Continuous Network Design

Hardness and Approximation

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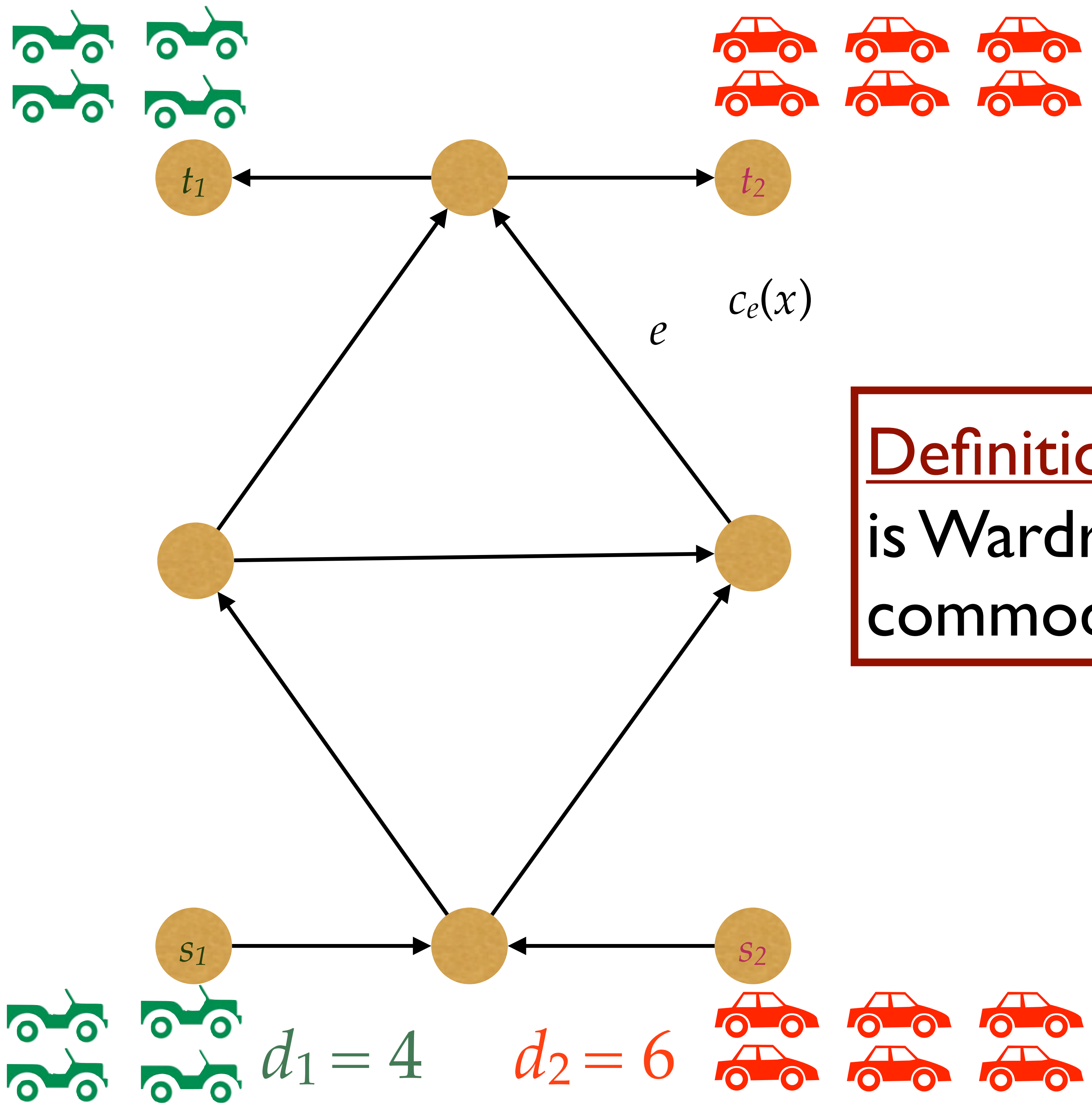
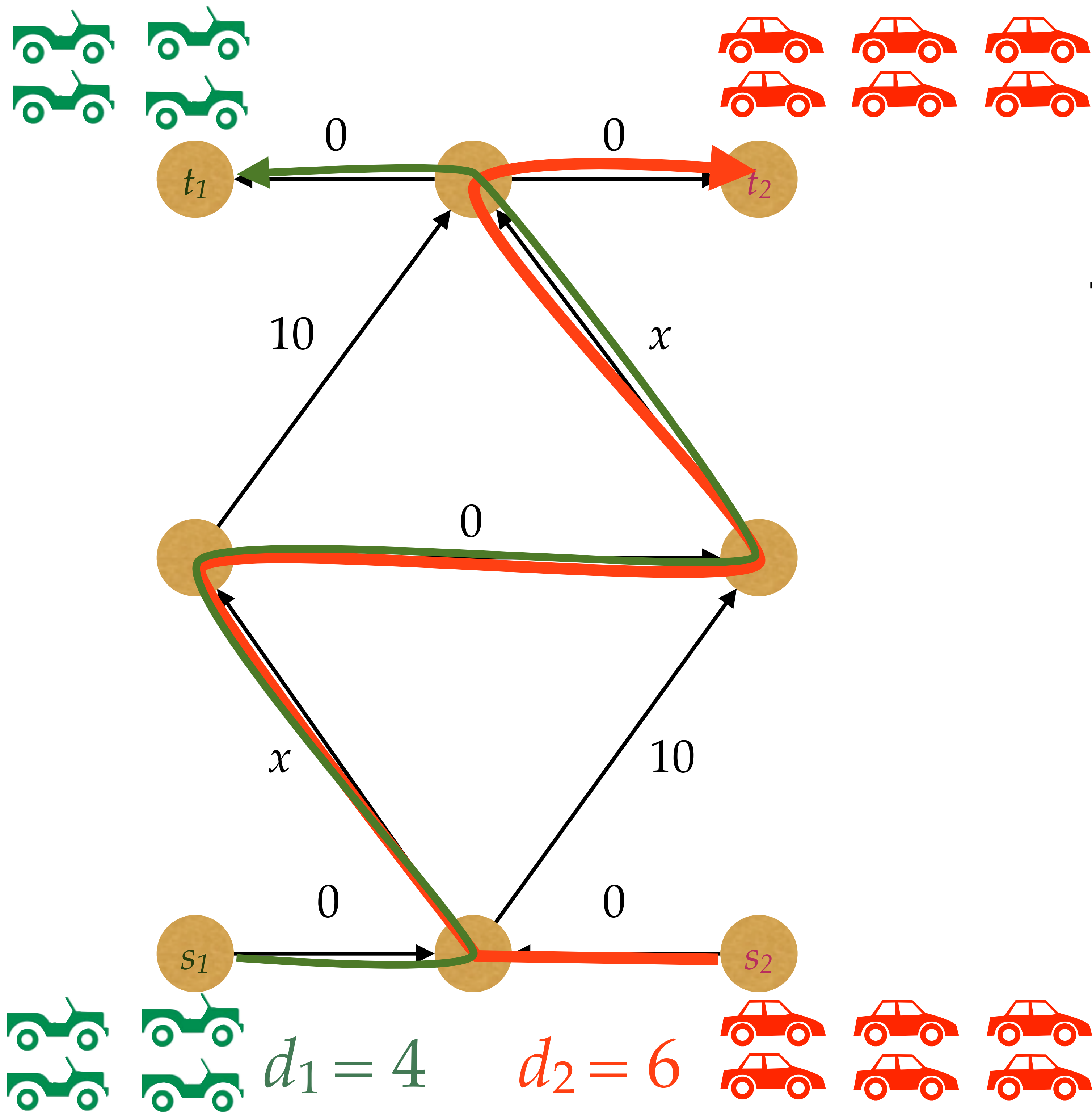
Wardrop equilibrium



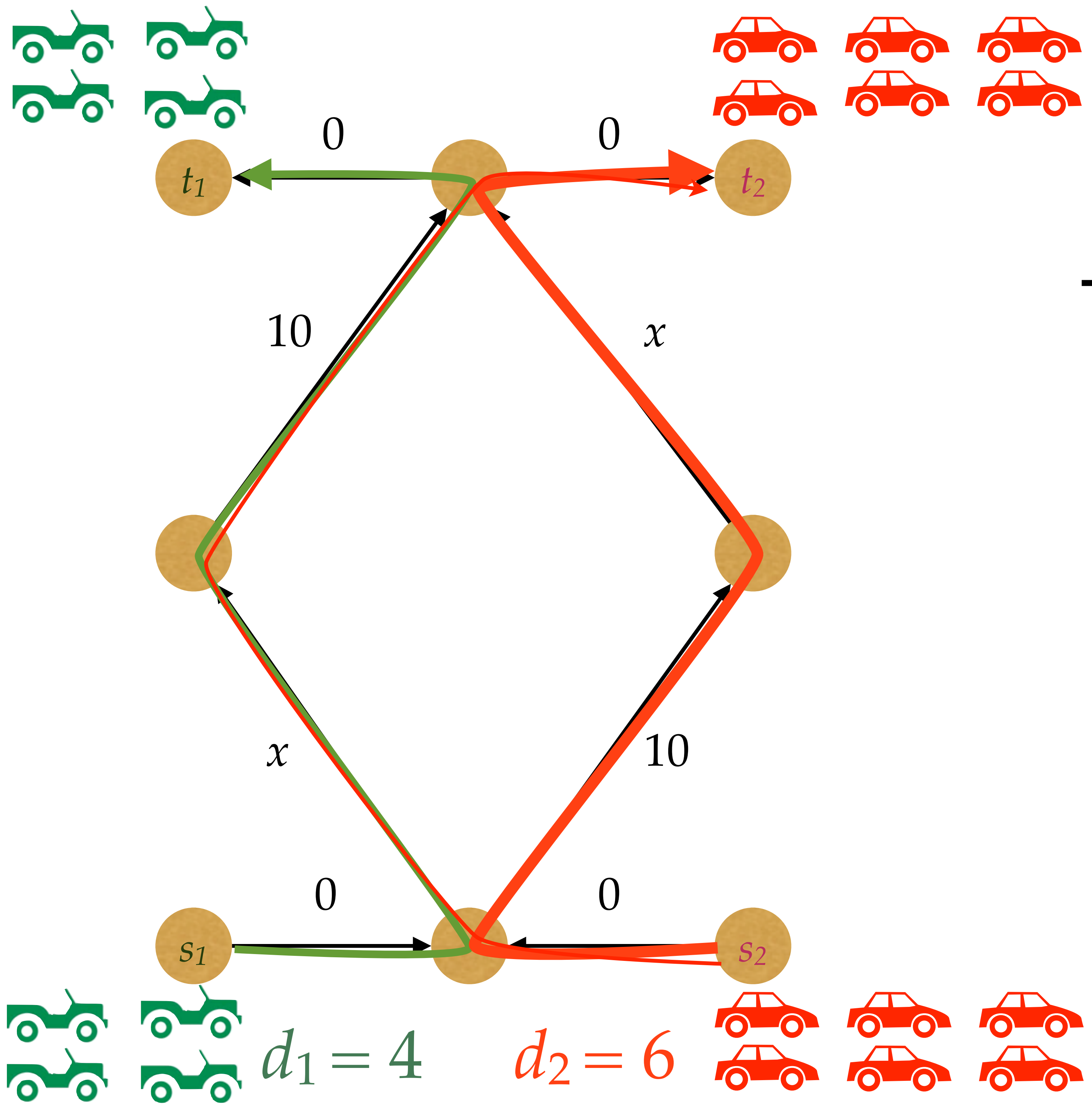
Definition Multi-commodity flow f is Wardrop equilibrium if commodities use shortest paths.

Wardrop equilibrium - example



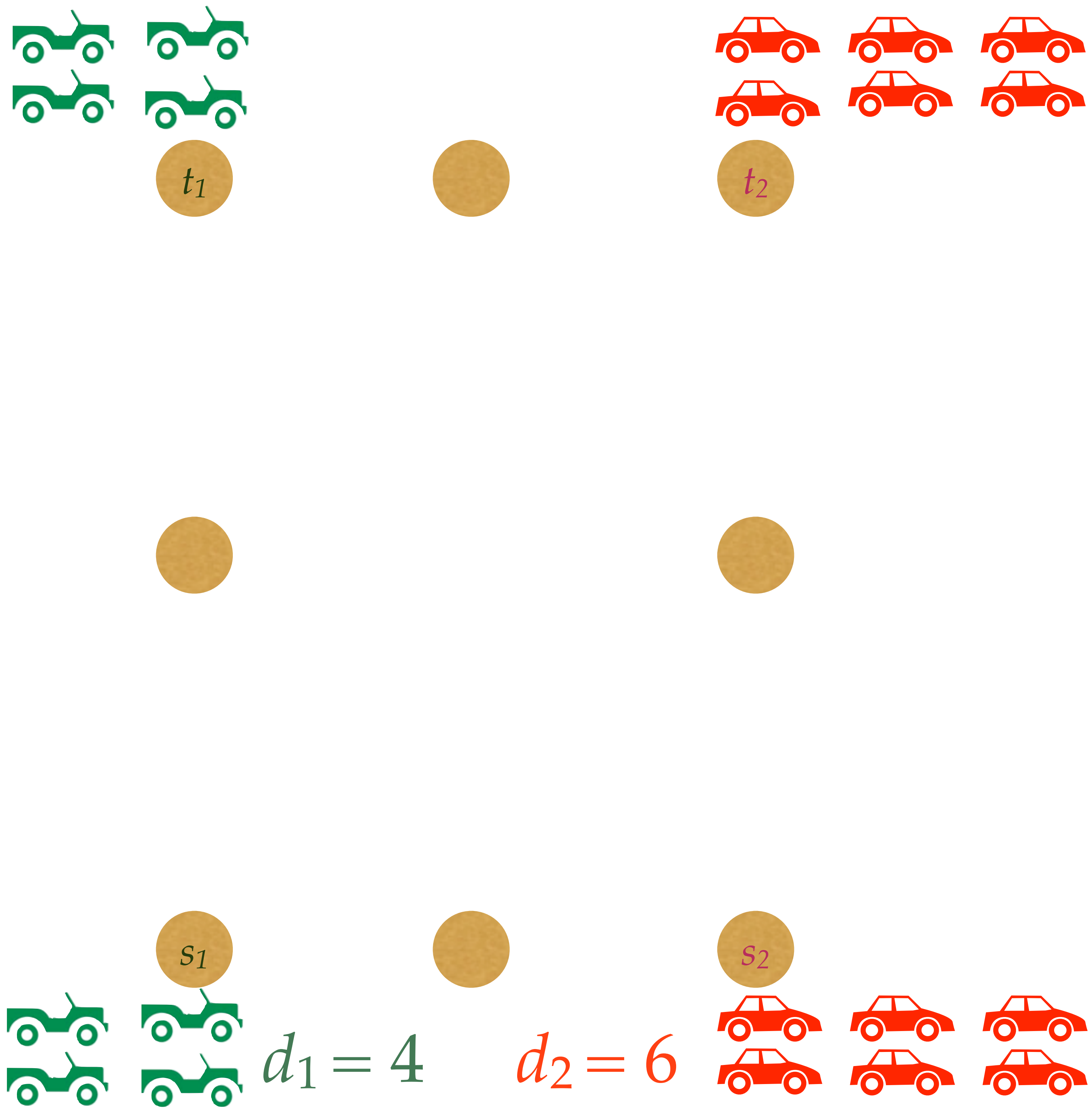
Total travel time = 200

Wardrop equilibrium - example

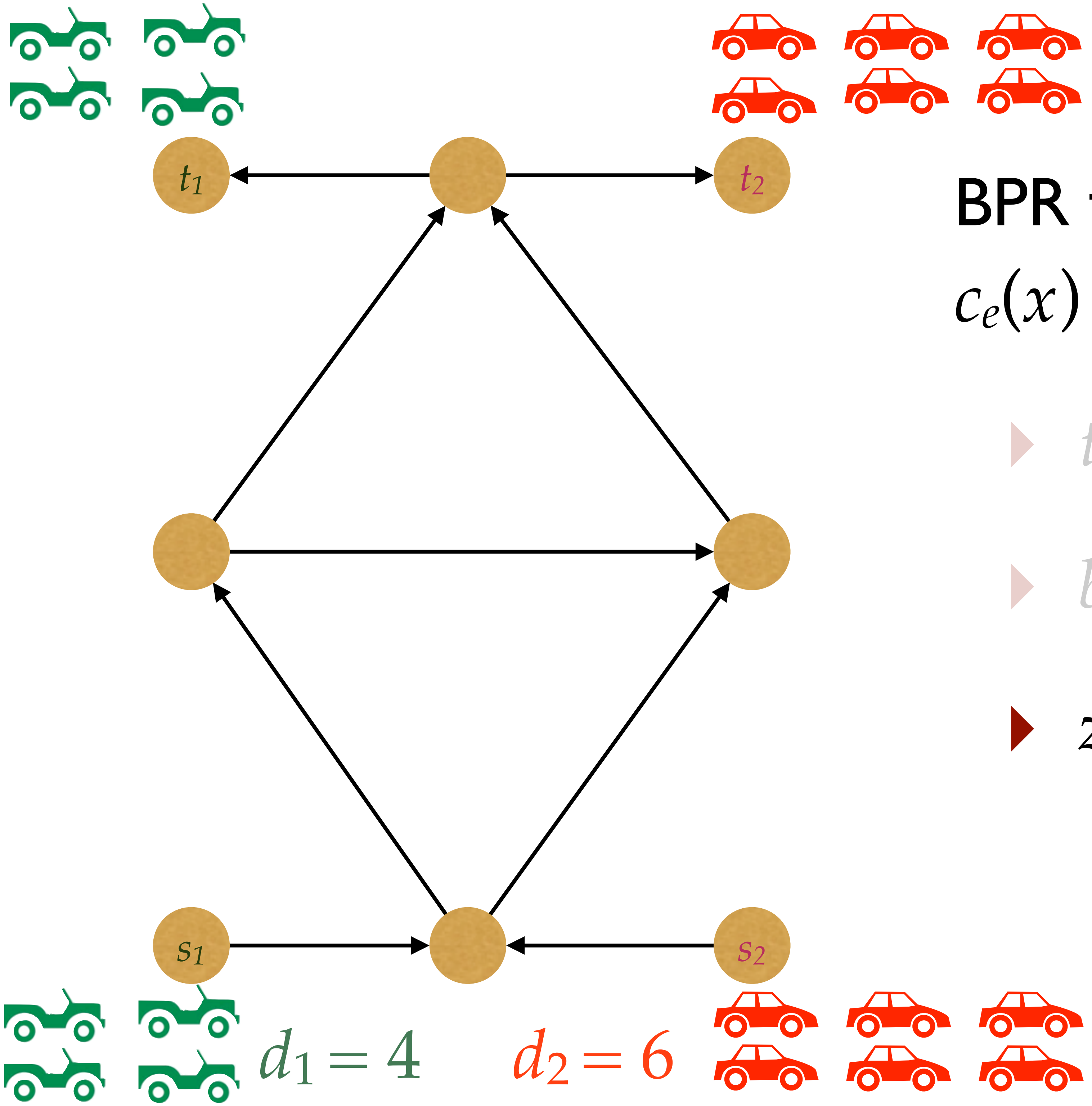


Total travel time = 150

A network design problem



A network design problem

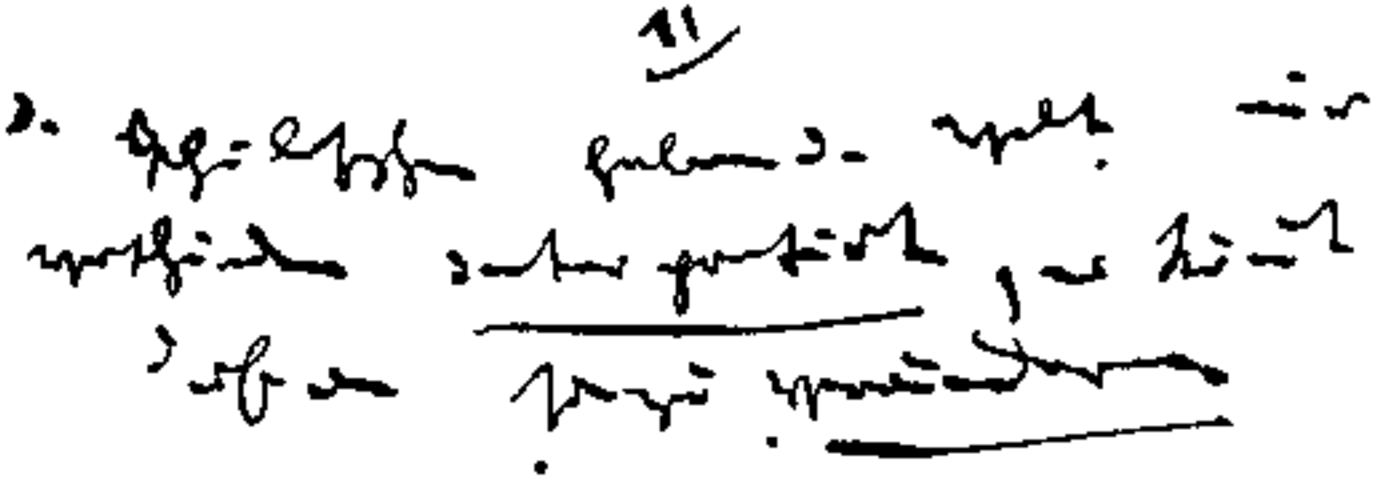


BPR function

$$c_e(x) = t_e\left(1 + b_e\,(f_e/z_e)^4\right)$$

- t_e : free flow travel time
- b_e : bias
- z_e : capacity

Descriptive vs. normative science



"Die Philosophen haben die Welt nur verschieden interpretiert; es kommt aber darauf an, sie zu verändern."

"Philosophers have hitherto only interpreted the world in various ways; the point is to change it."

Problem definition

- Directed graph $G = (V, E)$
  - Set K of commodities with $(s_k, t_k, d_k) \in V \times V \times \mathbb{R}_{\geq 0}$
  - Construction price (per unit) l_e
  - Cost functions $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, mapping v_e/z_e to latency $c_e(v_e/z_e)$
    (cont. differentiable, semi-convex, $c_e(x)$ and $x^2\, c_e'(x)$ non-decreasing and unbounded)

- Objective: Find a vector of capacities $(z_e)_{e \in E}$ minimizing

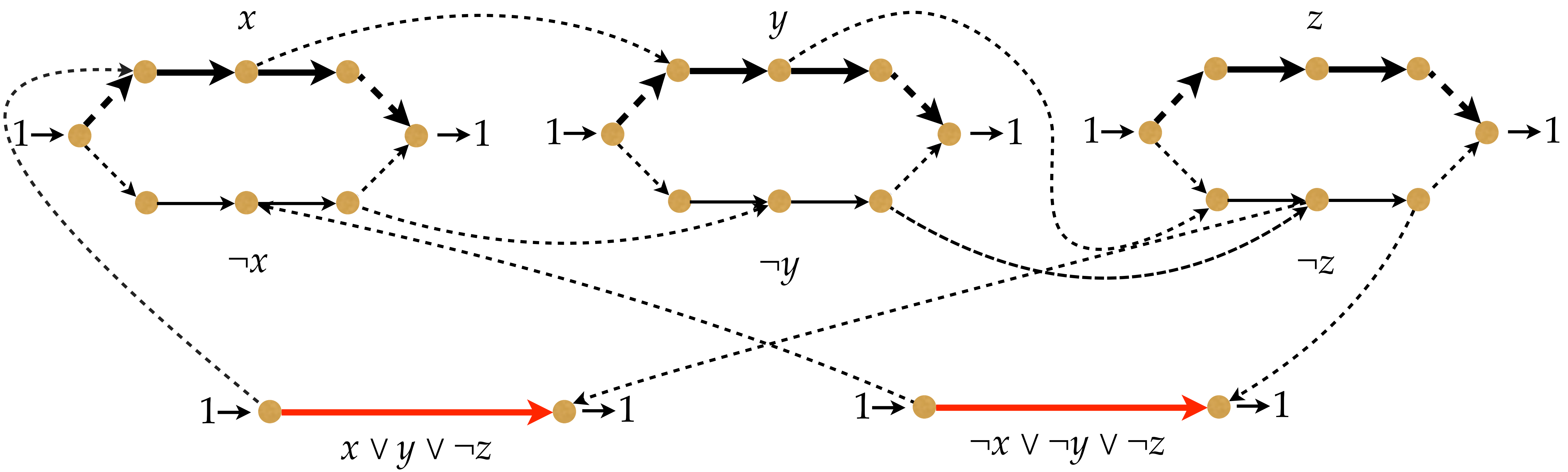
$$\sum_{e \in E} \left(c_e(v_e/z_e)\, v_e + z_e \cdot l_e \right) \qquad \text{(CNDP)}$$

s.t.: $v = (v_e)_{e \in E}$ is a Wardrop equilibrium

State-of-the-art

- Large body of work on heuristic approaches
  e.g. [Dafermos, TR `69][Dantzig et al., TR `79]
- Approximation Algorithm [Marcotte, MP `85]
  - 5/4 for affine cost functions $c(v/z) = a + b(v/z)$
  - Closed formula for monomials $c(v/z) = a + b(v/z)^d$ tending to 2 as d goes to ∞.
- "one of the most important, difficult and challenging problems in transport" [Yang and Bell, TRev `98]
- Discrete network design problem
  (Decide which edges to remove from the network)
  - Strongly NP-hard [Roughgarden, JCSS `06]

A hard instance



$\longrightarrow$ $c(v/z) = v/z,\ l = 1 \Rightarrow v^* = z^*$

$\dashrightarrow$ $c(v/z) = 0,$

$\longrightarrow$ (red) $c(v/z) = 4,$

- $\neg x, \neg y, \neg z$ satisfies the formula
- buy capacity for x, y, z
- total cost: $4\ \#\mathrm{cl} + 2\#\mathrm{cl} \cdot \#\mathrm{var}$

Solving a relaxation

- Find a vector of capacities $(z_e)_{e \in E}$ minimizing

$$\sum_{e \in E} \left(c_e(v_e/z_e)\, v_e + z_e \cdot l_e \right) \qquad \text{(CNDP)}$$

s.t.: $v = (v_e)_{e \in E}$ is a Wardrop equilibrium

[Marcotte, MP `85]

<u>Lemma</u> (CNDP‘) can be solved in polynomial time.

Proof:

- $\partial/\partial z_e \;\left(c_e(v_e/z_e)\, v_e + z_e \cdot l_e \right) = 0 \Leftrightarrow l_e = (v_e/z_e)^2\, c_e'(v_e/z_e)$
- If u_e solves $l_e = (x_e)^2\, c_e'(x_e)$, then $u_e = v_e/z_e$
- min $\sum_{e \in E} \left(c_e(u_e) + l_e/u_e\right) v_e$ s.t. v is flow
- compute a shortest path for each commodity

Approximation algorithm

Let $\lambda > 0$.

Solve the relaxation	Solve the relaxation
(v^*, z^*)	(v^*, z^*)
Choose z such that (v^*, z) is a Wardrop equilibrium	Compute a Wardrop equilibrium v for λz^*
(v^*, z)	$(v, \lambda z^*)$

take the best

The parameter p

Let (v^*, z^*) solve the relaxed problem (CNDP').

$$C(v^*, z^*) = \underbrace{\textstyle\sum_{e \in E} c_e(v^*_e / z^*_e)\, v^*_e}_{\substack{\text{Routing costs} \\ C^R(v^*, z^*) \\ := p\, C(v^*, z^*)}} + \underbrace{\textstyle\sum_{e \in E} z^*_e \cdot l_e}_{\substack{\text{Construction costs} \\ C^Z(v^*, z^*) \\ := (1-p)\, C(v^*, z^*)}}$$

Analysis of the approximation algorithm

[Correa, Schulz, Stir-Moses, MOR `04]

Definition For a set $\mathcal{C}$ of cost functions, let

$\mu(\mathcal{C}) = \sup_{c\in\mathcal{C}} \sup_{x\geq 0} \max_{\gamma\in[0,1]} \gamma(1\text{-}c(\gamma x)/c(x))$

and $\gamma(\mathcal{C})$ be the value for which the inner max is attained.

- Example: $\mathcal{C} = \{c : c(x) = a\,x + b\}$
  - $\mu(\mathcal{C}) = \max_{\gamma\in[0,1]} \gamma(1\text{-}\gamma)$

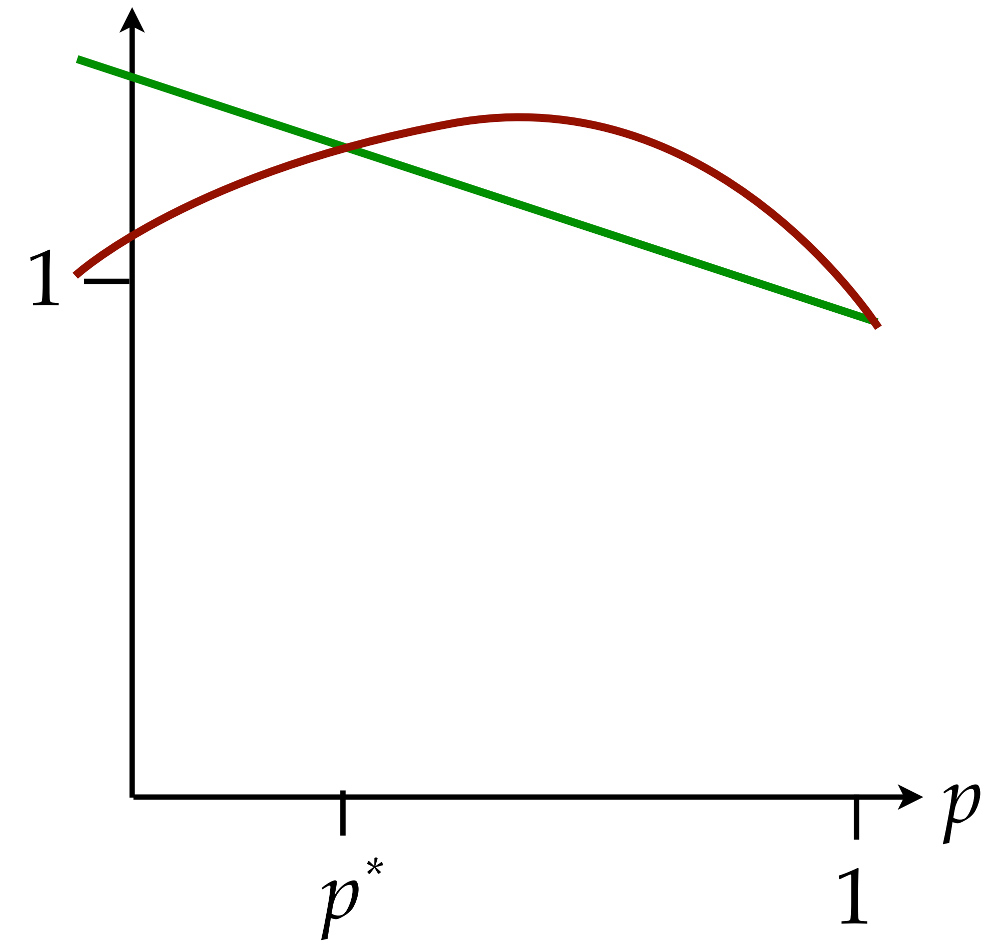
Analysis of best-of-two

- Choosing the best of the two, we obtain a guarantee of

$$\max_{p\in[0,1]} \min \left\{1+ \gamma(1\text{-}p), \left(\sqrt{p} + \sqrt{\mu(1\text{-}p)}\right)^2\right\}$$

- The approximation guarantee is

$$\frac{(\gamma+\mu+1)^2}{(\gamma+\mu+1)^2 - 4\mu\gamma}$$



Further analysis

- Our algorithm has

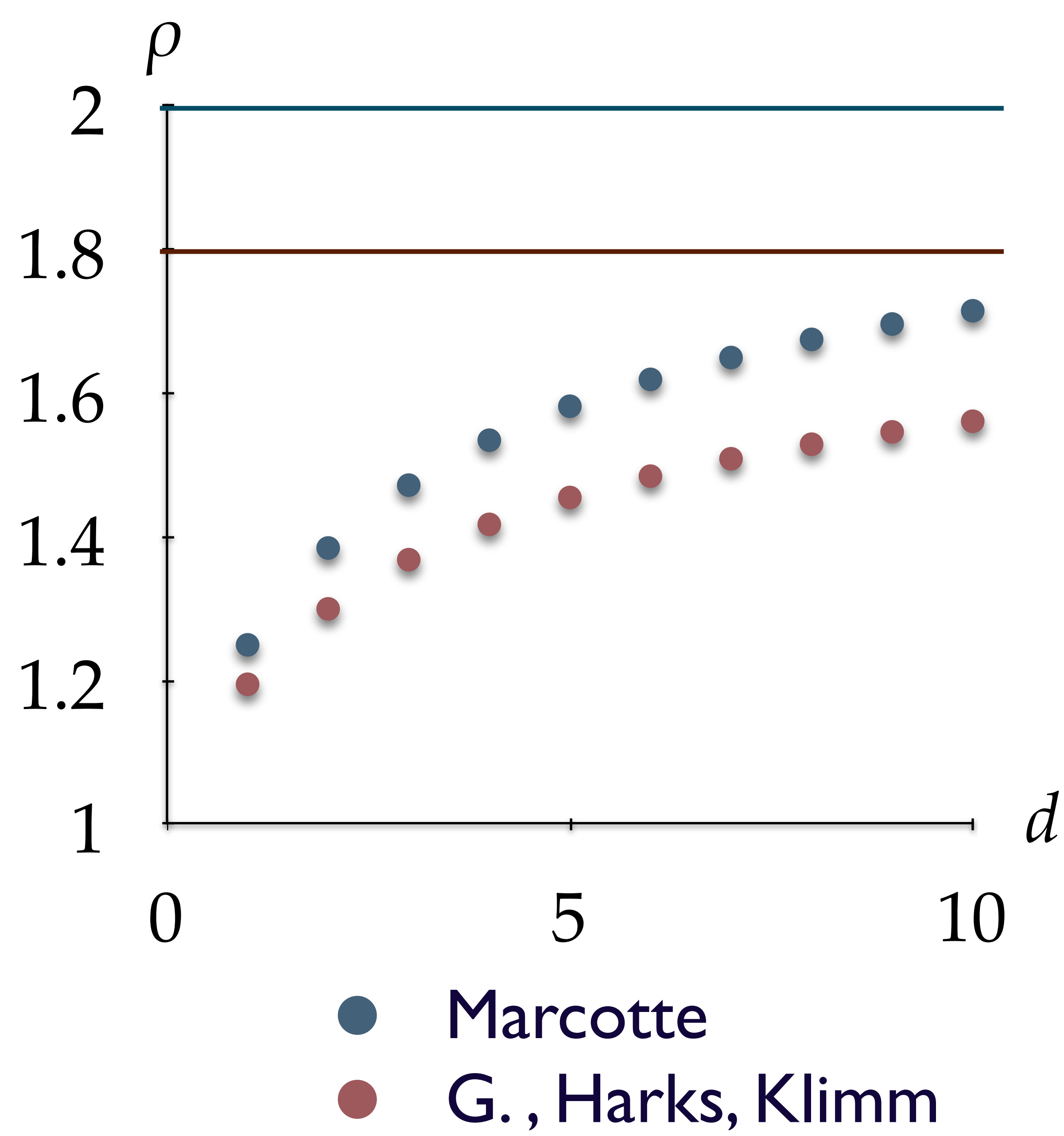
$$\rho^* = \frac{(\gamma+\mu+1)^2}{(\gamma+\mu+1)^2 - 4\mu\gamma} < 9/5$$

- Marcotte's algorithm has

$$\rho = 1 + \mu < 2$$

$$> \rho^*$$

Comparison for monomials of degree d



Conclusion

- Hardness for the continuous network design problem
- First approximation guarantee for arbitrary cost functions
- Improved approximation guarantees for monomials

Thank you.