A characterization of single-peaked single-crossing domain

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based on joint work with
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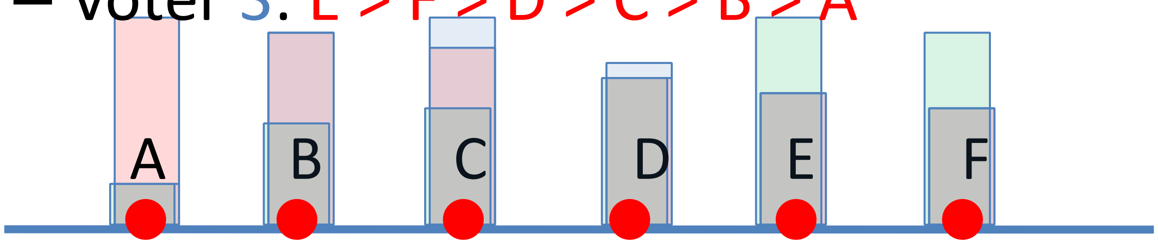
Voters and Their Preferences

- n voters, m candidates
- Each voter has a complete ranking of the candidates (his preference order)
- Problem:
  with no assumption on preference structure
  - counterintuitive behavior may occur
  - computational problems are often hard

A	B	C	D	B	C	A	B	C
B	C	A	A	C	D	B	C	A
C	D	B	B	D	A	C	A	B
D	A	D	C	A	B	D	D	D

Single-Peaked Preferences

- Definition: a preference profile is single-peaked (SP) wrt an ordering < of candidates (axis) if for each voter v there exists a candidate C such that:
  - v ranks C first
  - if C < D < E, v prefers D to E
  - if A < B < C, v prefers B to A
- Example:
  - voter 1: C > B > D > E > F > A
  - voter 2: A > B > C > D > E > F
  - voter 3: E > F > D > C > B > A



Single-Crossing Preferences

<u>Definition</u>: a profile is single-crossing (SC) wrt an ordering of voters $(v_1, \ldots, v_n)$ if for each pair of candidates A, B there exists an i $\{0, \ldots, n\}$ such that voters $v_1, \ldots, v_i$ prefer A to B, and voters $v_{i+1}, \ldots, v_n$ prefer B to A

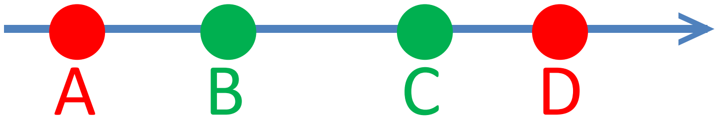
A	B	B	C	C	C	D
B	A	C	B	B	D	C
C	C	A	A	D	B	B
D	D	D	D	A	A	A

Single-Peaked vs. Single-Crossing Preferences

- Similarities:
  - both are motivated by the idea that the society is aligned along a single axis
  - both can be checked in poly-time
  - both ensure existence of a Condorcet winner
  - both enable efficient algorithms for many social choice problems
  - both admit forbidden minor characterization
- Differences:
  - order on candidates vs. order on voters

Single-Peaked Profile That Is Not Single-Crossing

B	B	C	C
C	C	B	B
A	D	A	D
D	A	D	A



- v_1 and v_2 have to be adjacent (because of B, C)
- v_3 and v_4 have to be adjacent (because of B, C)
- v_1 and v_3 have to be adjacent (because of A, D)
- v_2 and v_4 have to be adjacent (because of A, D)

a contradiction

Single-Crossing Profile That Is Not Single-Peaked

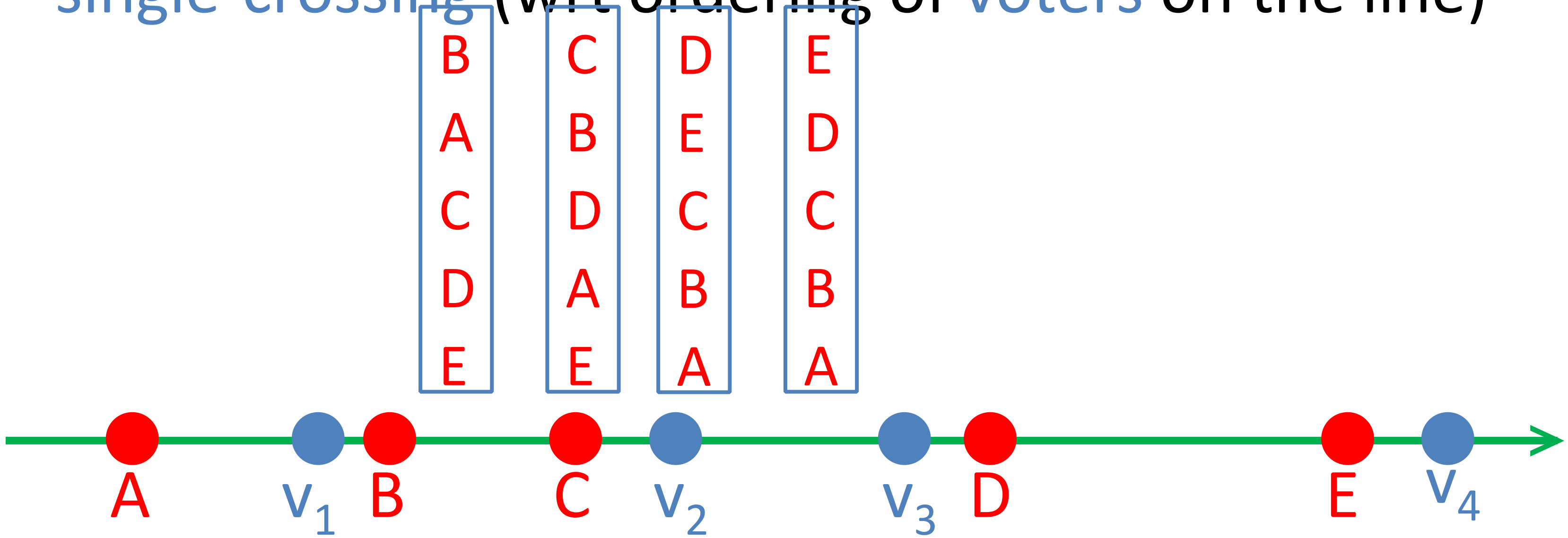
1	n	n		n	n
2	1	n-1		n-1	n-1
…	2	1		n-2	n-2
…	…	2	…	…	…
…	…	…		…	…
n-2	…	…		…	…
n-1	n-2	…		1	2
n	n-1	n-2		2	1

Each candidate is ranked last exactly once

Can we characterize preference profiles that are simultaneously single-peaked and single-crossing?

1D-Euclidean Preferences

- Both voters and candidates are points in R
- v prefers A to B if |v - A| < |v - B|
- Observation: 1D-Euclidean preferences are
  - single-peaked (wrt ordering of candidates on the line)
  - single-crossing (wrt ordering of voters on the line)



1-Euclidean Preferences: Bad News

- Proposition: There exists a preference profile that is SP and SC, but not 1-Euclidean

v_1: 1 $A_1A_2A_3$ 2 $B_1B_2B_3$ 3 $C_1C_2C_3$ $D_1D_2D_3$ 4 5

v_2: $A_2A_1A_3$ 2 $B_1B_2B_3$ 3 1 $C_1C_2C_3$ $D_1D_2D_3$ 4 5

v_3: $B_2B_1B_3$ 3 $C_1C_2C_3$ $D_1D_2D_3$ 4 2 $A_3A_2A_1$ 1 5

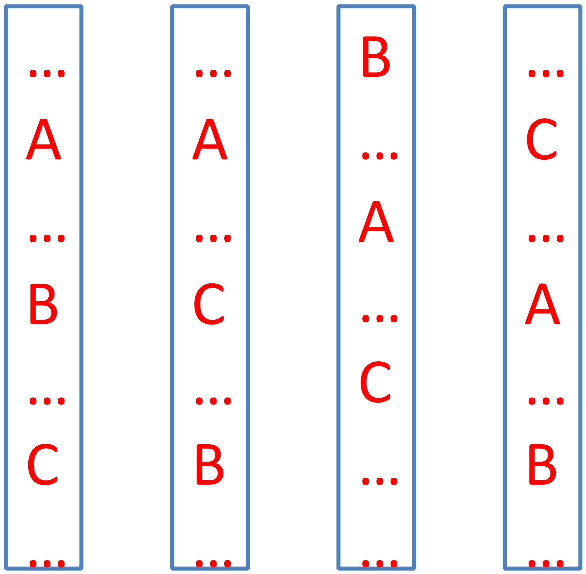
v_4: $C_2C_1C_3$ $D_1D_2D_3$ 4 3 $B_3B_2B_1$ 2 $A_3A_2A_1$ 1 5

v_5: $D_2D_1D_3$ $C_3C_2C_1$ 4 5 3 $B_3B_2B_1$ 2 $A_3A_2A_1$ 1

v_6: 5 4 $D_3D_2D_1$ $C_3C_2C_1$ 3 $B_3B_2B_1$ 2 $A_3A_2A_1$ 1

A Different Angle

- A preference profile is called narcissistic is every candidate is ranked 1st at least once
- Proposition: Every narcissistic SC profile is SP (axis = 1st vote)
- Proof:
  - suffices to show that if v_1 prefers A to B to C, then no voter ranks B last out of A, B, and C



Pre-NSC Preferences

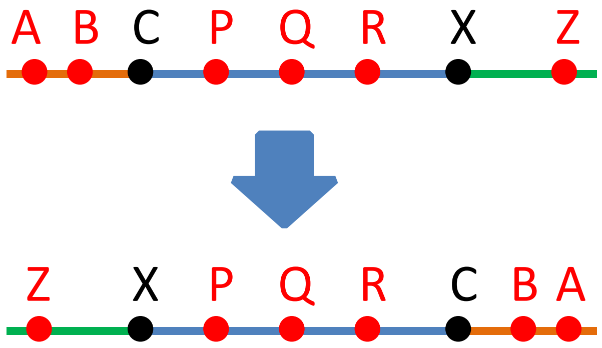
- Are all SP-SC profiles narcissistic?
  - obviously no: being SP and SC is robust to deletions, and being narcissistic is not
- Definition: a profile is called pre-NSC if it can be extended to a narcissistic SC profile by adding voters
  - every pre-NSC profile is SP and SC
- Main Theorem: the converse is also true

Characterization

- Theorem: every SP-SC profile is pre-NSC
- Proof idea:
  - constructive argument: extend a SP-SC profile to a narcissistic one
  - crucial lemma: given a SP-SC profile $V = (v_1, \ldots, v_n)$, there is a vote v_0 such that $(v_0, v_1, \ldots, v_n)$ is SP and SC, and v_0 is an axis witnessing that V is SP
  - by the lemma, can assume that the profile is SP wrt 1^{st} vote
  - use 1^{st} vote as a guiding order to insert votes

Lemma: Proof Idea

- Lemma: given a SP-SC profile $V = (v_1, \ldots, v_n)$, there is a vote v_0 such that $(v_0, v_1, \ldots, v_n)$ is SP and SC and v_0 is an axis witnessing that V is SP
- Proof idea:
  - try to add an arbitrary axis witnessing that V is SP
  - if this fails, pick a "minimal" pair of candidates that is at fault
  - modify the axis by swapping tails
  - argue that tail swap can be performed $\leq m$ times



Algorithmic Perspective

- Our proof implies a polynomial-time algorithm for
  (1) checking whether a given profile V is pre-NSC
  (2) finding a narcissistic profile extending it
- A simpler algorithm for (2) given (1):
  - for each missing candidate A, find possible positions in V to insert a vote v_A that ranks A first
  - turns out that there is ≤1 position for each candidate
  - if v_A is the only vote to be inserted between v_i and v_{i+1}, construct v_A by moving A to the top of v_i
  - if both v_A and v_B need to be inserted between v_i and v_{i+1}, v_A precedes v_B iff A precedes B in v_i

Applications to Fully Proportional Representation: Monroe's Rule

- n voters, m candidates
- Task: elect a k-member parliament
- Constraints:
  - candidates are explicitly assigned to voters
  - each elected candidate represents ≈ n/k voters
  - voter's dissatisfaction is determined by the rank of his representative in his vote (via a scoring rule)
- Objective: minimize
  - sum of voters' dissatisfactions (Monroe^{+}) , or
  - maximum dissatisfaction ($\text{Monroe}^{\max}$)
- Both Monroe^{+} and $\text{Monroe}^{\max}$ are NP-hard for general preferences

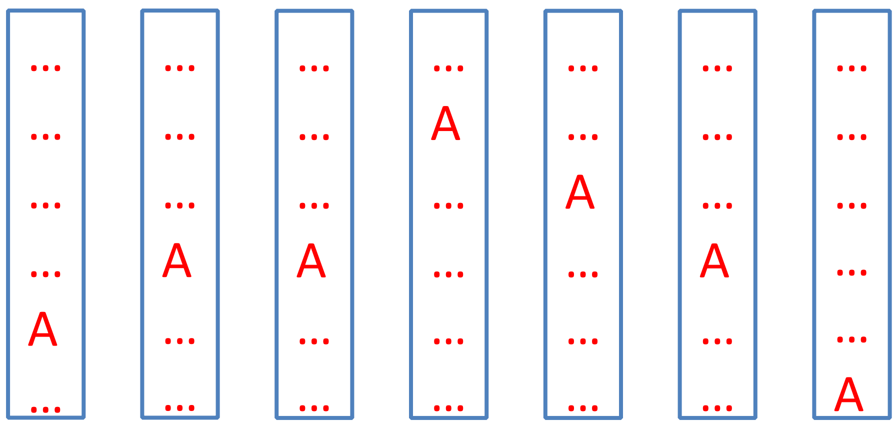
Monroe's Rule: Example

- k = 2, scoring rule = Borda
- A can be assigned to at most 5 voters
- For Monroe^{+}, we can assign B to v_1 -v_4 or C to v_2 - v_5
- For Monroe^{max} , the only solution is to assign C to 4 arbitrary voters

A	A	A	A	A	A	A	A	A
B	B	B	C	C	D	D	E	E
C	C	C	D	E	C	C	C	C
D	D	D	B	B	B	B	B	B
E	E	E	E	D	E	E	D	D

Single-Peaked Trajectories

- above (A, i): # of candidates v_i ranks above A
- <u>Definition</u>: a profile is said to have single-peaked trajectories property (SPTP) if for every candidate A there exists a voter v_i such that
  - above (A, j) ≥ above(A, k) whenever $j < k \leq i$
  - above (A, j) ≥ above(A, k) whenever $j > k \geq i$



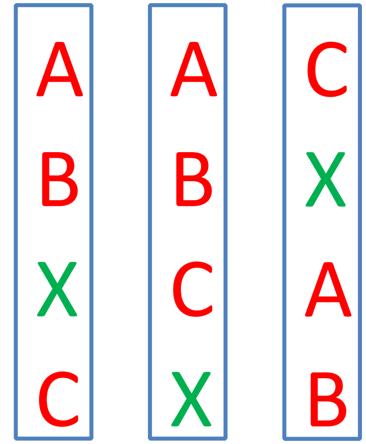
- <u>Claim</u>: pre-NSC profiles have SPTP

Monroe^{max} and SPTP

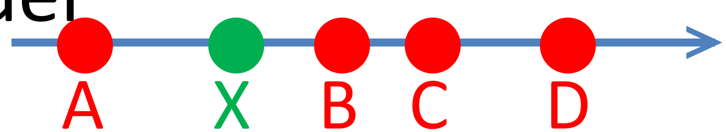
- Claim: if a profile has SPTP, then the set of voters matched to an elected candidate under Monroe^{max} is a contiguous segment of V
- Corollary: for pre-NSC preferences Monroe^{max} admits a very efficient DP algorithm
- [Betzler, Slinko, Uhlmann'13]: for single-peaked preferences Monroe^{max} admits a DP algorithm (but a much slower one)

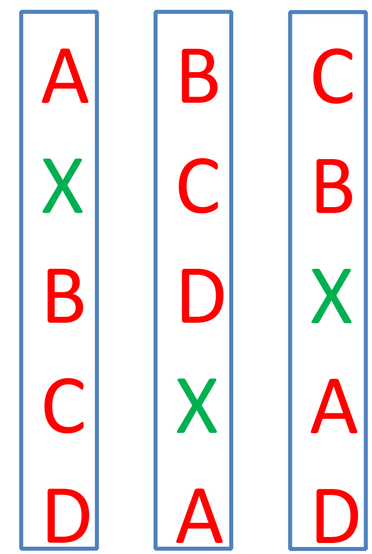
Comment: Single-Peaked and Single-Crossing Profiles and SPTP

- Observation:
  a single-crossing profile may fail to have SPTP



- Observation:
  a single-peaked profile may fail to have SPTP (wrt natural order of the voters)





Future Work: Other Applications

- Are there algorithmic problems that are
  - hard for single-peaked preferences
  - hard for single-crossing preferences
  - easy for pre-NSC preferences?
- I.e., the problem admits an algorithm that relies on SPTP
- Candidate problems:
  - manipulation of STV
  - certain questions about control and bribery

Future Work: Extensions

- Generalization: profiles that are single-peaked/single-crossing on a tree
- Definition: a profile V is single-peaked on a tree T if candidates can be matched to vertices of T so that the restriction of V to every path in T is single-peaked
- Definition: a profile V is single-crossing on a tree T if voters can be matched to vertices of T so that the restriction of V to every path in T is single-crossing
- Question: given T_1 and T_2, can we characterize elections that are single-peaked on T_1 and single-crossing on T_2?