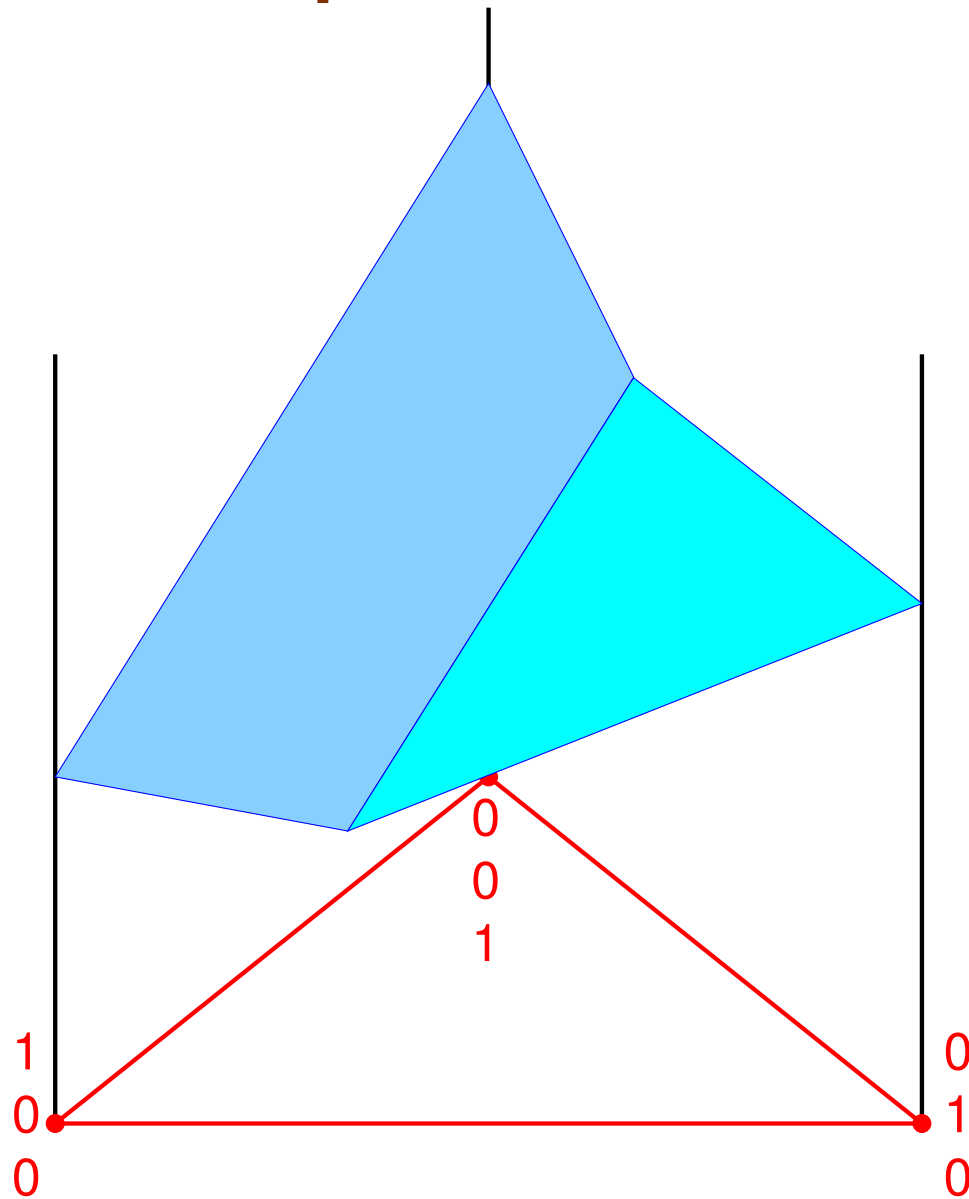
# Best responses to mixed strategy of player 1



	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to  
player II

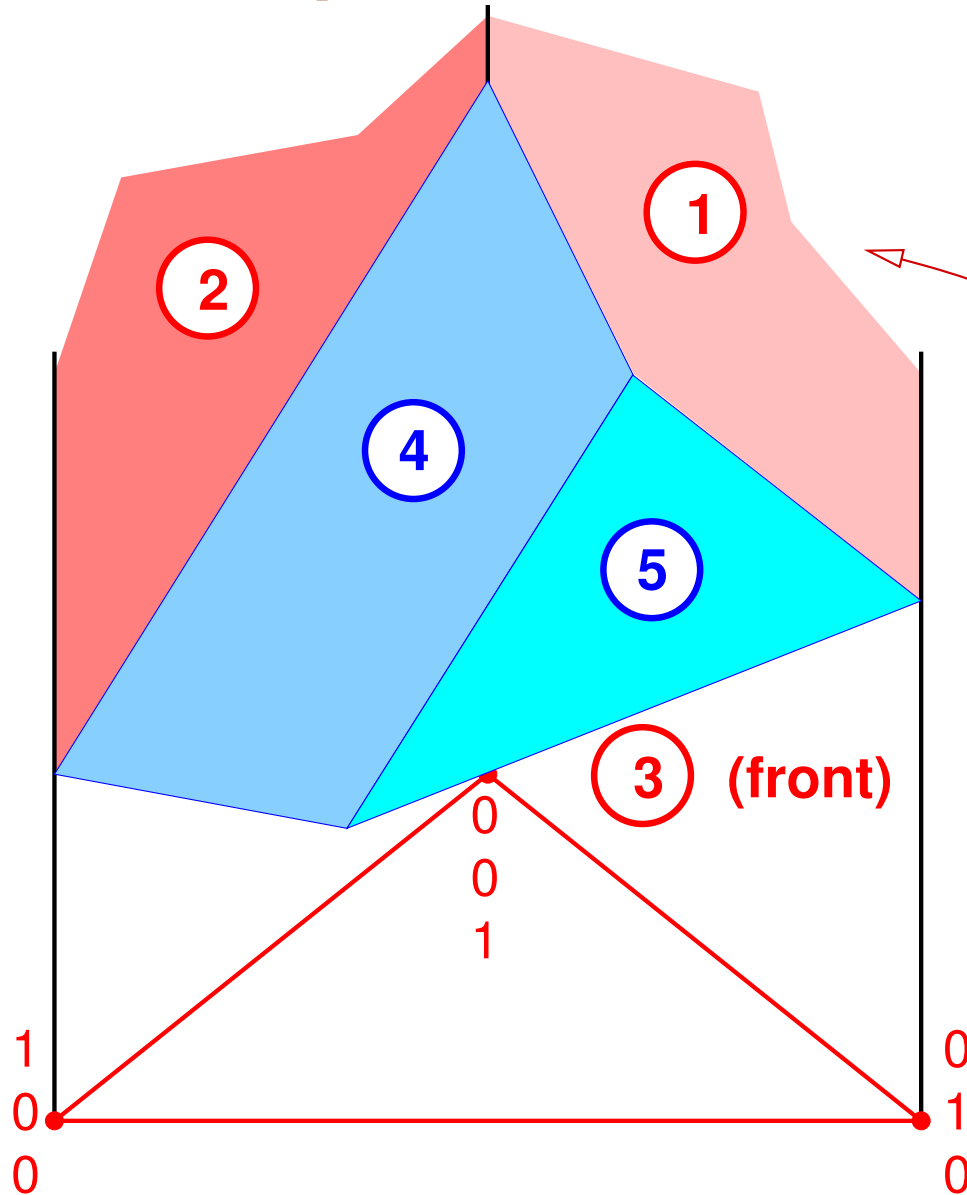
# Best responses to mixed strategy of player 1



	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to  
player II

# Best responses to mixed strategy of player 1



	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to player II

**best response polyhedron with facet labels**

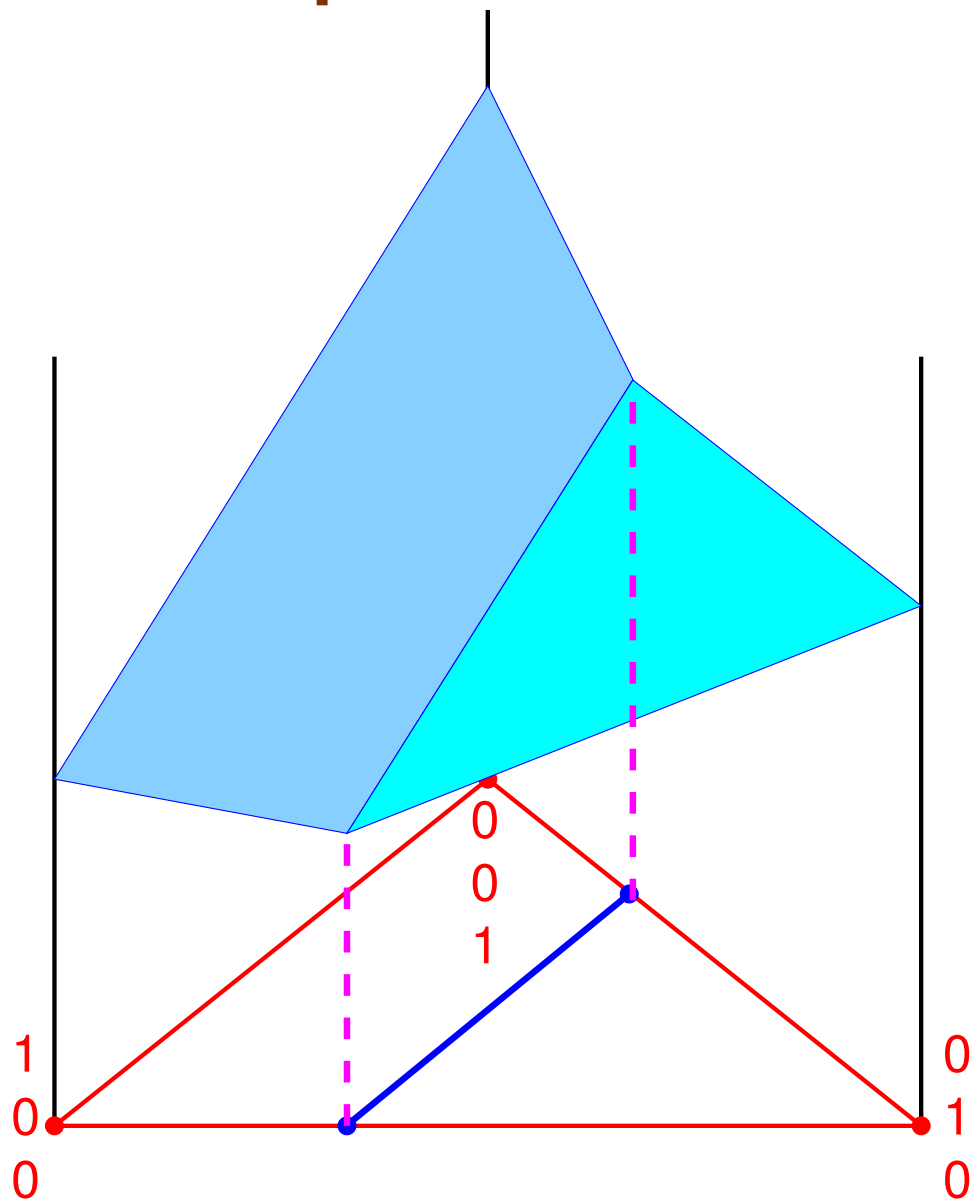
# Alternative view



# Alternative view



# Best responses to mixed strategy of player 1



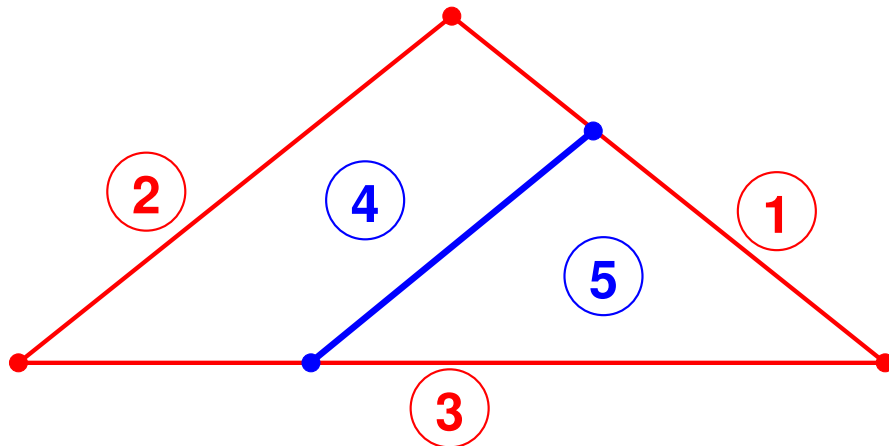
	4	5	
1	2	1	= B
2	1	3	
3	4	3	

payoffs to  
player II

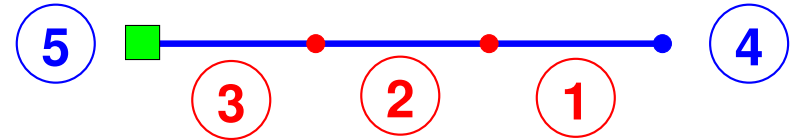
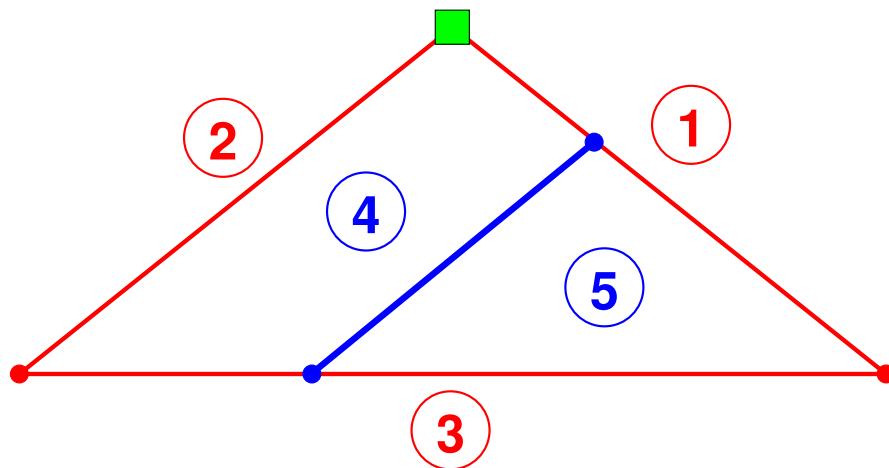
# Best responses to mixed strategy of player 1

	<b>4</b>	<b>5</b>	
<b>1</b>	<b>2</b>	<b>1</b>	= B
<b>2</b>	<b>1</b>	<b>3</b>	
<b>3</b>	<b>4</b>	<b>3</b>	

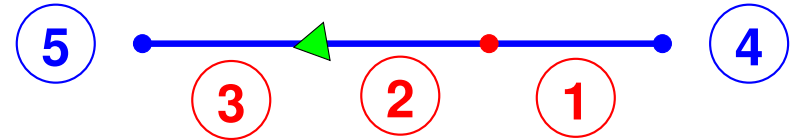
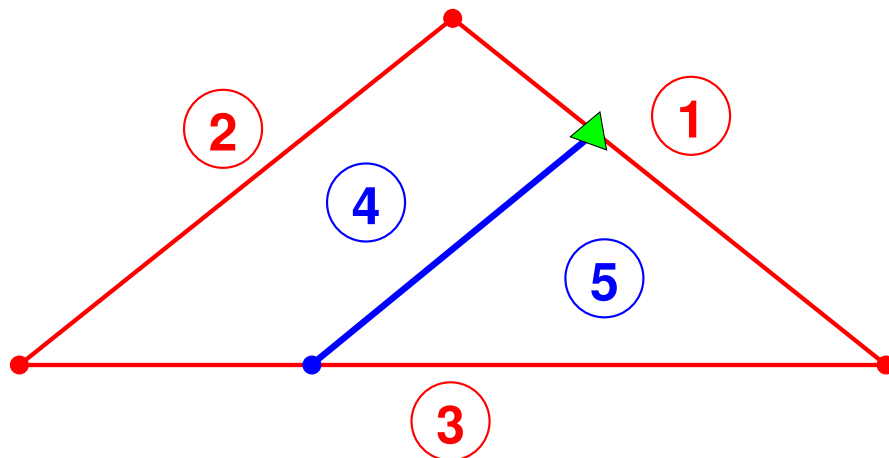
payoffs to  
player II



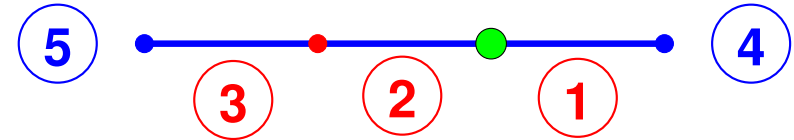
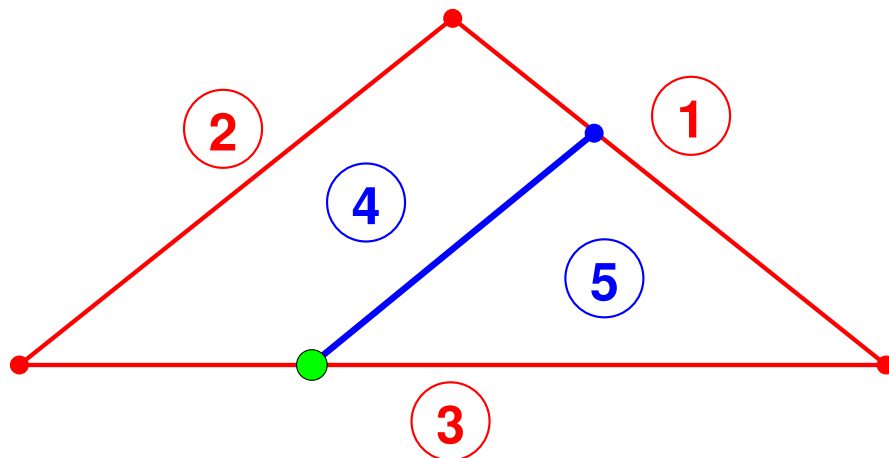
**Equilibrium = completely labeled strategy pair**



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**Equilibrium = completely labeled strategy pair**



# Constructing games using geometry

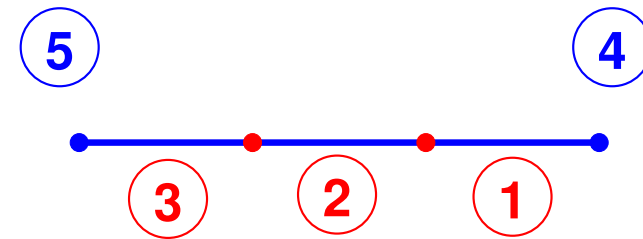
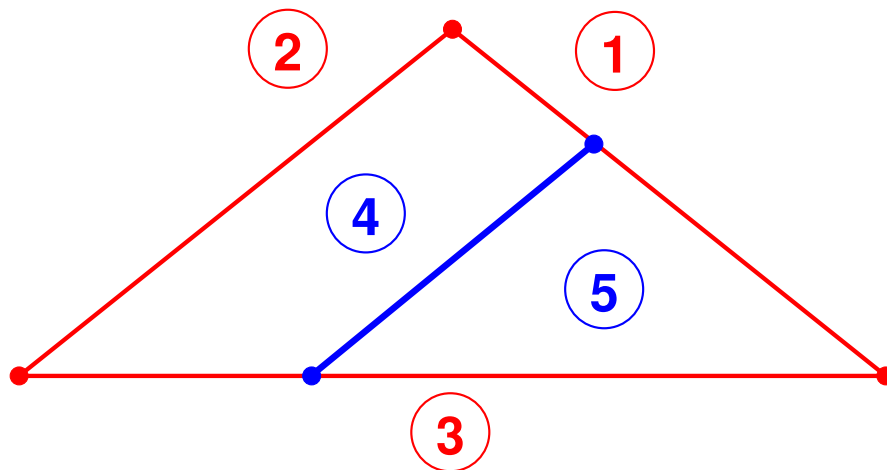
**low dimension:** 2, 3, (4) pure strategies:

subdivide mixed strategy simplex into response regions, label suitably

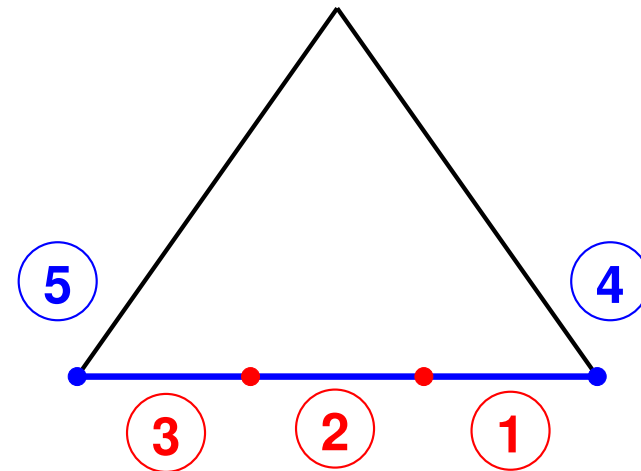
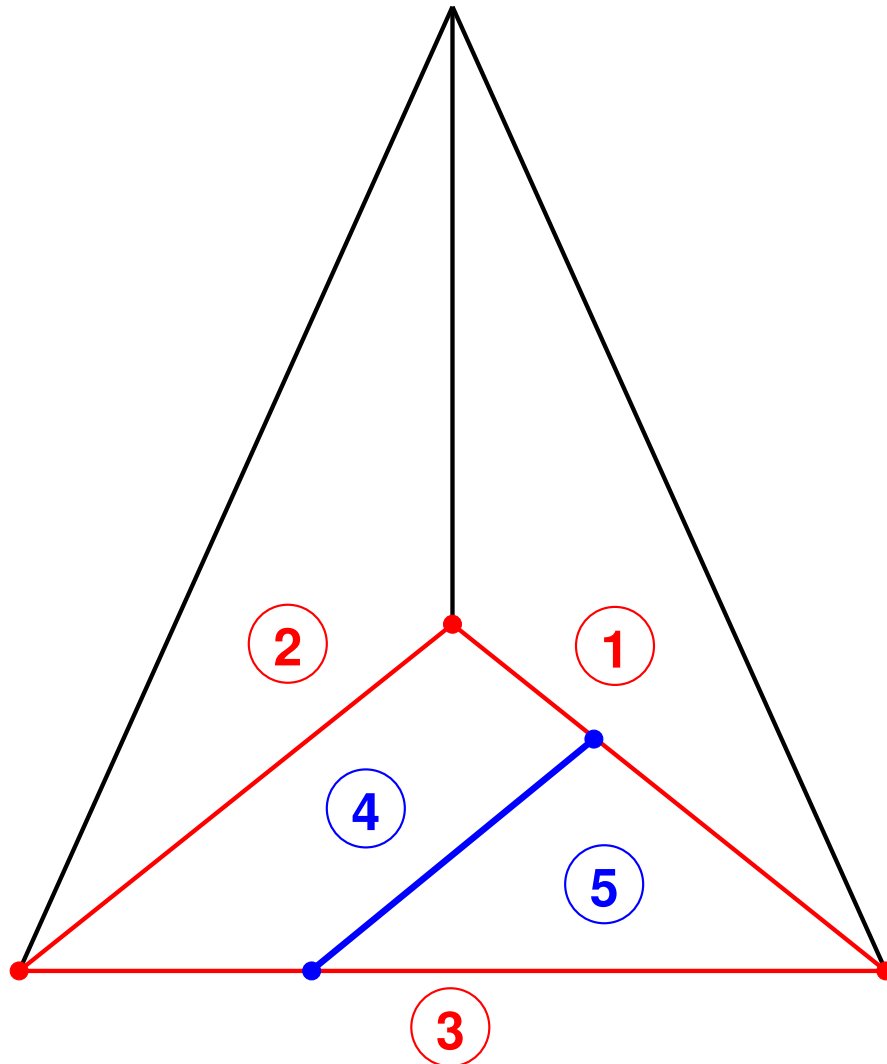
**high dimension:**

use polytopes with **known combinatorial structure**  
e.g. for constructing games with many equilibria,  
or long Lemke-Howson computations  
[Savani & von Stengel, *FOCS 2004*,  
*Econometrica 2006*]

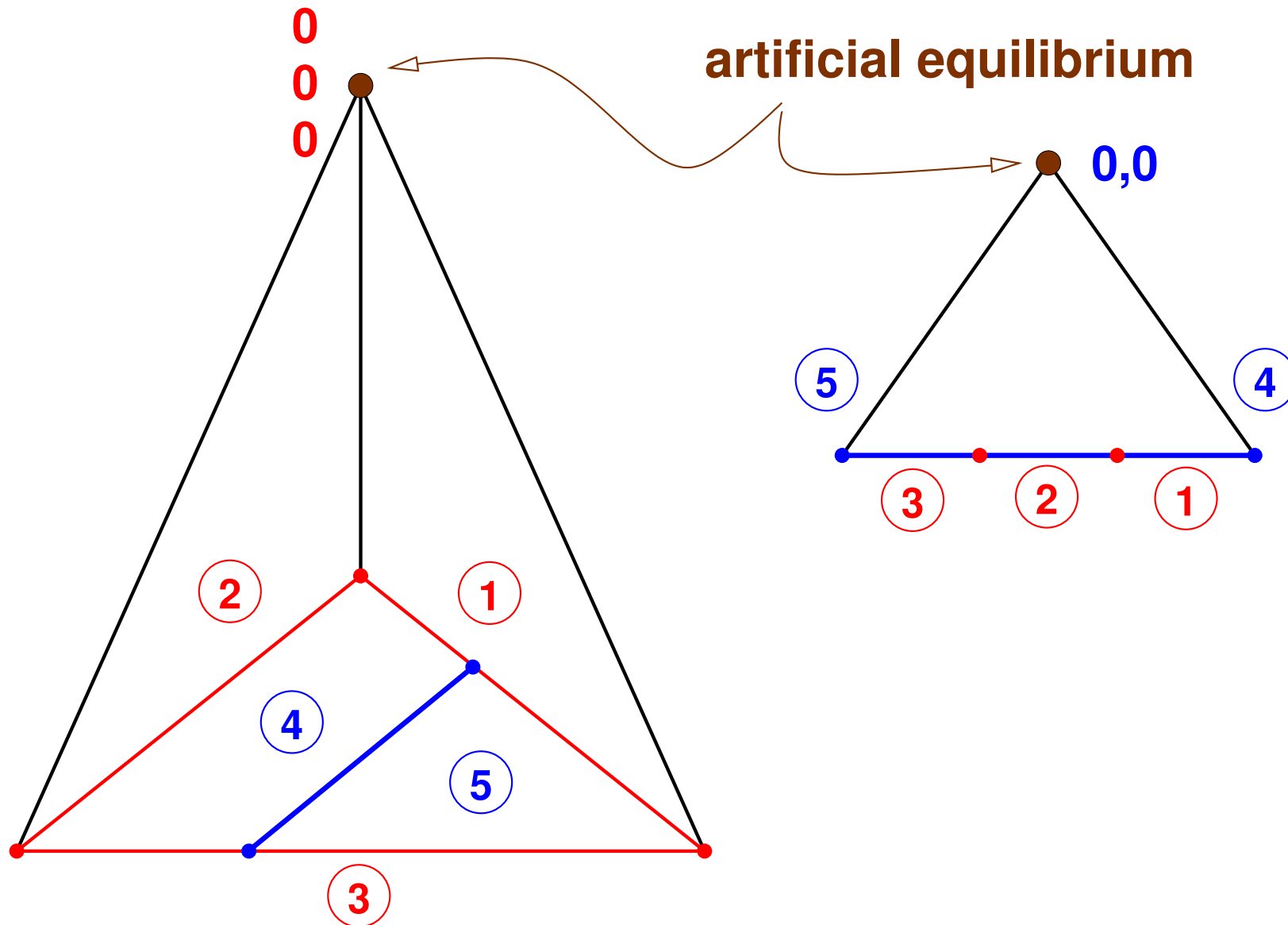
# The Lemke–Howson algorithm



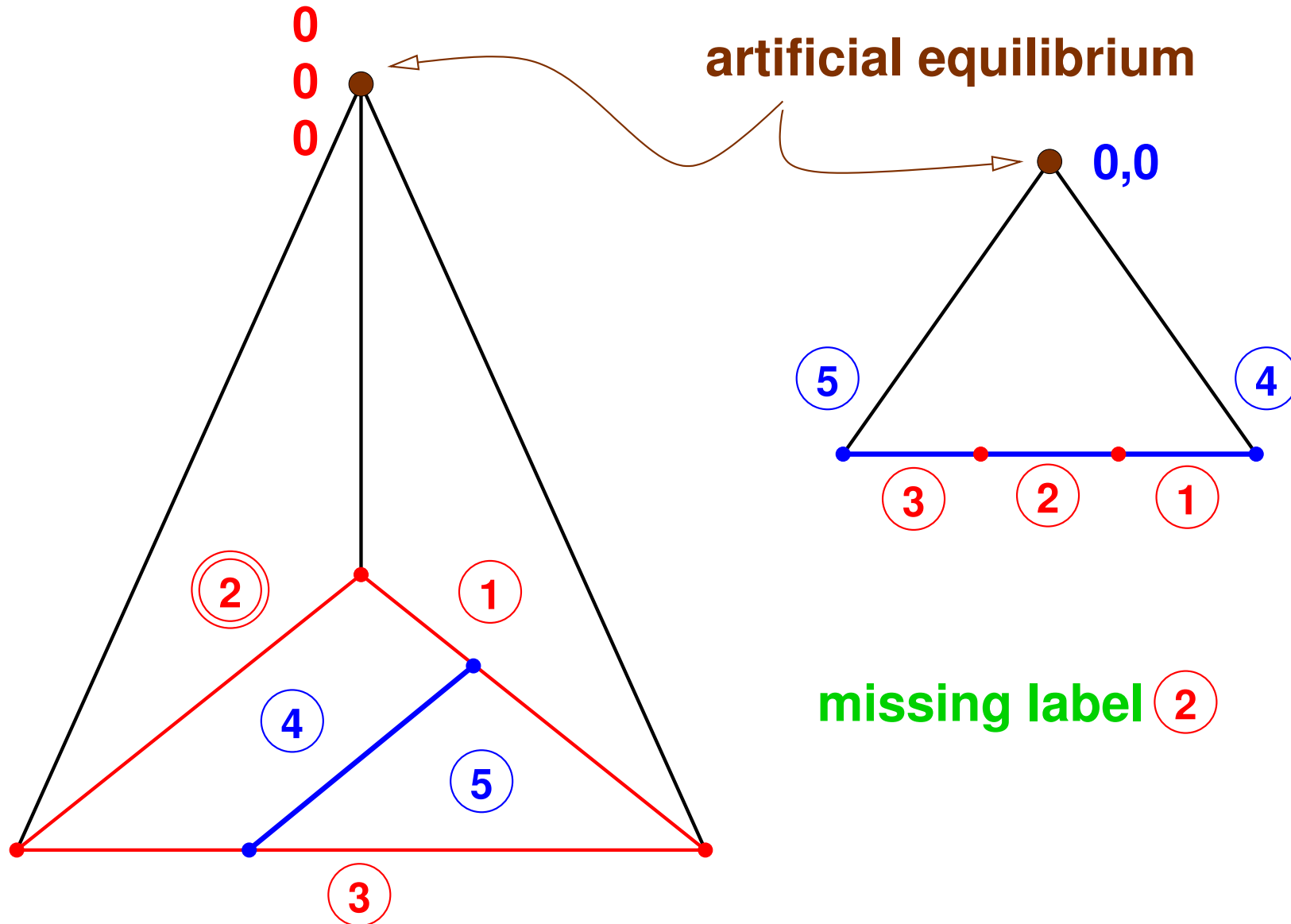
# The Lemke–Howson algorithm



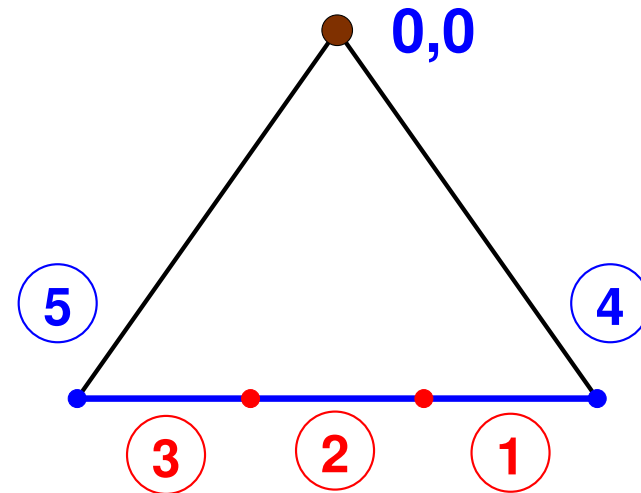
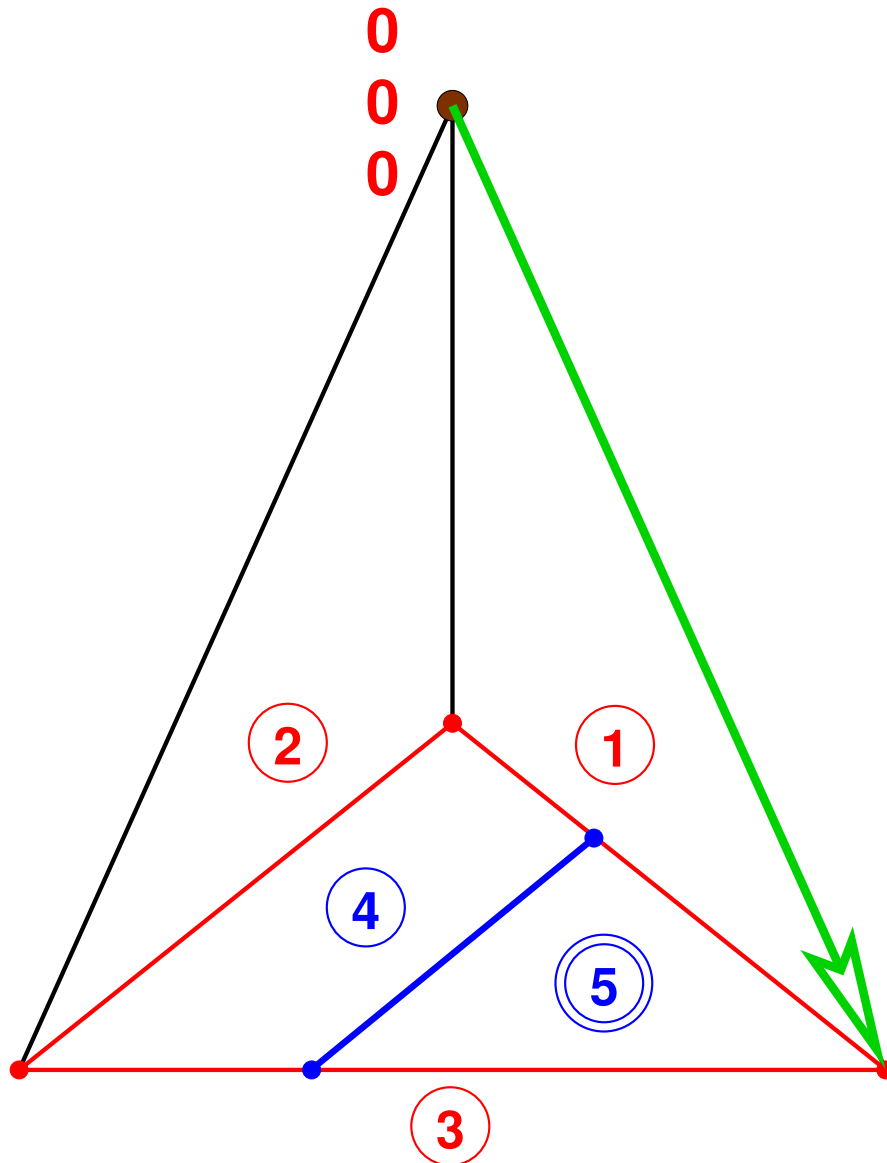
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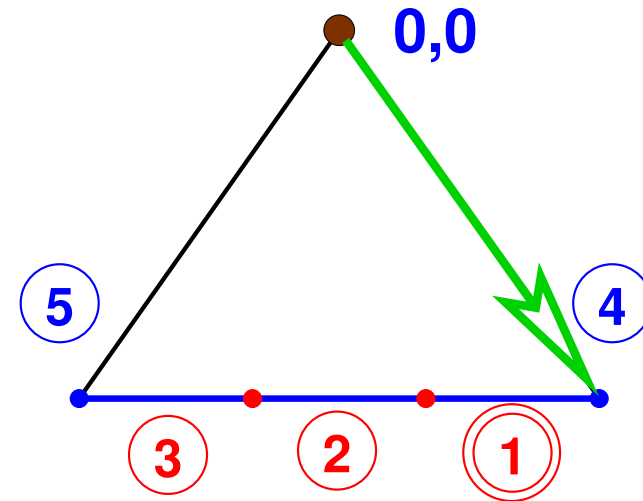
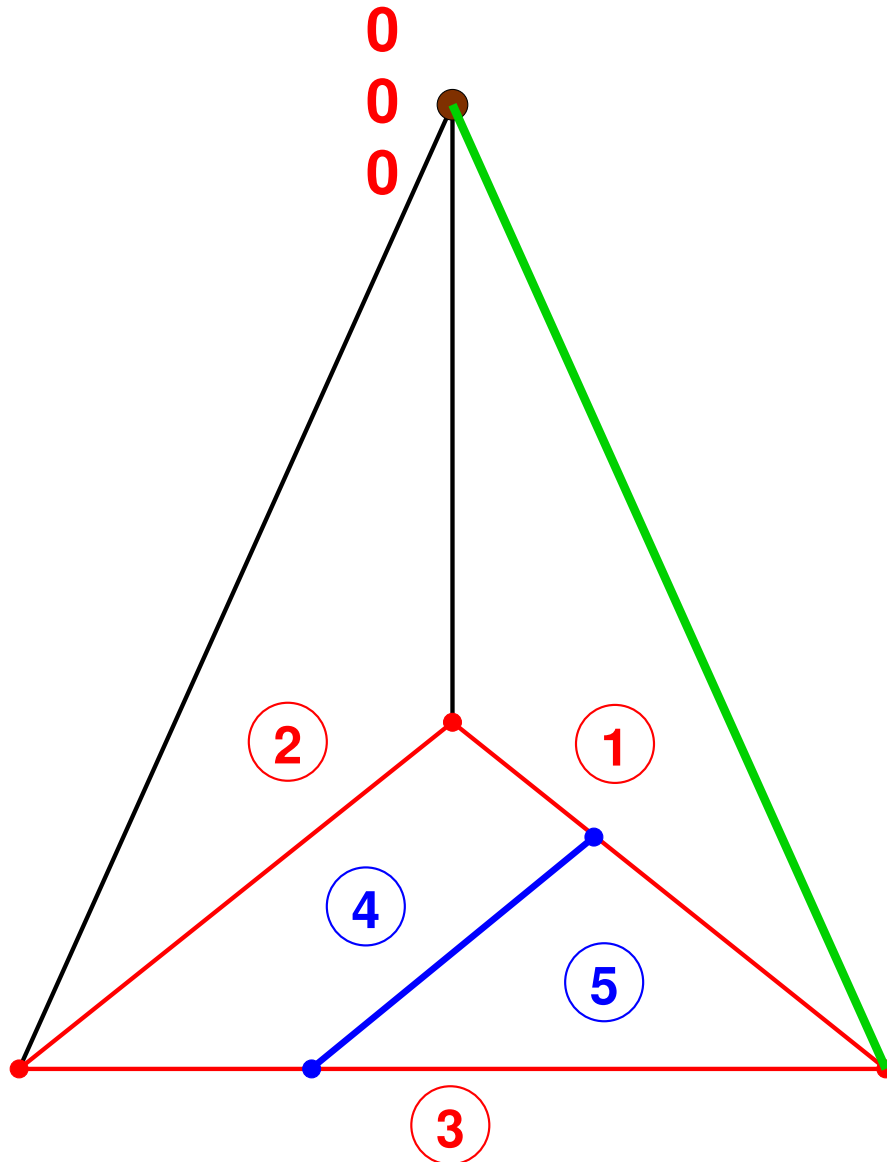


# The Lemke-Howson algorithm



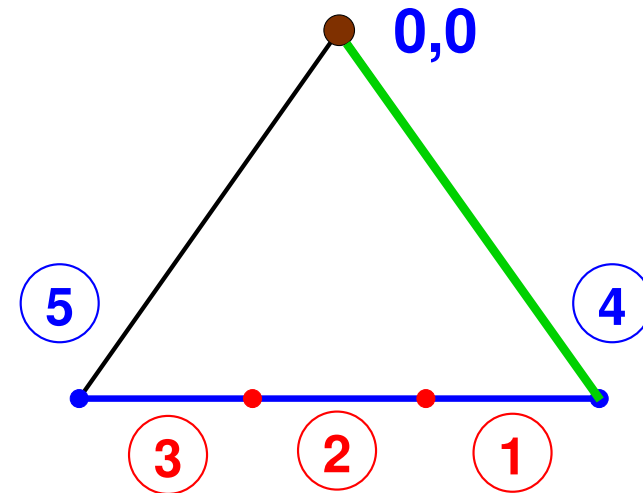
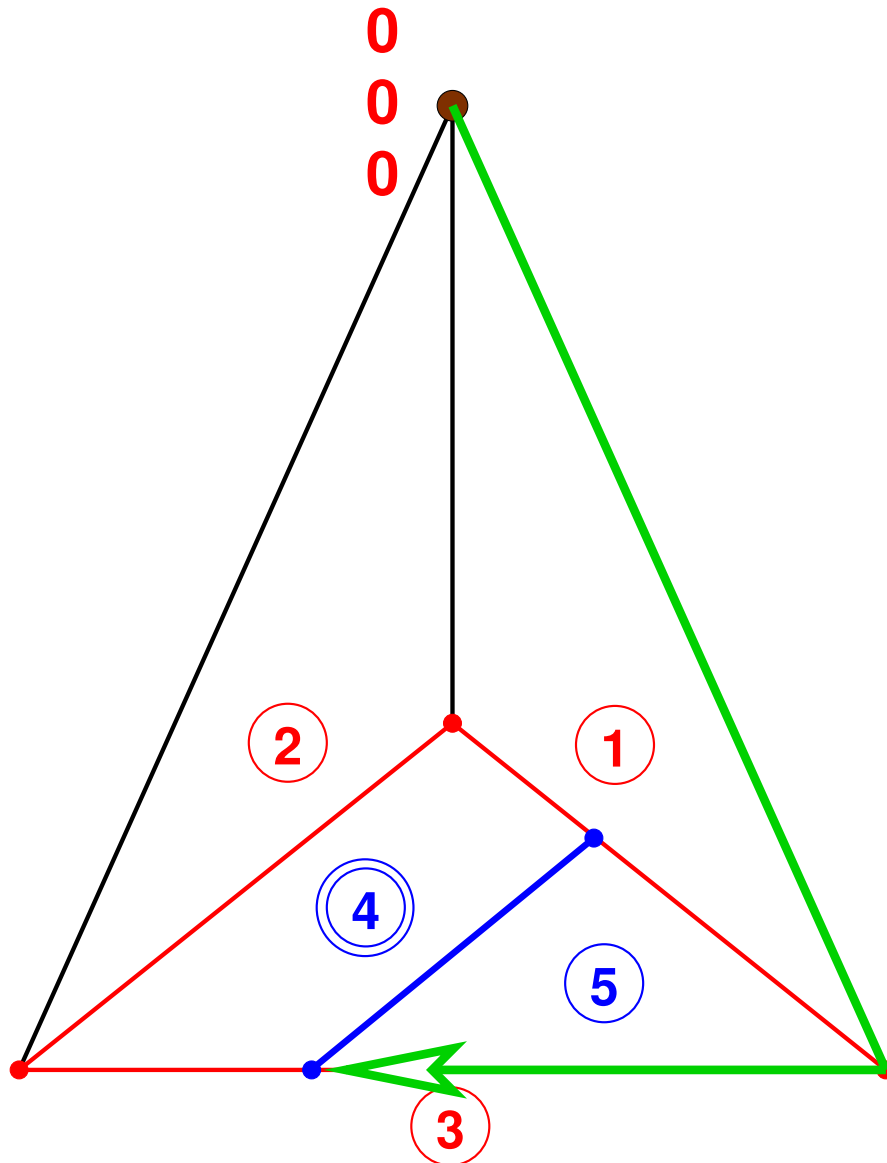
missing label 2

# The Lemke-Howson algorithm



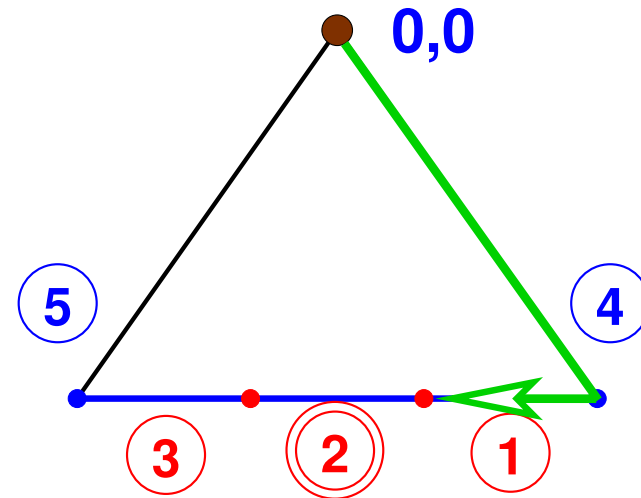
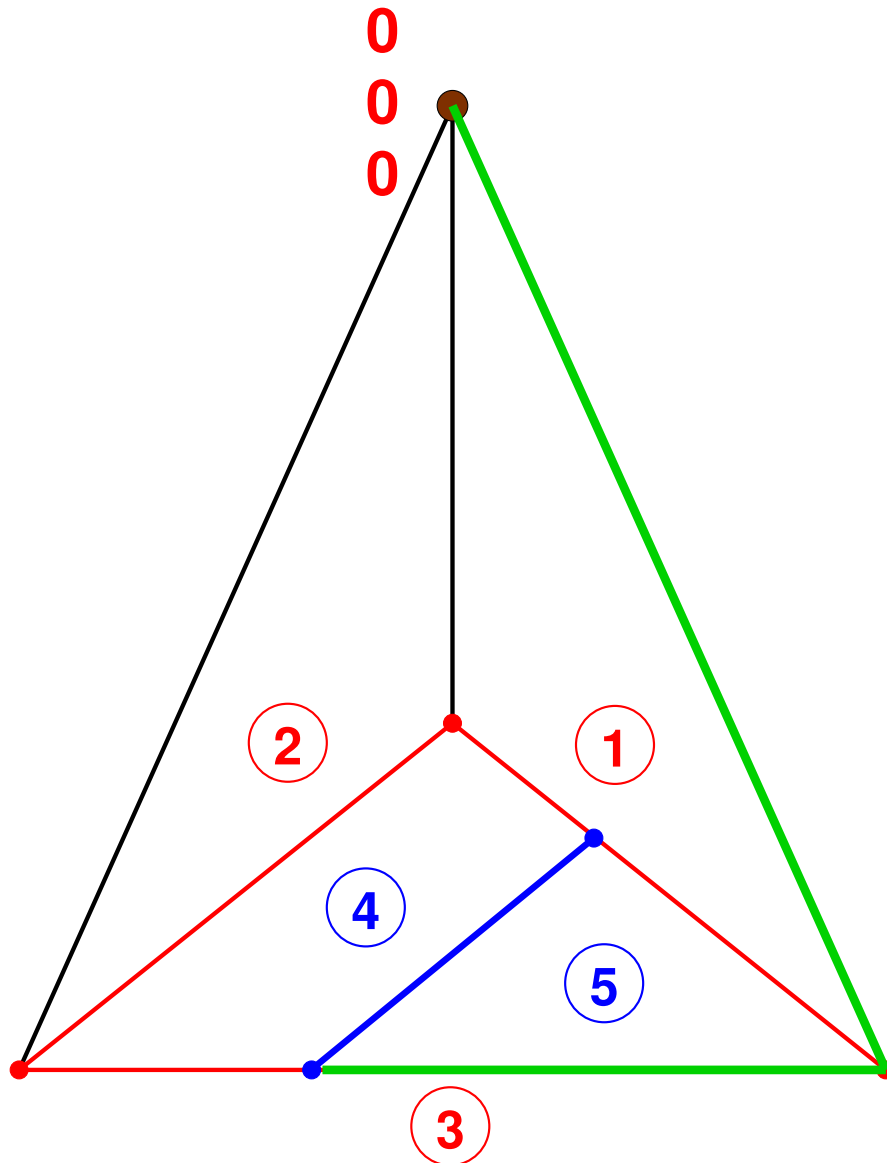
missing label 2

# The Lemke-Howson algorithm



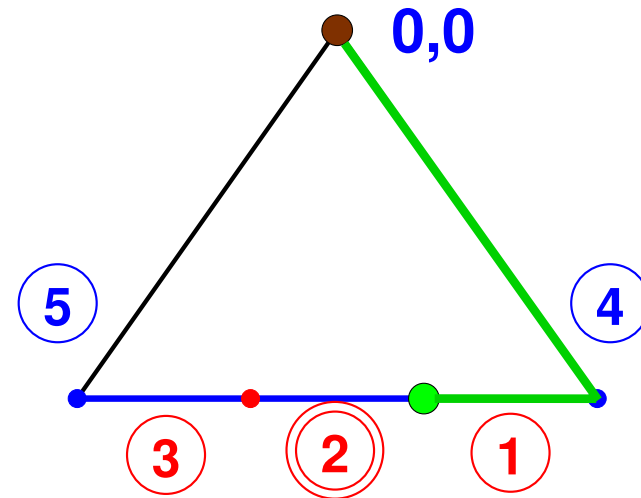
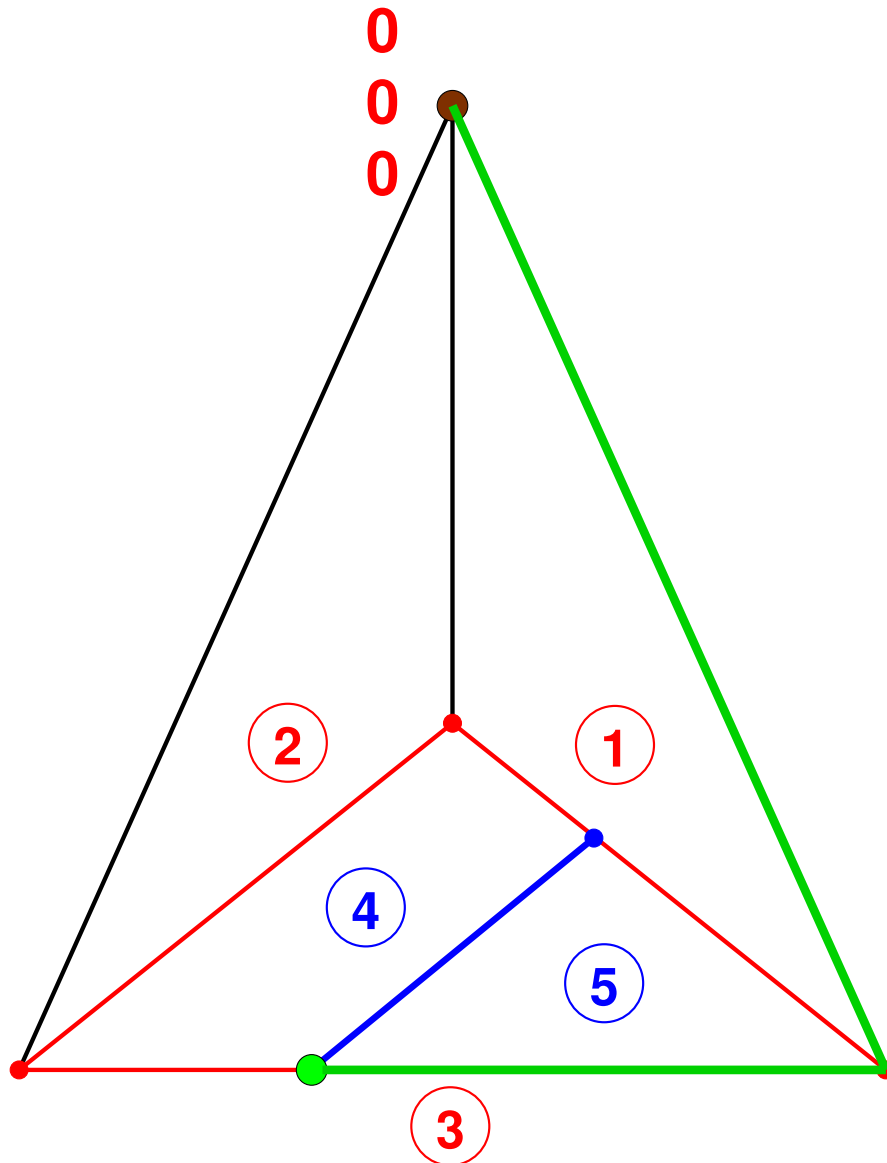
missing label 2

# The Lemke-Howson algorithm



missing label (2)

# The Lemke-Howson algorithm



found label **2**

# Why Lemke-Howson works

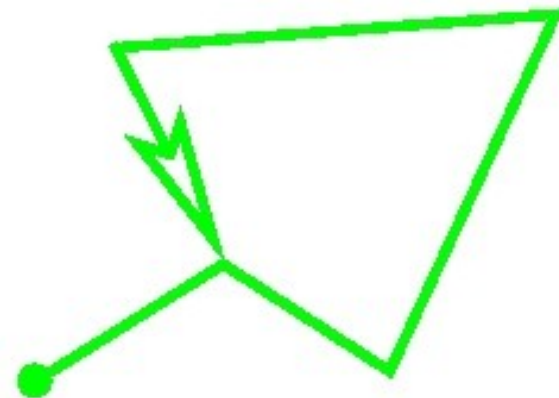
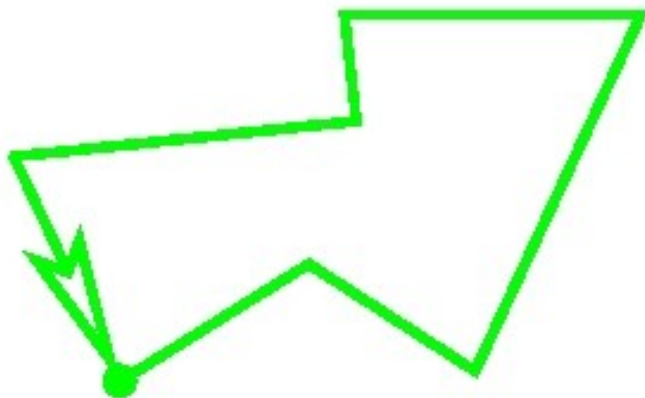
LH finds at least one Nash equilibrium because

- **finitely many** "vertices"

for nondegenerate (generic) games:

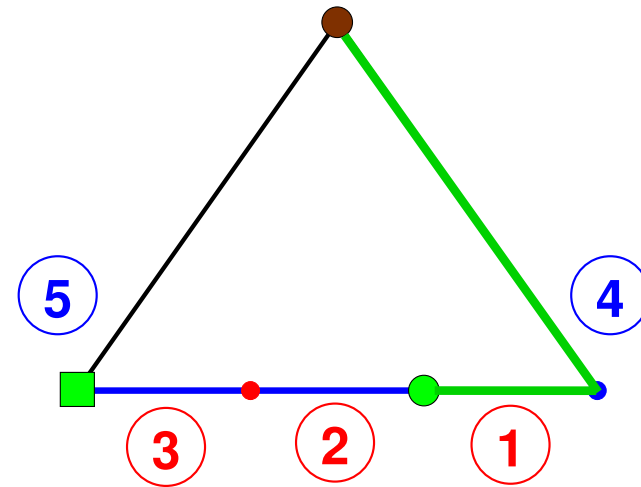
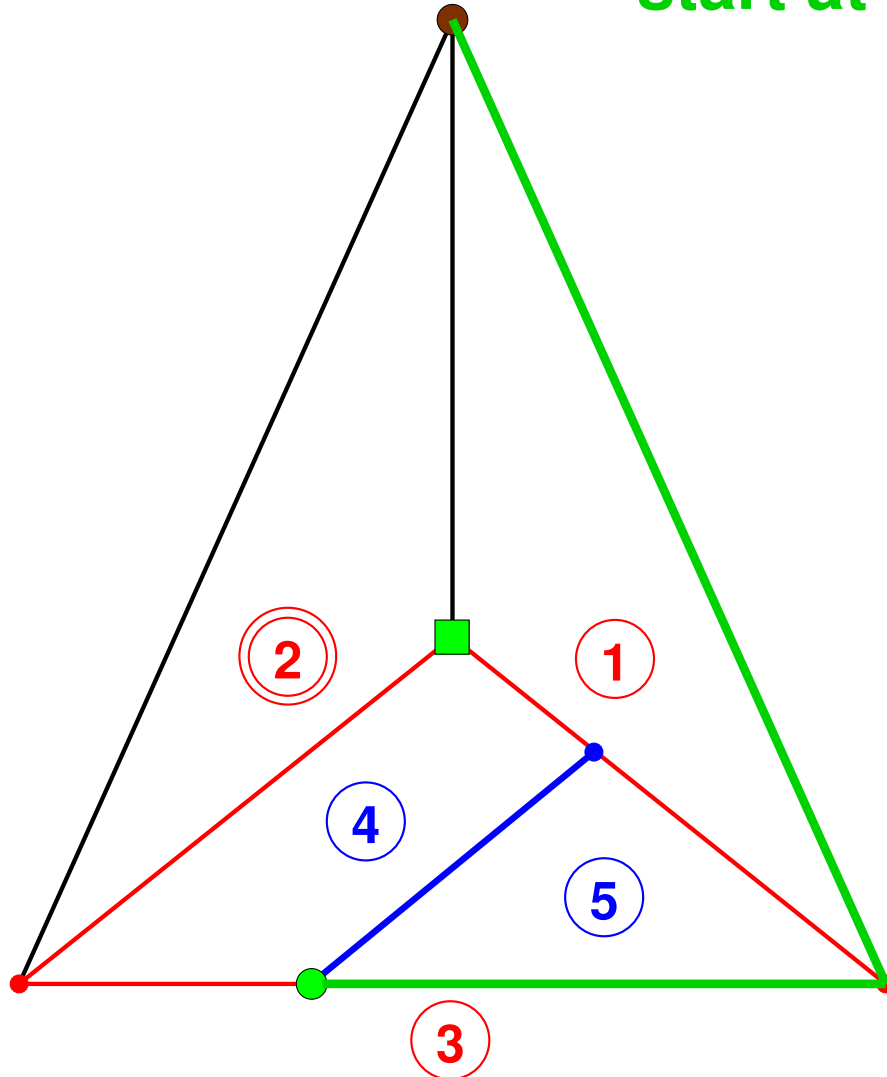
- **unique** starting edge given missing label
- **unique** continuation

⇒ precludes "coming back" like here:



# The Lemke–Howson algorithm

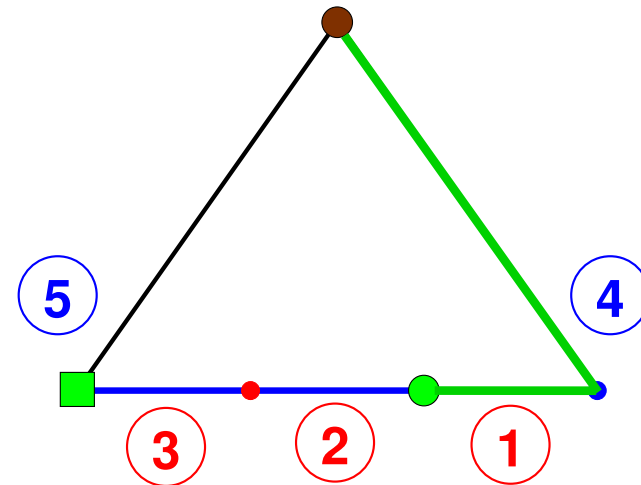
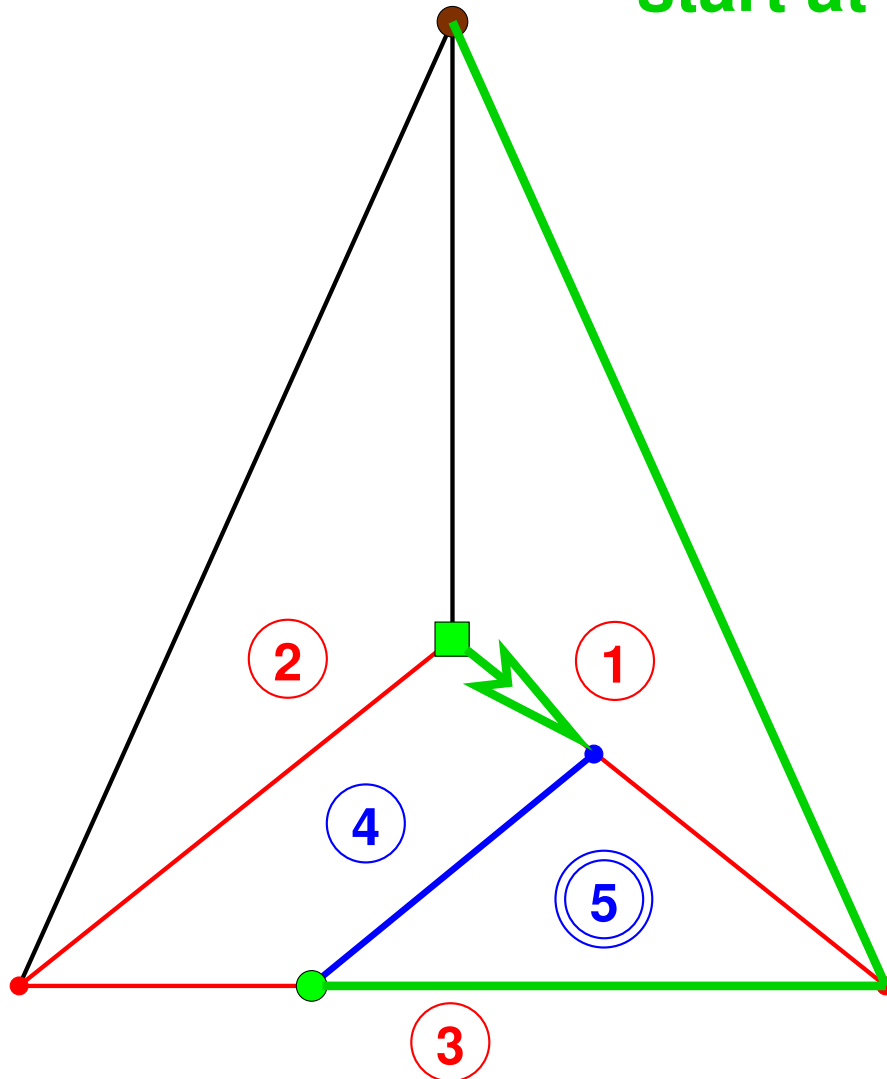
start at Nash equilibrium ■



missing label 2

# The Lemke–Howson algorithm

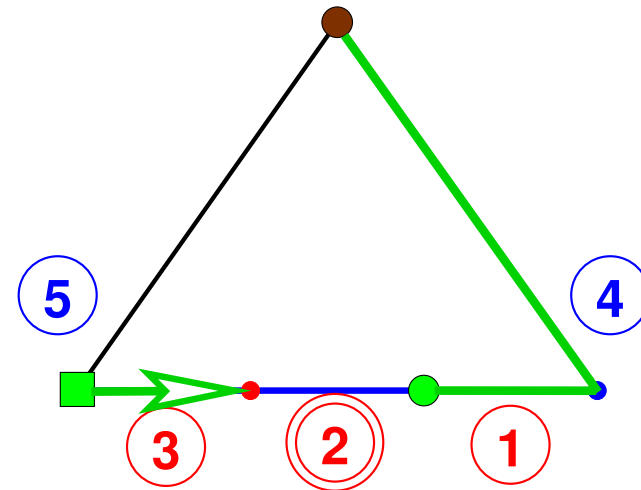
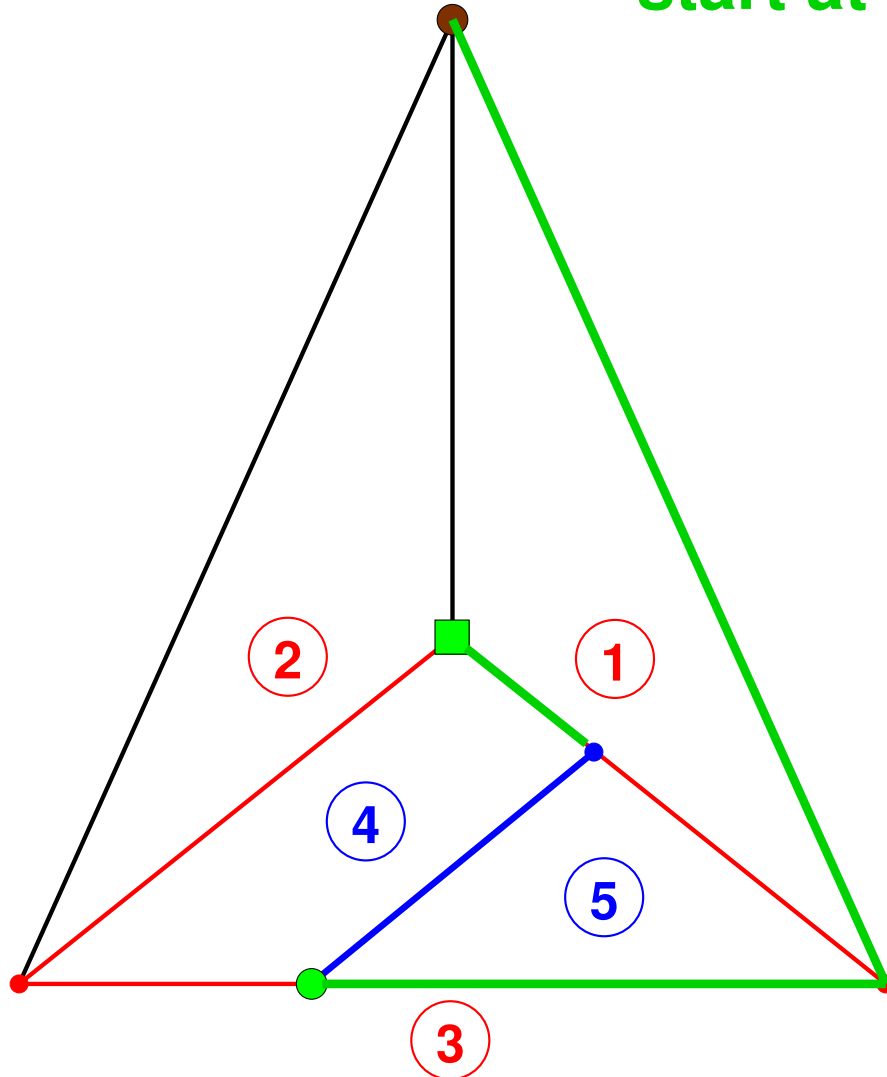
start at Nash equilibrium ■



missing label 2

# The Lemke–Howson algorithm

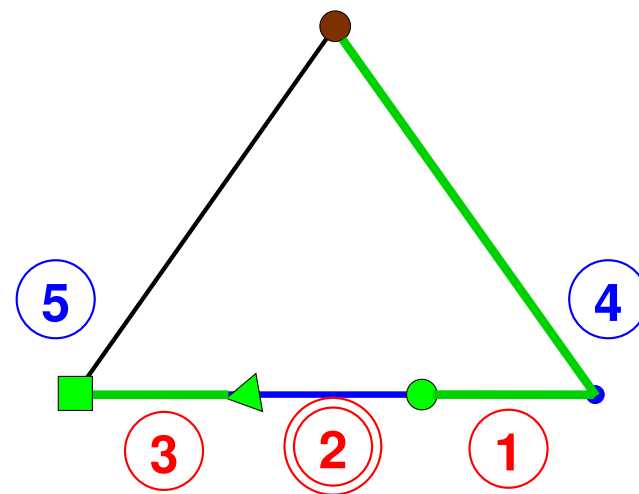
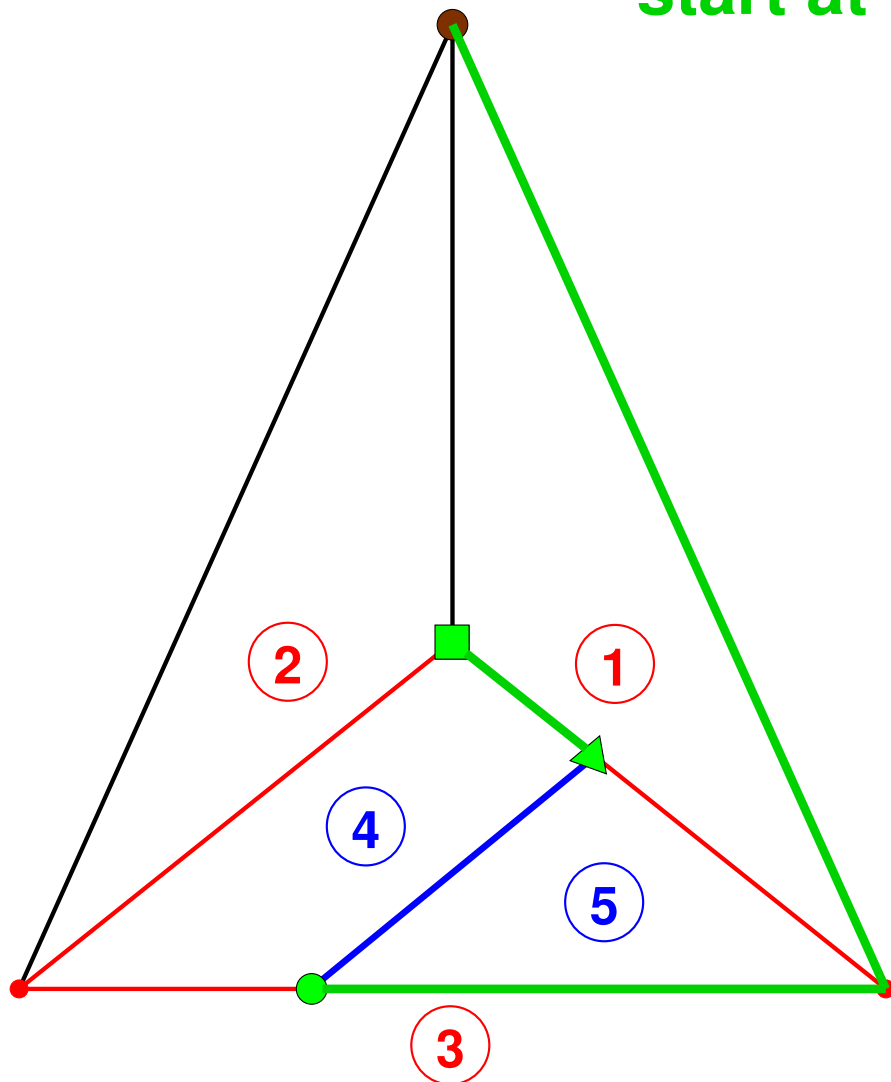
start at Nash equilibrium ■



missing label 2

# Odd number of Nash equilibria!

start at Nash equilibrium ■



found label 2

# Nondegenerate bimatrix games

Given:  $m \times n$  bimatrix game  $(A, B)$

$$X = \{ \mathbf{x} \in \mathbf{R}^m \mid \mathbf{x} \geq \mathbf{0}, x_1 + \dots + x_m = 1 \}$$

$$Y = \{ \mathbf{y} \in \mathbf{R}^n \mid \mathbf{y} \geq \mathbf{0}, y_1 + \dots + y_n = 1 \}$$

$$\text{supp}(\mathbf{x}) = \{ i \mid x_i > 0 \}$$

$$\text{supp}(\mathbf{y}) = \{ j \mid y_j > 0 \}$$

$(A, B)$  nondegenerate  $\iff \forall \mathbf{x} \in X, \mathbf{y} \in Y:$

$$| \{ j \mid j \text{ best response to } \mathbf{x} \} | \leq | \text{supp}(\mathbf{x}) |,$$

$$| \{ i \mid i \text{ best response to } \mathbf{y} \} | \leq | \text{supp}(\mathbf{y}) |.$$

# Nondegeneracy via labels

$m \times n$  bimatrix game  $(A, B)$  **nondegenerate**

$\Leftrightarrow$  no  $x \in X$  has more than  $m$  labels,  
no  $y \in Y$  has more than  $n$  labels.

E.g.  $x$  with  $> m$  labels,

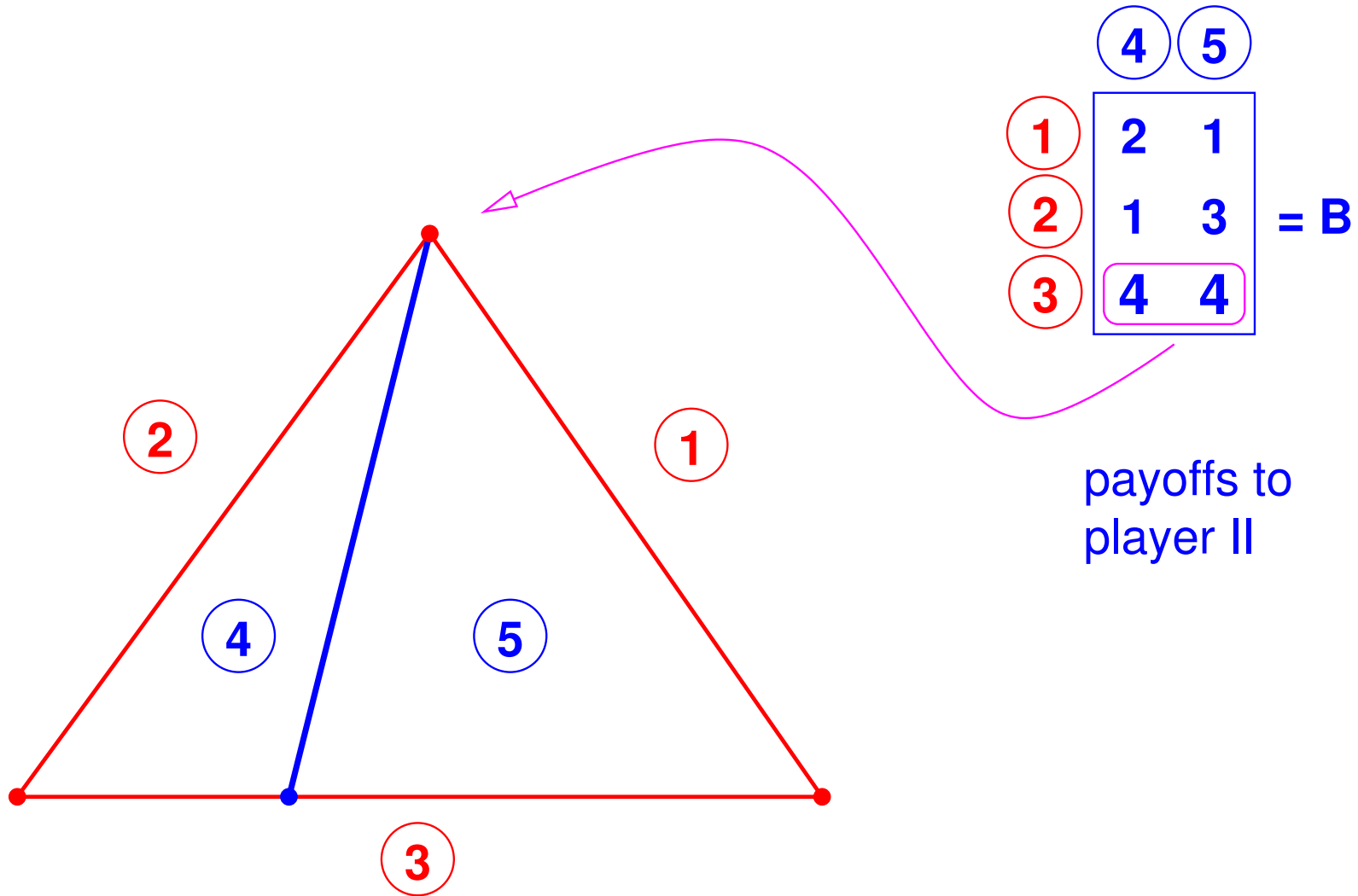
$s$  labels from  $\{1, \dots, m\}$ ,

$\Rightarrow > m-s$  labels from  $\{m+1, \dots, m+n\}$

$\Leftrightarrow > |\text{supp}(x)|$  **best responses** to  $x$ .

$\Rightarrow$  degenerate.

# Example of a degenerate game



## Handling degenerate games

Lemke–Howson implemented by pivoting, i.e., changing from one *basic feasible solution* of a linear system to another by choosing an entering and a leaving variable.

Choice of entering variable via complementarity (only difference to simplex algorithm for linear programming).

Leaving variable is *unique* in nondegenerate games.

In degenerate games: *perturb* system by adding  $(\varepsilon, \dots, \varepsilon^n)^\top$ , creates nondegenerate system.

Implemented *symbolically* by lexicographic rule.