Nash equilibria of bimatrix games

$$A = \begin{pmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{pmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix} \qquad x^T B = \begin{bmatrix} 5/3 & 5/3 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \qquad y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

only **pure best responses** can have probability > 0

Best response condition

Let $\mathbf{x}$ and $\mathbf{y}$ be mixed strategies of player I and II, respectively. Then $\mathbf{x}$ is a best response to $\mathbf{y}$
$\iff$ for all pure strategies i of player I:

$$x_i > 0 \implies (\mathbf{Ay})_i = u = \max\{(\mathbf{Ay})_k \mid 1 \le k \le m\}.$$

Here, $(\mathbf{Ay})_i$ is the ith component of $\mathbf{Ay}$, which is the expected payoff to player I when playing row i.

Proof.

$$\mathbf{xAy} = \sum_{i=1}^{m} \mathbf{x}_i (\mathbf{Ay})_i = \sum_{i=1}^{m} \mathbf{x}_i (u - (u - (\mathbf{Ay})_i)$$

$$= \sum_{i=1}^{m} \mathbf{x}_i u - \sum_{i=1}^{m} \mathbf{x}_i (u - (\mathbf{Ay})_i) = u - \sum_{i=1}^{m} \mathbf{x}_i (u - (\mathbf{Ay})_i) \le u,$$

because $\mathbf{x}_i \ge 0$ and $u - (\mathbf{Ay})_i \ge 0$ for all i. Furthermore, $\mathbf{xAy} = u \iff \mathbf{x}_i > 0$ implies $(\mathbf{Ay})_i = u$, as claimed.

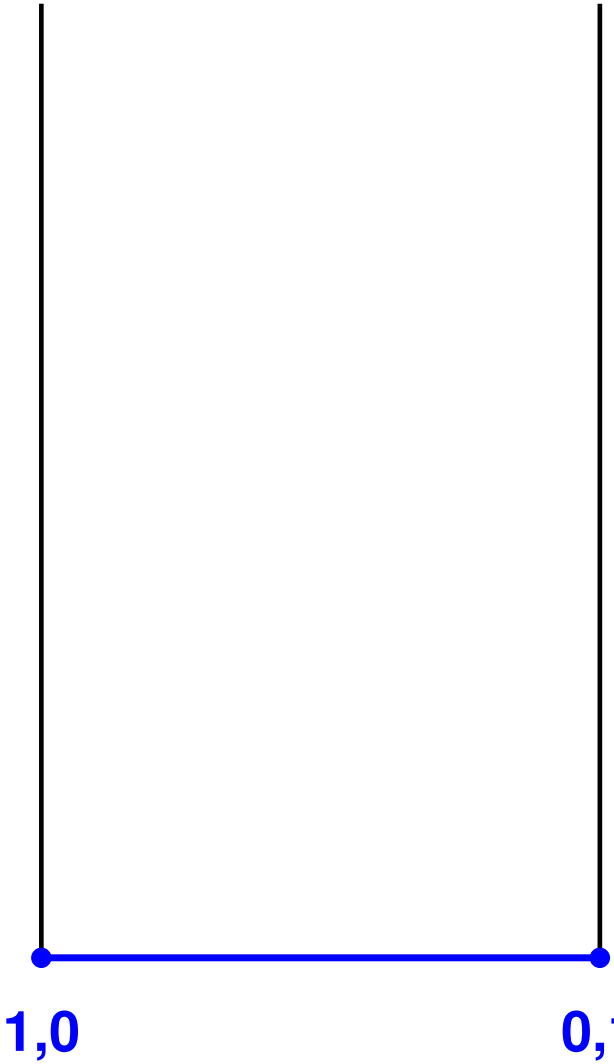
Best responses to mixed strategy of player 2

	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

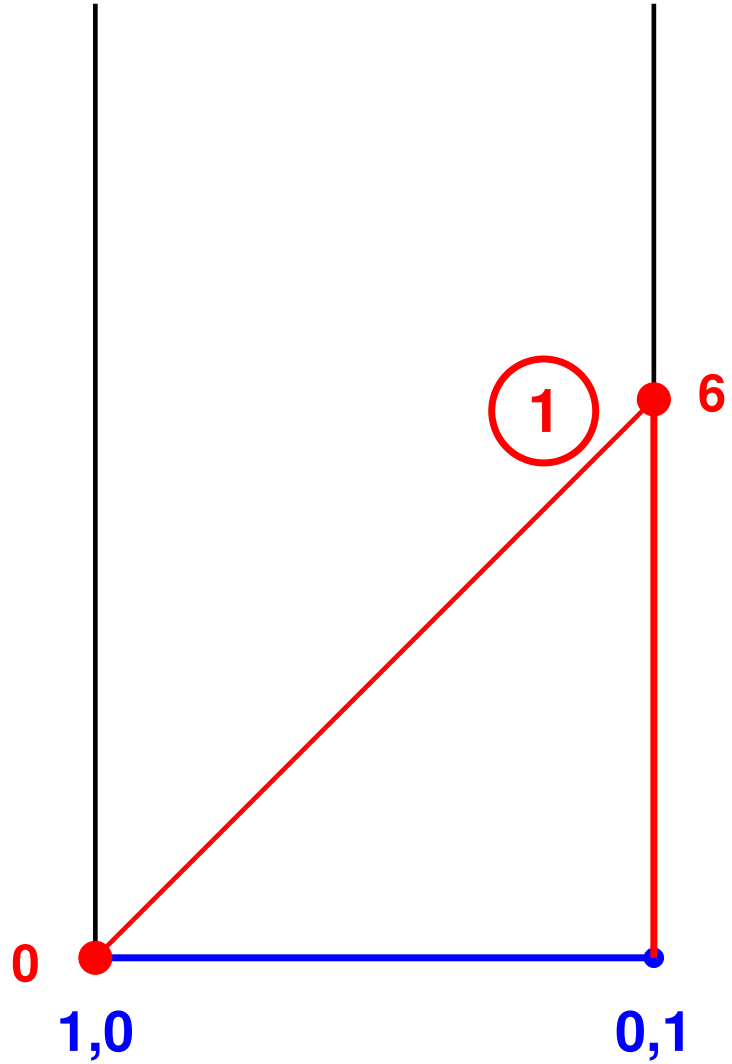


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

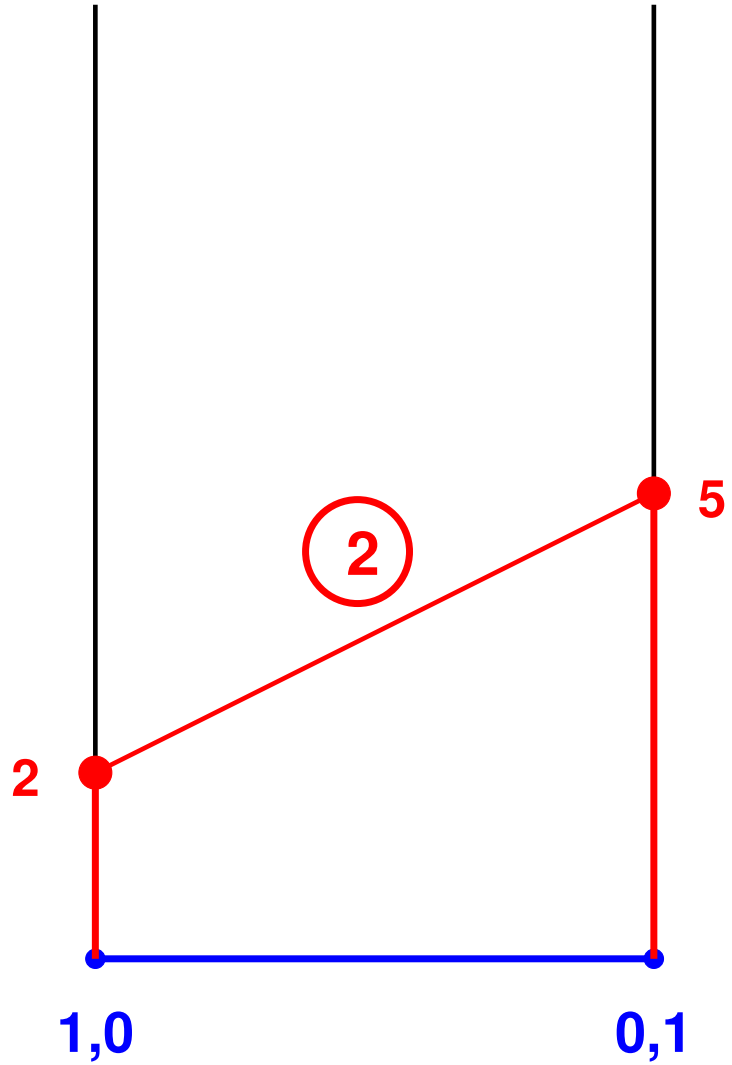


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

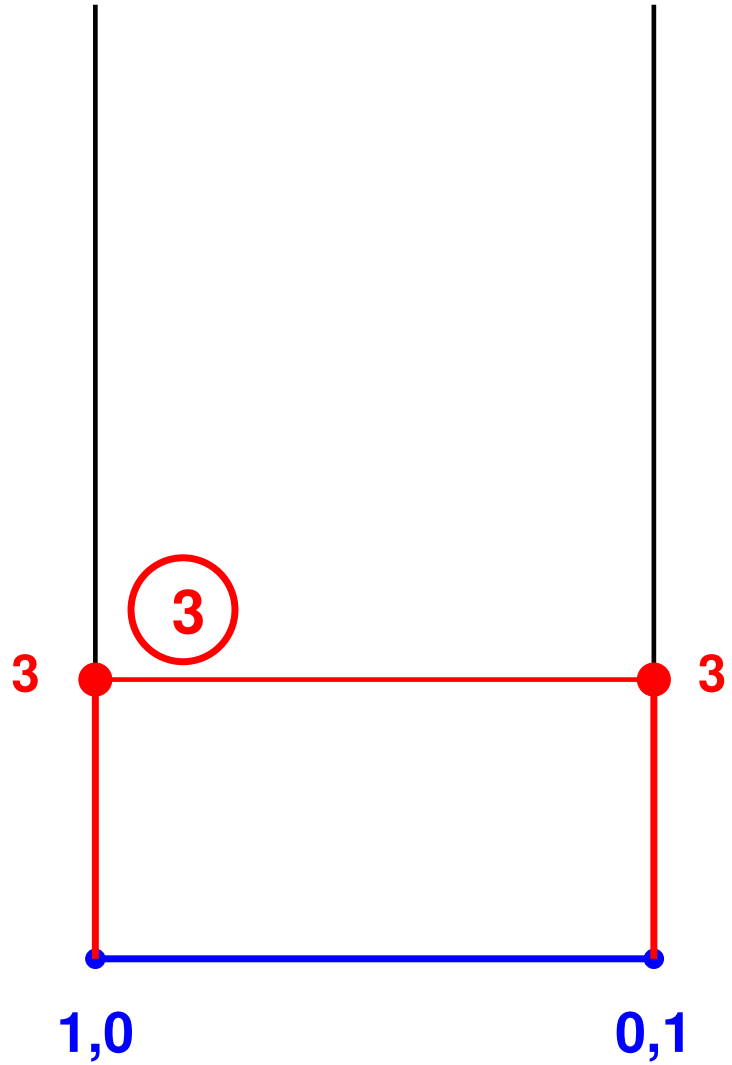


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

Best responses to mixed strategy of player 2

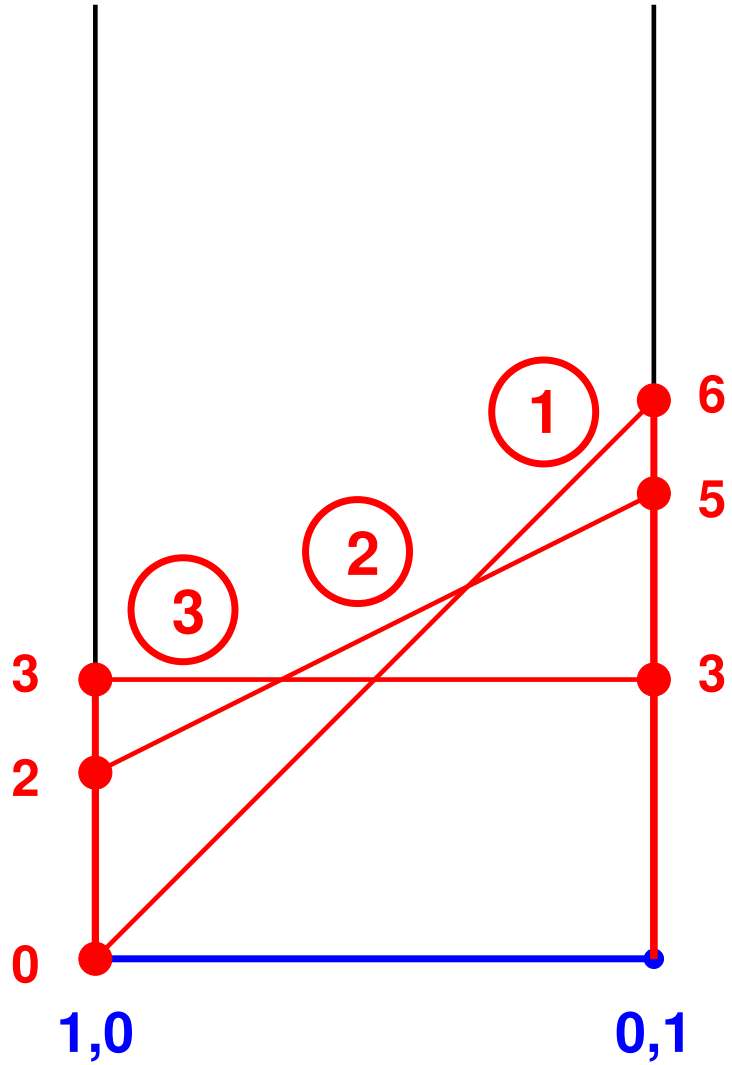


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

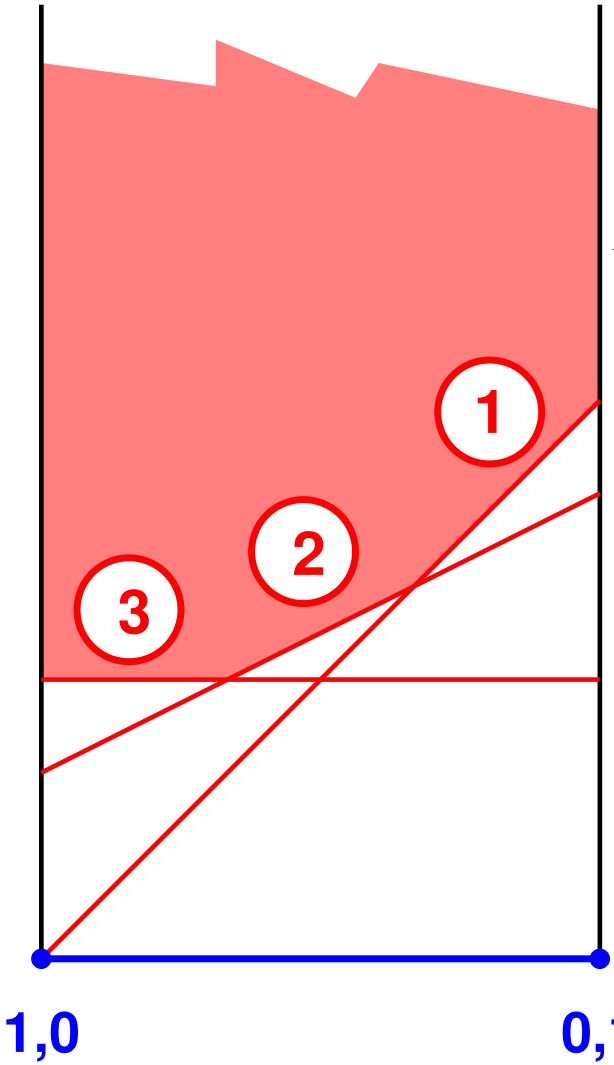


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

Best responses to mixed strategy of player 2



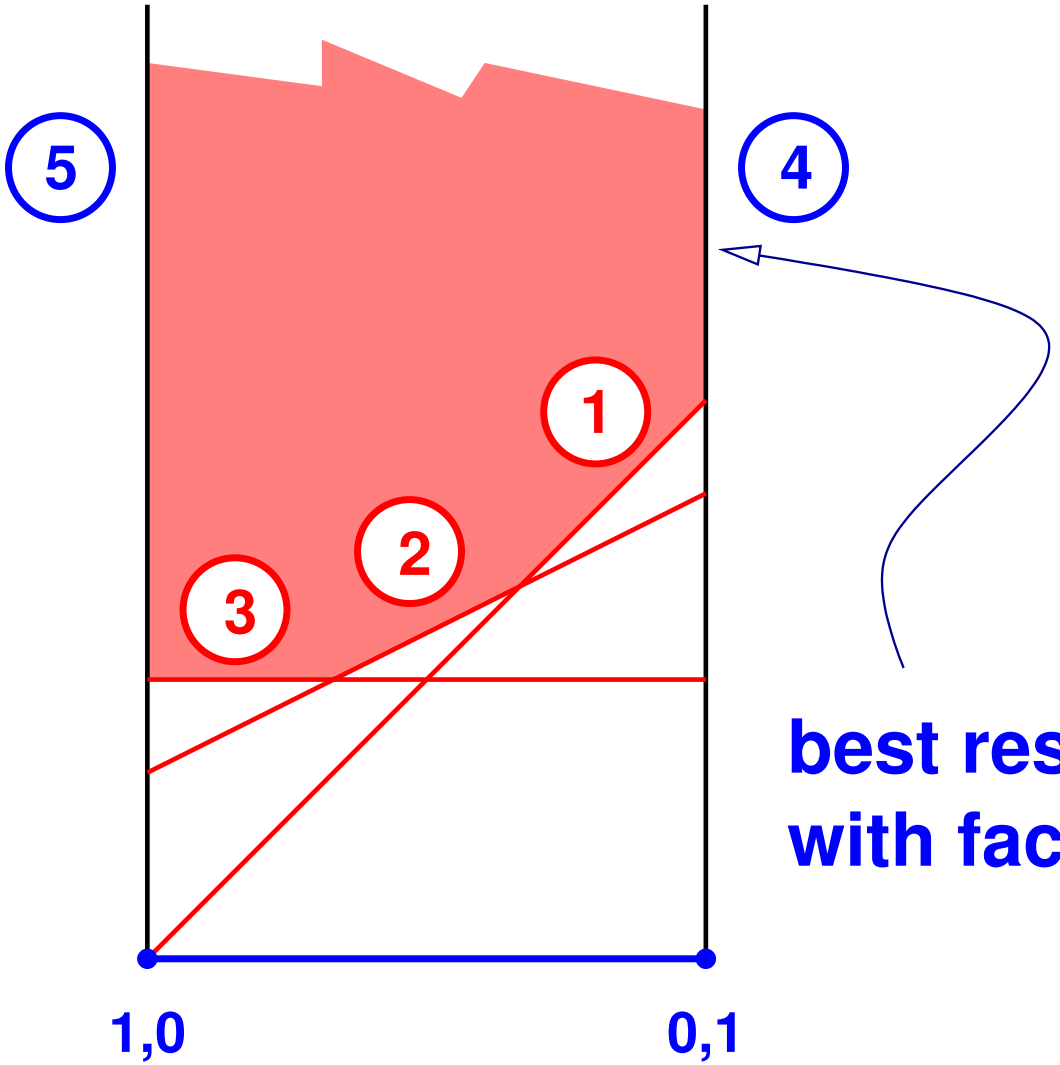
	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

best response polyhedron

Best responses to mixed strategy of player 2



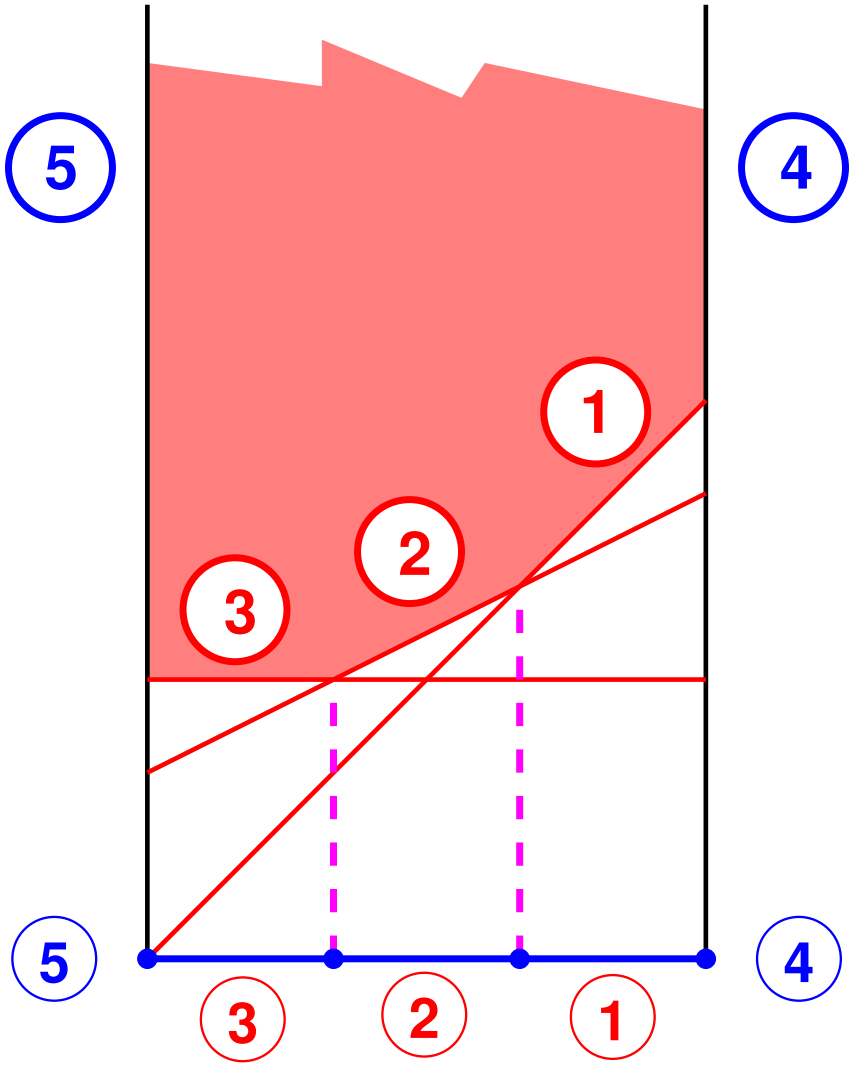
	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

best response polyhedron with facet labels

Best responses to mixed strategy of player 2



	4	5
1	0	6
2	2	5
3	3	3

= A

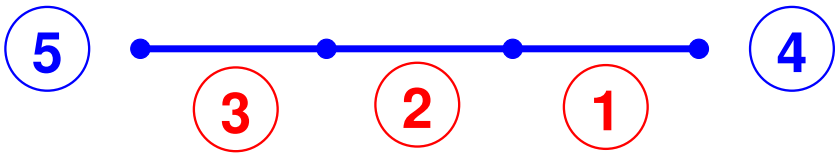
payoffs to player I

Best responses to mixed strategy of player 2

	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

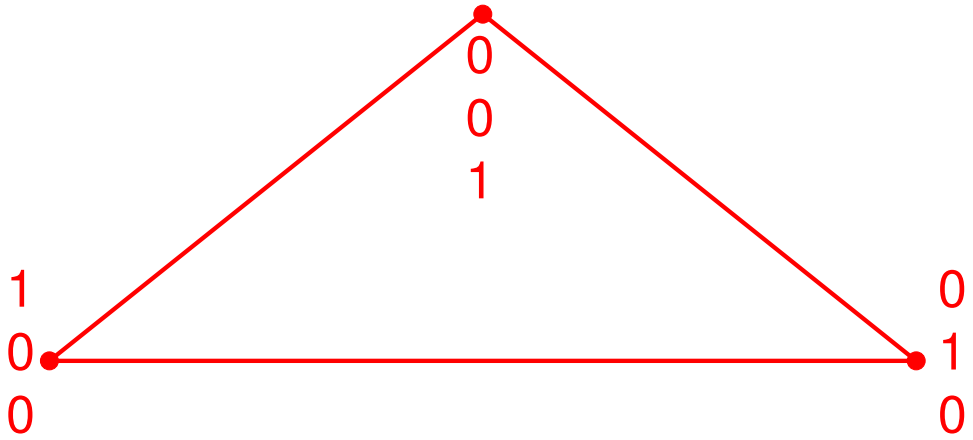


Best responses to mixed strategy of player 1

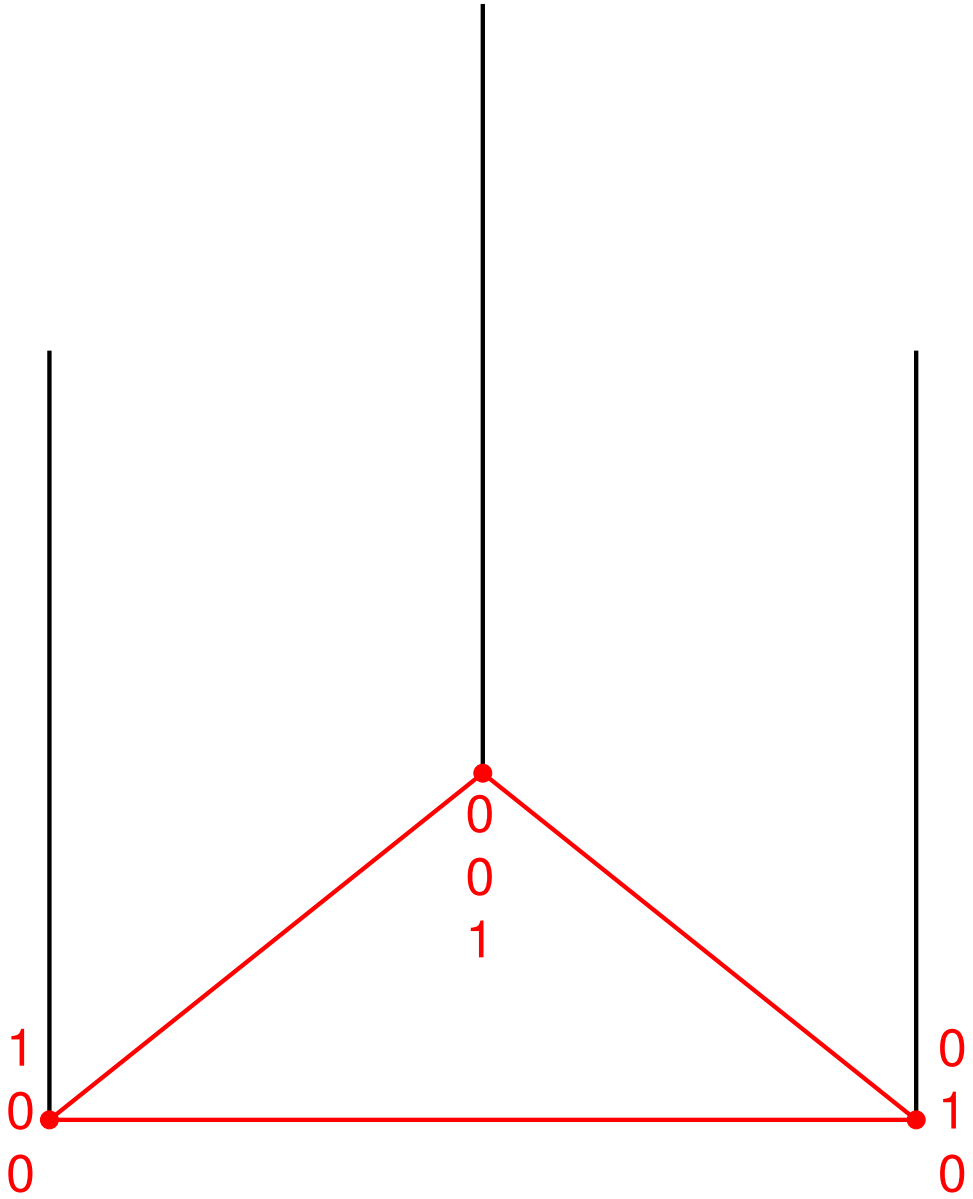
	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II



Best responses to mixed strategy of player 1

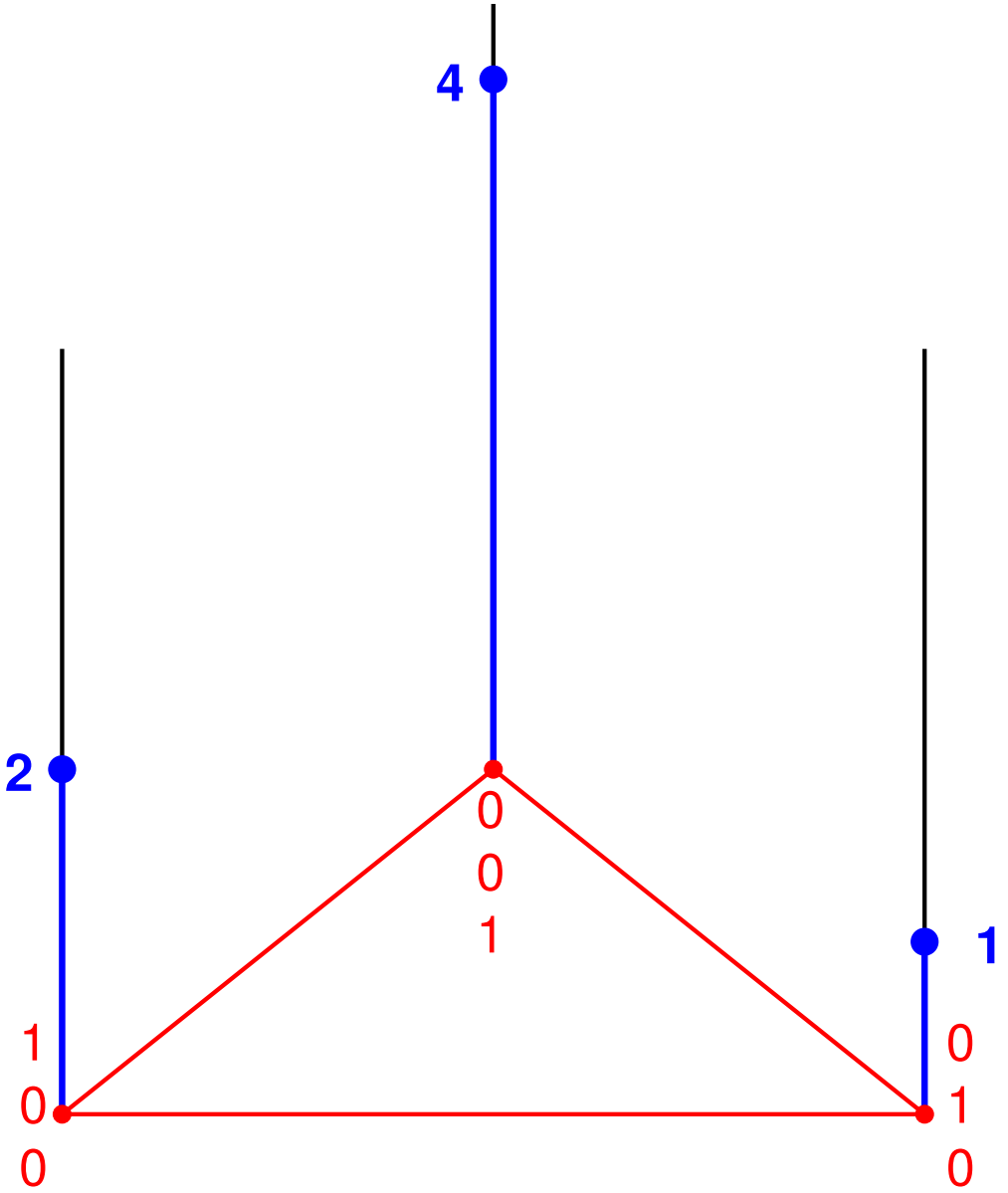


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

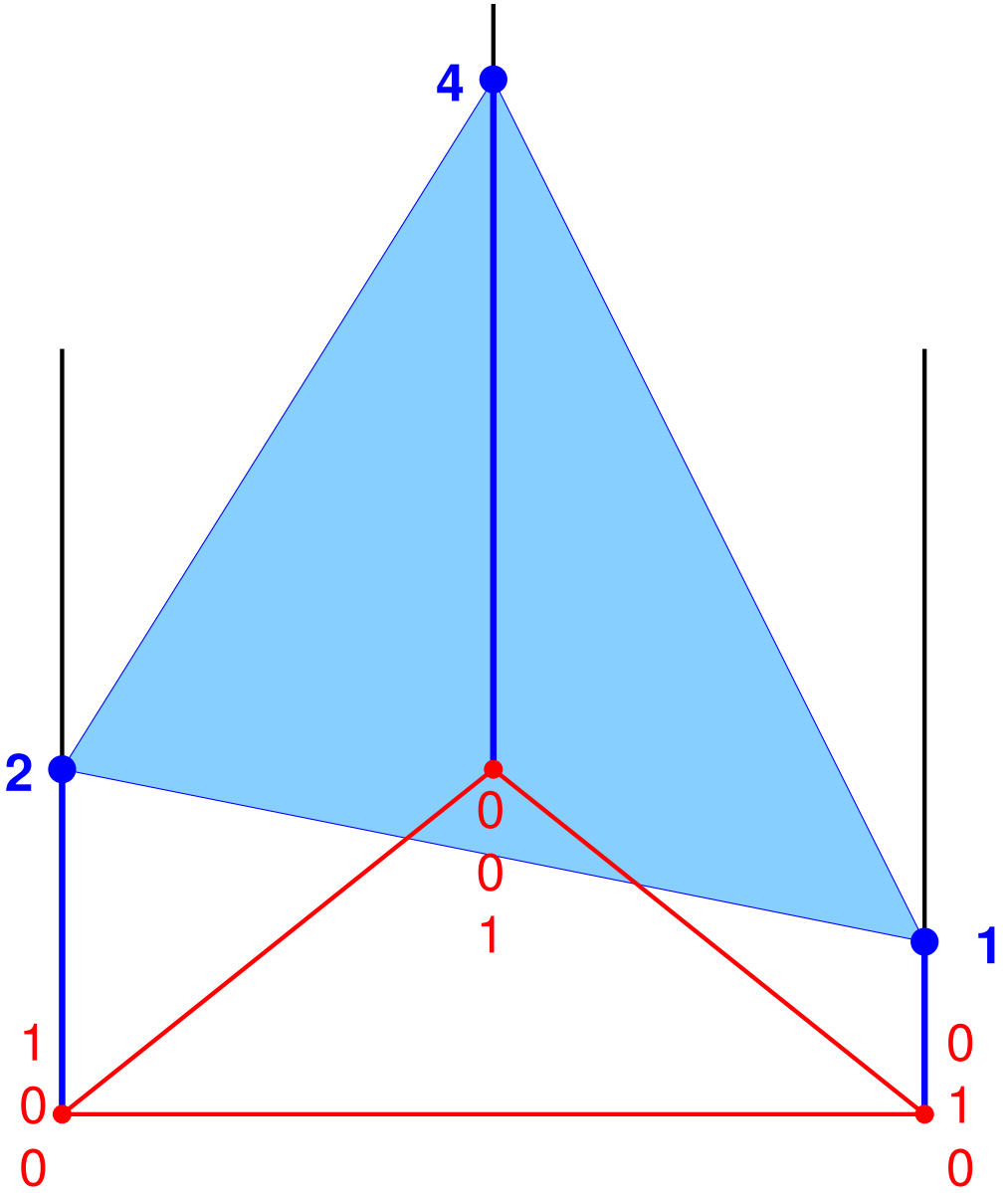


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

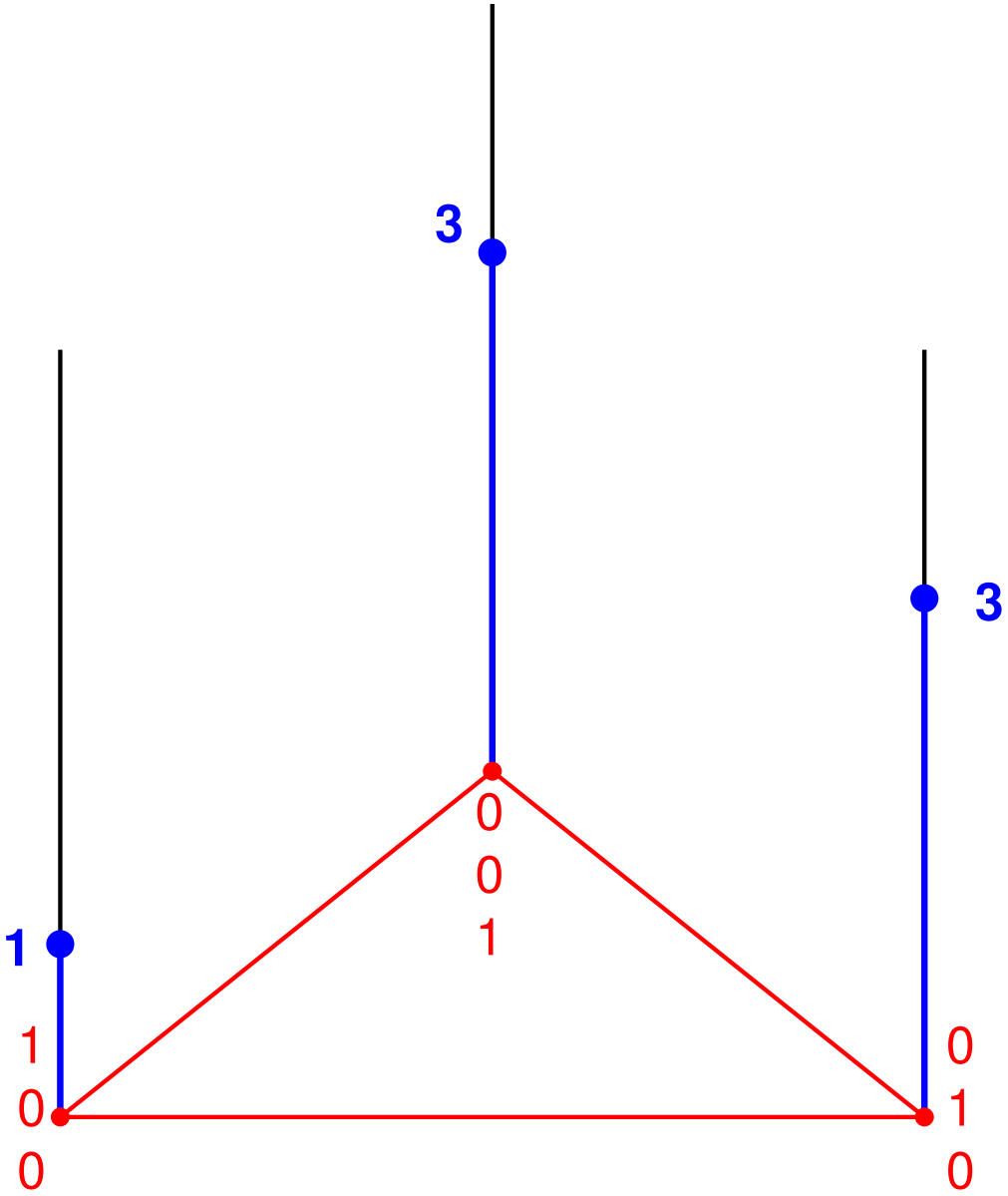


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

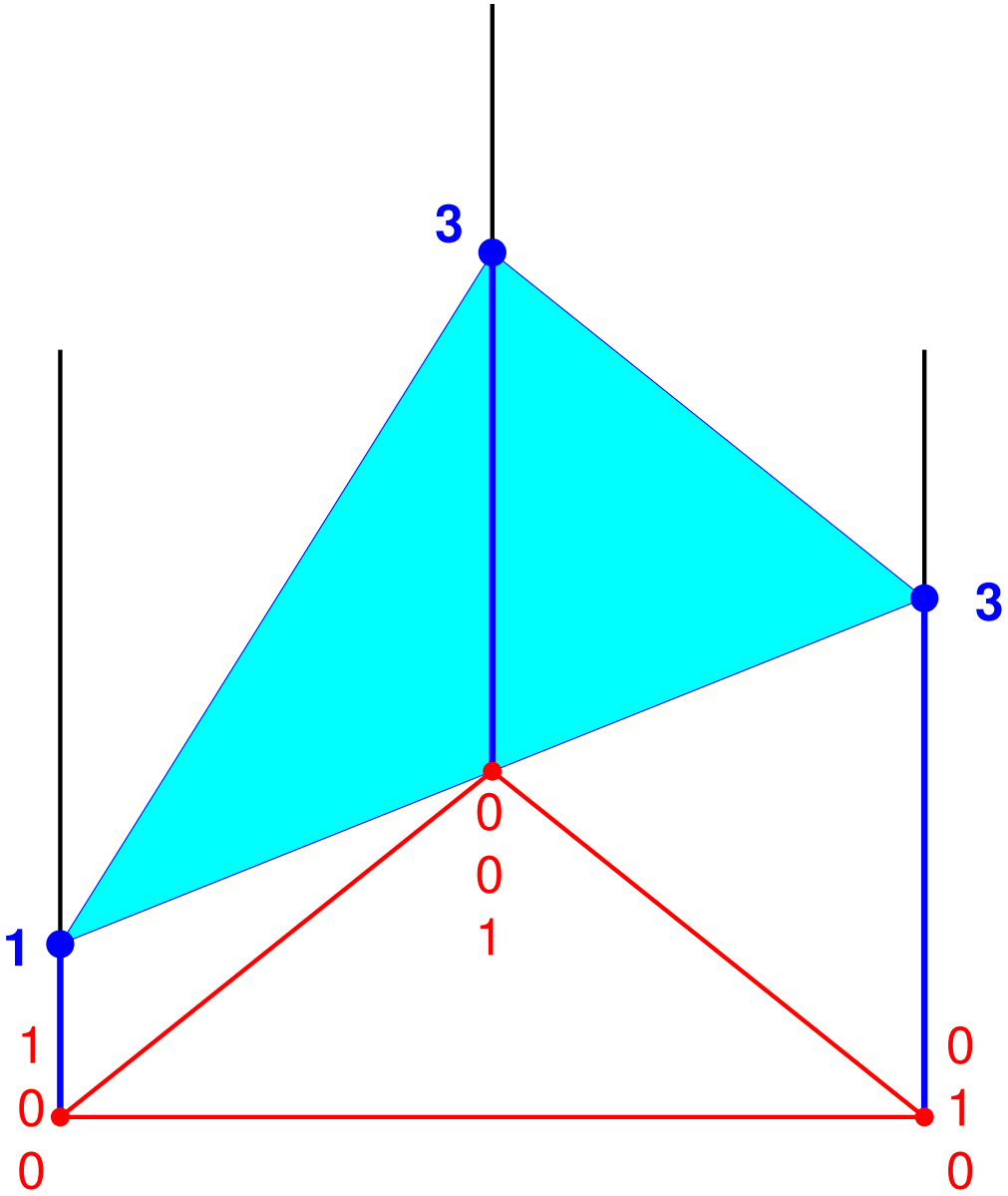


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

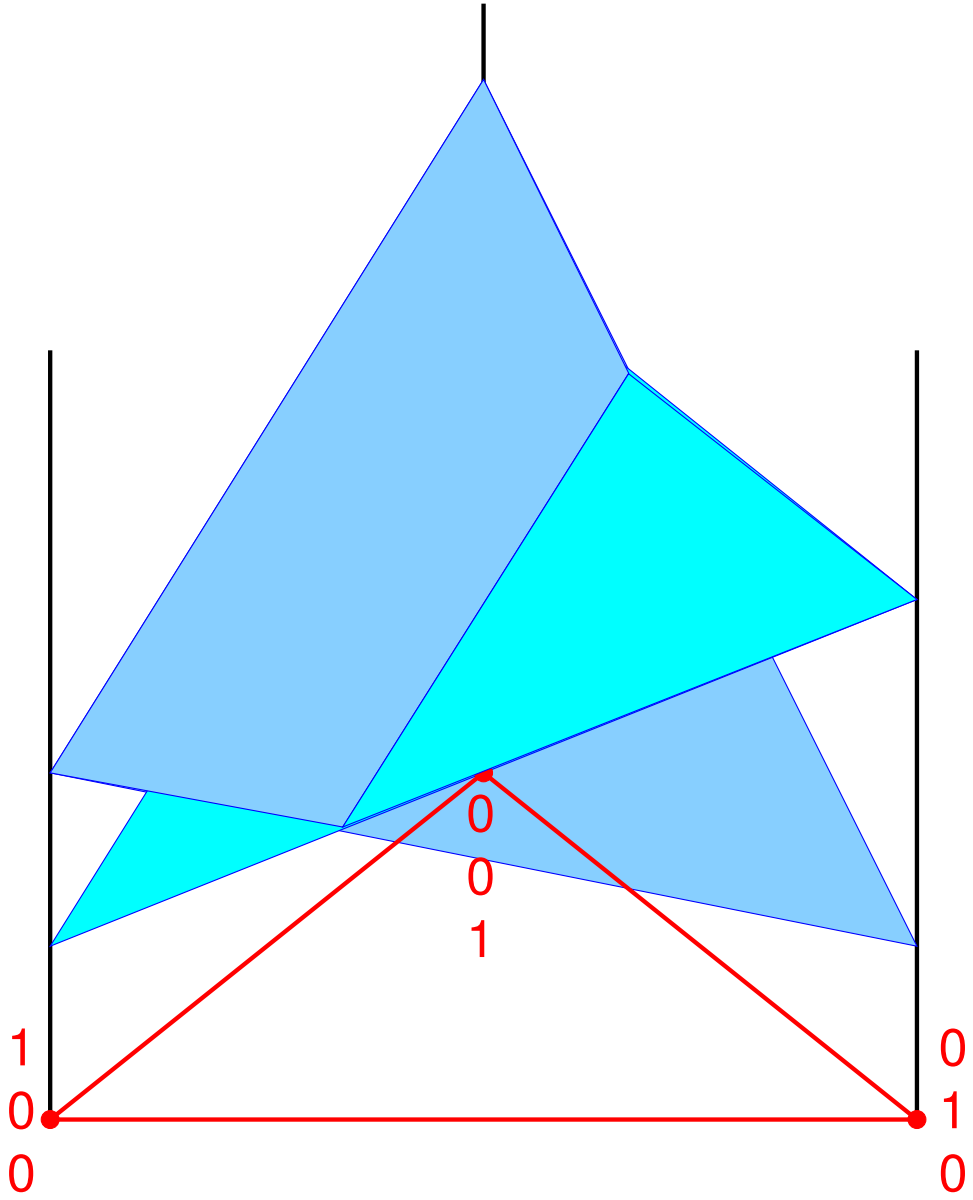


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

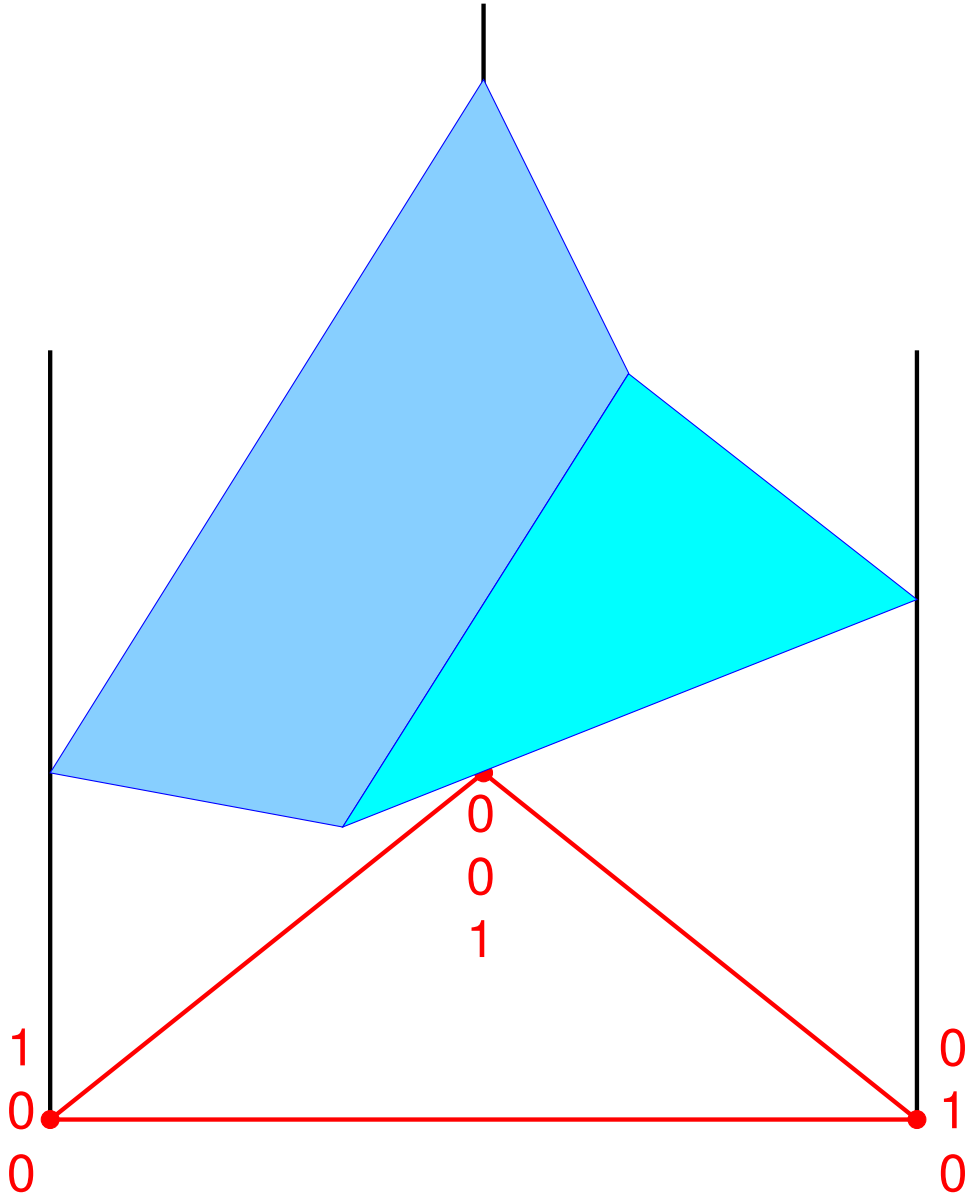


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

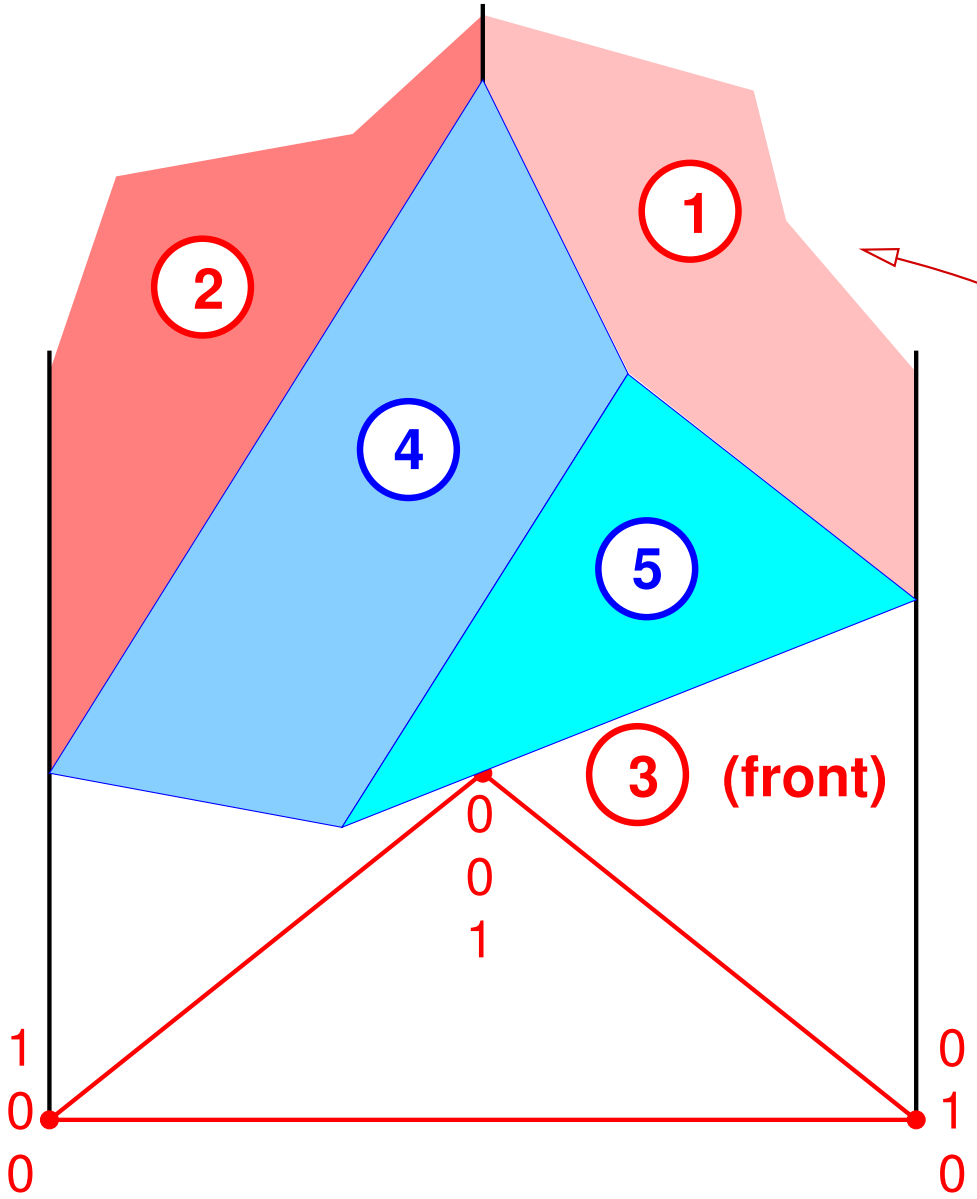


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to player II

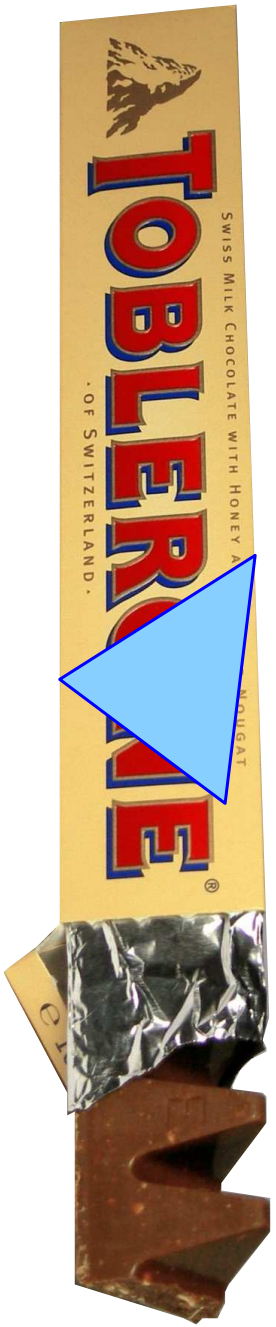
best response polyhedron with facet labels

Alternative view

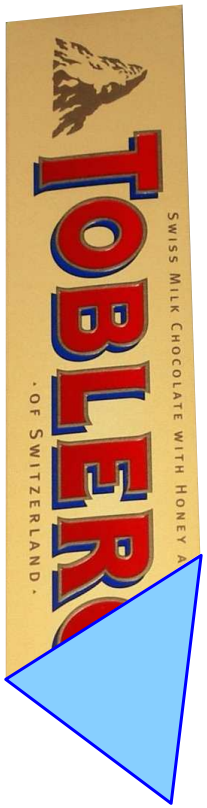
Chop off Toblerone prism

Chop off Toblerone prism

Chop off Toblerone prism

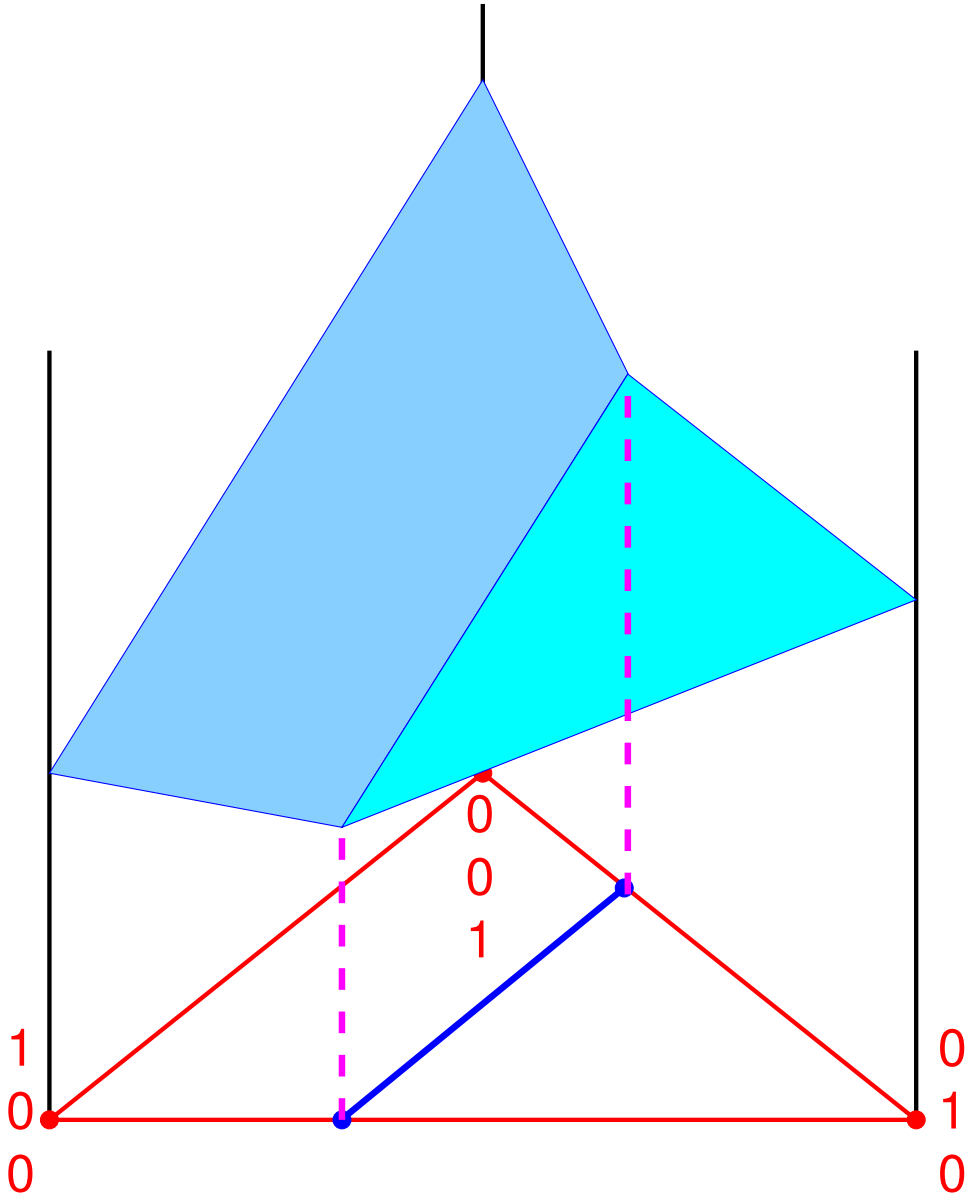


Chop off Toblerone prism



Chop off Toblerone prism

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

= B

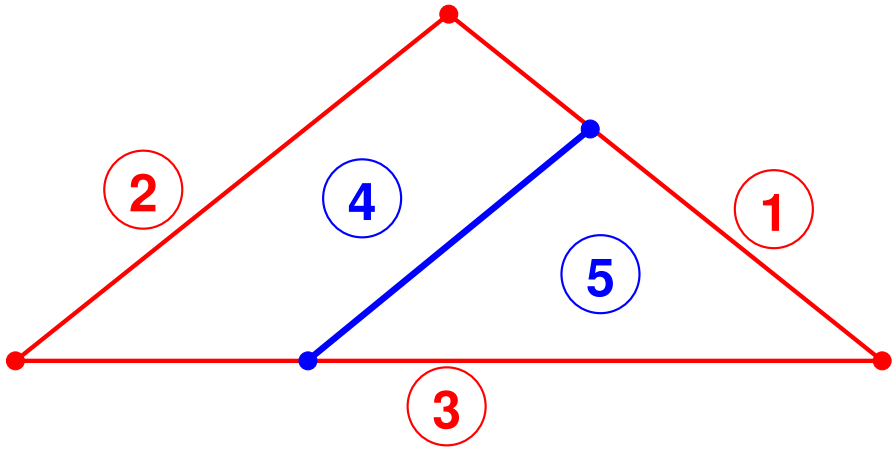
payoffs to
player II

Best responses to mixed strategy of player 1

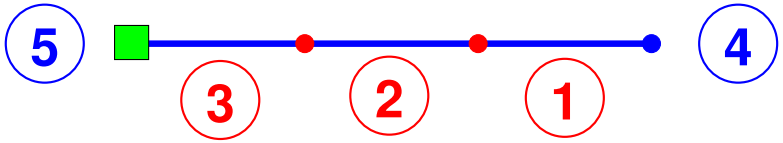
	4	5
1	2	1
2	1	3
3	4	3

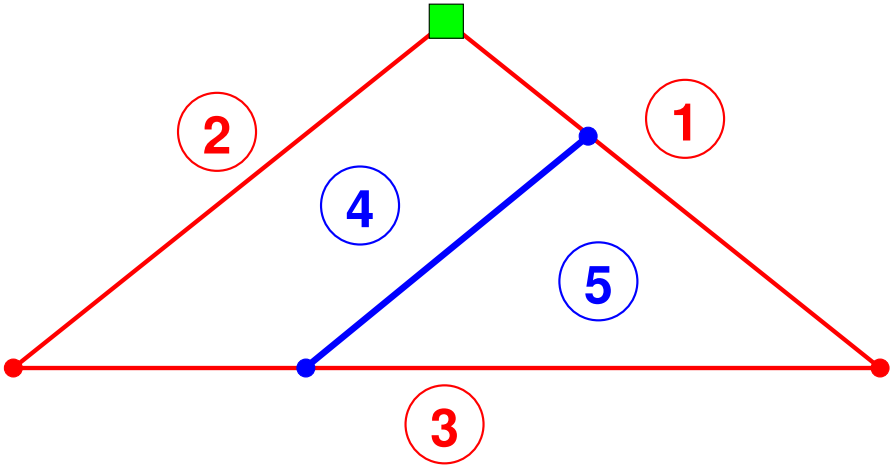
= B

payoffs to
player II

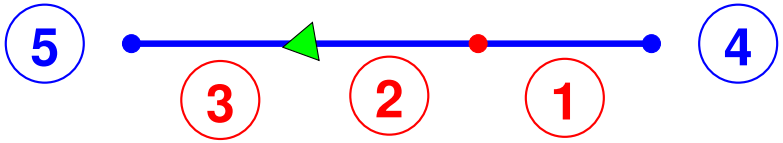


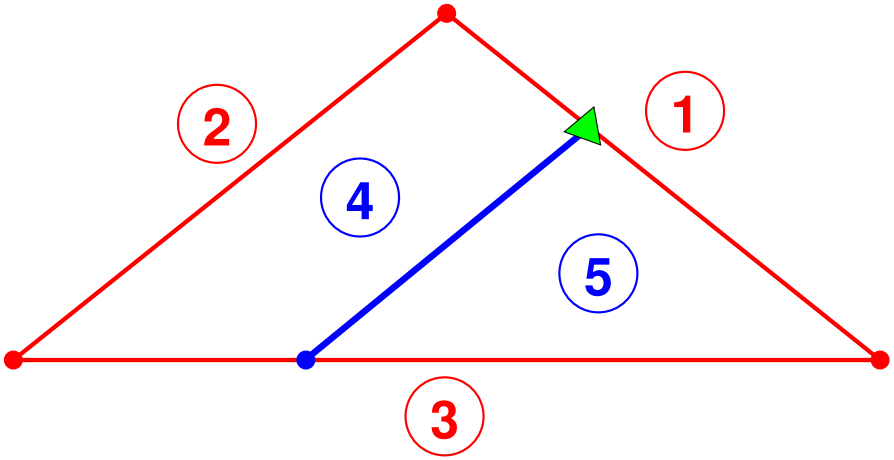
Equilibrium = completely labeled strategy pair



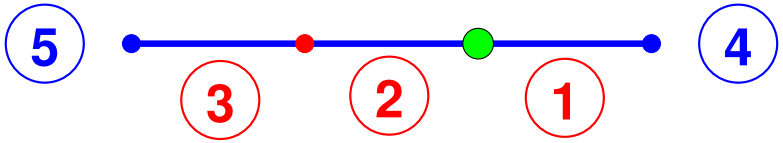


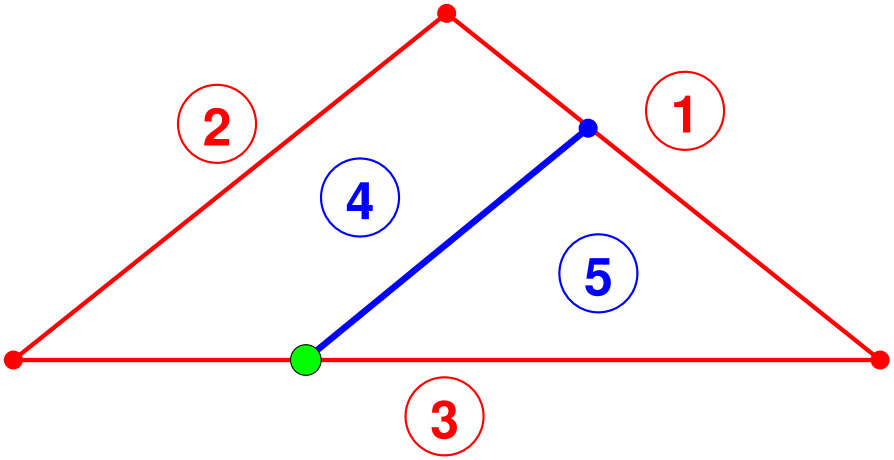
Equilibrium = completely labeled strategy pair





Equilibrium = completely labeled strategy pair





Constructing games using geometry

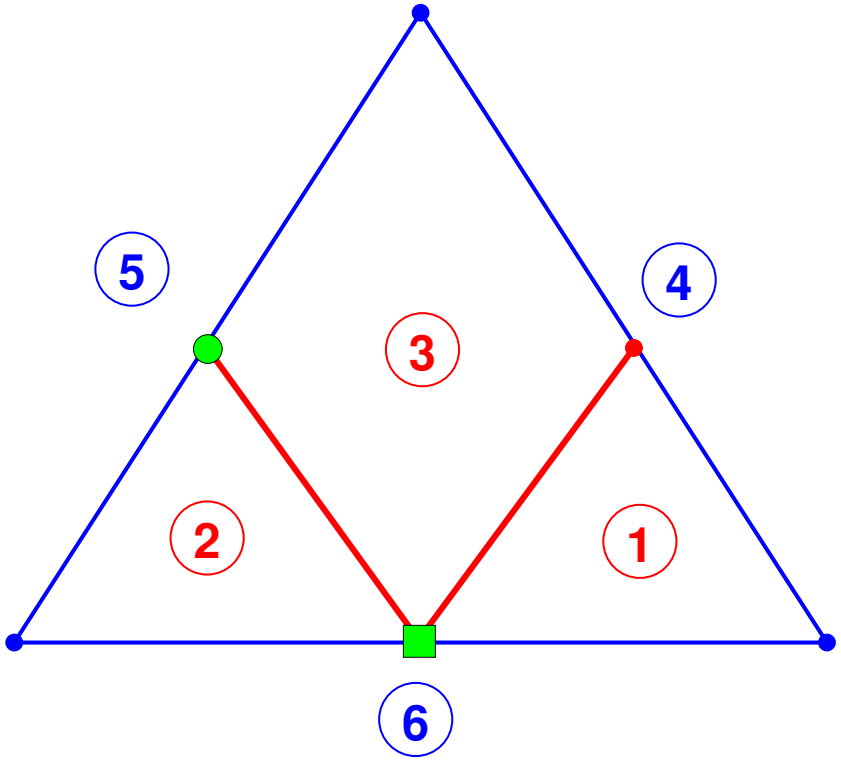
low dimension: 2, 3, (4) pure strategies:

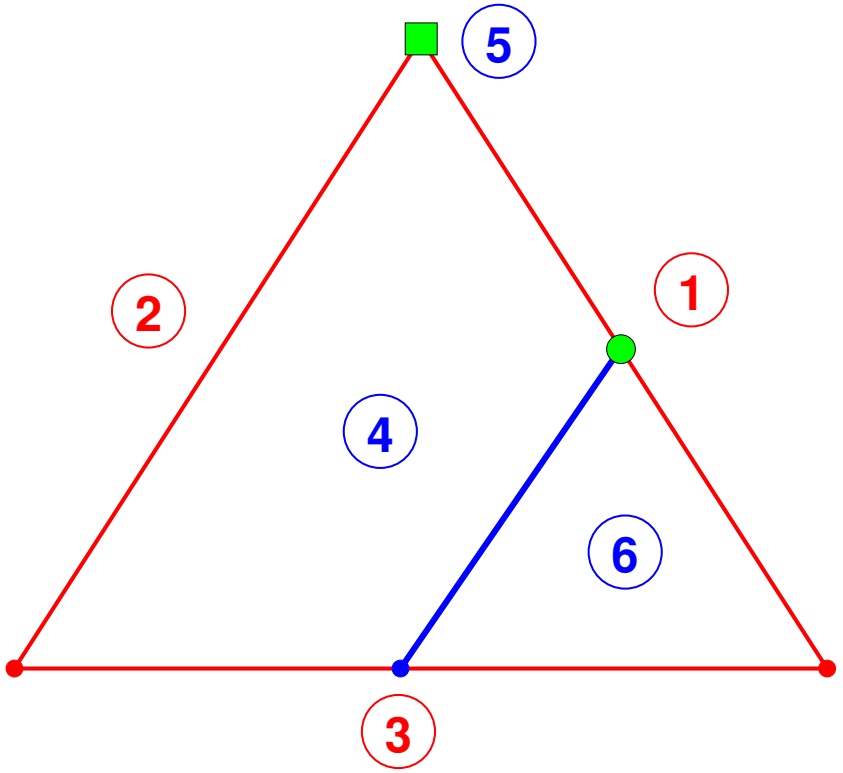
subdivide mixed strategy simplex into response regions, label suitably

high dimension:

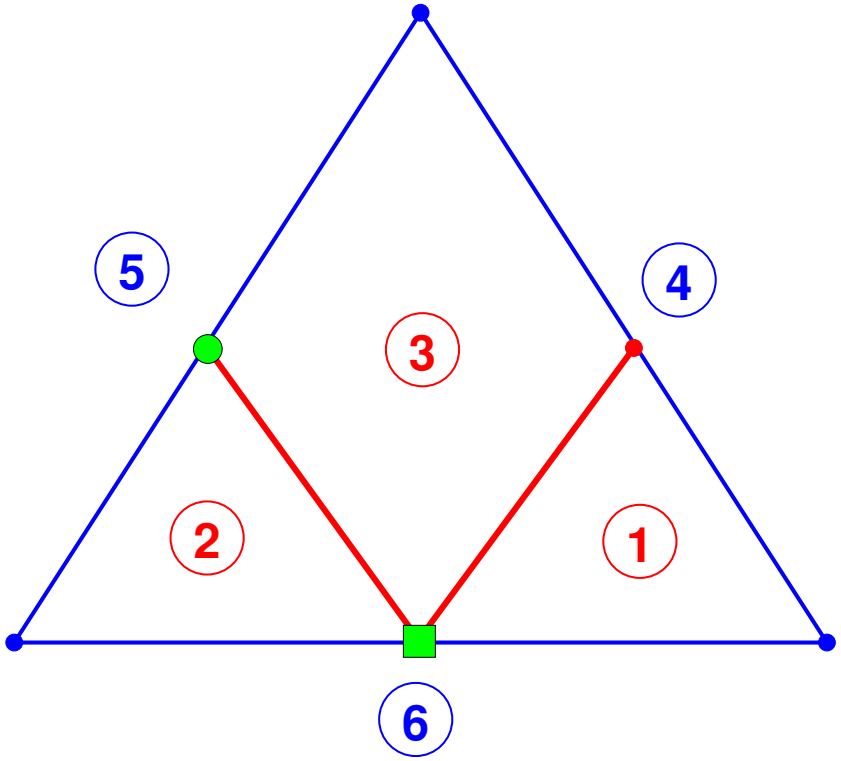
use polytopes with known combinatorial structure
e.g. for constructing games with many equilibria, or long Lemke-Howson computations
[Savani & von Stengel, *FOCS 2004*, *Econometrica* 2006]

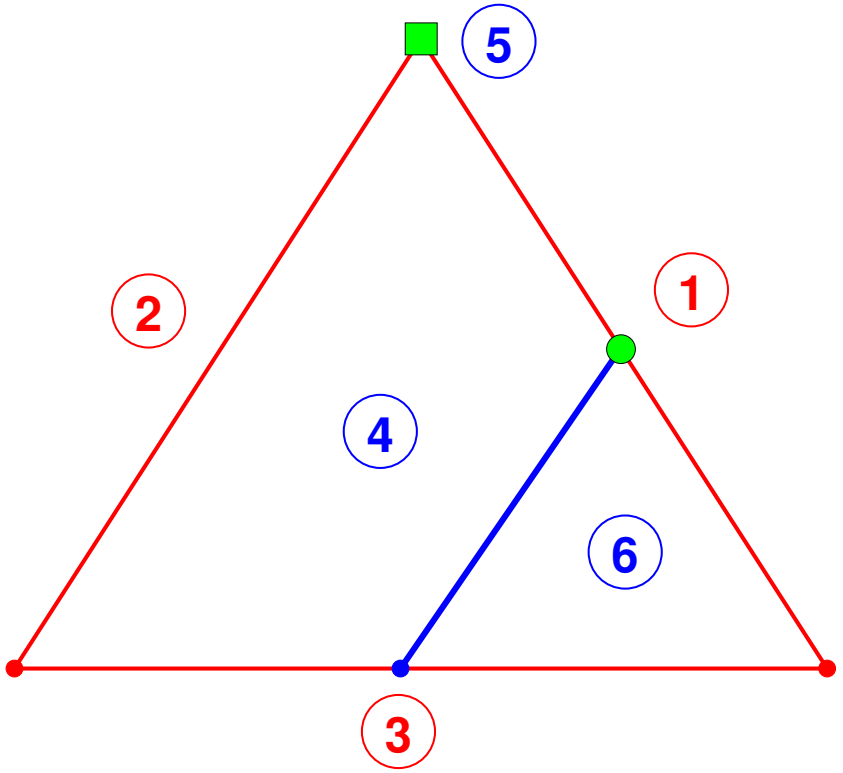
Construct isolated non–quasi–strict equilibrium





Construct isolated non–quasi–strict equilibrium

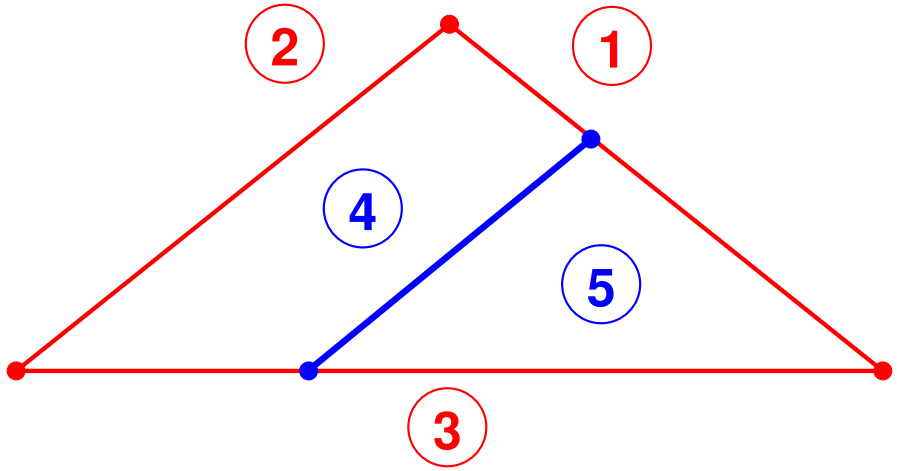


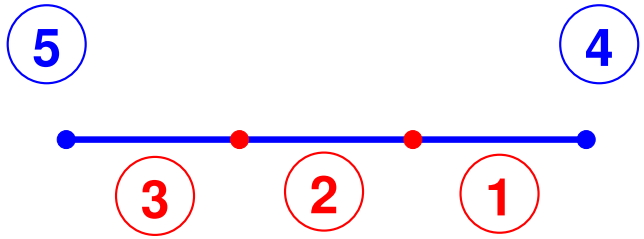


$$A = \begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

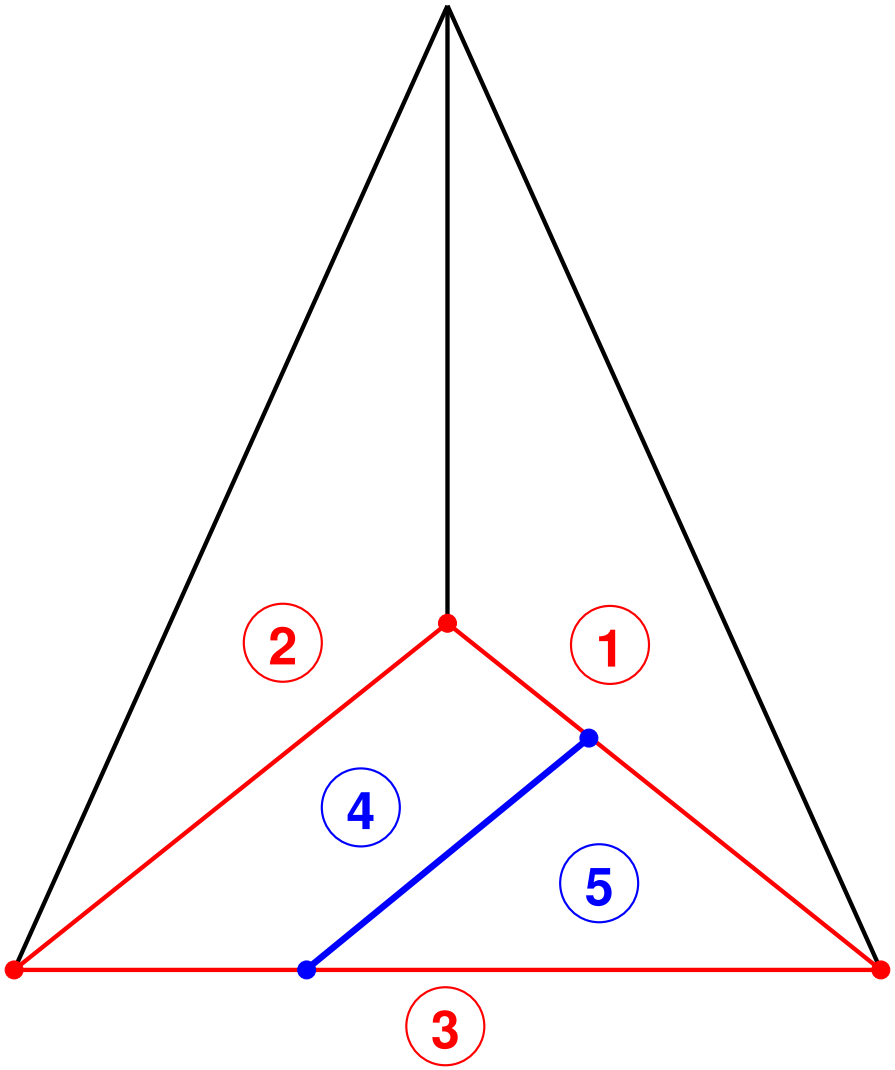
$$B = \begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ 3 & 1 & 1 & 0 \end{array}$$

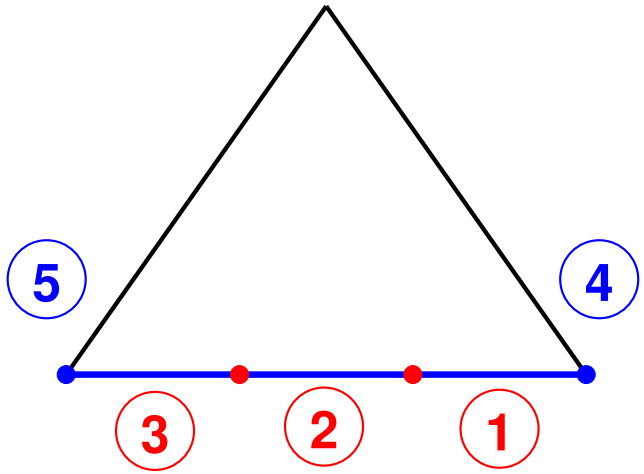
The Lemke–Howson algorithm



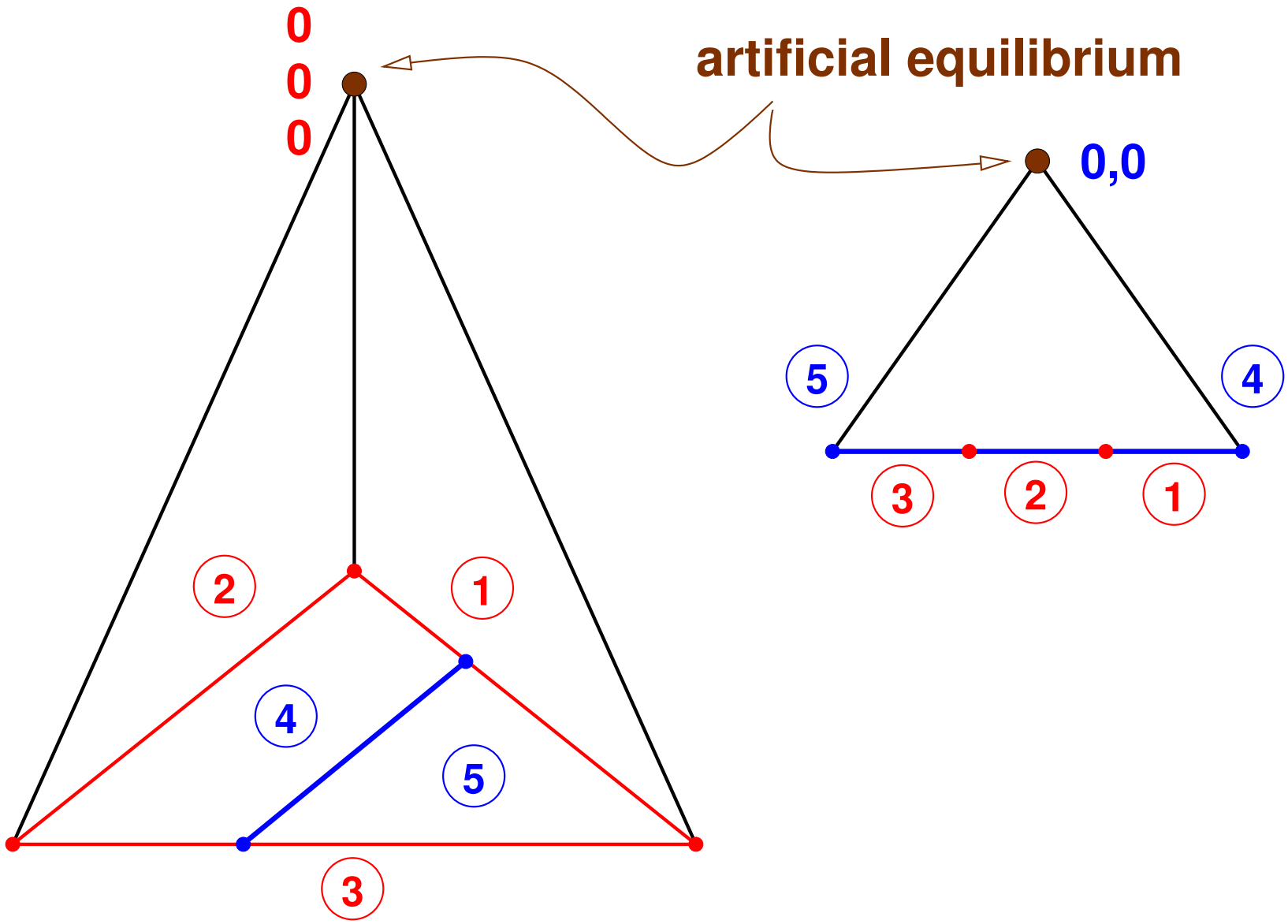


The Lemke–Howson algorithm

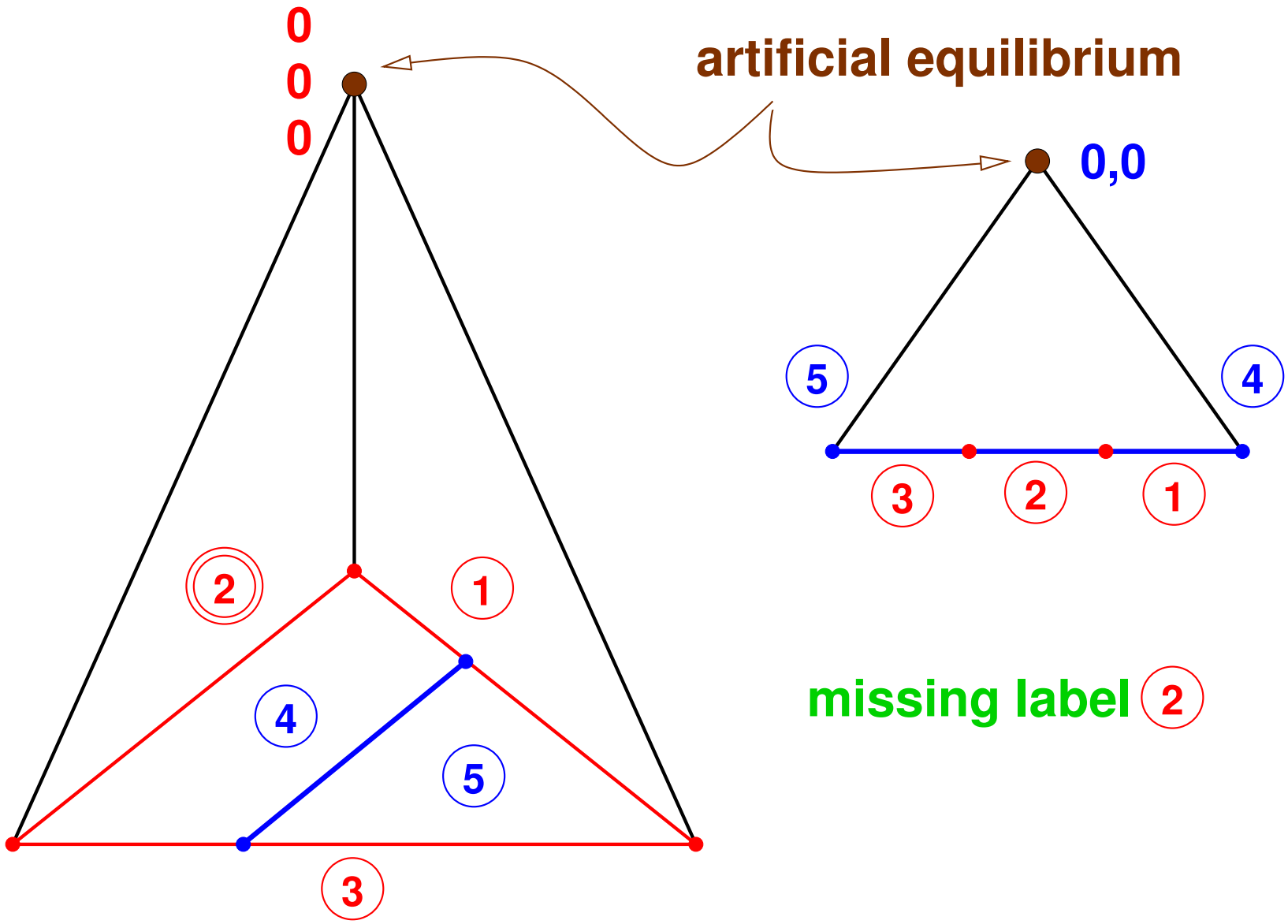




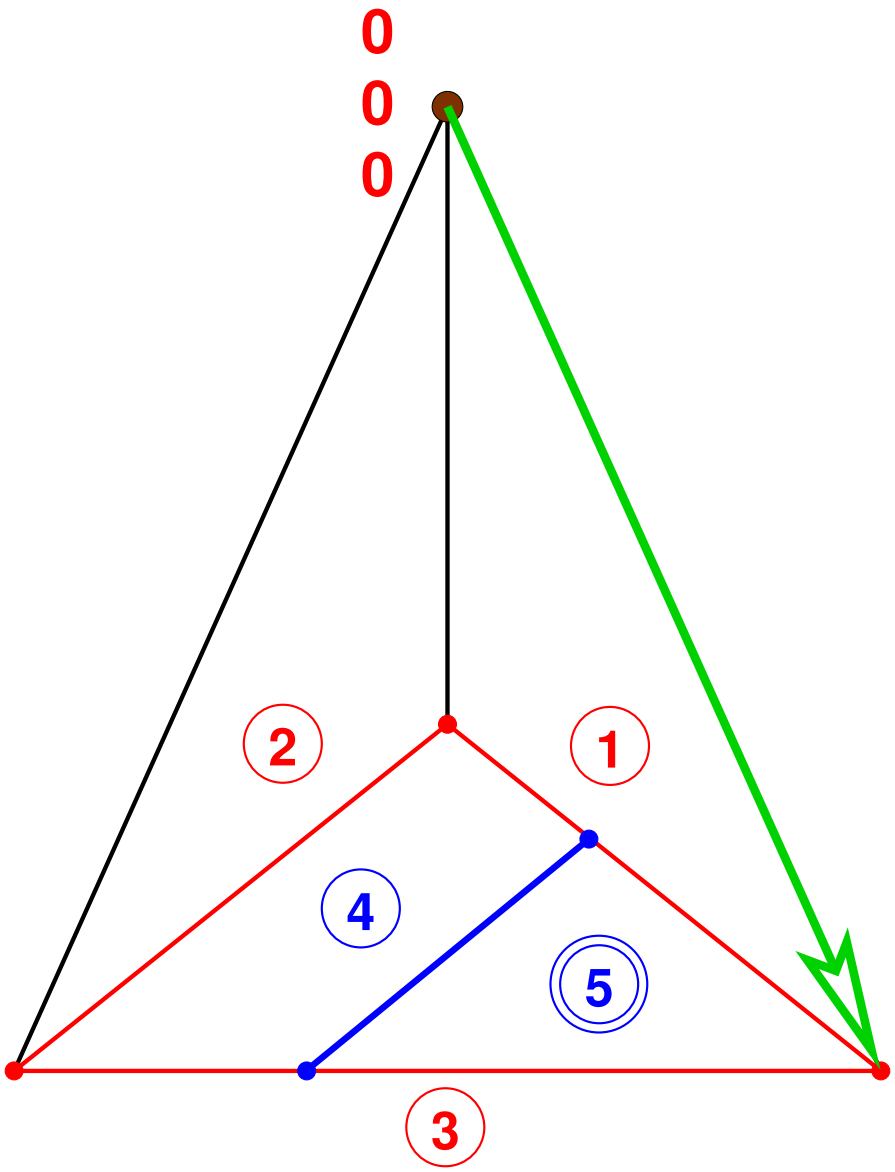
The Lemke–Howson algorithm

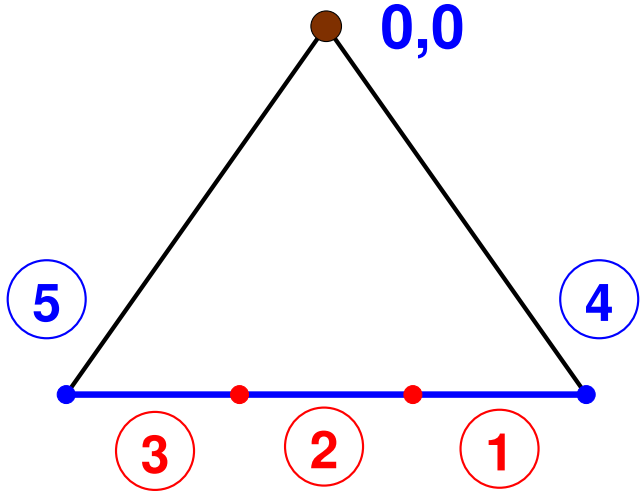


The Lemke–Howson algorithm



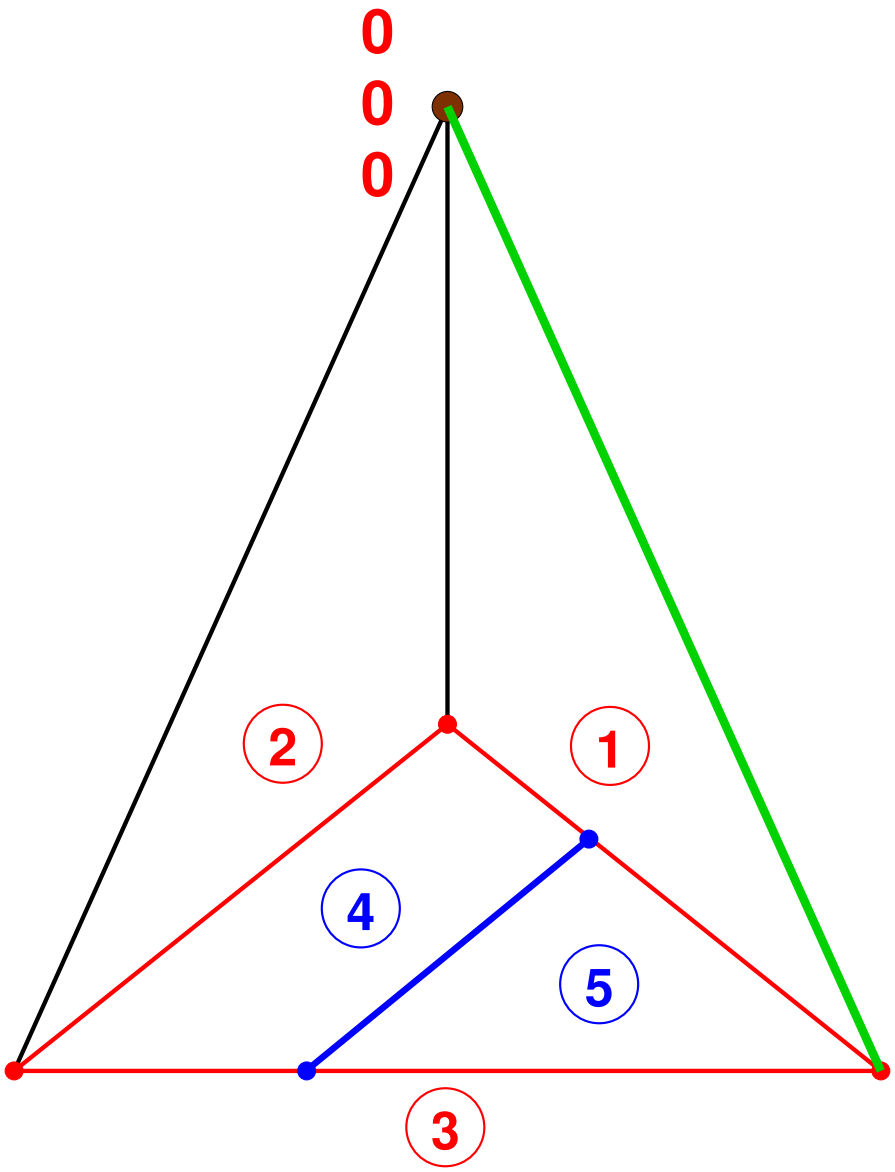
The Lemke–Howson algorithm

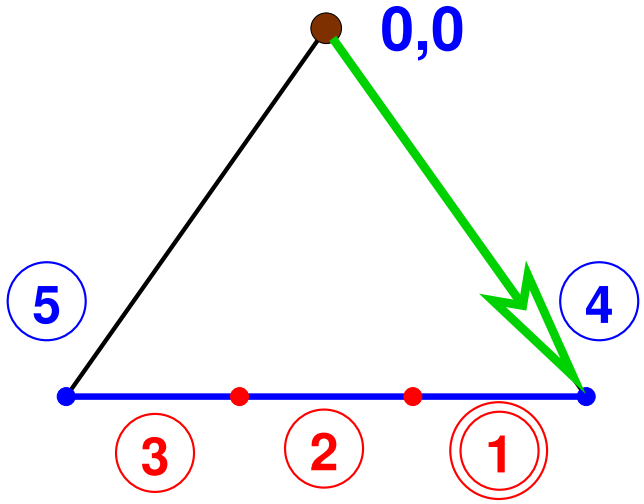




missing label 2

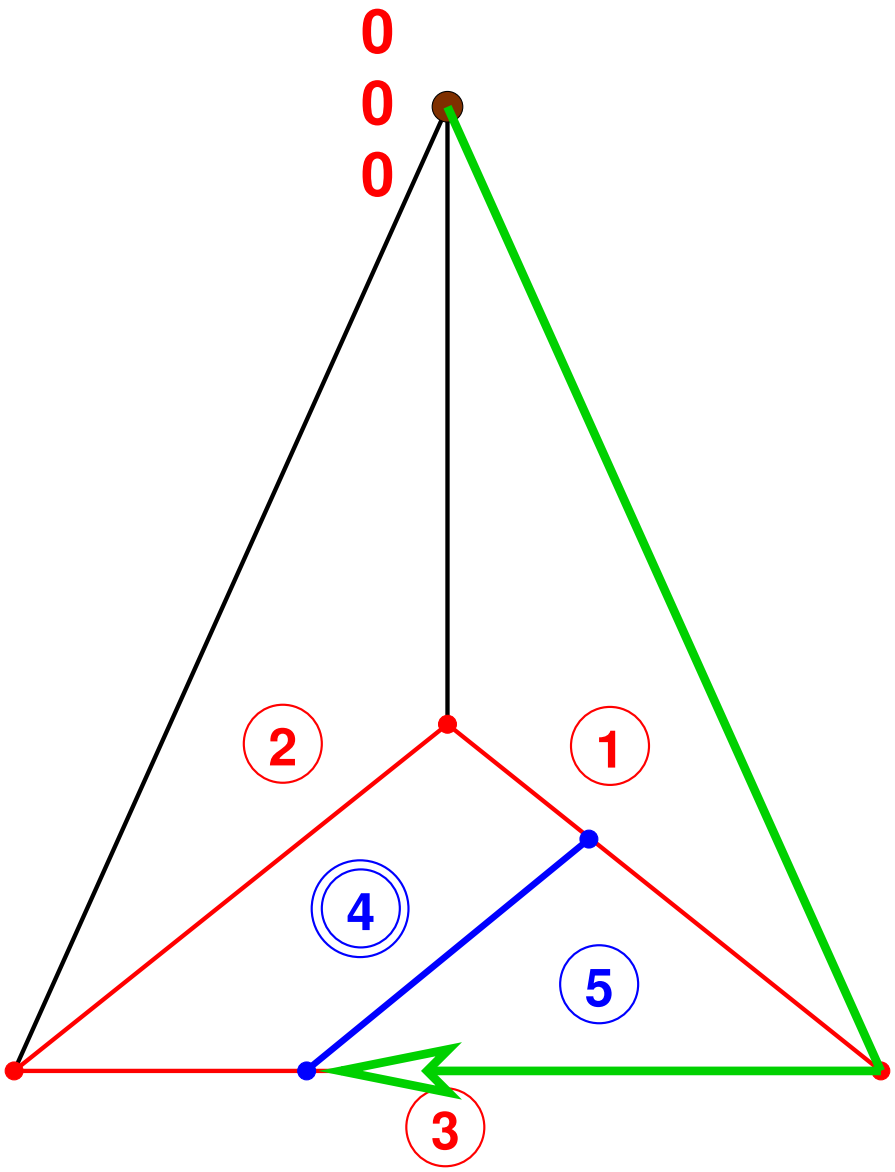
The Lemke–Howson algorithm

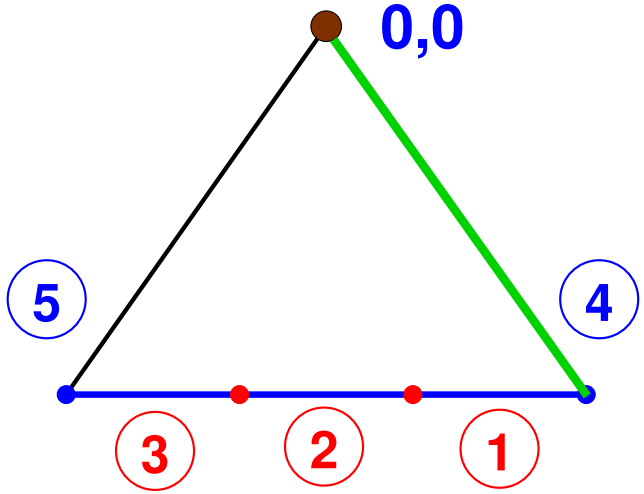




missing label 2

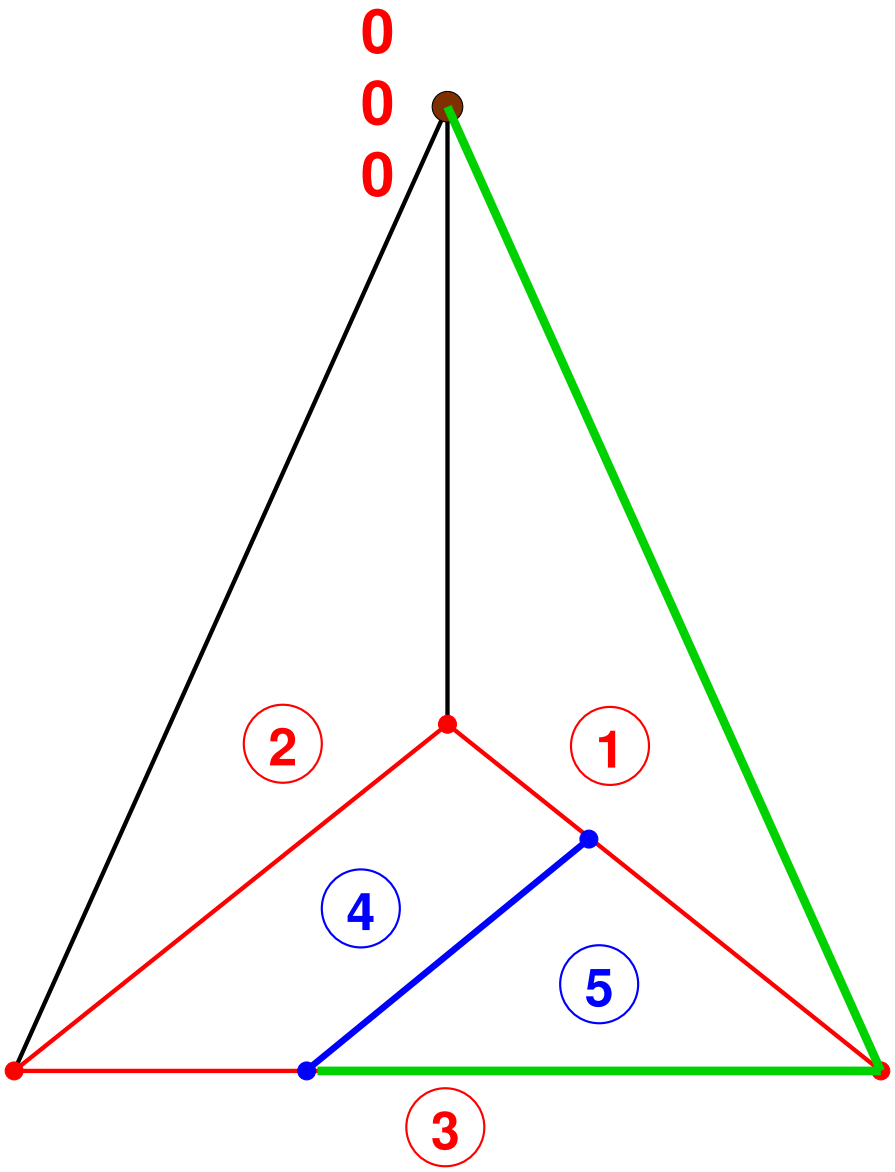
The Lemke–Howson algorithm

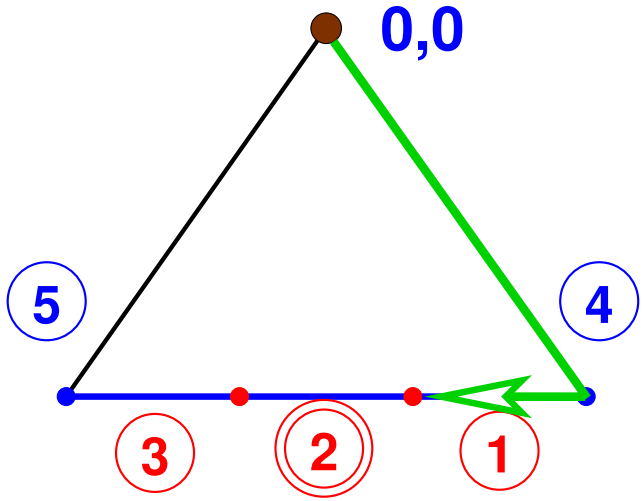




missing label 2

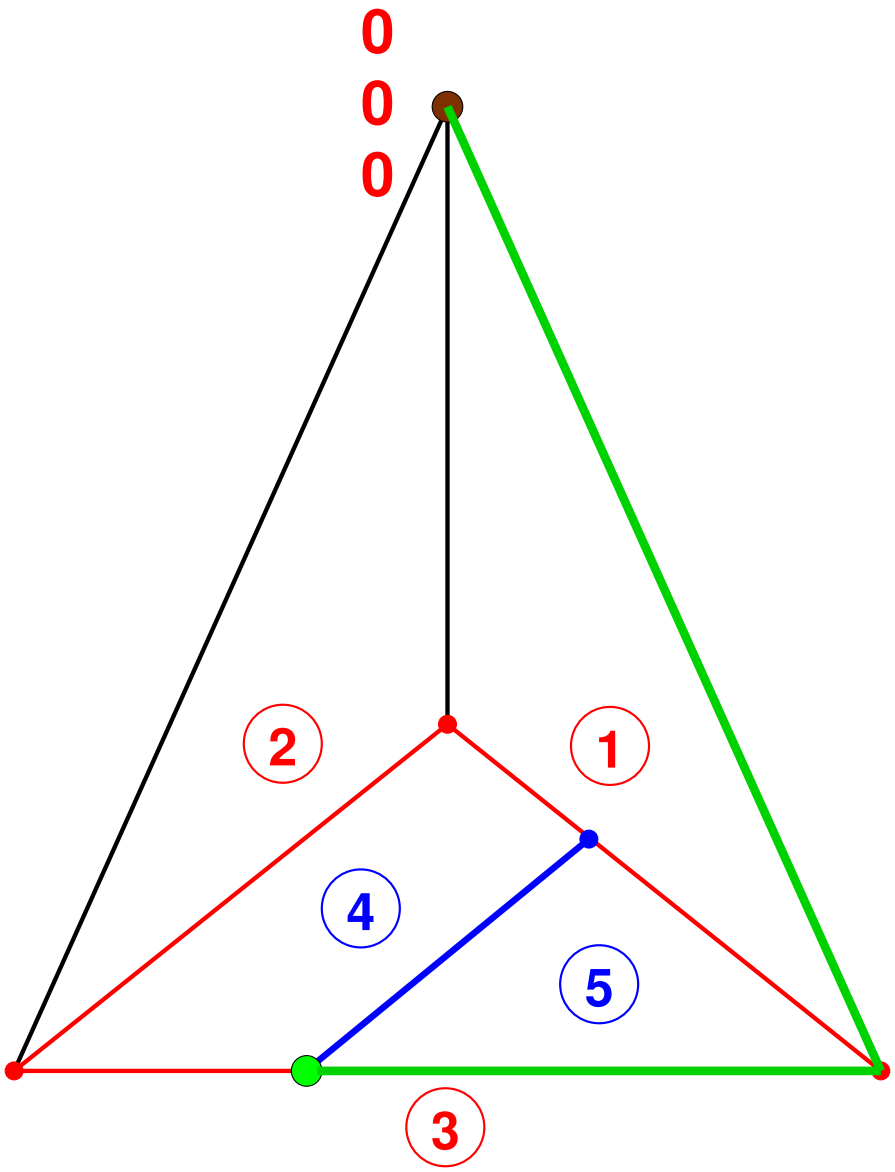
The Lemke–Howson algorithm

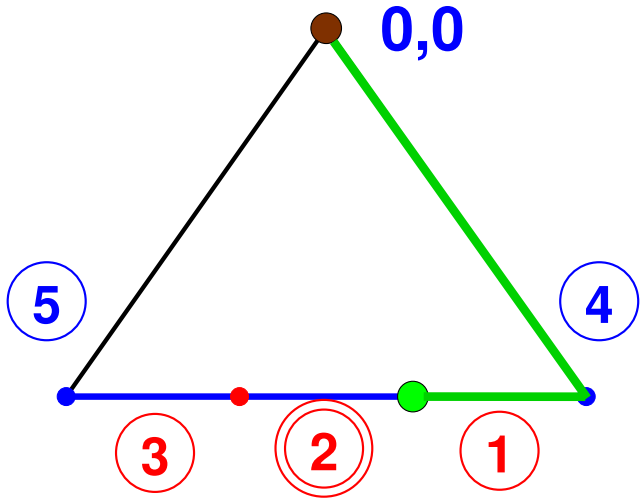




missing label 2

The Lemke–Howson algorithm





found label 2

Why Lemke-Howson works

LH finds at least one Nash equilibrium because

- finitely many "vertices"

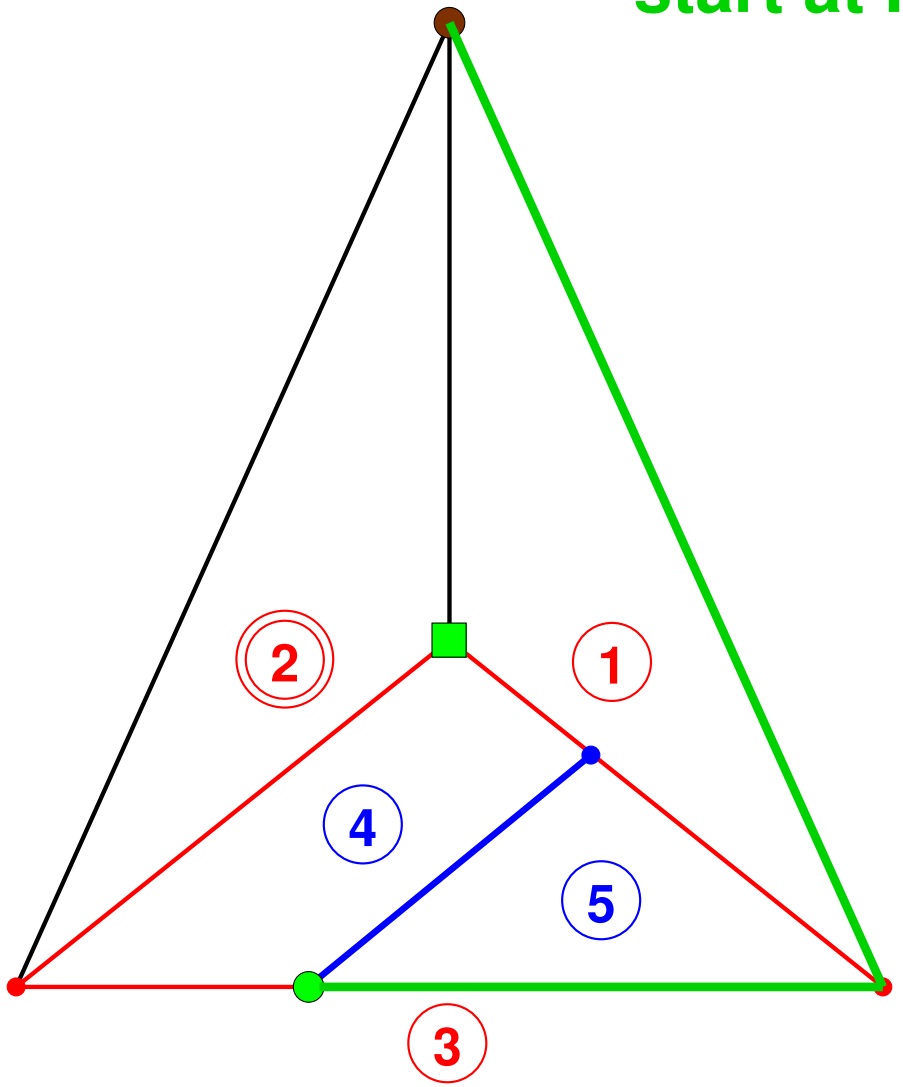
for nondegenerate (generic) games:

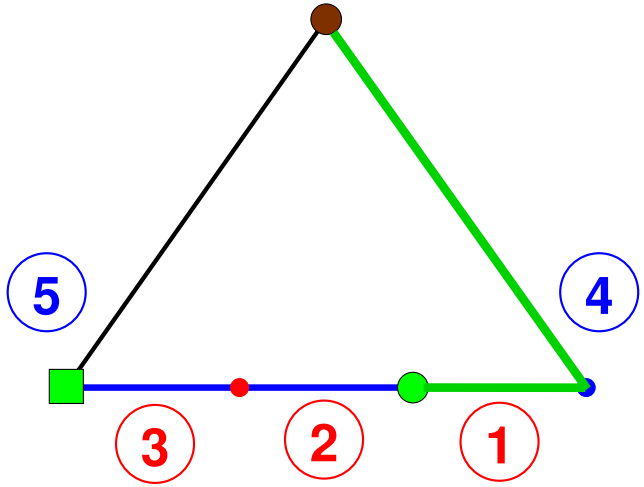
- **unique** starting edge given missing label
- **unique** continuation

$\Rightarrow$ precludes "coming back" like here:

The Lemke–Howson algorithm

start at Nash equilibrium

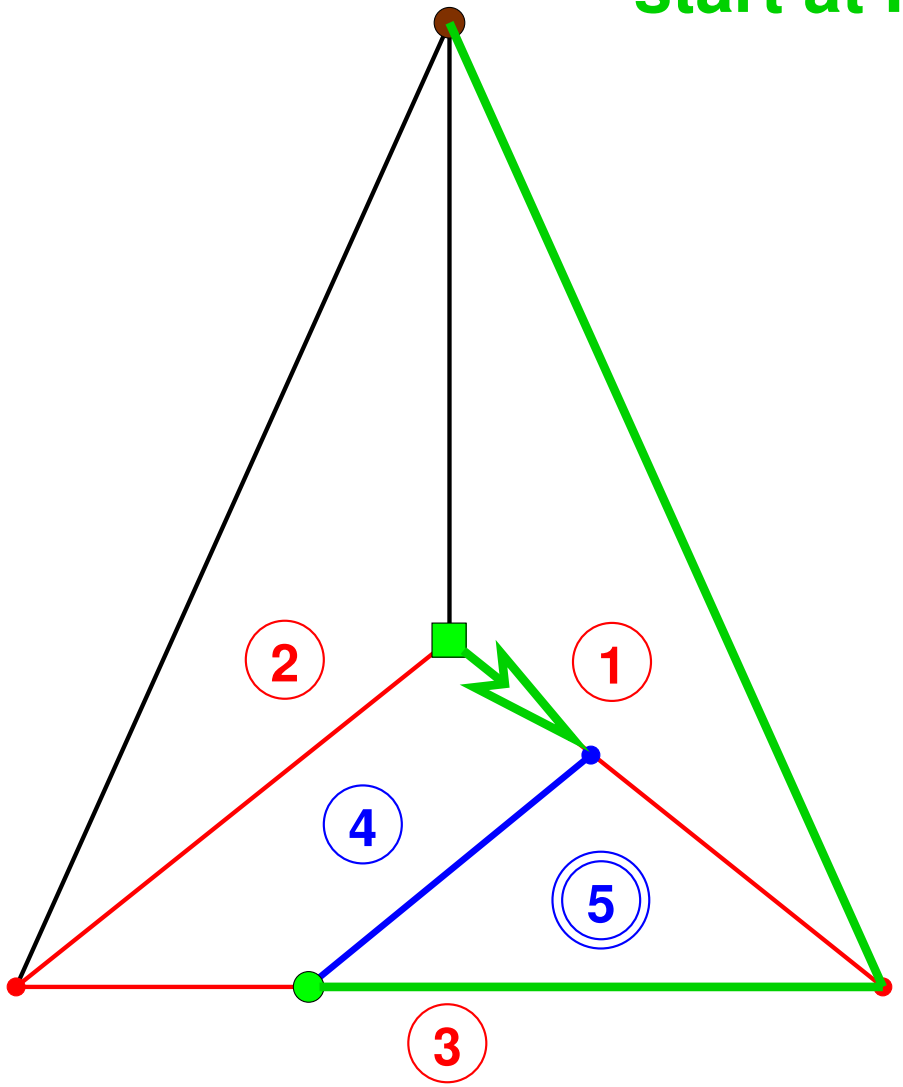


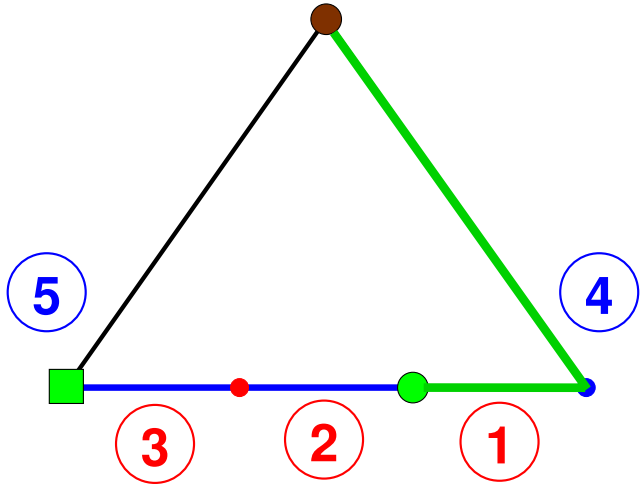


missing label 2

The Lemke–Howson algorithm

start at Nash equilibrium

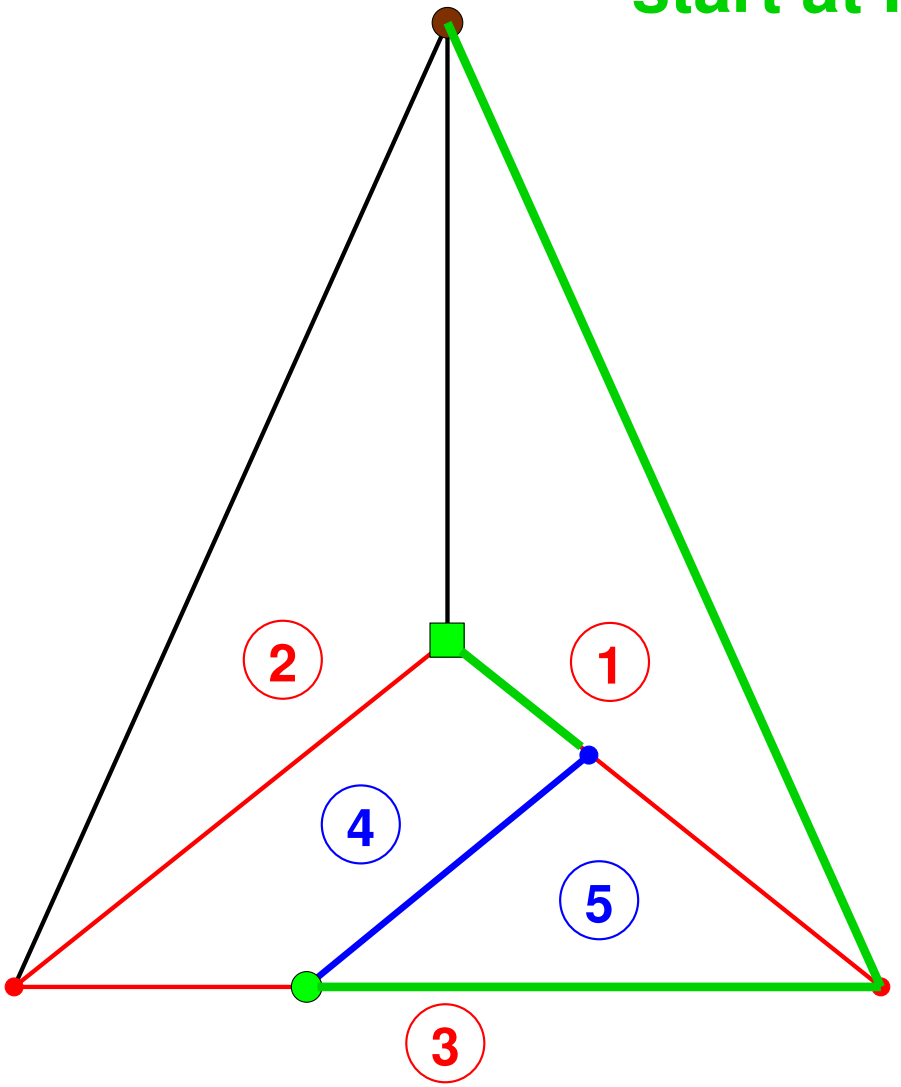


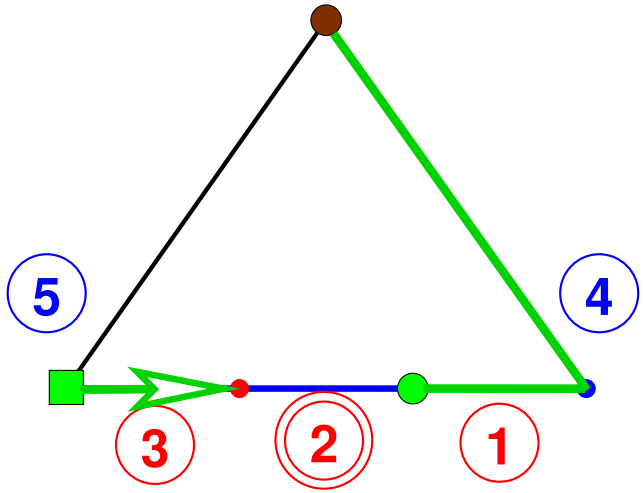


missing label 2

The Lemke–Howson algorithm

start at Nash equilibrium

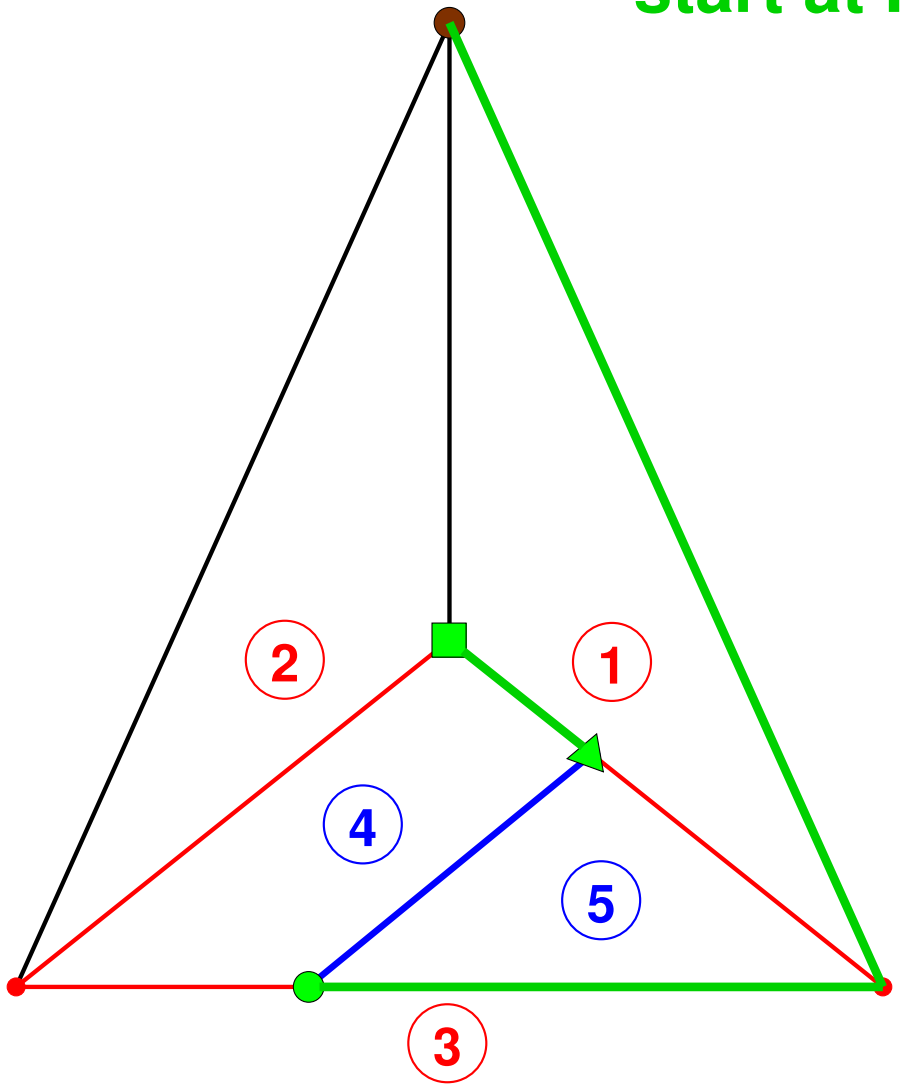


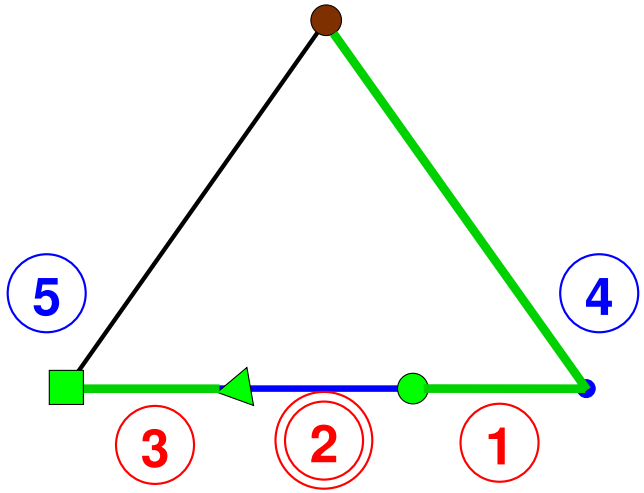


missing label 2

Odd number of Nash equilibria!

start at Nash equilibrium





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Nondegenerate bimatrix games

Given: $m \times n$ bimatrix game (A,B)

$$X = \{ x \in \mathbf{R}^m \mid x \geq \mathbf{0},\ x_1 + \ldots + x_m = 1 \}$$
$$Y = \{ y \in \mathbf{R}^n \mid y \geq \mathbf{0},\ y_1 + \ldots + y_n = 1 \}$$

$$\mathrm{supp}(x) = \{ i \mid x_i > 0 \}$$
$$\mathrm{supp}(y) = \{ j \mid y_j > 0 \}$$

(A,B) **nondegenerate** $\Leftrightarrow$ $\forall\, x \in X,\ y \in Y$:

$$|\{ j \mid j \text{ best response to } x \}| \leq |\mathrm{supp}(x)|,$$
$$|\{ i \mid i \text{ best response to } y \}| \leq |\mathrm{supp}(y)|.$$

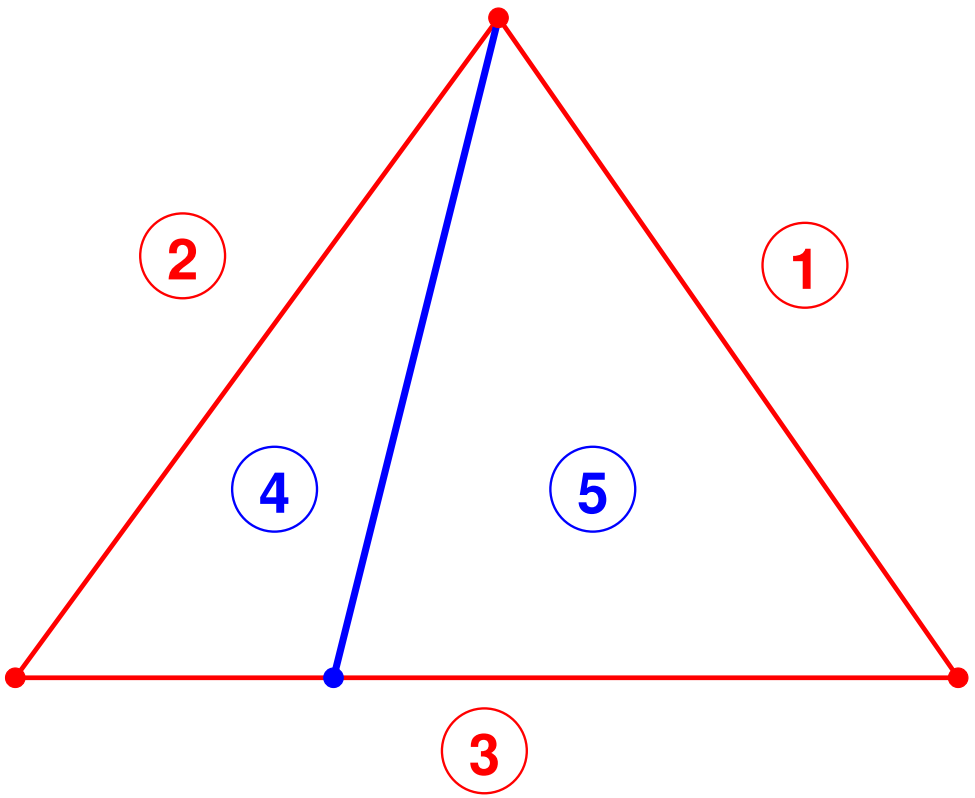
Nondegeneracy via labels

$m \times n$ bimatrix game (A,B) **nondegenerate**

$\Leftrightarrow$ no $x \in X$ has more than m labels,
no $y \in Y$ has more than n labels.

E.g. x with $> m$ labels,
s labels from $\{ 1 , \ldots , m \}$,
$\Rightarrow$ $> m–s$ labels from $\{ m+1 , \ldots , m+n \}$
$\Leftrightarrow$ $>$ |supp(x)| best responses to x.
$\Rightarrow$ degenerate.

Example of a degenerate game



	4	5
1	2	1
2	1	3
3	4	4

= B

payoffs to
player II

Handling degenerate games

Lemke–Howson implemented by pivoting, i.e., changing from one *basic feasible solution* of a linear system to another by choosing an entering and a leaving variable.

Choice of entering variable via complementarity (only difference to simplex algorithm for linear programming).

Leaving variable is *unique* in nondegenerate games.

In degenerate games: *perturb* system by adding $(\varepsilon, \ldots, \varepsilon^n)^\top$, creates nondegenerate system.
Implemented *symbolically* by lexicographic rule.