# Pathways to Equilibria, Pretty Pictures and Diagrams (PPAD)

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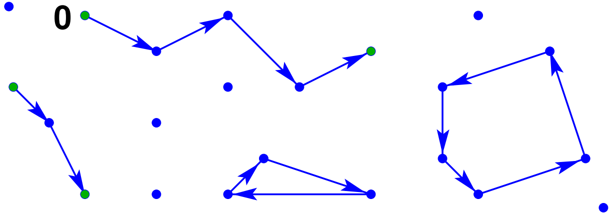
# 2-player game: find one Nash equilibrium

2-NASH $\in$ PPAD (Polynomial Parity Argument with Direction)

Implicit digraph with indegrees and outdegrees $\leq$ **1** is a set of [nodes], paths and cycles:



Parity argument: number of **sources** of paths = number of **sinks**

Comput. problem: given one source **0**, find another source or sink

[Chen/Deng 2006] 2-NASH is PPAD-complete.

# Symmetric Nash equilibria of symmetric games

square game matrix $A$ = payoffs to row player

$$A = \boxed{\begin{matrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{matrix}}$$

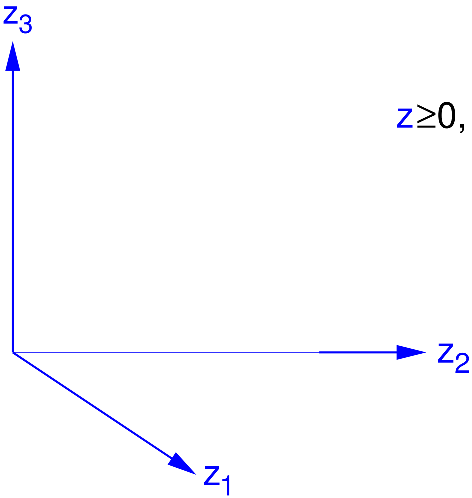
# Symmetric Nash equilibria of symmetric games

**equilibrium**: only optimal strategies are played

$$A = \begin{array}{ccc|c} 1/3 & 2/3 & \mathbf{0} & \\ \hline 0 & 3 & 0 & \mathbf{2} \\ 2 & 2 & 2 & \mathbf{2} \\ 3 & 0 & 0 & 1 \end{array}$$

# Symmetric Nash equilibria of symmetric games

plot polytope with strategy weights $z_1$, $z_2$, $z_3$

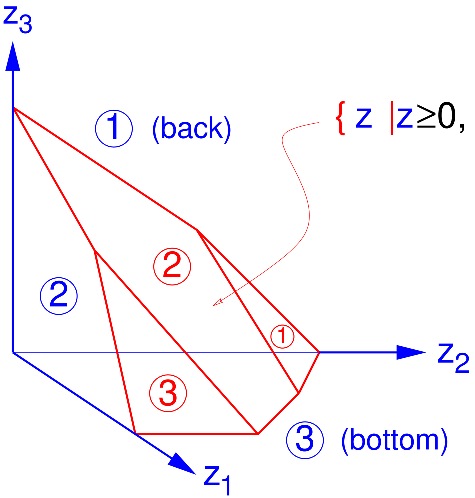


$z \geq 0,$

$$A = \begin{array}{|ccc|} \multicolumn{1}{c}{z_1} & z_2 & \multicolumn{1}{c}{z_3} \\ \hline 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \\ \hline \end{array}$$

# Symmetric Nash equilibria of symmetric games

with payoffs (scaled to 1) and **labels** for binding inequalities

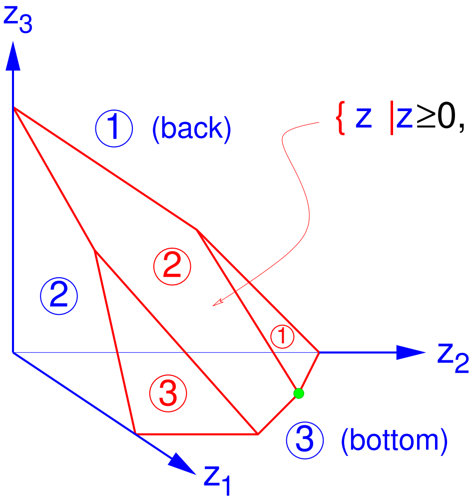


$\{\, z \mid z \geq 0,\ Az \leq 1 \,\}$

$$A = \begin{array}{ccc} z_1 & z_2 & z_3 \\ \hline 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{array}$$

# Symmetric Nash equilibria of symmetric games

**equilibrium** = **completely labeled** point

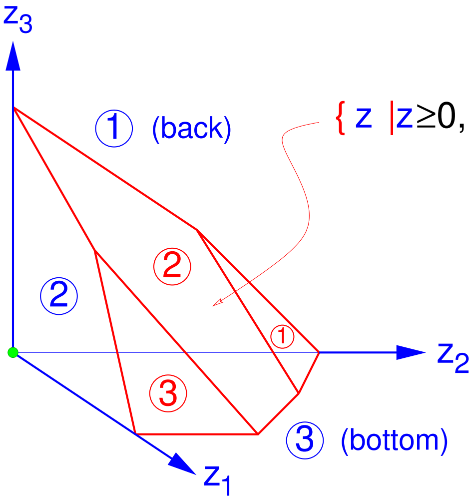


$\{\, z \mid z \geq 0,\ Az \leq \mathbf{1} \,\}$

$$A = \begin{array}{ccc|c} 1/6 & 1/3 & \mathbf{0} & \\ \hline 0 & 3 & 0 & \mathbf{1} \\ 2 & 2 & 2 & \mathbf{1} \\ 3 & 0 & 0 & 1/2 \end{array}$$

# Symmetric Nash equilibria of symmetric games

start path with **artificial equilibrium** $z$=**0**

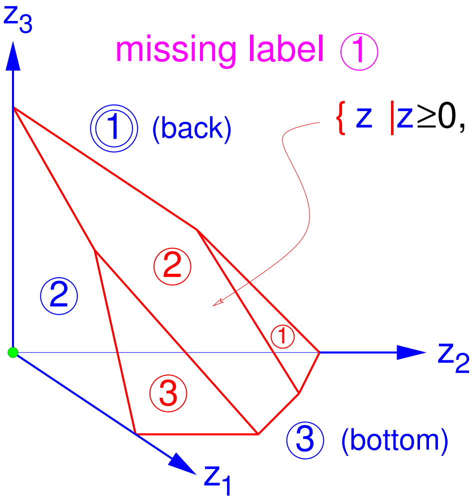


$\{ z \mid z \geq 0,\ Az \leq 1 \}$

$$A = \boxed{\begin{matrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{matrix}}$$

# Symmetric Nash equilibria of symmetric games

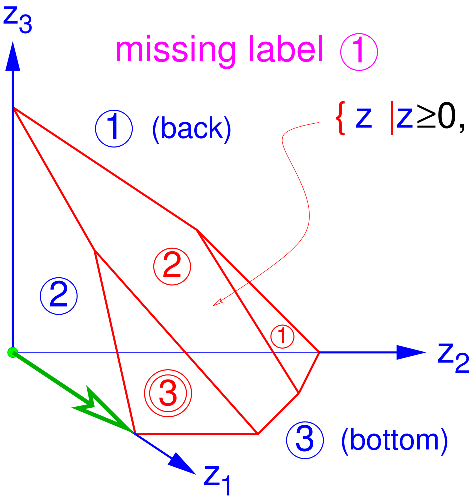
start path with **artificial equilibrium** $z=\mathbf{0}$, choose e.g.



$$A = \boxed{\begin{matrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{matrix}}$$

# Symmetric Nash equilibria of symmetric games

leave facet with label **1**, find duplicate label **3**



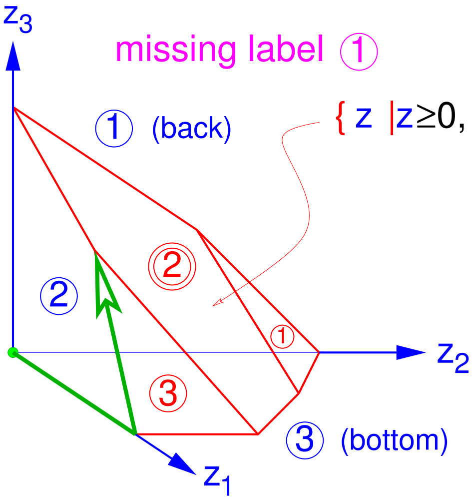
$\{ z \mid z \geq 0, Az \leq 1 \}$

$$A = \boxed{\begin{matrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{matrix}}$$

# Symmetric Nash equilibria of symmetric games

leave facet with old label **3**, find duplicate label **2**

missing label ①

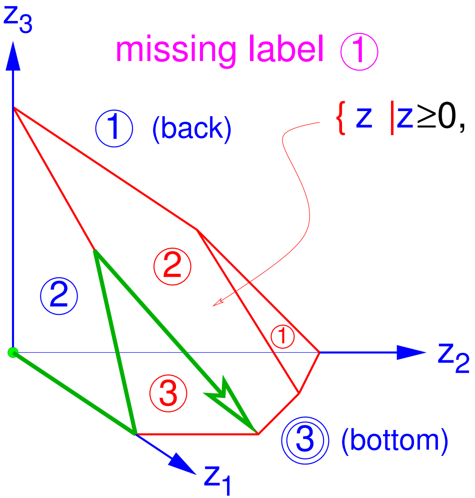


$\{\, z \mid z \geq 0,\ Az \leq 1 \,\}$

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

# Symmetric Nash equilibria of symmetric games

leave facet with old label **2**, find duplicate label **3**

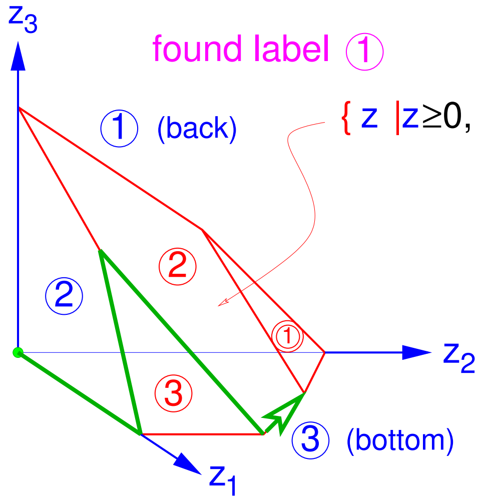


$\{ z \mid z \geq 0,\ Az \leq 1 \}$

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

# Symmetric Nash equilibria of symmetric games

leave facet with old label **3**, find missing label **1**



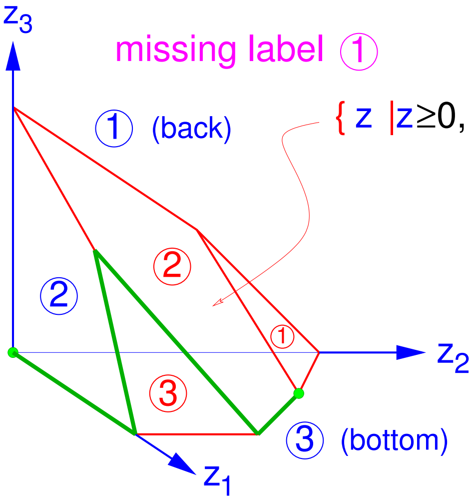
$\{ z \mid z \geq 0, Az \leq 1 \}$

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

# Symmetric Nash equilibria of symmetric games

equilibria (including artificial equilibrium) = endpoints of paths

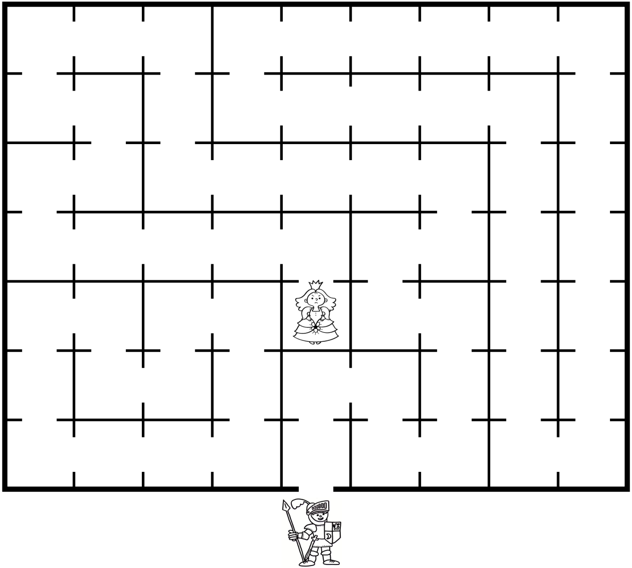
missing label ①



$\{\ z\ |\ z \geq 0,\ Az \leq \mathbf{1}\ \}$

$$A = \begin{array}{ccc|c} 1/6 & 1/3 & \mathbf{0} & \\ \hline 0 & 3 & 0 & \mathbf{1} \\ 2 & 2 & 2 & \mathbf{1} \\ 3 & 0 & 0 & 1/2 \end{array}$$

# The castle where each room has at most two doors



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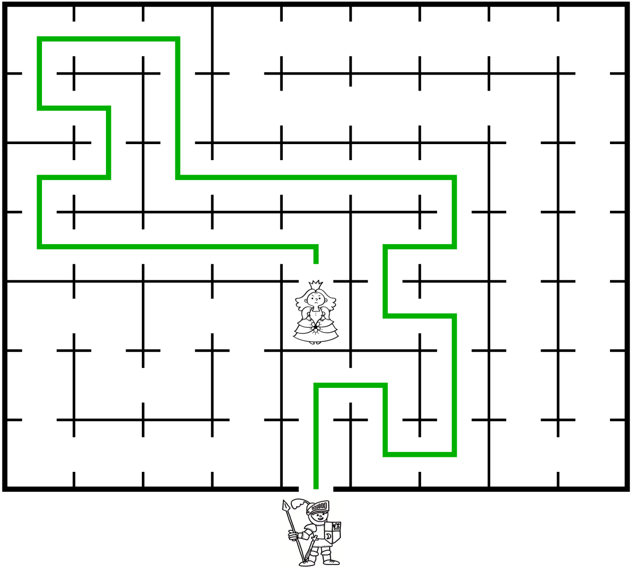

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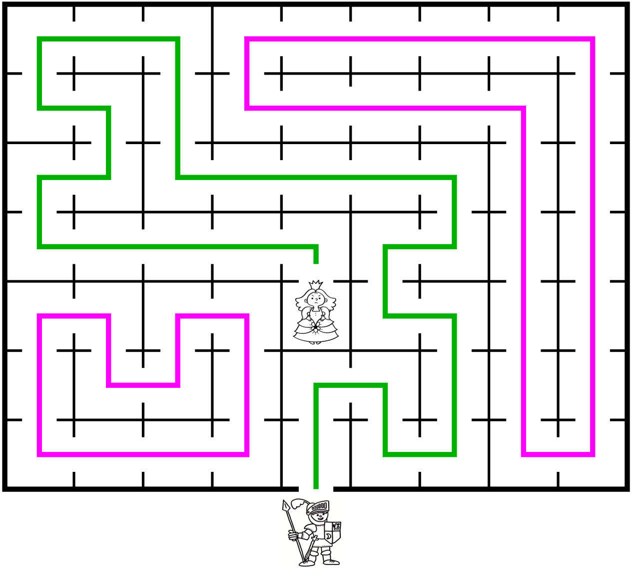

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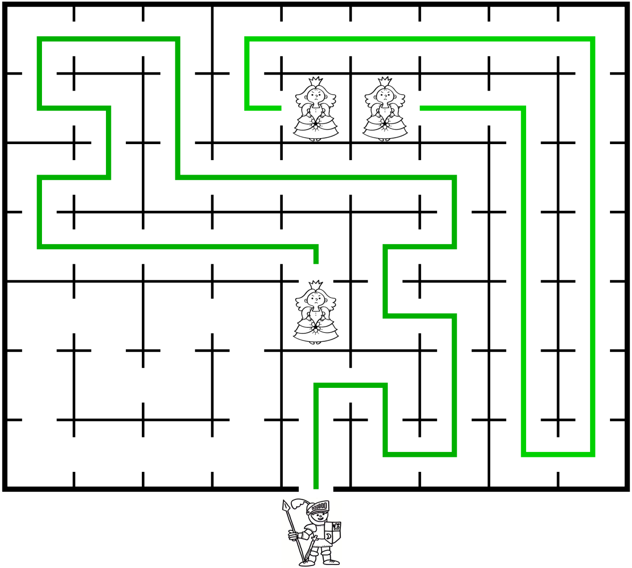
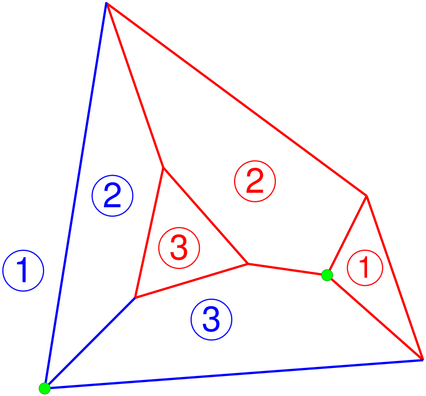


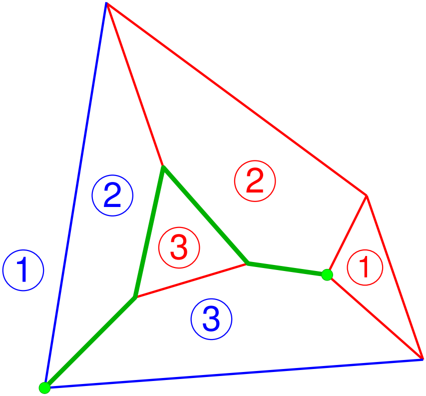
# Path of “almost completely labeled” edges

two completely labeled vertices



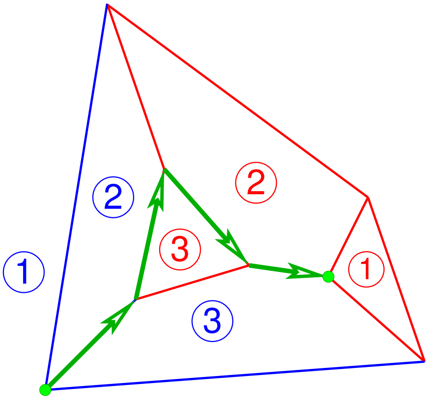
# Path of “almost completely labeled” edges

path because at most two neighbours (“doors” in castle)



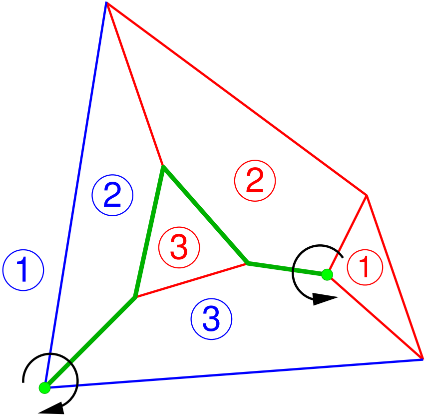
# Path of “almost completely labeled” edges

orientation of edges: **2** on left, **3** on right



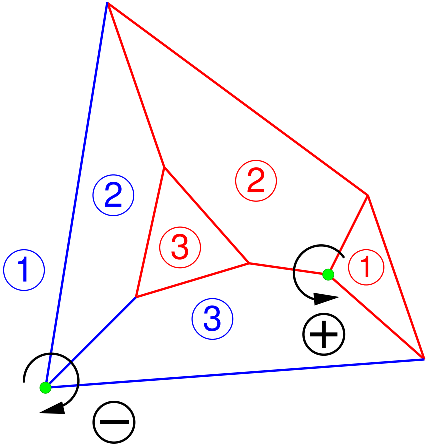
# Path of “almost completely labeled” edges

opposite orientation (“sign”) of endpoints



# Path of “almost completely labeled” edges

equilibrium **sign** ⊖ or ⊕ does not depend on path



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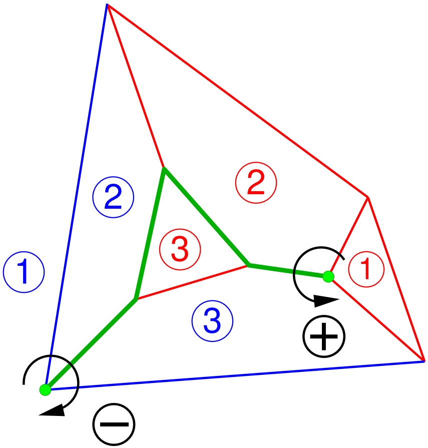

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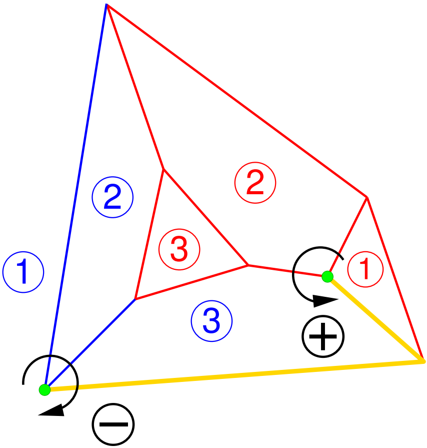

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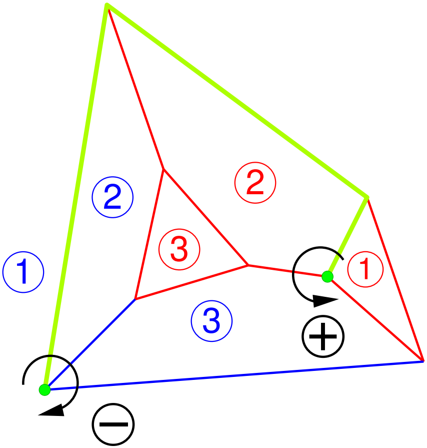

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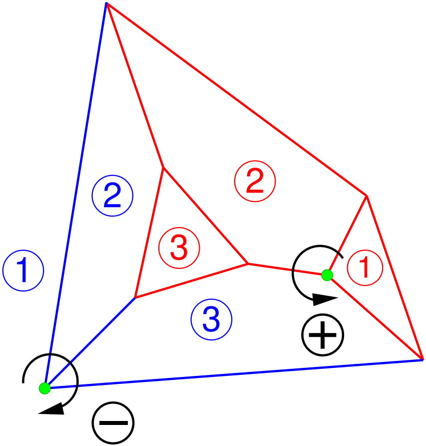

# Labeled polytope $P$

Let $a_j \in \mathbb{R}^m$, $\beta_j \in \mathbb{R}$,

$$\boxed{P = \{x \in \mathbb{R}^m \mid a_j x \le \beta_j, \;\; 1 \le j \le n\},}$$

let **facet** $\quad F_j = \{\, x \in P \mid a_j x = \beta_j \,\}$ have

**label** $\quad l(j) \in \{1, \ldots, m\}$.

Assume $P$ is a **simple** polytope (no $x \in P$ on $> m$ facets)

$\Rightarrow$ each vertex $x$ on $m$ facets = $m$ linearly independent equations.

$x$ **completely labeled** $\Leftrightarrow \{l(j) \mid x \in F_j\} = \{1, \ldots, m\}$.

# Completely labeled points come in pairs

**Theorem** [ Parity Argument ]

Let $\boldsymbol{P}$ be a labeled polytope.

Then $\boldsymbol{P}$ has an **even** number of completely labeled vertices.

# Completely labeled points come in pairs of opposite sign

**Theorem** [ Parity Argument with Direction ]

Let ***P*** be a labeled polytope.

Then ***P*** has an **even** number of completely labeled vertices.
Half of these have **sign** ⊖, half have sign ⊕.

# Completely labeled points come in pairs of opposite sign

**Theorem** [ Parity Argument with Direction ]

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Half of these have **sign** ⊖, half have sign ⊕.

---

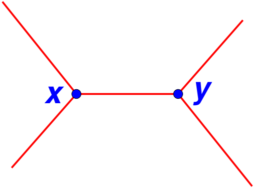
**sign** of completely labeled $\boldsymbol{x}$ is **sign of determinant** of facet normal vectors in order of their labels: if (e.g.) facet $\boldsymbol{a_i x} = \boldsymbol{\beta_i}$ has label $\boldsymbol{i = 1, 2, ..., m}$, then

$$\boldsymbol{sign(x) = sign\,|a_1\ a_2 \cdots a_m|}$$

# Pivoting changes signs

**Lemma**

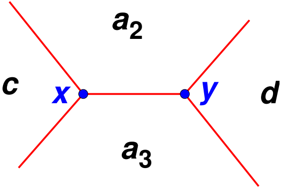
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}$ be adjacent vertices of a simple polytope $\boldsymbol{P}$



# Pivoting changes signs

**Lemma**

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{m}$ be adjacent vertices of a simple polytope $\boldsymbol{P}$ with facet normals $\boldsymbol{c}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{x}$ and $\boldsymbol{d}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{y}$.



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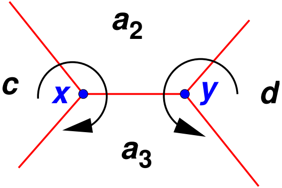
Then $|\boldsymbol{c}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_m|$ and $|\boldsymbol{d}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_m|$ have opposite sign.



# Pivoting changes signs

**Proof :**

$$\begin{array}{rcl} \boldsymbol{c}\boldsymbol{x} = \beta_0 & & \\ & & \boldsymbol{d}\boldsymbol{y} = \beta_1 \\ \boldsymbol{a}_2\boldsymbol{x} = \beta_2 & & \boldsymbol{a}_2\boldsymbol{y} = \beta_2 \\ \vdots & & \vdots \\ \boldsymbol{a}_m\boldsymbol{x} = \beta_m & & \boldsymbol{a}_m\boldsymbol{y} = \beta_m \end{array}$$

# Pivoting changes signs

**Proof :**

$$\begin{array}{rr} cx = \beta_0 & \\ & dy = \beta_1 \\ a_2 x = \beta_2 & a_2 y = \beta_2 \\ \vdots & \vdots \\ a_m x = \beta_m & a_m y = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

# Pivoting changes signs

**Proof :**

$$\begin{array}{rcl} cx = \beta_0 & & \\ & & dy = \beta_1 \\ a_2x = \beta_2 & & a_2y = \beta_2 \\ \vdots & & \vdots \\ a_mx = \beta_m & & a_my = \beta_m \end{array}$$

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$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d)x = (\gamma c + \delta d)y$$

# Pivoting changes signs

**Proof :**

$$\begin{array}{cc} cx = \beta_0 & \boxed{cy < \beta_0} \\ \boxed{dx < \beta_1} & dy = \beta_1 \\ a_2x = \beta_2 & a_2y = \beta_2 \\ \vdots & \vdots \\ a_mx = \beta_m & a_my = \beta_m \end{array}$$

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$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d)x = (\gamma c + \delta d)y, \qquad \gamma(cx - cy) = \delta(dy - dx)$$

# Pivoting changes signs

**Proof :**

$$\begin{array}{cc} cx = \beta_0 & \boxed{cy < \beta_0} \\ \boxed{dx < \beta_1} & dy = \beta_1 \\ a_2x = \beta_2 & a_2y = \beta_2 \\ \vdots & \vdots \\ a_mx = \beta_m & a_my = \beta_m \end{array}$$

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$\Rightarrow \quad \gamma$ and $\delta$ have same sign

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**Proof :**

$$\begin{array}{cc} cx = \beta_0 & \boxed{cy < \beta_0} \\ \boxed{dx < \beta_1} & dy = \beta_1 \\ a_2x = \beta_2 & a_2y = \beta_2 \\ \vdots & \vdots \\ a_mx = \beta_m & a_my = \beta_m \end{array}$$

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$\Rightarrow \quad \gamma$ and $\delta$ have same sign,

$$|(\gamma c + \delta d)\ a_2 \cdots a_m| = \gamma\ |c\ a_2 \cdots a_m| + \delta\ |d\ a_2 \cdots a_m| = 0$$

# Pivoting changes signs

**Proof :**

$$\begin{array}{cc} cx = \beta_0 & \boxed{cy < \beta_0} \\ \boxed{dx < \beta_1} & dy = \beta_1 \\ a_2 x = \beta_2 & a_2 y = \beta_2 \\ \vdots & \vdots \\ a_m x = \beta_m & a_m y = \beta_m \end{array}$$

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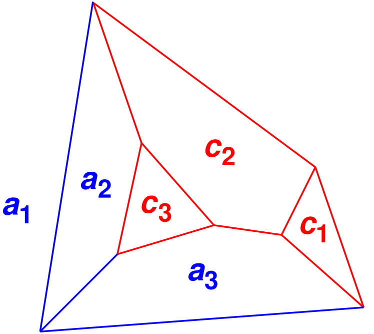
$\Rightarrow \quad \gamma$ and $\delta$ have same sign,

$$|(\gamma c + \delta d)\ a_2 \cdots a_m| = \gamma\ |c\ a_2 \cdots a_m| + \delta\ |d\ a_2 \cdots a_m| = 0$$

$\Rightarrow \quad |c\ a_2 \cdots a_m|$ and $|d\ a_2 \cdots a_m|$ have opposite sign, QED.

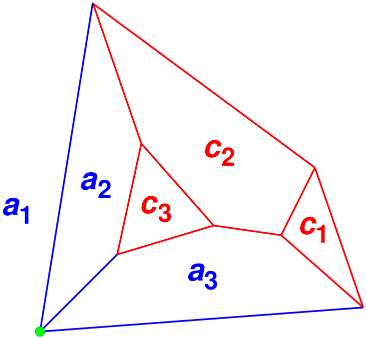
# General Parity Argument with Direction

Facet normal vectors $\mathbf{a}_1$ $\mathbf{a}_2$ $\mathbf{a}_3$ $\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$, labels **1 2 3 1 2 3**



# General Parity Argument with Direction

Start with $a_1$ $a_2$ $a_3$, sign ⊖

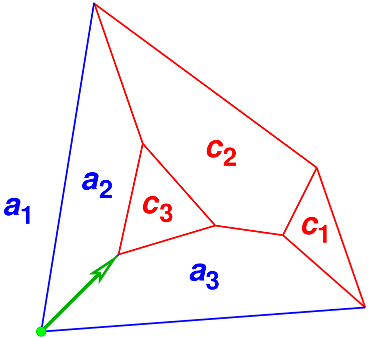


⊖

$$\left|\, \boxed{a_1}\ a_2\ a_3 \,\right|$$

# General Parity Argument with Direction

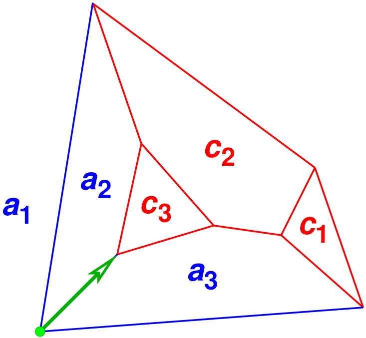
Start with $a_1\ a_2\ a_3$, sign $\ominus$, label **1** missing, $a_1 \to c_3$ gives sign $\oplus$



$$\overset{\ominus}{\left|\,\boxed{a_1}\ a_2\ a_3\,\right|} \longrightarrow \overset{\oplus}{\left|\,\boxed{c_3}\ a_2\ a_3\,\right|}$$

# General Parity Argument with Direction

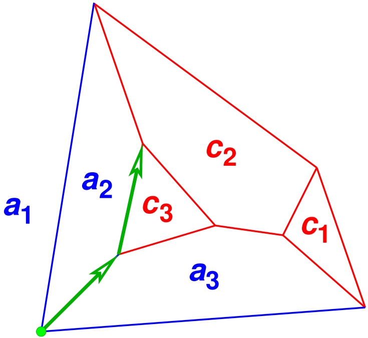
**Switch columns** $c_3$ and $a_3$ in determinant: back to sign ⊖



$$\begin{tikzcd} \ominus & \oplus \\ \left|\boxed{a_1}\ a_2\ a_3\right| \arrow[r] & \left|\boxed{c_3}\ a_2\ a_3\right| \\ \left|\boxed{a_3}\ a_2\ c_3\right| \arrow[ur, leftrightarrow] & \end{tikzcd}$$

# General Parity Argument with Direction

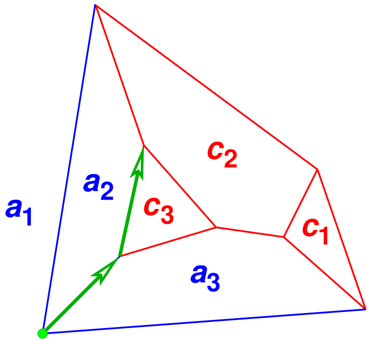
**next pivot** $a_3 \to c_2$ gives sign $\oplus$



$$\begin{tikzcd} \ominus & \oplus \\ \left|\boxed{a_1}\ a_2\ a_3\right| \arrow[r] & \left|\boxed{c_3}\ a_2\ a_3\right| \arrow[dl, leftrightarrow] \\ \left|\boxed{a_3}\ a_2\ c_3\right| \arrow[r] & \left|\boxed{c_2}\ a_2\ c_3\right| \end{tikzcd}$$

# General Parity Argument with Direction

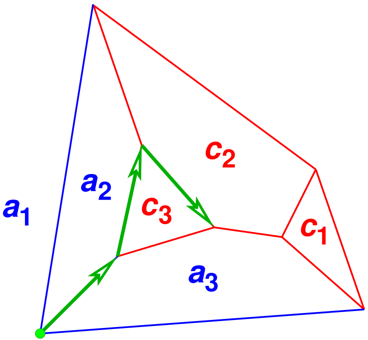
**Switch columns $c_2$** and $a_2$ in determinant: back to sign ⊖

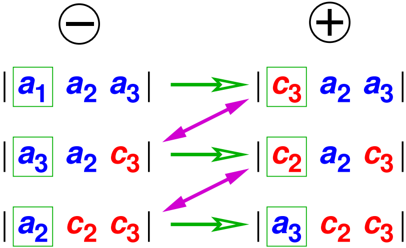


$$\begin{tikzcd}
\ominus & \oplus \\
\left|\boxed{a_1}\ a_2\ a_3\right| \arrow[r] & \left|\boxed{c_3}\ a_2\ a_3\right| \\
\left|\boxed{a_3}\ a_2\ c_3\right| \arrow[r] \arrow[ur, leftrightarrow] & \left|\boxed{c_2}\ a_2\ c_3\right| \\
\left|\boxed{a_2}\ c_2\ c_3\right| \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

# General Parity Argument with Direction

**next pivot** $a_2 \to a_3$ gives sign ⊕





⊖ ⊕

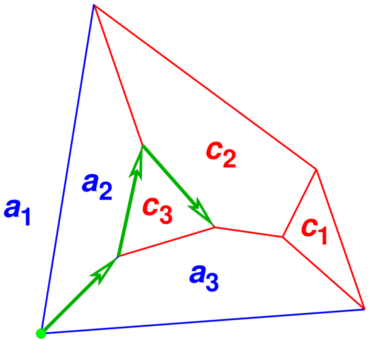
$|\boxed{a_1}\ a_2\ a_3| \longrightarrow |\boxed{c_3}\ a_2\ a_3|$

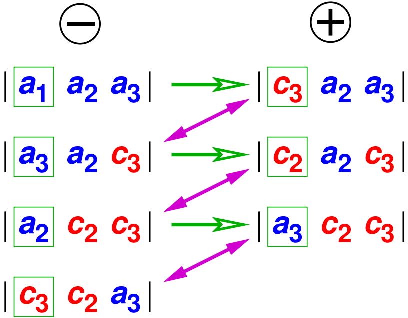
$|\boxed{a_3}\ a_2\ c_3| \longrightarrow |\boxed{c_2}\ a_2\ c_3|$

$|\boxed{a_2}\ c_2\ c_3| \longrightarrow |\boxed{a_3}\ c_2\ c_3|$

# General Parity Argument with Direction

**Switch columns** $a_3$ and $c_3$ in determinant: back to sign ⊖





⊖

$|\boxed{a_1}\ a_2\ a_3|$

$|\boxed{a_3}\ a_2\ c_3|$

$|\boxed{a_2}\ c_2\ c_3|$

$|\boxed{c_3}\ c_2\ a_3|$

⊕

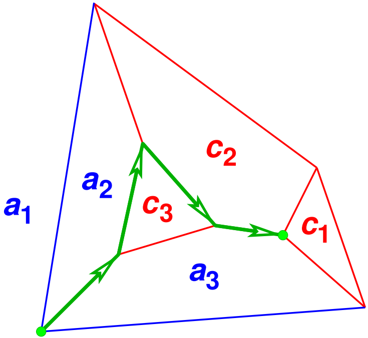
$|\boxed{c_3}\ a_2\ a_3|$

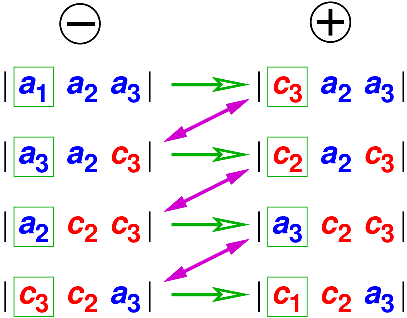
$|\boxed{c_2}\ a_2\ c_3|$

$|\boxed{a_3}\ c_2\ c_3|$

# General Parity Argument with Direction

**Last pivot** $c_3 \to c_1$ gives sign ⊕, opposite to starting sign ⊖.

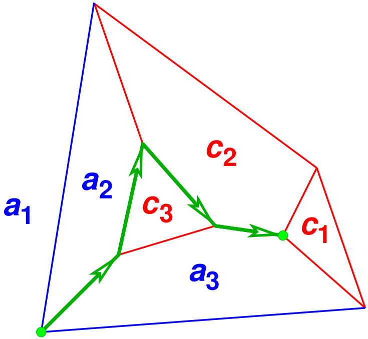




$$
\begin{array}{ccc}
\ominus & & \oplus \\
|\boxed{a_1}\ a_2\ a_3| & \longrightarrow & |\boxed{c_3}\ a_2\ a_3| \\
|\boxed{a_3}\ a_2\ c_3| & \longrightarrow & |\boxed{c_2}\ a_2\ c_3| \\
|\boxed{a_2}\ c_2\ c_3| & \longrightarrow & |\boxed{a_3}\ c_2\ c_3| \\
|\boxed{c_3}\ c_2\ a_3| & \longrightarrow & |\boxed{c_1}\ c_2\ a_3|
\end{array}
$$

# General Parity Argument with Direction

Only need: sign-switching of **pivots** and **column exchanges**



$$\begin{tikzcd}
\ominus & \oplus \\
\left|\boxed{a_1}\ a_2\ a_3\right| \arrow[r] & \left|\boxed{c_3}\ a_2\ a_3\right| \arrow[dl, leftrightarrow] \\
\left|\boxed{a_3}\ a_2\ c_3\right| \arrow[r] & \left|\boxed{c_2}\ a_2\ c_3\right| \arrow[dl, leftrightarrow] \\
\left|\boxed{a_2}\ c_2\ c_3\right| \arrow[r] & \left|\boxed{a_3}\ c_2\ c_3\right| \arrow[dl, leftrightarrow] \\
\left|\boxed{c_3}\ c_2\ a_3\right| \arrow[r] & \left|\boxed{c_1}\ c_2\ a_3\right|
\end{tikzcd}$$

# A more abstract example

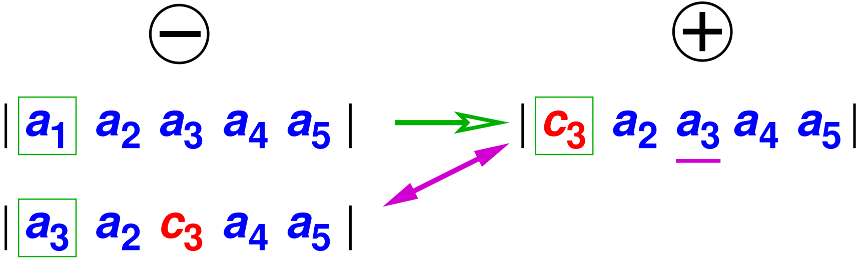
⊖

$$\left|\ \boxed{a_1}\ a_2\ a_3\ a_4\ a_5\ \right|$$

# A more abstract example

$$|\boxed{a_1}\ a_2\ \overset{\ominus}{a_3}\ a_4\ a_5| \longrightarrow |\boxed{c_3}\ a_2\ \overset{\oplus}{\underline{a_3}}\ a_4\ a_5|$$

# A more abstract example



$$\begin{tikzcd}
\ominus & \oplus \\
\left|\boxed{a_1}\ a_2\ a_3\ a_4\ a_5\right| \arrow[r] & \left|\boxed{c_3}\ a_2\ \underline{a_3}\ a_4\ a_5\right| \\
\left|\boxed{a_3}\ a_2\ c_3\ a_4\ a_5\right| \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

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$$\begin{tikzcd}
\ominus & \oplus \\
|\boxed{a_1}\ a_2\ a_3\ a_4\ a_5| \arrow[r] & |\boxed{c_3}\ a_2\ \underline{a_3}\ a_4\ a_5| \arrow[dl, leftrightarrow] \\
|\boxed{a_3}\ a_2\ c_3\ a_4\ a_5| \arrow[r] & |\boxed{c_4}\ a_2\ c_3\ \underline{a_4}\ a_5|
\end{tikzcd}$$

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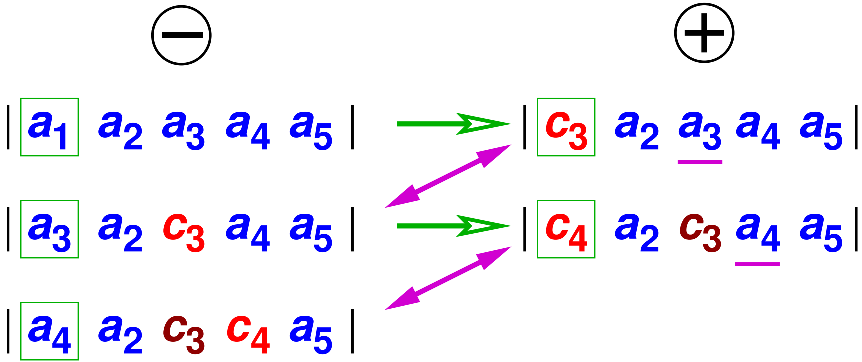

$$\ominus \qquad\qquad\qquad\qquad \oplus$$

$$|\boxed{a_1}\; a_2\; a_3\; a_4\; a_5| \longrightarrow |\boxed{c_3}\; a_2\; \underline{a_3}\; a_4\; a_5|$$

$$|\boxed{a_3}\; a_2\; c_3\; a_4\; a_5| \longrightarrow |\boxed{c_4}\; a_2\; c_3\; \underline{a_4}\; a_5|$$

$$|\boxed{a_4}\; a_2\; c_3\; c_4\; a_5|$$

# A more abstract example

$$\begin{tikzcd}
\ominus & \oplus \\
\left| \boxed{a_1} \; a_2 \; a_3 \; a_4 \; a_5 \right| \arrow[r] & \left| \boxed{c_3} \; a_2 \; \underline{a_3} \; a_4 \; a_5 \right| \\
\left| \boxed{a_3} \; a_2 \; c_3 \; a_4 \; a_5 \right| \arrow[r] \arrow[ur, leftrightarrow] & \left| \boxed{c_4} \; a_2 \; c_3 \; \underline{a_4} \; a_5 \right| \\
\left| \boxed{a_4} \; a_2 \; c_3 \; c_4 \; a_5 \right| \arrow[r] \arrow[ur, leftrightarrow] & \left| \boxed{c_5} \; a_2 \; c_3 \; c_4 \; \underline{a_5} \right|
\end{tikzcd}$$

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$$\begin{tikzcd}
\ominus & \oplus \\
\left| \boxed{a_1}\ a_2\ a_3\ a_4\ a_5 \right| \arrow[r] & \left| \boxed{c_3}\ a_2\ \underline{a_3}\ a_4\ a_5 \right| \arrow[dl, leftrightarrow] \\
\left| \boxed{a_3}\ a_2\ c_3\ a_4\ a_5 \right| \arrow[r] & \left| \boxed{c_4}\ a_2\ c_3\ \underline{a_4}\ a_5 \right| \arrow[dl, leftrightarrow] \\
\left| \boxed{a_4}\ a_2\ c_3\ c_4\ a_5 \right| \arrow[r] & \left| \boxed{c_5}\ a_2\ c_3\ c_4\ \underline{a_5} \right| \arrow[dl, leftrightarrow] \\
\left| \boxed{a_5}\ a_2\ c_3\ c_4\ c_5 \right| &
\end{tikzcd}$$

# A more abstract example

$$\begin{tikzcd}
\ominus & & \oplus \\
\left| \boxed{a_1}\ a_2\ a_3\ a_4\ a_5 \right| \arrow[rr, Rightarrow] & & \left| \boxed{c_3}\ a_2\ \underline{a_3}\ a_4\ a_5 \right| \arrow[dll, leftrightarrow] \\
\left| \boxed{a_3}\ a_2\ c_3\ a_4\ a_5 \right| \arrow[rr, Rightarrow] & & \left| \boxed{c_4}\ a_2\ c_3\ \underline{a_4}\ a_5 \right| \arrow[dll, leftrightarrow] \\
\left| \boxed{a_4}\ a_2\ c_3\ c_4\ a_5 \right| \arrow[rr, Rightarrow] & & \left| \boxed{c_5}\ a_2\ c_3\ c_4\ \underline{a_5} \right| \arrow[dll, leftrightarrow] \\
\left| \boxed{a_5}\ a_2\ c_3\ c_4\ c_5 \right| \arrow[rr, Rightarrow] & & \left| \boxed{c_1}\ a_2\ c_3\ c_4\ c_5 \right|
\end{tikzcd}$$

# Nash equilibria of bimatrix games

Recall: $m \times m$ matrix $C$,

$$P = \{z \in \mathbb{R}^m \mid -z \leq \mathbf{0}, \quad Cz \leq \mathbf{1} \}$$

with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$.

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Completely labeled $z \neq 0 \quad \Leftrightarrow$

Nash equilibrium $(z, z)$ of game $(C, C^\top)$

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Nash equilibrium $(z, z)$ of game $(C, C^\top)$

**Normalize** sign of "artificial equilibrium" $\mathbf{0}$ to ⊖, in general

$$index(z) = sign(z) \cdot (-1)^{m+1}$$

# Nash equilibria of bimatrix games

Recall: $\boldsymbol{m} \times \boldsymbol{m}$ matrix $\boldsymbol{C}$,

$$\boldsymbol{P} = \{\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{m}} \mid -\boldsymbol{z} \leq \mathbf{0},\quad \boldsymbol{Cz} \leq \mathbf{1}\ \}$$

with $\mathbf{2}\boldsymbol{m}$ inequalities labeled $\mathbf{1}, \ldots, \boldsymbol{m}, \mathbf{1}, \ldots, \boldsymbol{m}$.

**bimatrix game $(\boldsymbol{A}, \boldsymbol{B})$**:

$$\boldsymbol{C} = \begin{pmatrix} \mathbf{0} & \boldsymbol{A} \\ \boldsymbol{B}^\top & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) :$$

Completely labeled $(\boldsymbol{x}, \boldsymbol{y}) \neq (\mathbf{0}, \mathbf{0}) \quad \Leftrightarrow$

Nash equilibrium $(\boldsymbol{x}, \boldsymbol{y})$ of game $(\boldsymbol{A}, \boldsymbol{B})$

# Index of an equilibrium

**Theorem** [Shapley 1974]

A nondegenerate bimatrix game **(*A*, *B*)** has an odd number of equilibria, one more of index ⊕ than of index ⊖.

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[*Proof:* Endpoints of pivoting paths have opposite index ⊖ and ⊕.]

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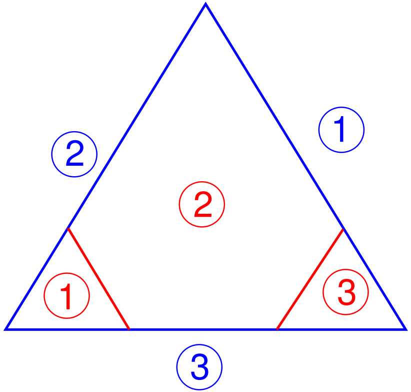
[*Proof:* Endpoints of pivoting paths have opposite index ⊖ and ⊕.]

Equilibria of index ⊕ include every

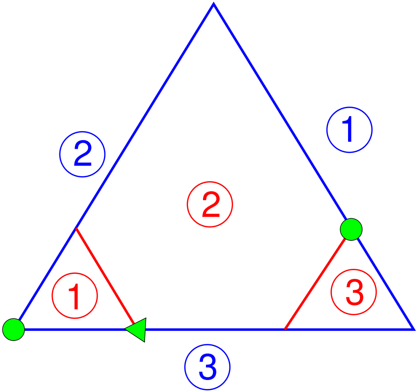
- pure-strategy equilibrium
- unique equilibrium
- **dynamically stable** equilibrium [Hofbauer 2003]

# Dynamically stable equilibrium: only if index ⊕



| | ① | ② | ③ |
|---|---|---|---|
| ① | **3** | **0** | **0** |
| ② | **2** | **2** | **2** |
| ③ | **0** | **3** | **0** |

# Dynamically stable equilibrium: only if index ⊕



| | ① | ② | ③ |
|---|---|---|---|
| ① | **3** | **0** | **0** |
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| ③ | **0** | **3** | **0** |

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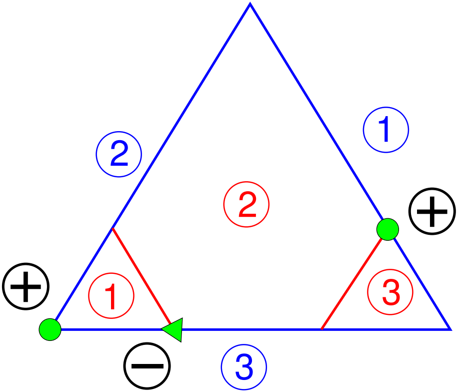

$$\begin{array}{c|ccc} & \textcircled{1} & \textcircled{2} & \textcircled{3} \\ \hline \textcircled{1} & \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \textcircled{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \textcircled{3} & \mathbf{0} & \mathbf{3} & \mathbf{0} \end{array}$$

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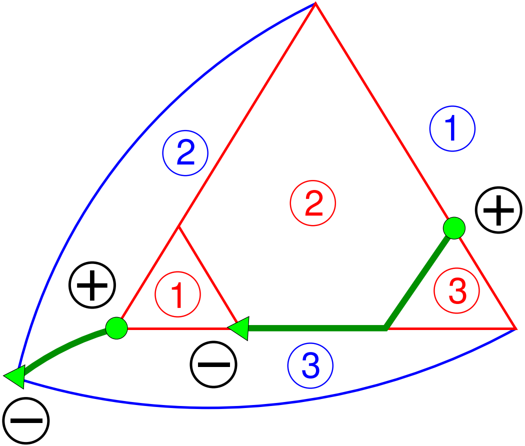

| | ① | ② | ③ |
|---|---|---|---|
| ① | **3** | **0** | **0** |
| ② | **2** | **2** | **2** |
| ③ | **0** | **3** | **0** |

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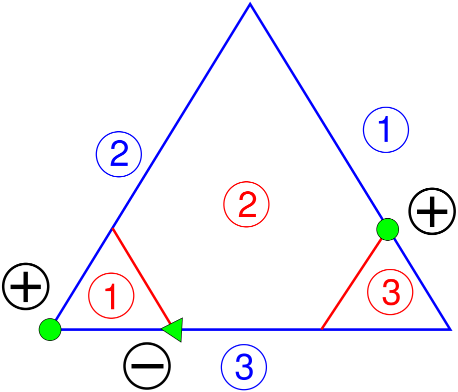
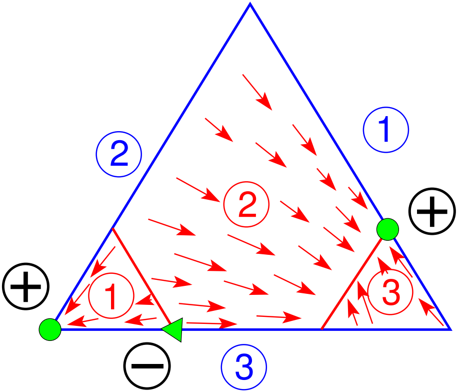


| | ① | ② | ③ |
|---|---|---|---|
| ① | **3** | **0** | **0** |
| ② | **2** | **2** | **2** |
| ③ | **0** | **3** | **0** |

# Dynamically stable equilibrium: only if index ⊕



$$\begin{array}{c|ccc} & \text{①} & \text{②} & \text{③} \\ \hline \text{①} & \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \text{②} & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \text{③} & \mathbf{0} & \mathbf{3} & \mathbf{0} \end{array}$$

# Strategic characterization of the index

**Theorem** [von Schemde / von Stengel 2004]

An equilibrium of a nondegenerate bimatrix game has index ⊕

⇔ it is the **unique** equilibrium in a larger game that has suitable additional strategies for one player.

# Strategic characterization of the index

**Theorem** [von Schemde / von Stengel 2004]

An equilibrium of a nondegenerate bimatrix game has index $\oplus$

$\Leftrightarrow$ it is the **unique** equilibrium in a larger game that has suitable additional strategies for one player.

**Theorem** [Balthasar / von Stengel 2009]

A *symmetric* equilibrium of a nondegenerate *symmetric* bimatrix game has *symmetric* index $\oplus$

$\Leftrightarrow$ it is the **unique** equilibrium in a larger *symmetric* game that has suitable additional strategies for both players.

# Signed perfect matchings

- Graph $G = (V, E)$, $V = \{1, \ldots, n\}$
- orient each edge $ab \in E$ as $(a, b)$ or $(b, a)$
- **perfect matching** $M \subset E$ of $G$
- for the edges $ab$ of $M$ (in any sequence), write down endpoints $a$, $b$ in the order of the orientation of the edge. Define

  $sign(M) =$ **parity** of the resulting **permutation** of $1, \ldots, n$ .

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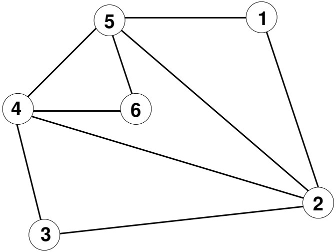

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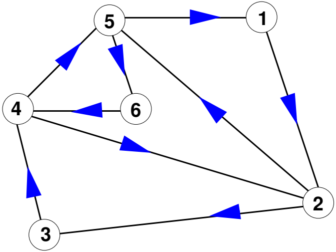

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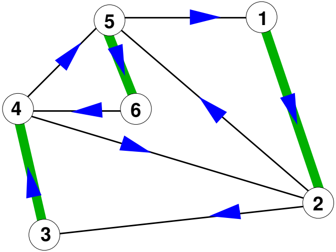

⊕ **1 2  3 4  5 6**

# Signed perfect matchings

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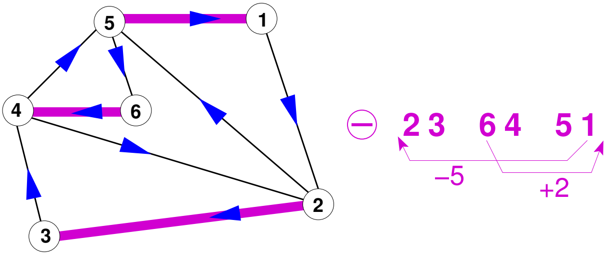

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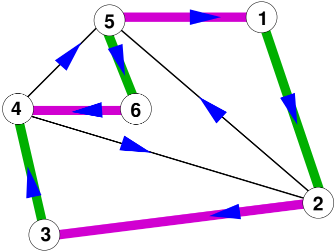
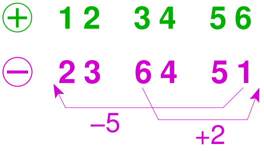




⊕ 1 2 3 4 5 6

⊖ 2 3 6 4 5 1

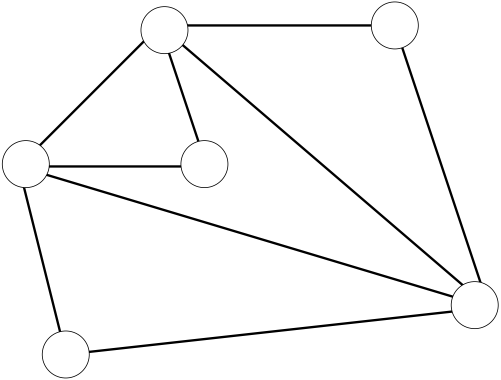
–5

+2

# Euler graphs

Euler graph

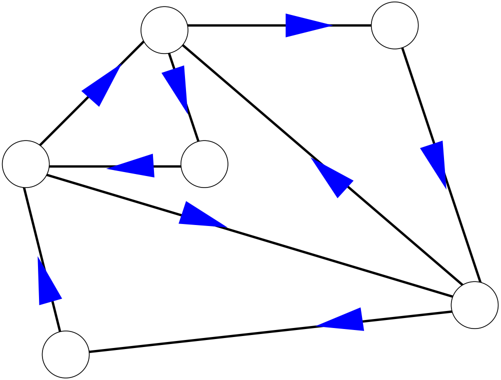
- every node has even degree (= number of neighbours)



# Euler graphs

Euler graph

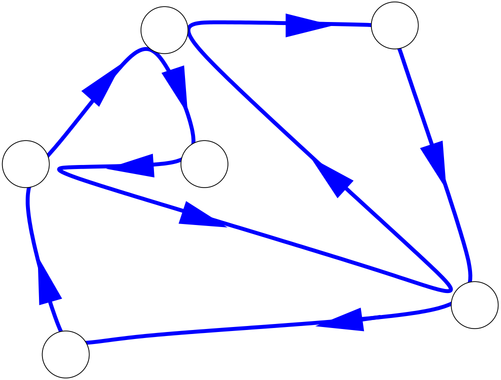
- every node has even degree (= number of neighbours)
- has Eulerian orientation (indegree = outdegree)



# Euler graphs ... have tours

Euler graph

- every node has even degree (= number of neighbours)
- has Eulerian orientation (indegree = outdegree) ... and tour



# Signs of matchings in Euler graphs

**Theorem**

A graph with an Eulerian orientation has as many perfect matchings of sign $\oplus$ as of sign $\ominus$.

# Signs of matchings in Euler graphs

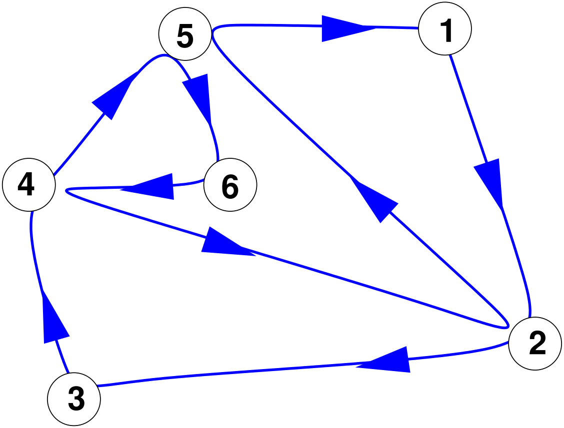
**Theorem**

A graph with an Eulerian orientation has as many perfect matchings of sign ⊕ as of sign ⊖.

**Proof :**

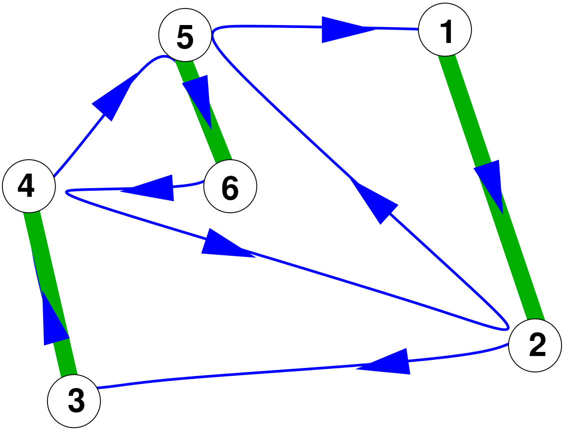
Any two perfect matchings are connected by a **pivoting path** which connects matchings of opposite sign.

# Finding a second perfect matching in an Euler graph



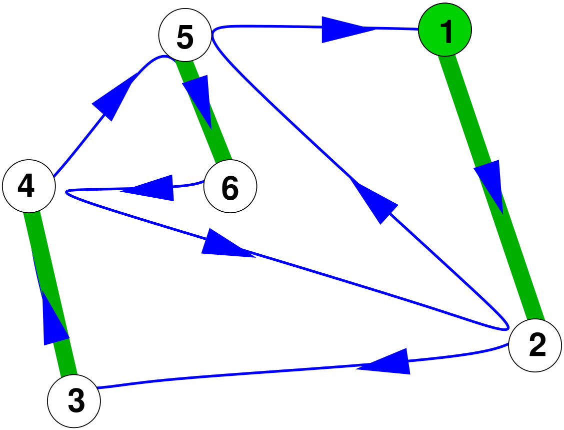
# Finding a second perfect matching in an Euler graph

**1 2　3 4　5 6**



# Finding a second perfect matching in an Euler graph

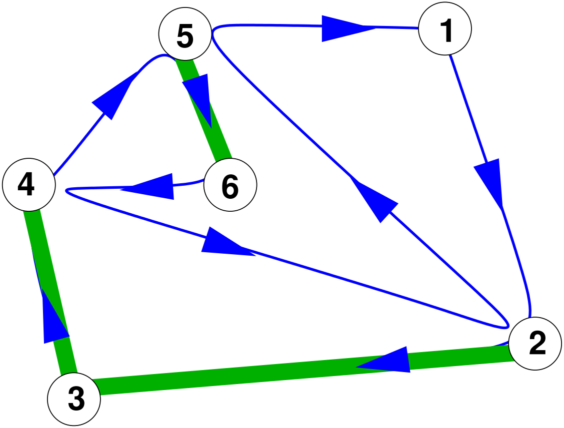
$\boxed{1}\ 2 \quad 3\ 4 \quad 5\ 6$



# Finding a second perfect matching in an Euler graph

1 2   3 4   5 6

2 3   3 4   5 6



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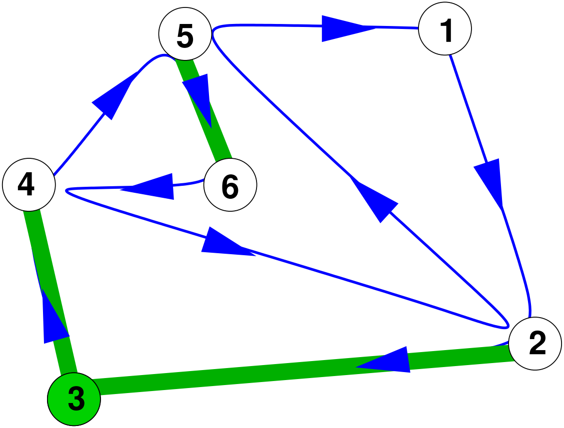

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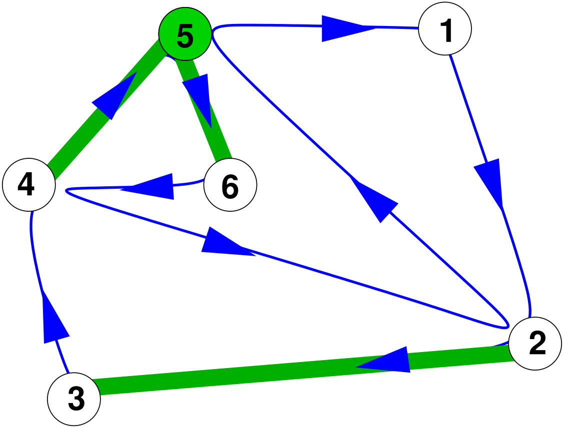

$\boxed{1}\,2 \quad 3\;4 \quad 5\;6$

$2\;\underline{3} \quad \boxed{3}\,4 \quad 5\;6$

$2\;3 \quad 4\;\underline{5} \quad \boxed{5}\,6$

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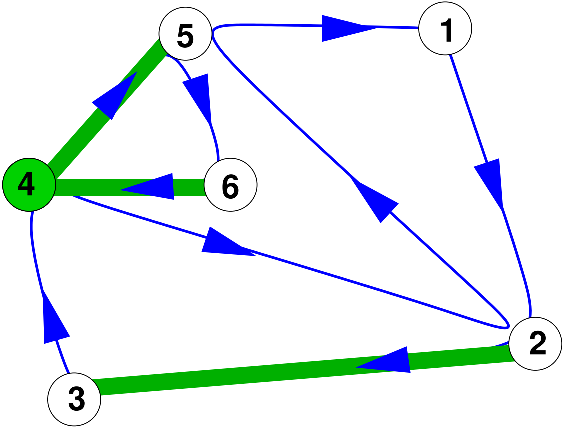

| | | |
|---|---|---|
| $\boxed{1}$ 2 | 3 4 | 5 6 |
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| 2 3 | 4 $\underline{5}$ | $\boxed{5}$ 6 |
| 2 3 | $\boxed{4}$ 5 | 6 $\underline{4}$ |

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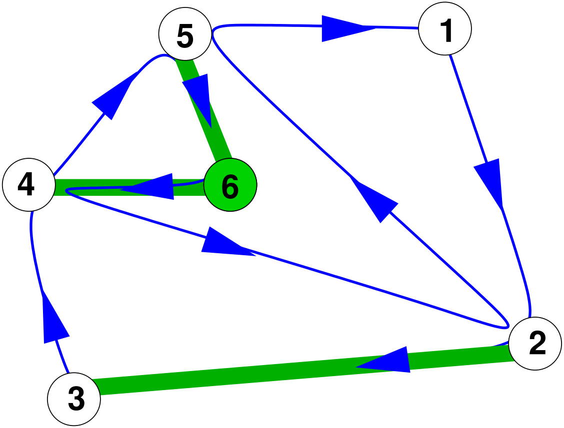

| | | |
|---|---|---|
| $\boxed{1}$ 2 | 3 4 | 5 6 |
| 2 $\underline{3}$ | $\boxed{3}$ 4 | 5 6 |
| 2 3 | 4 $\underline{5}$ | $\boxed{5}$ 6 |
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| 2 3 | 5 $\underline{6}$ | $\boxed{6}$ 4 |

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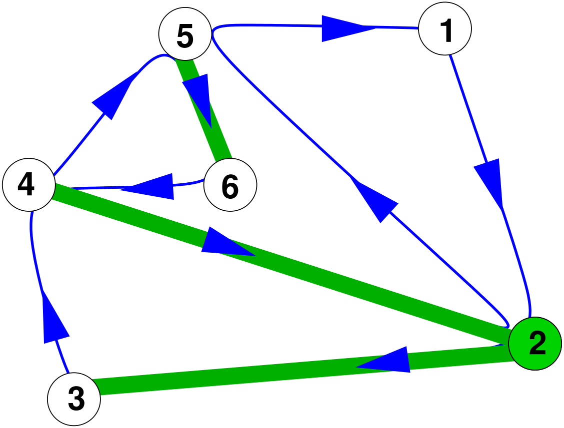

| | | |
|---|---|---|
| $\boxed{1}$ 2 | 3 4 | 5 6 |
| 2 $\underline{3}$ | $\boxed{3}$ 4 | 5 6 |
| 2 3 | 4 $\underline{5}$ | $\boxed{5}$ 6 |
| 2 3 | $\boxed{4}$ 5 | 6 $\underline{4}$ |
| 2 3 | 5 $\underline{6}$ | $\boxed{6}$ 4 |
| $\boxed{2}$ 3 | 5 6 | 4 $\underline{2}$ |

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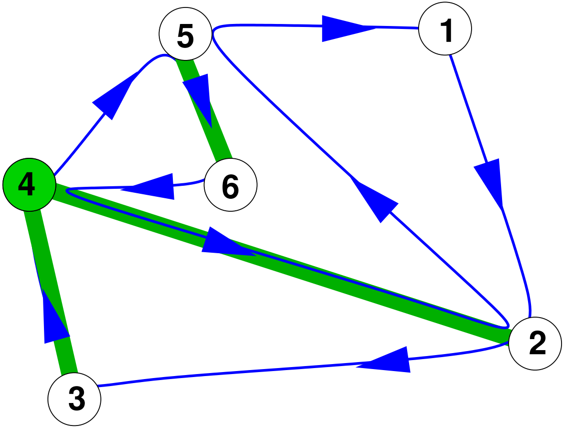

| | | |
|---|---|---|
| $\boxed{1}$ 2 | 3 4 | 5 6 |
| 2 $\underline{3}$ | $\boxed{3}$ 4 | 5 6 |
| 2 3 | 4 $\underline{5}$ | $\boxed{5}$ 6 |
| 2 3 | $\boxed{4}$ 5 | 6 $\underline{4}$ |
| 2 3 | 5 $\underline{6}$ | $\boxed{6}$ 4 |
| $\boxed{2}$ 3 | 5 6 | 4 $\underline{2}$ |
| $\underline{3}$ 4 | 5 6 | $\boxed{4}$ 2 |

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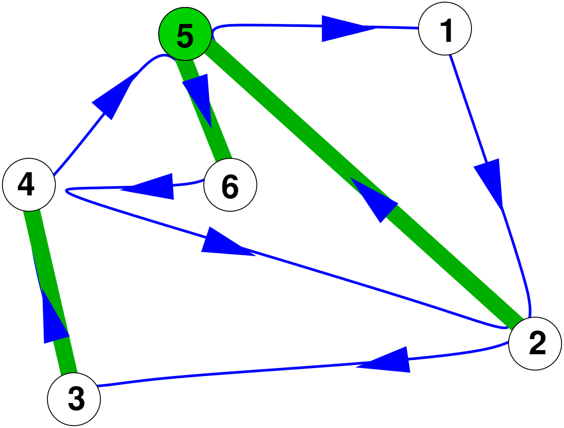

| | | |
|---|---|---|
| $\boxed{1}$ 2 | 3 4 | 5 6 |
| 2 $\underline{3}$ | $\boxed{3}$ 4 | 5 6 |
| 2 3 | 4 $\underline{5}$ | $\boxed{5}$ 6 |
| 2 3 | $\boxed{4}$ 5 | 6 $\underline{4}$ |
| 2 3 | 5 $\underline{6}$ | $\boxed{6}$ 4 |
| $\boxed{2}$ 3 | 5 6 | 4 $\underline{2}$ |
| $\underline{3}$ 4 | 5 6 | $\boxed{4}$ 2 |
| 3 4 | $\boxed{5}$ 6 | 2 $\underline{5}$ |

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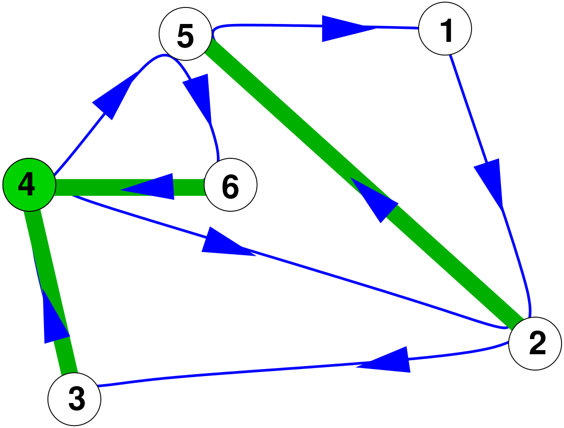

| | | |
|---|---|---|
| $\boxed{1}\ 2$ | $3\ 4$ | $5\ 6$ |
| $2\ \underline{3}$ | $\boxed{3}\ 4$ | $5\ 6$ |
| $2\ 3$ | $4\ \underline{5}$ | $\boxed{5}\ 6$ |
| $2\ 3$ | $\boxed{4}\ 5$ | $6\ \underline{4}$ |
| $2\ 3$ | $5\ \underline{6}$ | $\boxed{6}\ 4$ |
| $\boxed{2}\ 3$ | $5\ 6$ | $4\ \underline{2}$ |
| $\underline{3}\ 4$ | $5\ 6$ | $\boxed{4}\ 2$ |
| $3\ 4$ | $\boxed{5}\ 6$ | $2\ \underline{5}$ |
| $3\ \boxed{4}$ | $6\ \underline{4}$ | $2\ 5$ |

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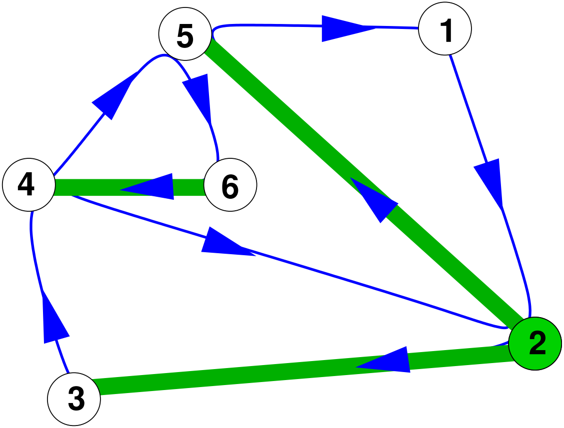

| | | |
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| $\boxed{1}\ 2$ | $3\ 4$ | $5\ 6$ |
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| $\boxed{2}\ 3$ | $5\ 6$ | $4\ \underline{2}$ |
| $\underline{3}\ 4$ | $5\ 6$ | $\boxed{4}\ 2$ |
| $3\ 4$ | $\boxed{5}\ 6$ | $2\ \underline{5}$ |
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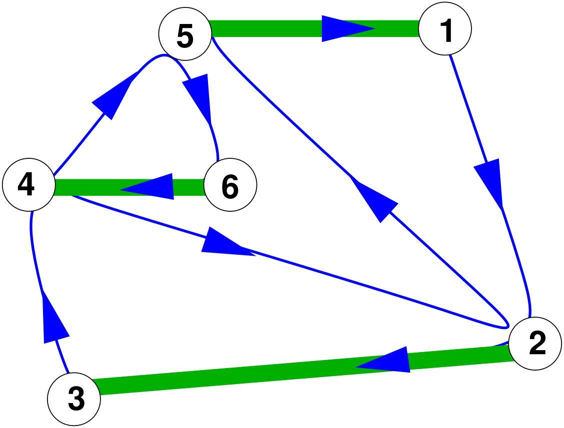

| | | |
|---|---|---|
| $\boxed{1}\ 2$ | $3\ 4$ | $5\ 6$ |
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| $2\ 3$ | $\boxed{4}\ 5$ | $6\ \underline{4}$ |
| $2\ 3$ | $5\ \underline{6}$ | $\boxed{6}\ 4$ |
| $\boxed{2}\ 3$ | $5\ 6$ | $4\ \underline{2}$ |
| $\underline{3}\ 4$ | $5\ 6$ | $\boxed{4}\ 2$ |
| $3\ 4$ | $\boxed{5}\ 6$ | $2\ \underline{5}$ |
| $3\ \boxed{4}$ | $6\ \underline{4}$ | $2\ 5$ |
| $\underline{2}\ 3$ | $6\ 4$ | $\boxed{2}\ 5$ |
| $2\ 3$ | $6\ 4$ | $5\ \underline{1}$ |

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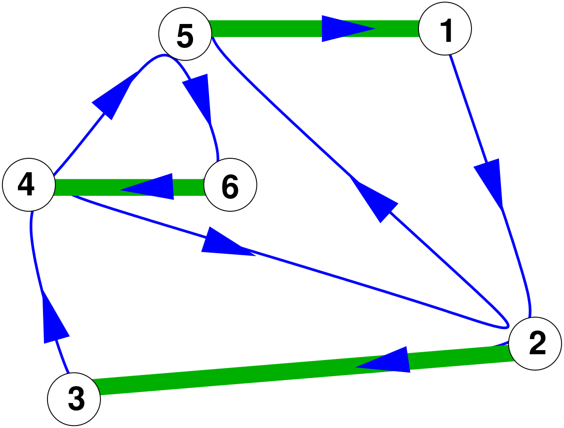

| | | | |
|---|---|---|---|
| ⊕ | $\boxed{1}\,2$ | $3\,4$ | $5\,6$ |
| | $2\,\underline{3}$ | $\boxed{3}\,4$ | $5\,6$ |
| | $2\,3$ | $4\,\underline{5}$ | $\boxed{5}\,6$ |
| | $2\,3$ | $\boxed{4}\,5$ | $6\,\underline{4}$ |
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| | $\boxed{2}\,3$ | $5\,6$ | $4\,\underline{2}$ |
| | $\underline{3}\,4$ | $5\,6$ | $\boxed{4}\,2$ |
| | $3\,4$ | $\boxed{5}\,6$ | $2\,\underline{5}$ |
| | $3\,\boxed{4}$ | $6\,\underline{4}$ | $2\,5$ |
| | $\underline{2}\,3$ | $6\,4$ | $\boxed{2}\,5$ |
| ⊖ | $2\,3$ | $6\,4$ | $5\,\underline{1}$ |

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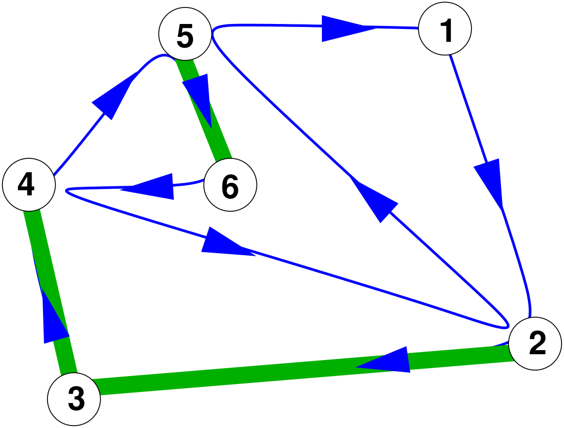

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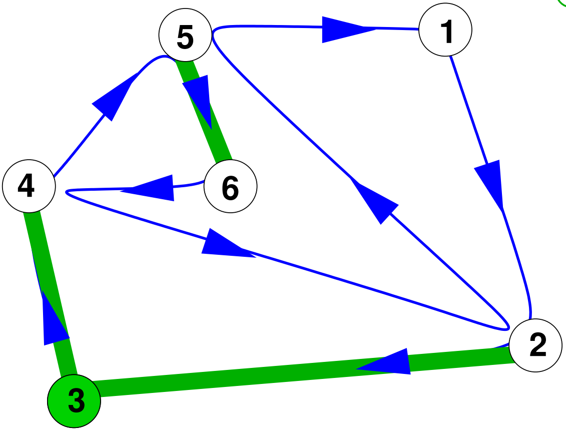
⊕ $\boxed{1}\,2 \quad 3\,4 \quad 5\,6$

⊕ ⊖ $2\,\underline{3} \quad \boxed{1}\,4 \quad 5\,6$



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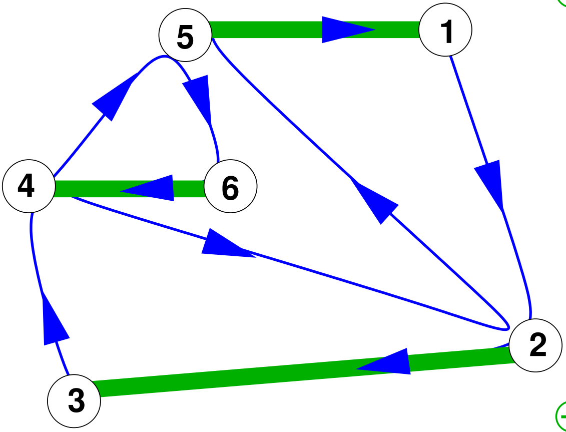
⊕ ⊖ 2 $\underline{3}$ $\boxed{1}$ 4 5 6



⊕ $\underline{2}$ 3 6 4 $\boxed{2}$ 5

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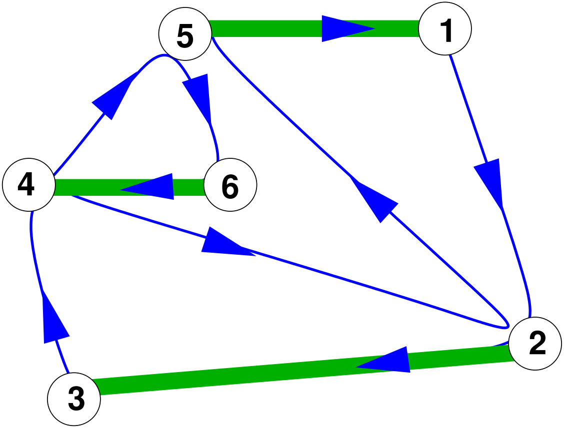

| | | | |
|---|---|---|---|
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| | $\boxed{2}$ 3 | 5 6 | 4 $\underline{2}$ |
| | $\underline{3}$ 4 | 5 6 | $\boxed{4}$ 2 |
| | 3 4 | $\boxed{5}$ 6 | 2 $\underline{5}$ |
| | 3 $\boxed{4}$ | 6 $\underline{4}$ | 2 5 |
| | $\underline{2}$ 3 | 6 4 | $\boxed{2}$ 5 |
| ⊖ | 2 3 | 6 4 | 5 $\underline{1}$ |

# A computational problem

**Input**: Graph $\boldsymbol{G = (V, E)}$ with Eulerian orientation and perfect matching of sign $\oplus$.

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- but may take exponential time in general [Morris 1994]

Note: A second matching can be found in polynomial time [Edmonds 1965], but not with sign $\ominus$.

Related difficult problem: Pfaffian orientations of graphs.

# Finding a second matching of opposite sign

**Theorem** [Végh / von Stengel 2014]

Given a graph $\boldsymbol{G = (V, E)}$ with an Eulerian orientation and a perfect matching of sign $\oplus$, a matching of sign $\ominus$ can be found in time near-linear$^*$ in $|E|$.

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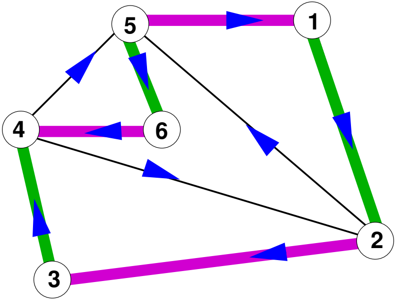
Suffices to find **sign-switching cycle.**

* up to factor given by inverse Ackermann function $\alpha$ .

# Sign-switching cycle (SSC)

Given an oriented graph and a perfect matching ***M***, a **sign-switching cycle** is a cycle ***C*** with every other edge in ***M*** and an **even** number of forward-pointing edges.

$\Rightarrow$ $M \triangle C$ is a matching of opposite sign to $M$.



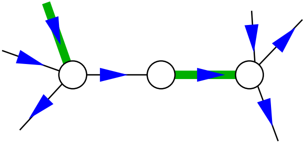
$\oplus$ $\boxed{1}\,2 \quad 3\,4 \quad 5\,6$

$\ominus$ $2\,3 \quad 6\,4 \quad 5\,\underline{1}$

# Finding a SSC in near-linear time

Two **reductions** which preserve Euler and matching property:

**1. contract** node of indegree = outdegree = **1** with its two edges



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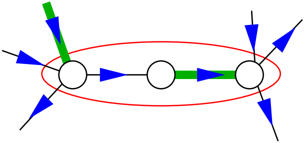

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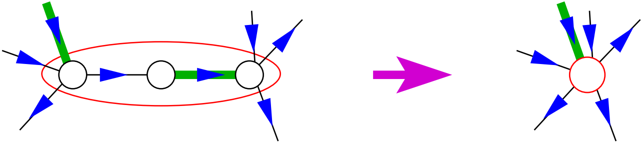

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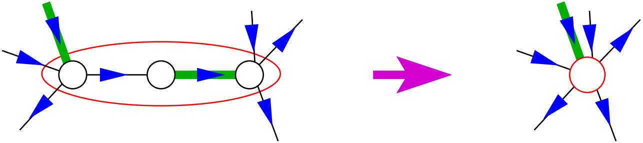
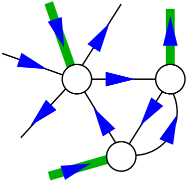


**2. delete** directed cycle of **unmatched** edges



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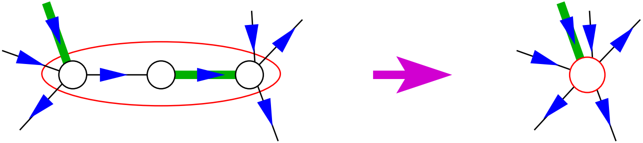

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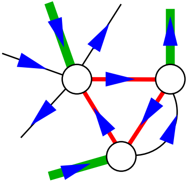

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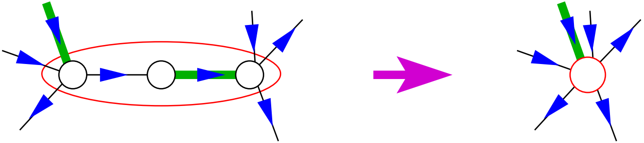

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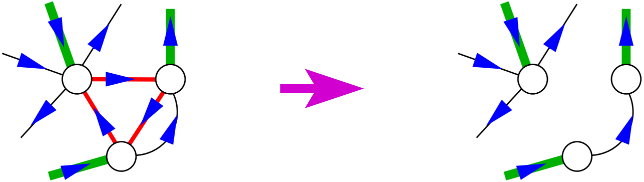

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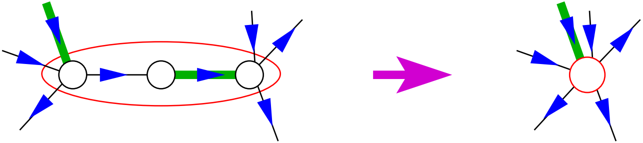

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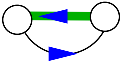
until trivial SSC found



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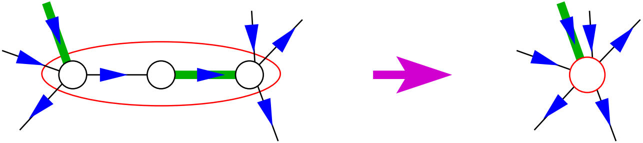

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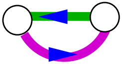

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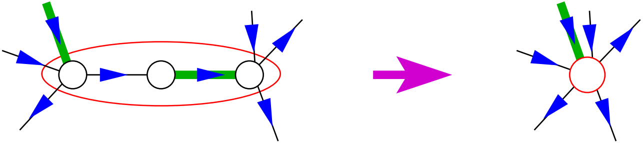
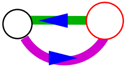


**2. delete** directed cycle of **unmatched** edges

until trivial SSC found, re-insert contracted edge pairs



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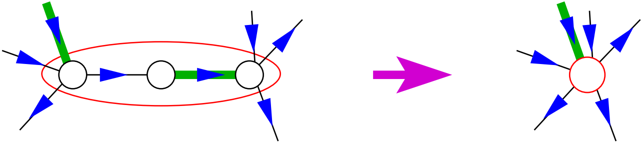

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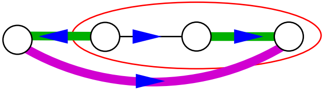

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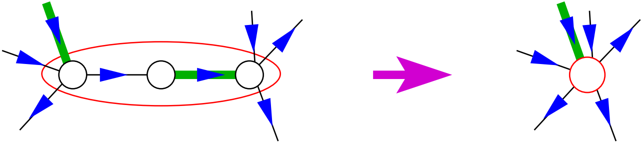
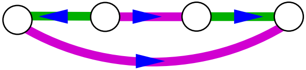


**2. delete** directed cycle of **unmatched** edges

until trivial SSC found, re-insert contracted edge pairs, switch.



# Bimatrix games and signed matchings

- Lemke–Howson for bimatrix games can take exponential time [Savani / von Stengel 2004], like complementary pivoting on labeled polytopes [Morris 1994].

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- Proved using dual **cyclic polytopes**, defined by $n$ inequalities in $d$-space with vertices characterized by **Gale evenness**:

  **Gale string** = bitstring of length $n$ with $d$ bits **1** with **forbidden** odd runs of **1**'s such as **010**, **01110**, **0111110**, . . .

  Examples: **111111000**, **011011110**, **111001101**.

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- Lemke–Howson for bimatrix games can take exponential time [Savani / von Stengel 2004], like complementary pivoting on labeled polytopes [Morris 1994].
- Proved using dual **cyclic polytopes**, defined by ***n*** inequalities in ***d***-space with vertices characterized by **Gale evenness**:

  **Gale string** = bitstring of length ***n*** with ***d*** bits **1** with **forbidden** odd runs of **1**'s such as **010**, **01110**, **0111110**, . . .

  Examples: **111111000**, **011011110**, **111001101**.
- Gale string = vertex, bit **1** = facet (tight inequality).

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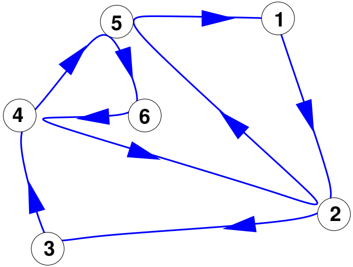

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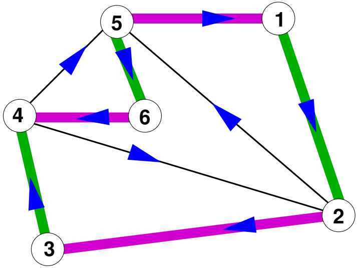

# Oiks and pivoting

**Definition** [Edmonds 2009] $(V, \mathcal{R})$ ***d*-oik** (Euler complex)

$\Leftrightarrow$ $V$ finite set of **nodes**, $\mathcal{R}$ multiset of **rooms** $R$ with $|R| = d$,

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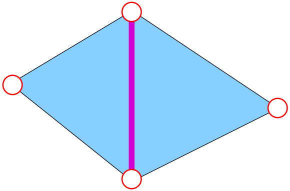

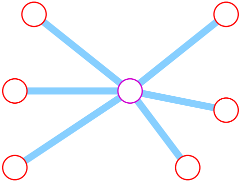

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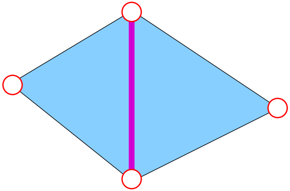
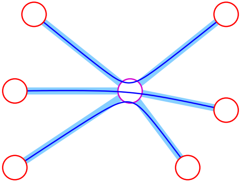




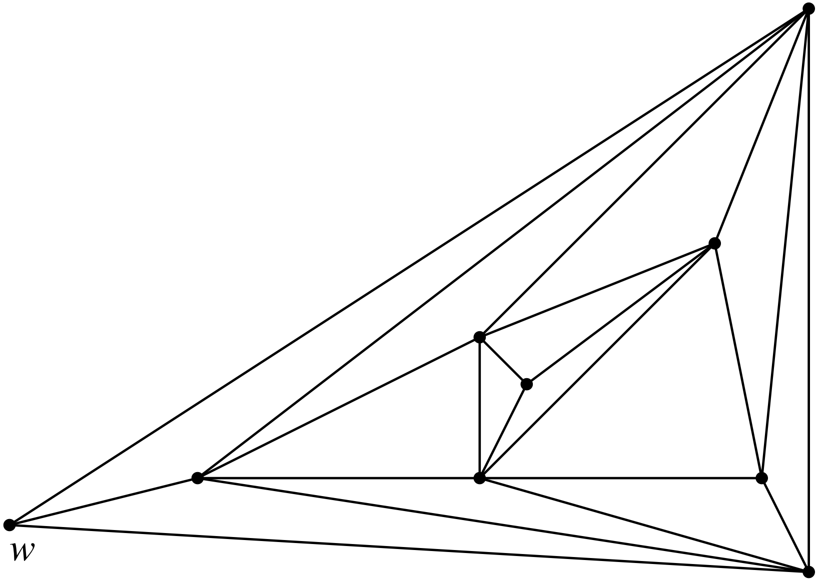
# Room partitions come in pairs

Given an oik $\mathcal{R}$ with node set $V$,
a **room partition** is a partition of $V$ into rooms.

**Theorem** [Edmonds 2009]

The number of room partitions is **even.**

# Room partition for **3**-manifold



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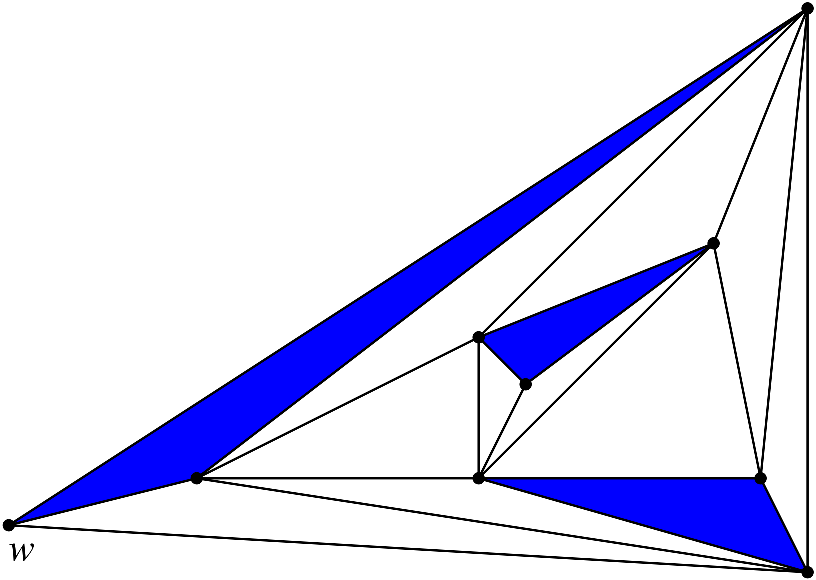
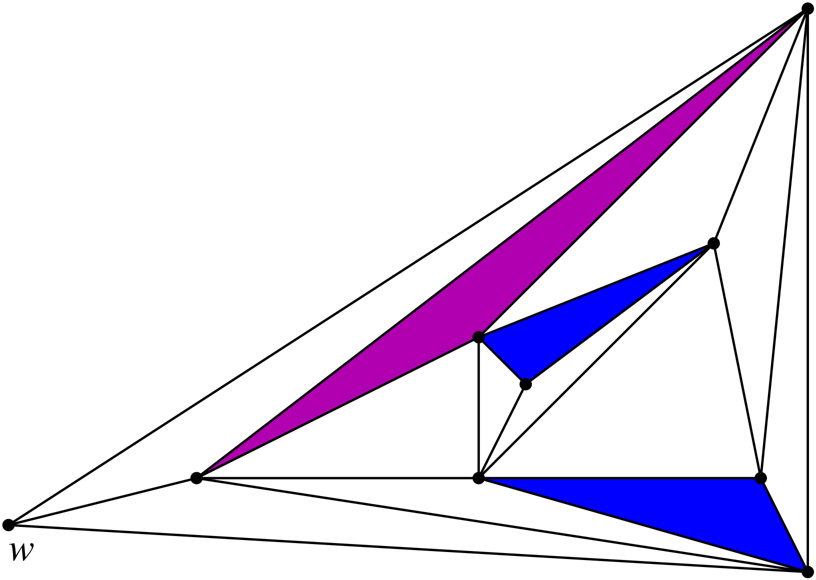


# ***w***-almost room partition



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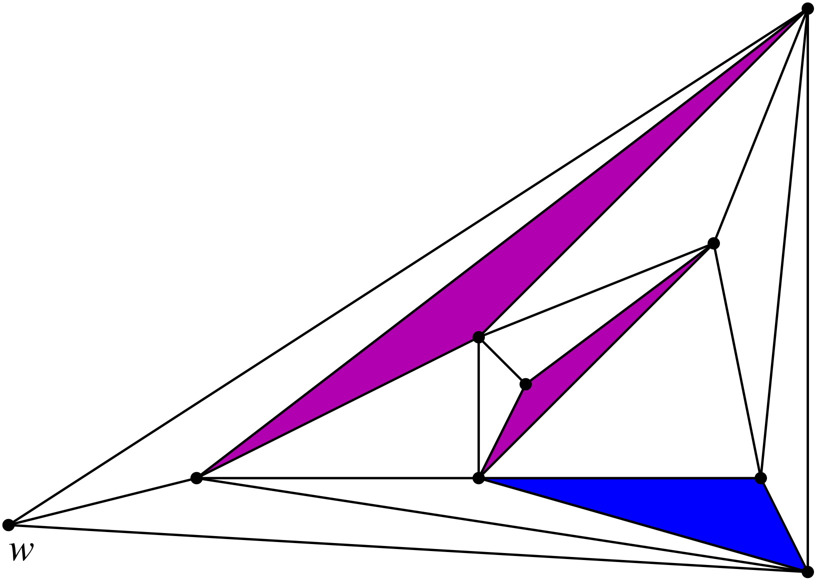

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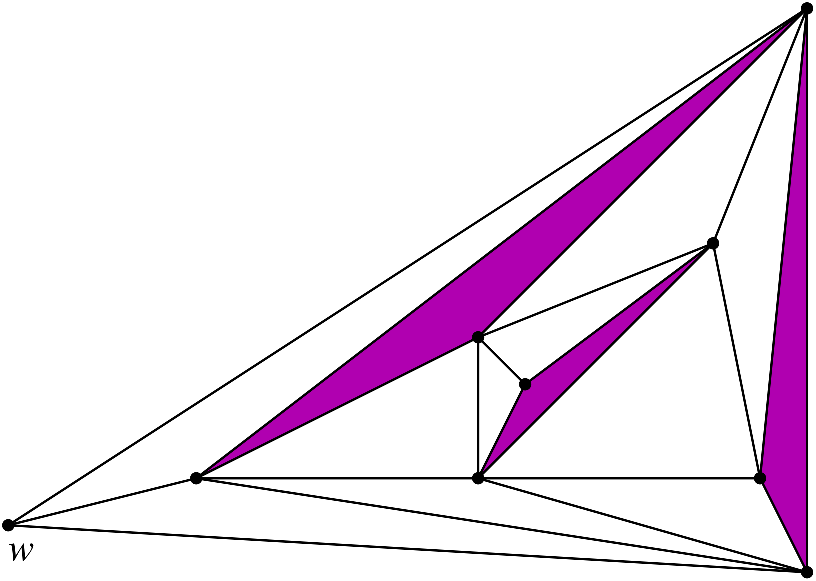

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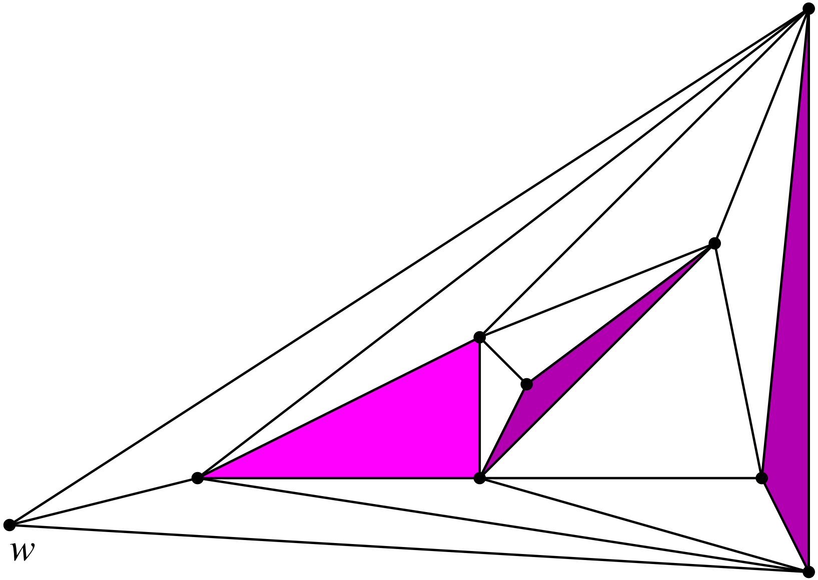

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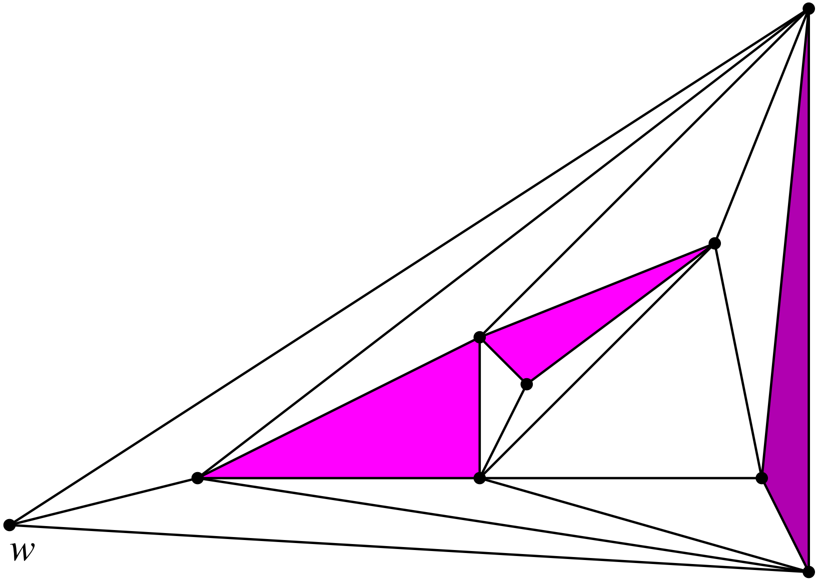

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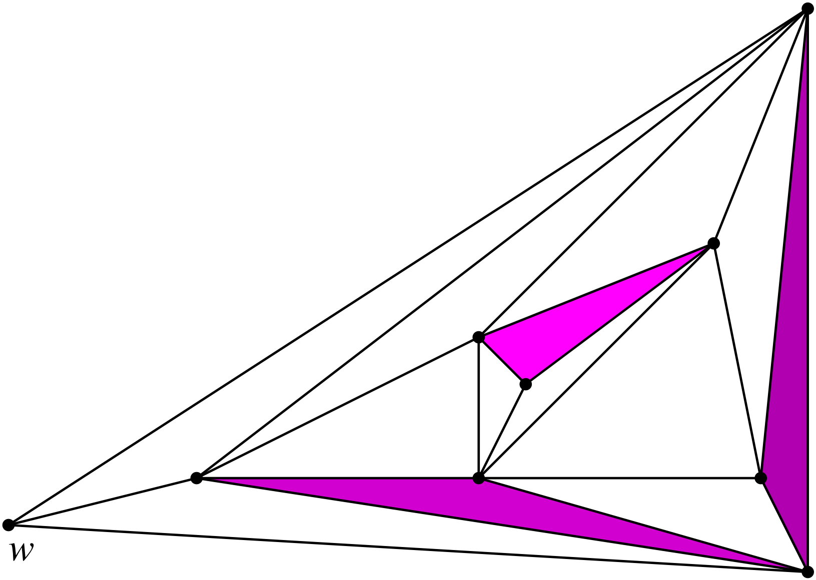

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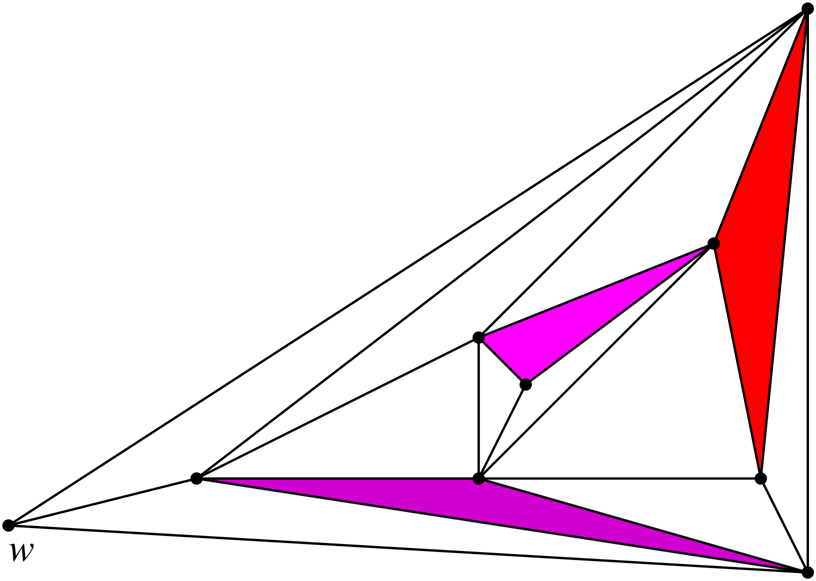

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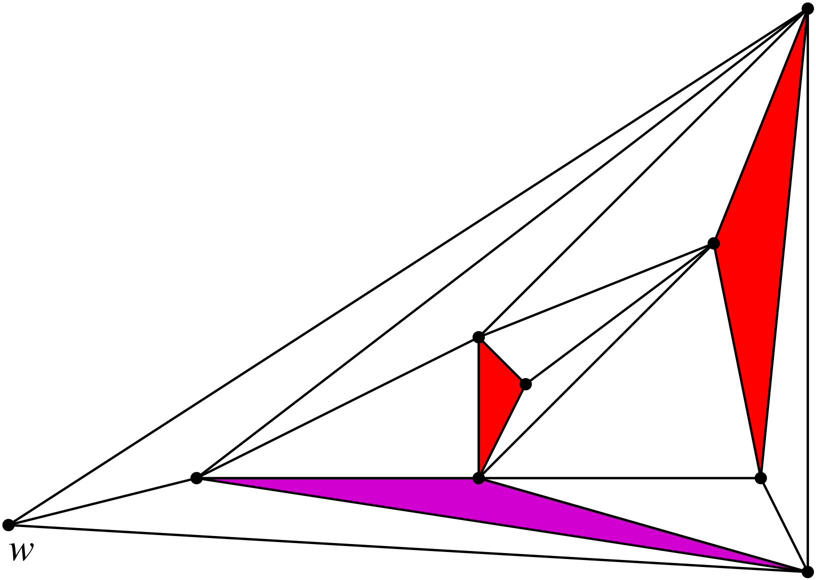
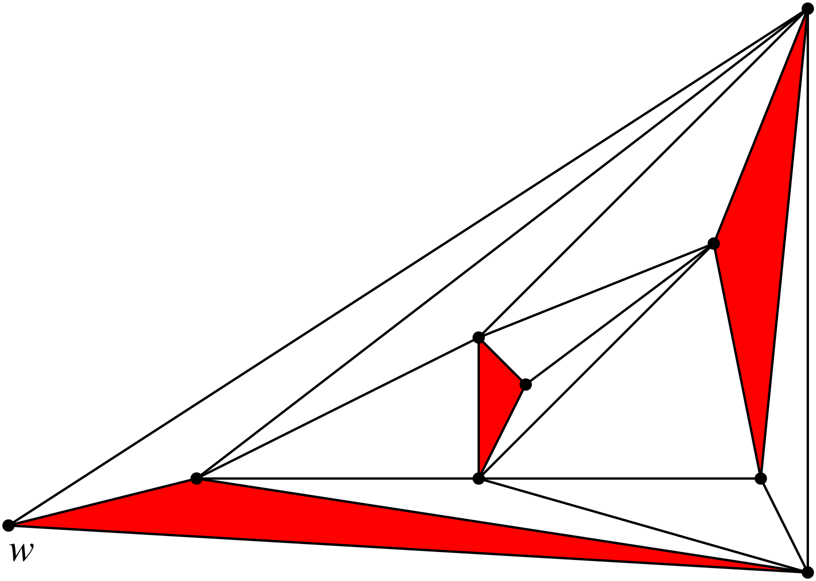
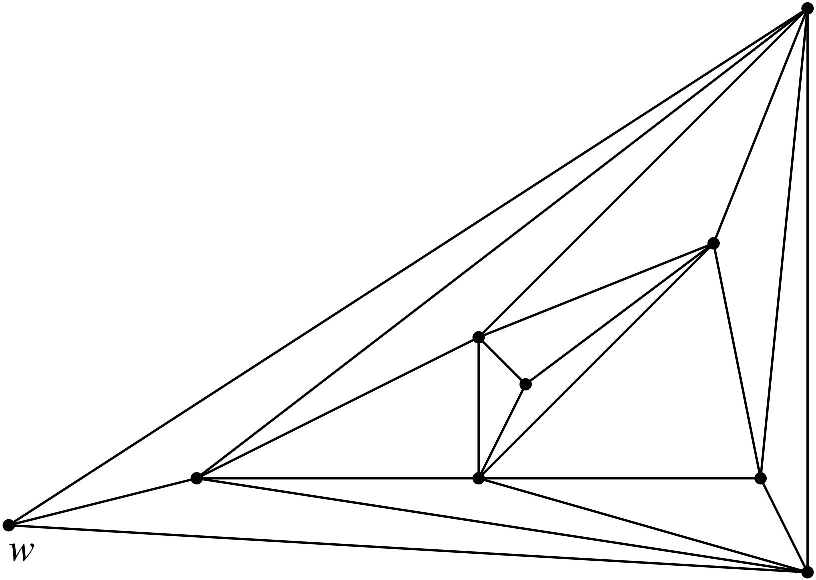


# Found second room partition



# [Edmonds / Sanità 2010]: exponentially long path



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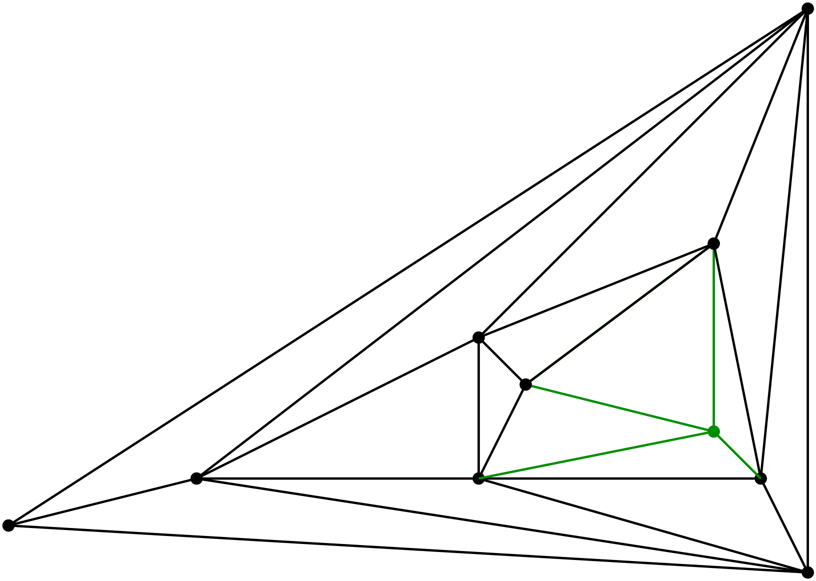

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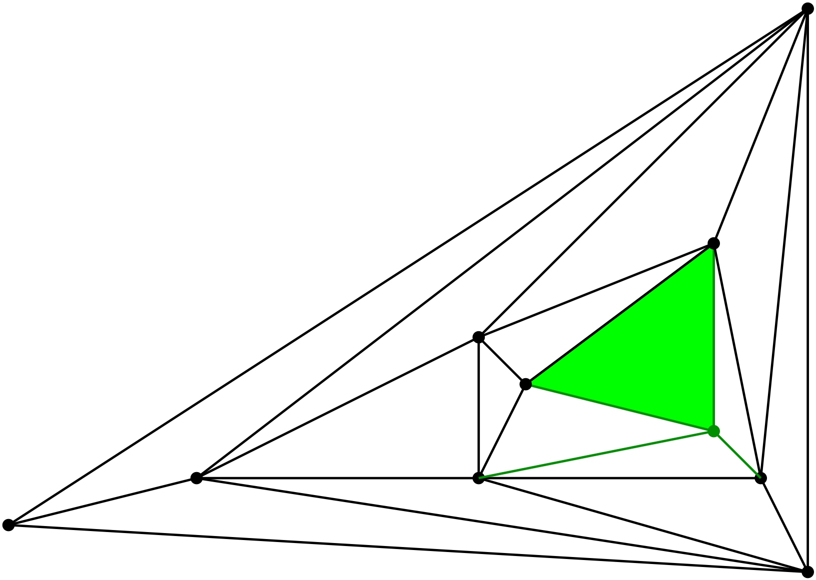

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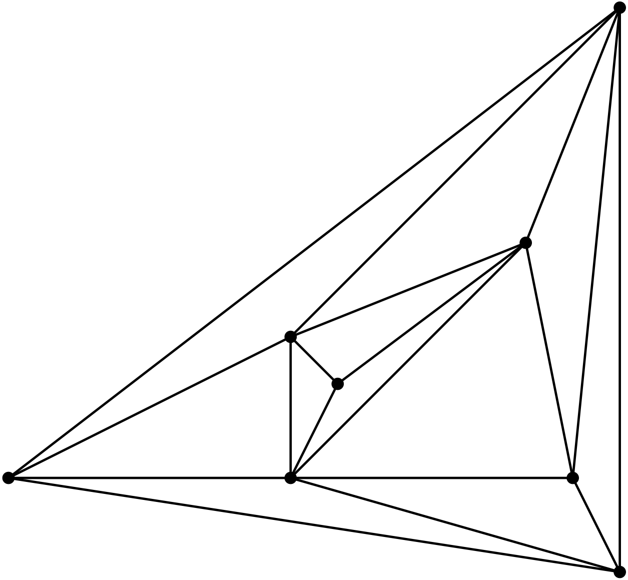

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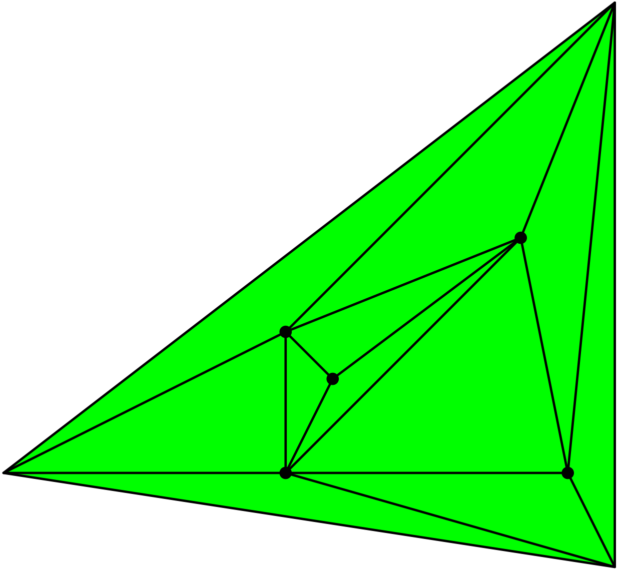

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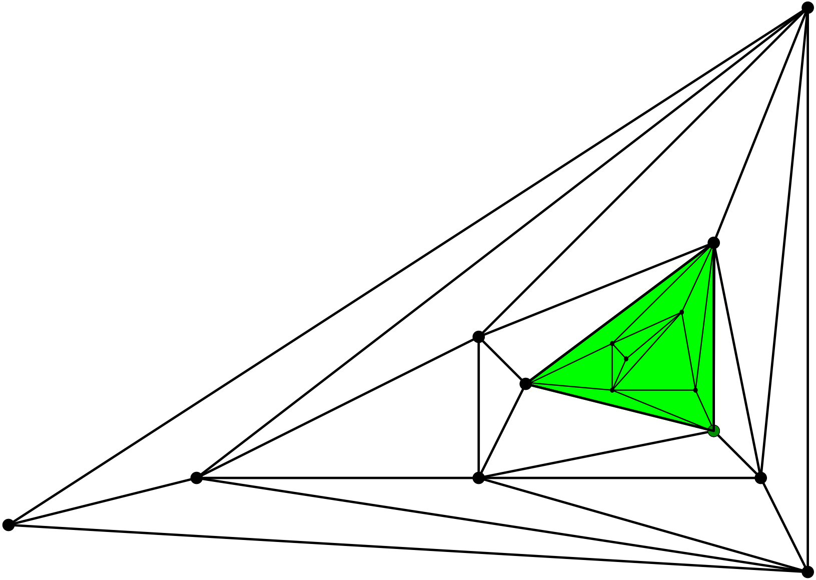
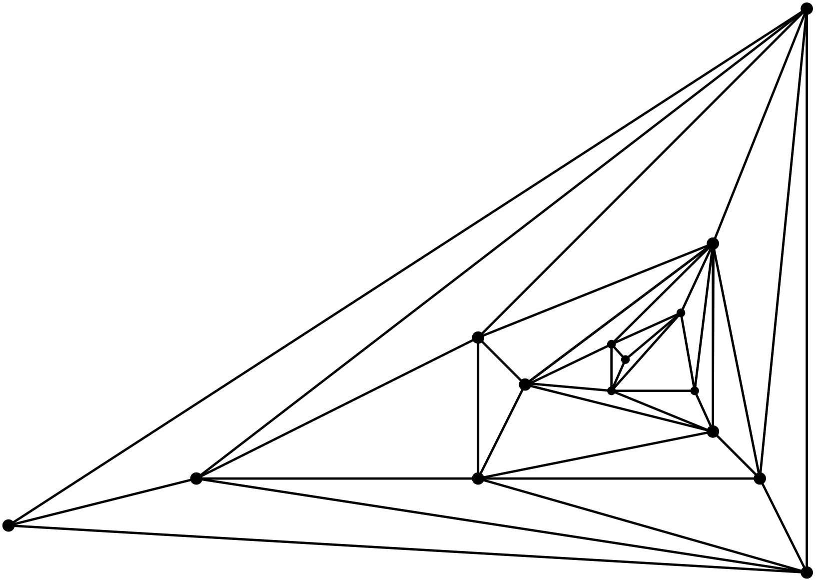
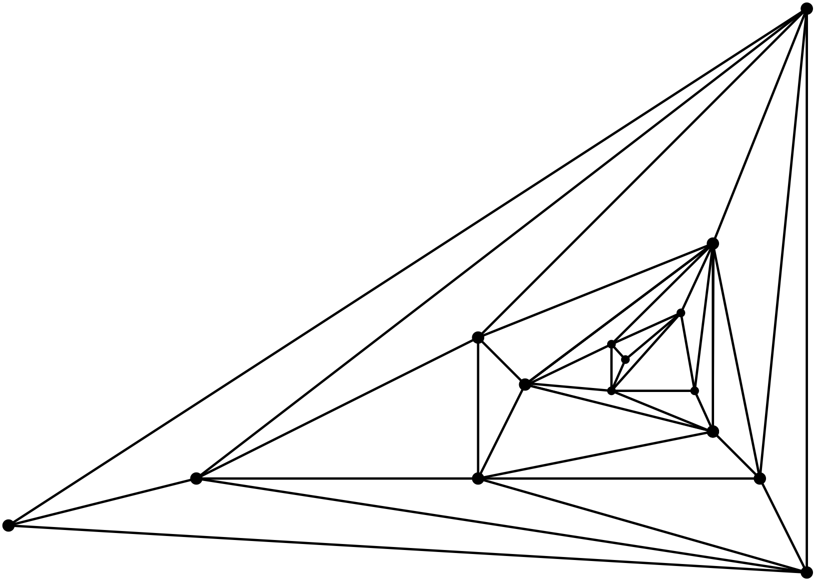


# **6** extra nodes, **12** extra rooms



# Path length more than doubles



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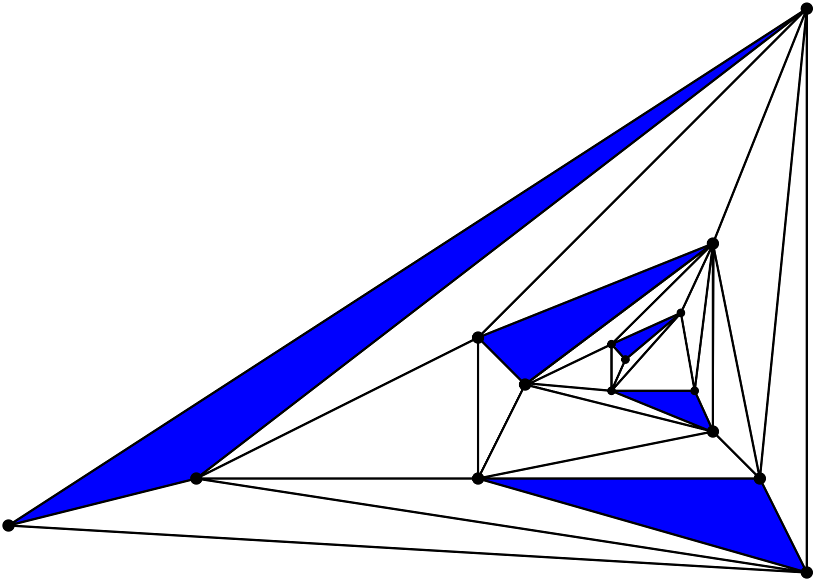

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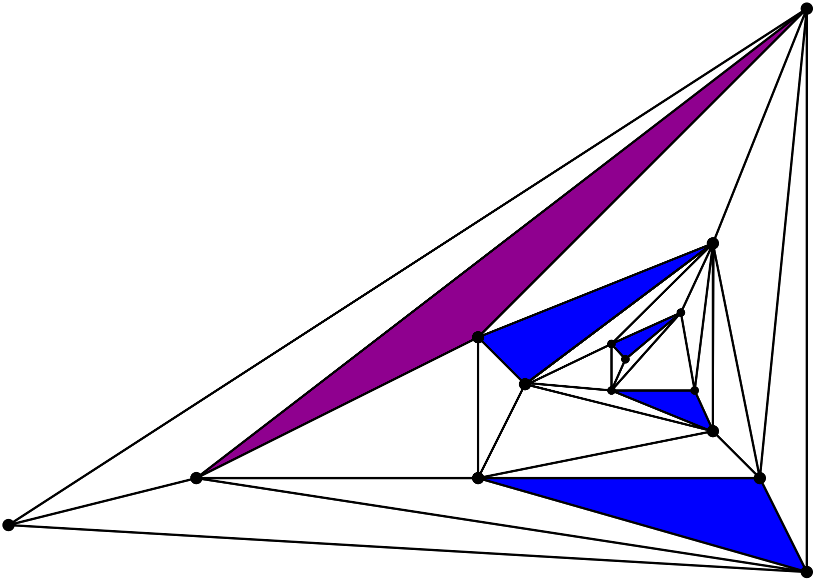

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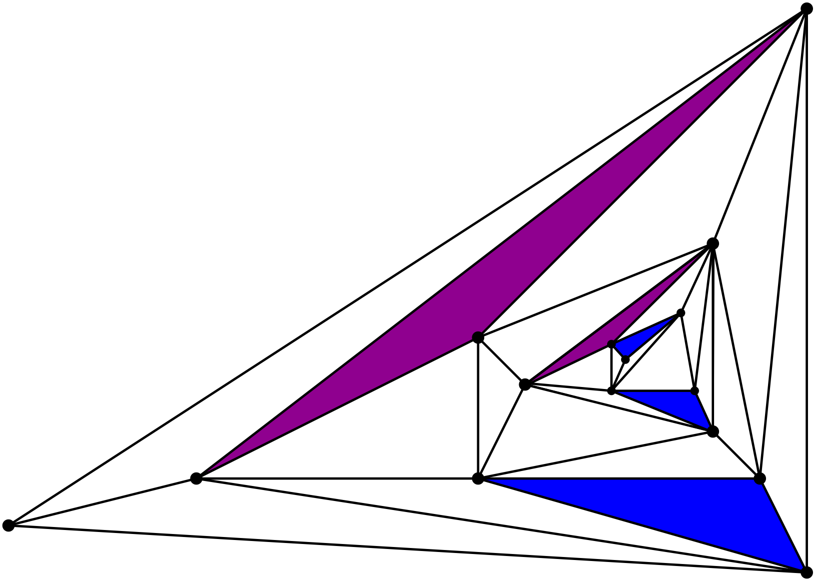

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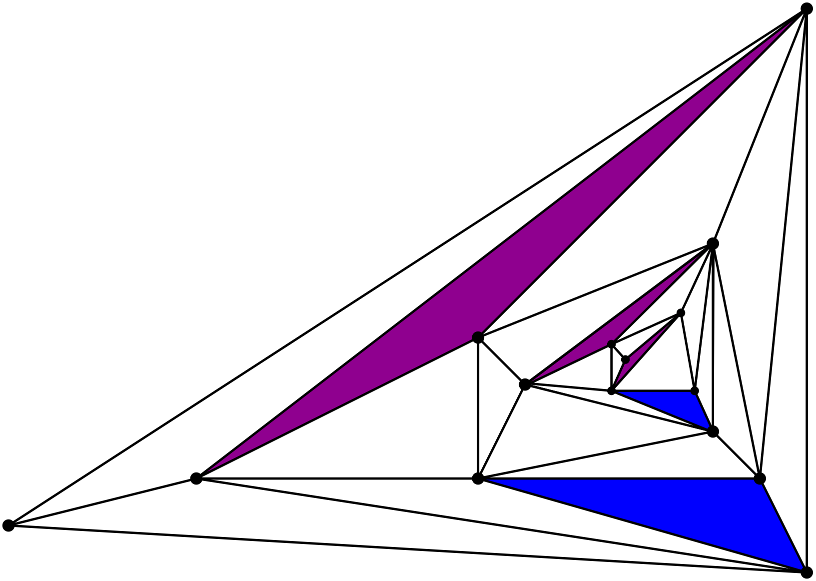

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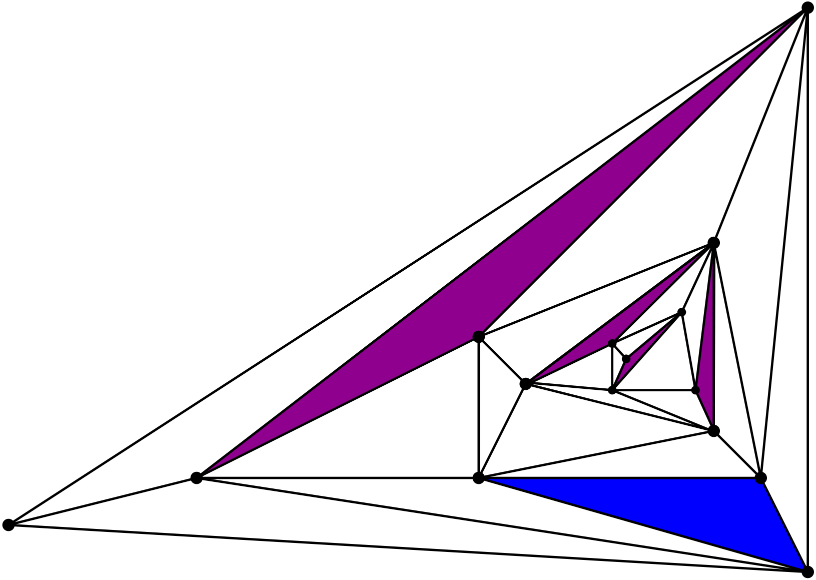

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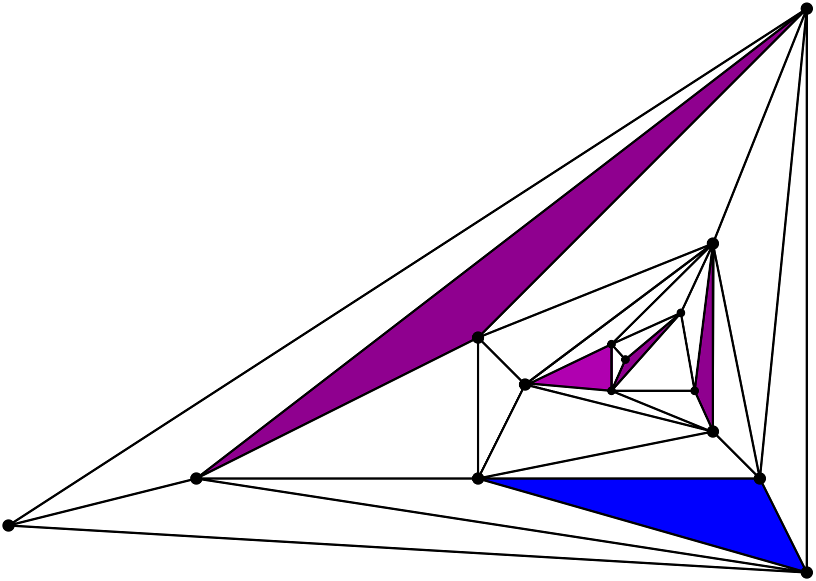

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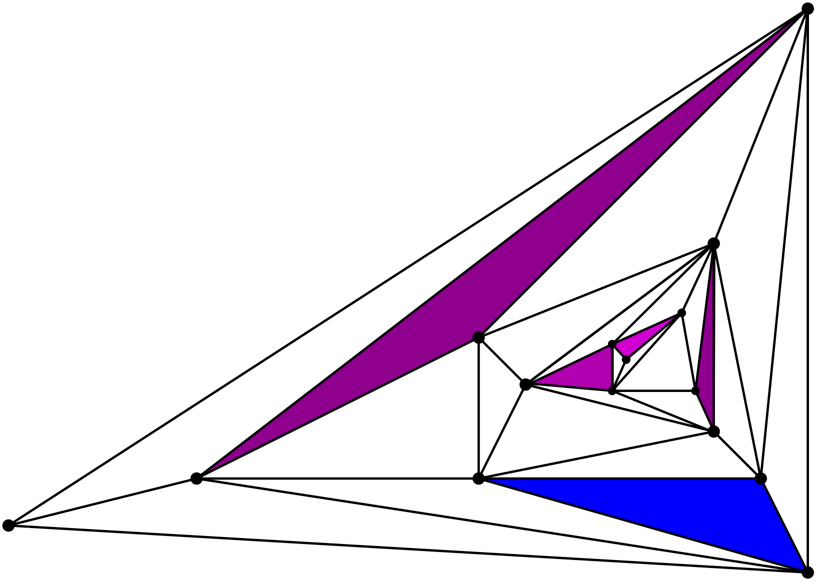

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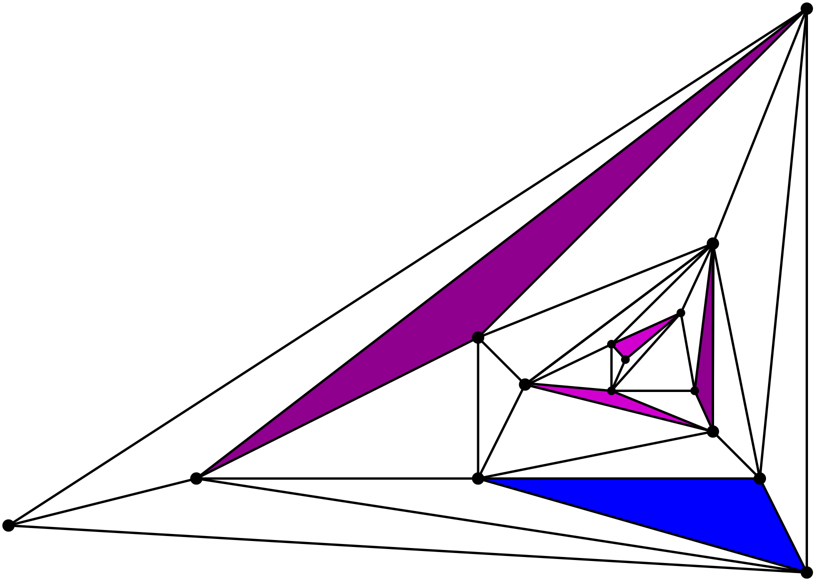

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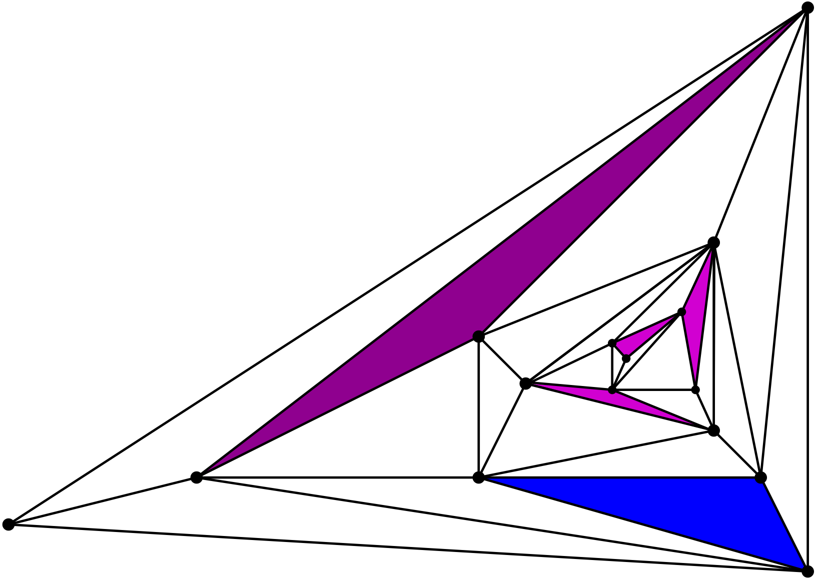

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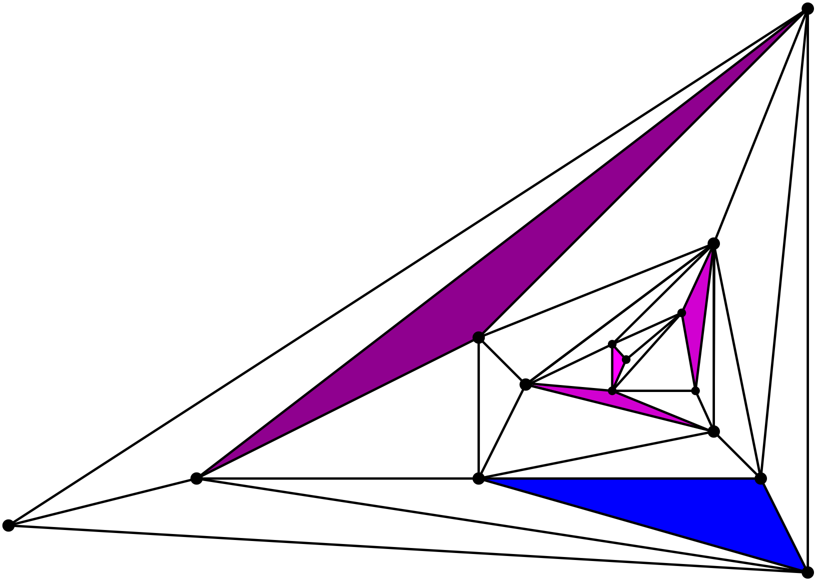

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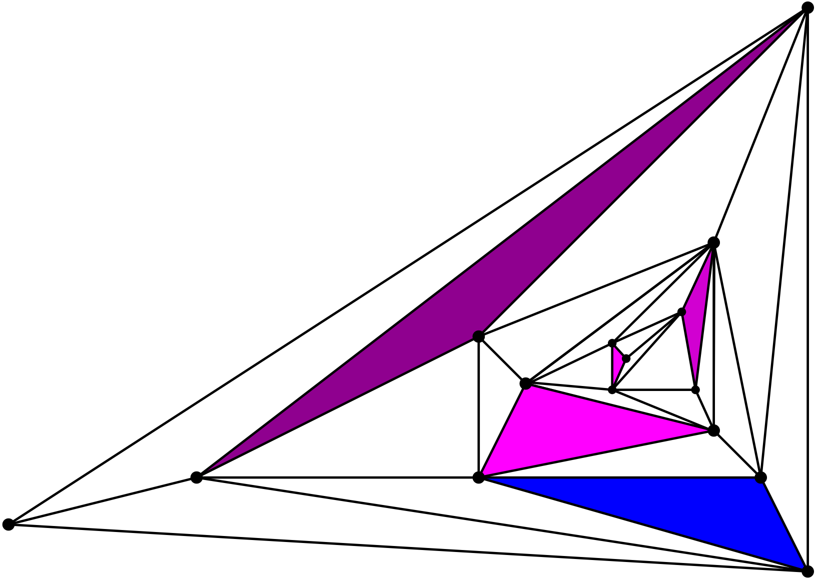

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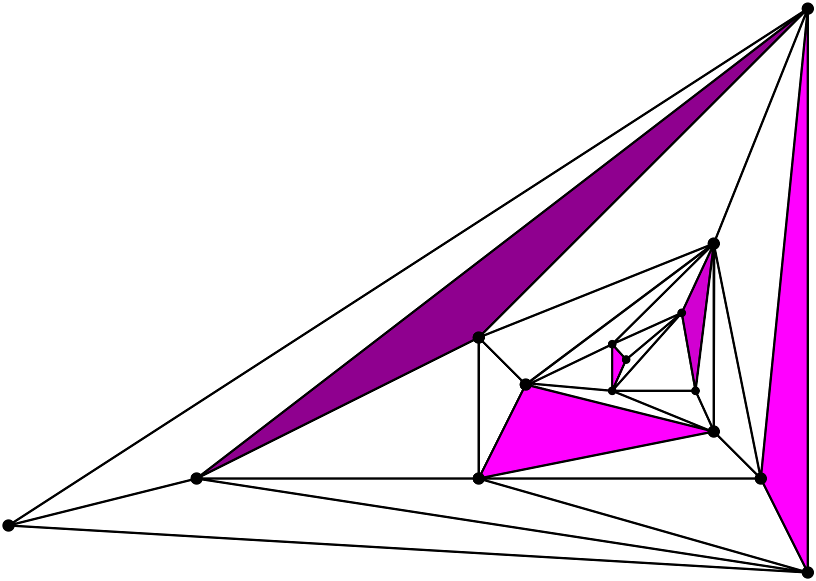

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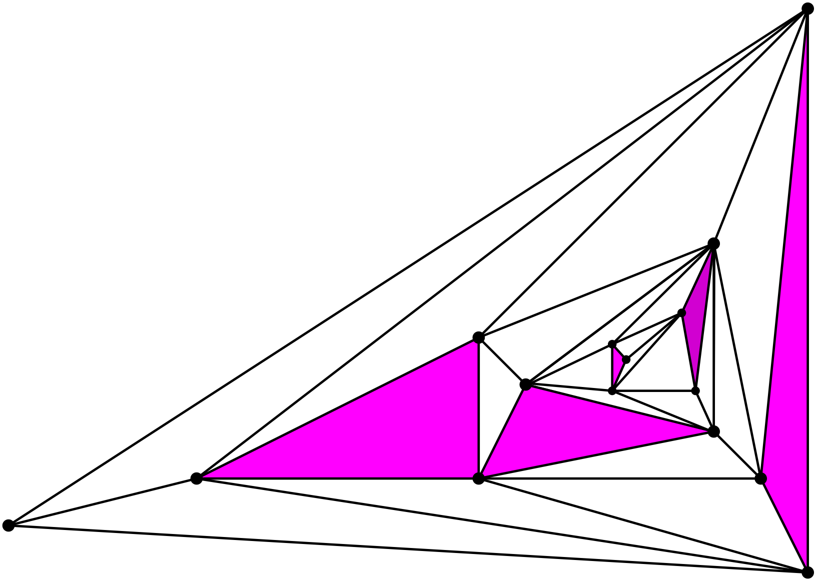

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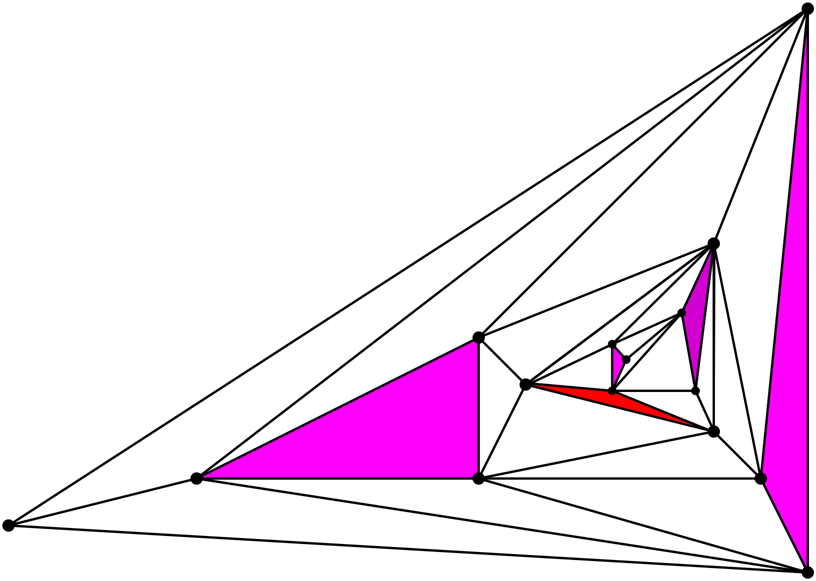

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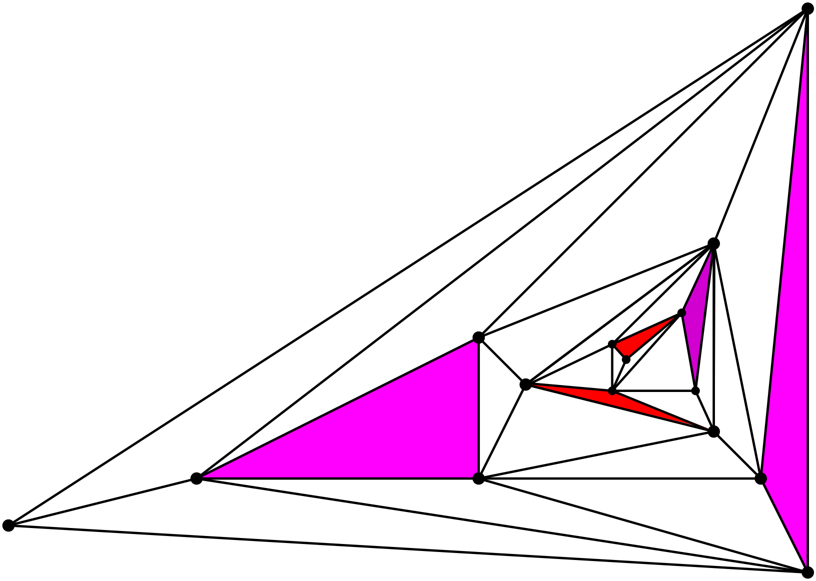

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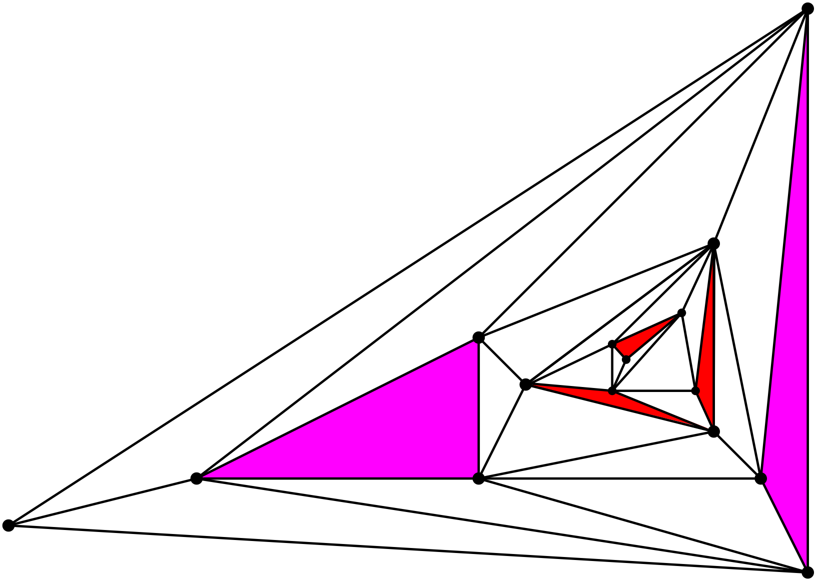

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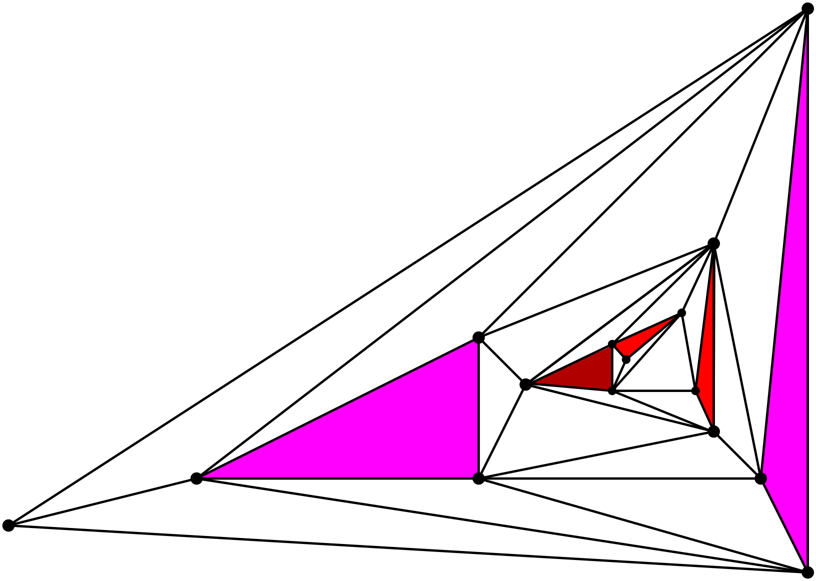

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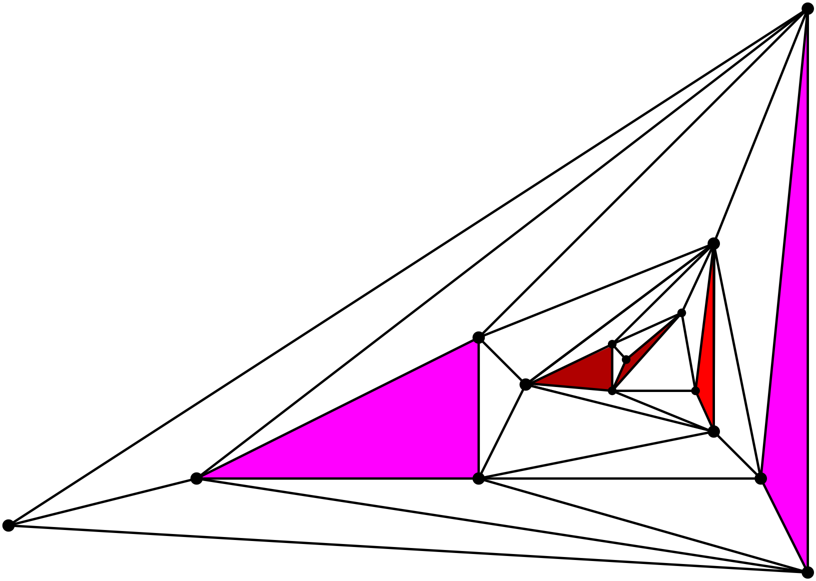

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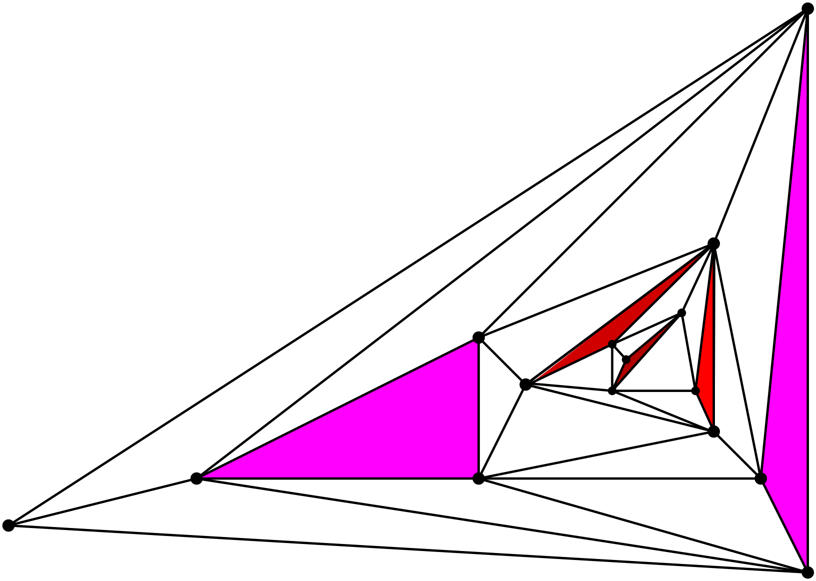

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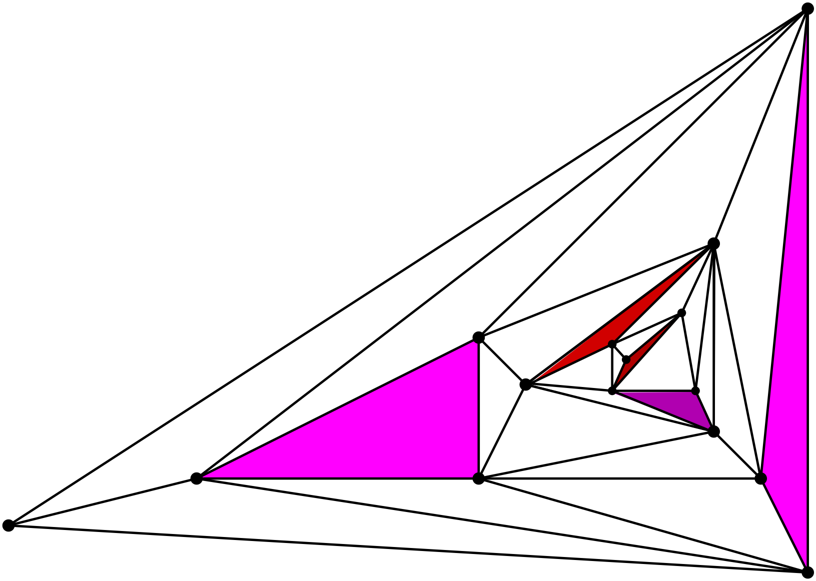

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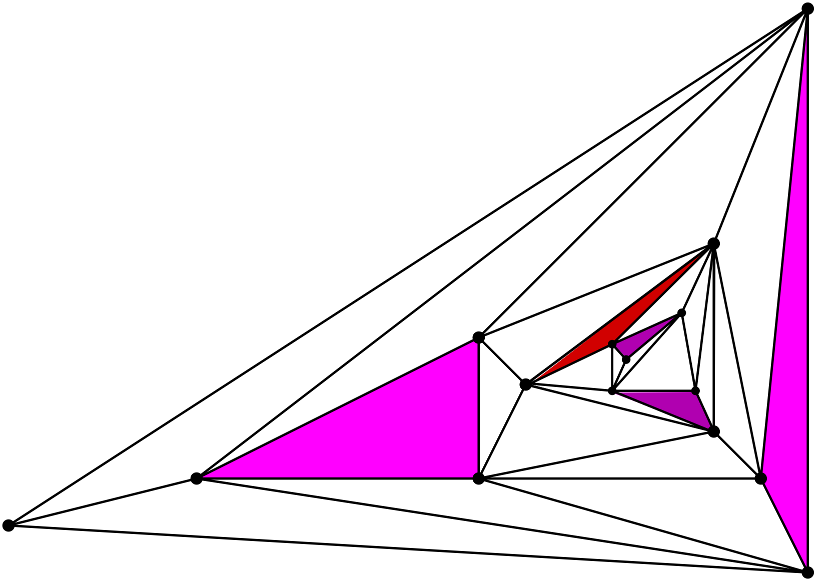

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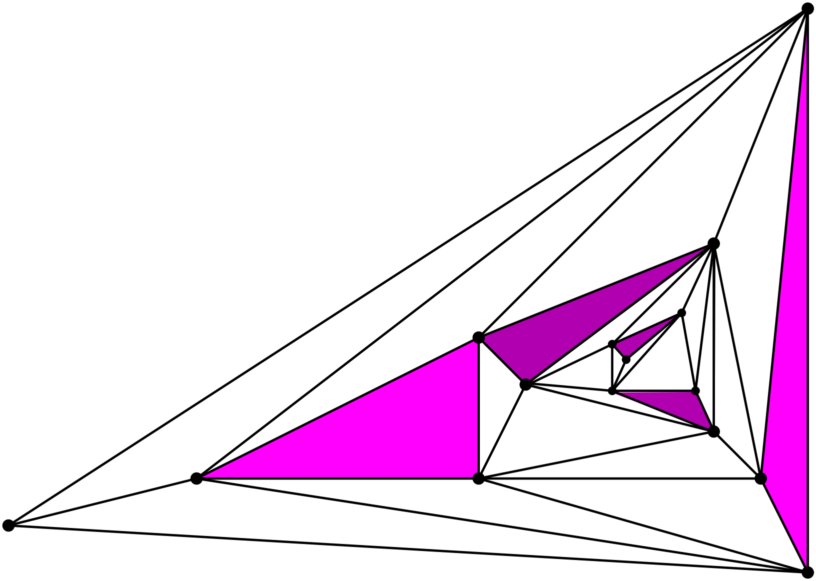

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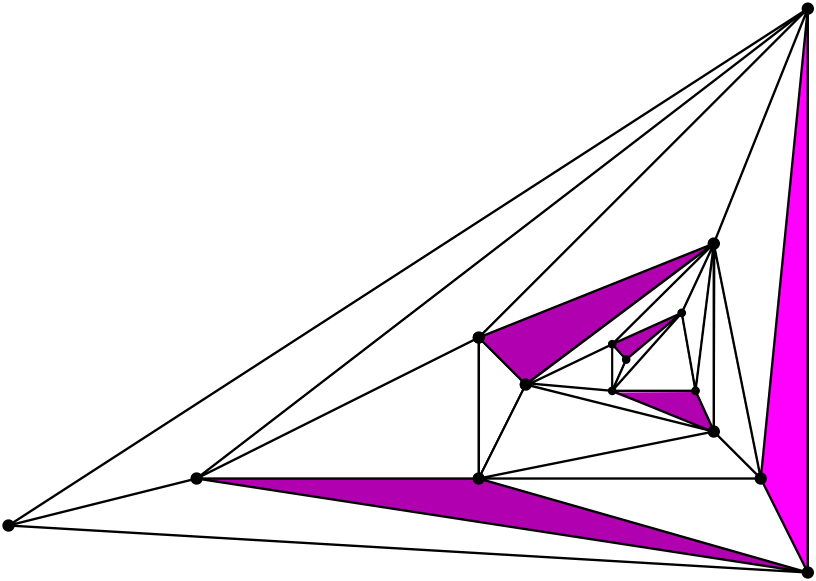

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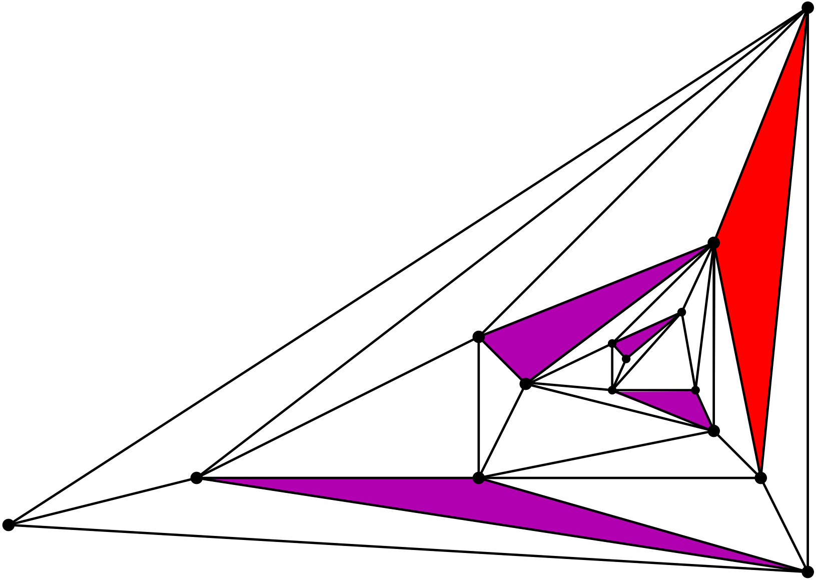

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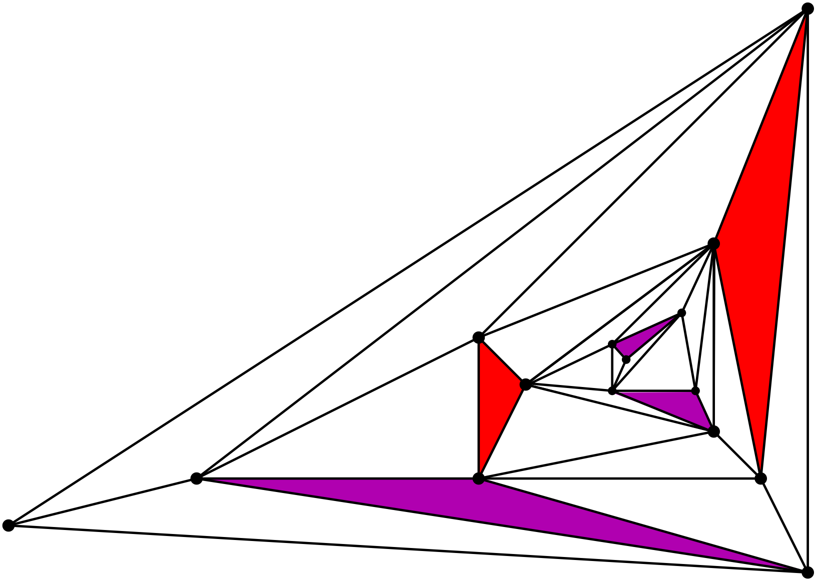
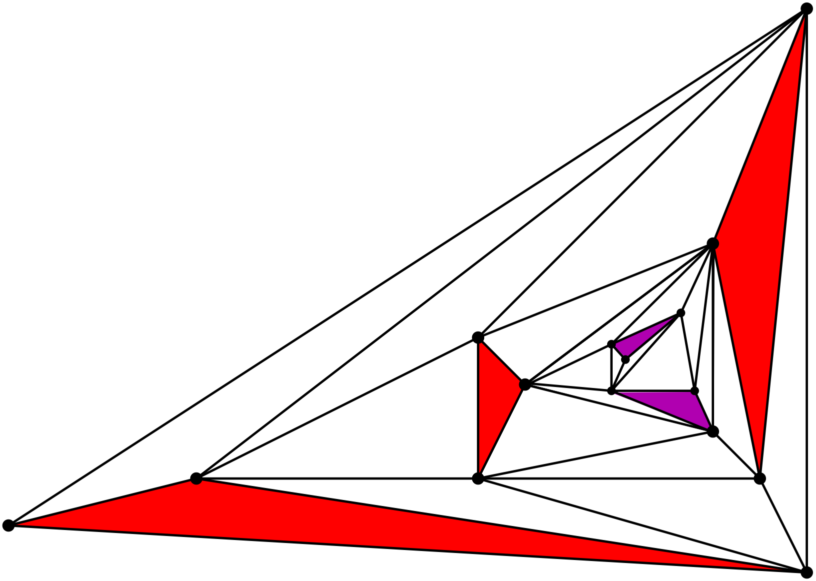


# General construction: exponentially long path



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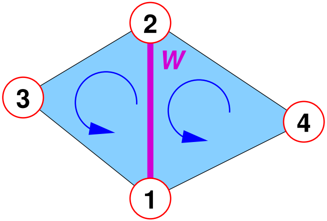
A $d$-manifold is **orientable** if each room has a sign $\oplus$ or $\ominus$ so that any two rooms with a common wall $W$ induce **opposite** orientation on $W$ **($\Leftrightarrow$ pivoting changes sign)**.



**1 2 3** $\oplus$ **1 4 2** $\oplus$

**= 1 2 4** $\ominus$

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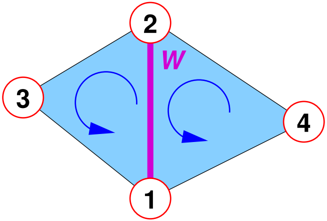
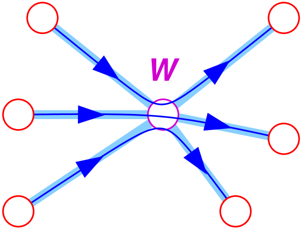


**1 2 3** ⊕ **1 4 2** ⊕

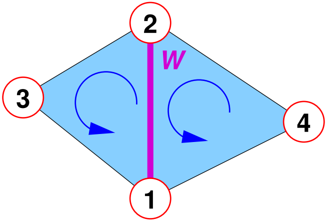
**= 1 2 4** ⊖



# Orienting oiks

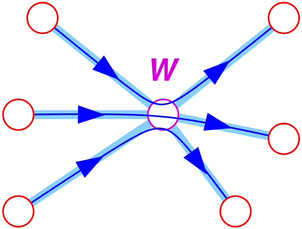
$W = R - \{v\}$ for $v \in R$ is called a **wall** of a room $R$

A $d$-manifold is **orientable** if each room has a sign ⊕ or ⊖ so that any two rooms with a common wall $W$ induce **opposite** orientation on $W$ **(⇔ pivoting changes sign)**.



**1 2 3** ⊕ **1 4 2** ⊕

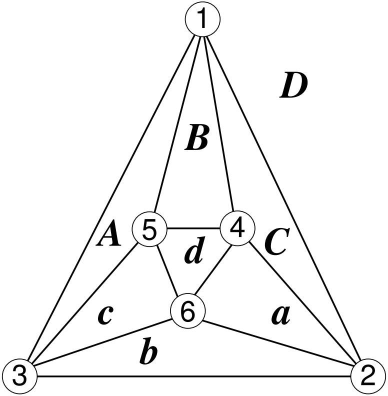
**= 1 2 4** ⊖



A $d$-**oik** is **orientable** if half of the rooms with a common wall $W$ induce sign ⊕ on $W$, the other half sign ⊖ on $W$.

# How to orient room partitions?

Example: orientable manifold



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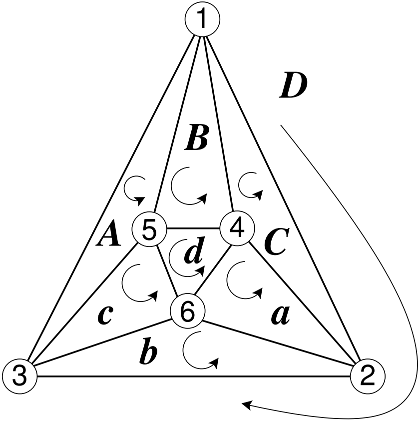
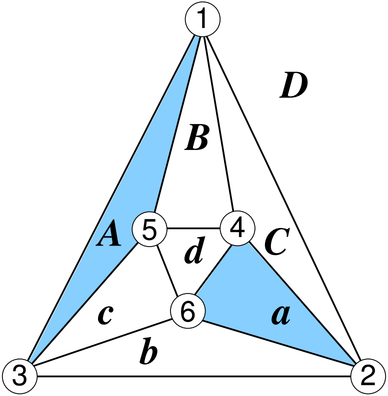
Example: orientable manifold



# How to orient room partitions?

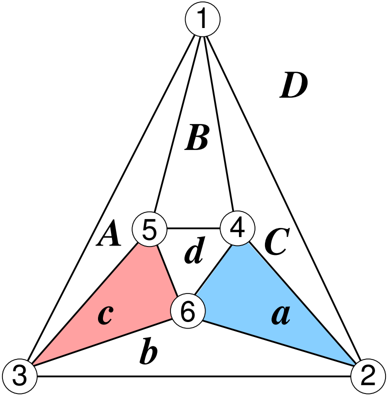
Room partition $\boldsymbol{A, a} \ = \ \{\mathbf{1, 3, 5}\}, \ \{\mathbf{2, 4, 6}\}$

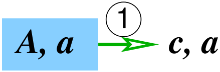


***A, a***

# How to orient room partitions?

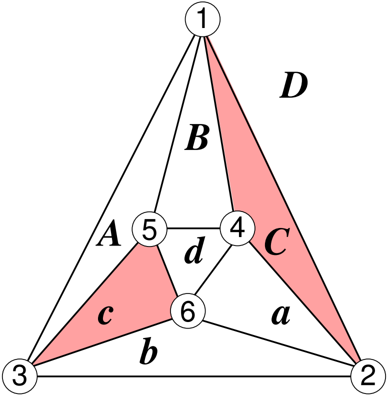
Room partition ***A, a*** : drop node **1**

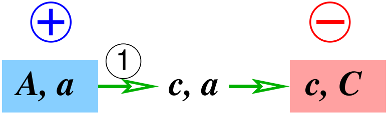




# How to orient room partitions?

Room partition ***A*, *a***, sign ⊕: drop node **1** leads to ***c*, *C***, sign ⊖

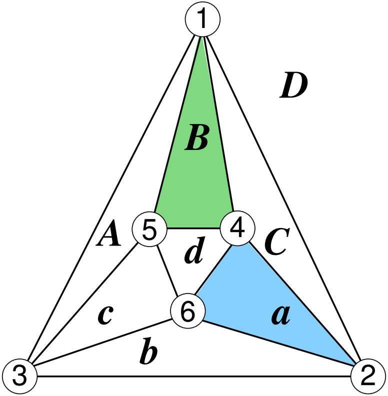


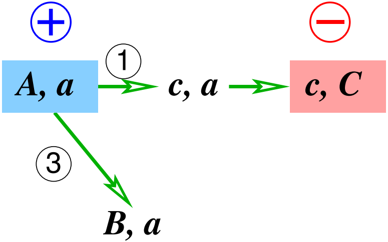


***A*, *a*** → ***c*, *a*** → ***c*, *C***

# How to orient room partitions?

Room partition ***A*, *a***, sign ⊕: drop node **3**





⊕

⊖

*A*, *a*

①

*c*, *a*

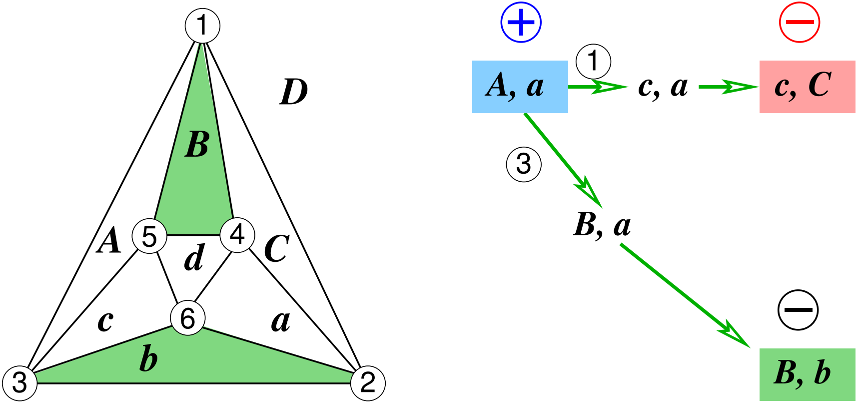
*c*, *C*

③

*B*, *a*

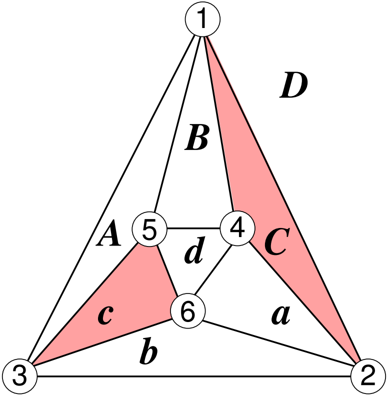
# How to orient room partitions?

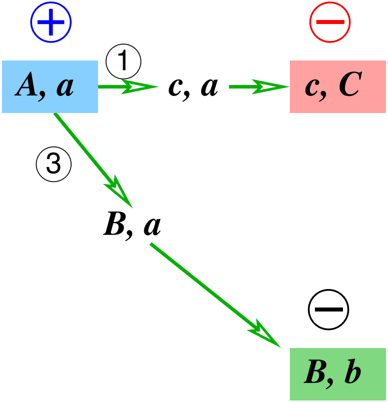
Room partition ***A*, *a***, sign ⊕: drop node **3** leads to ***B*, *b***, sign ⊖



# How to orient room partitions?

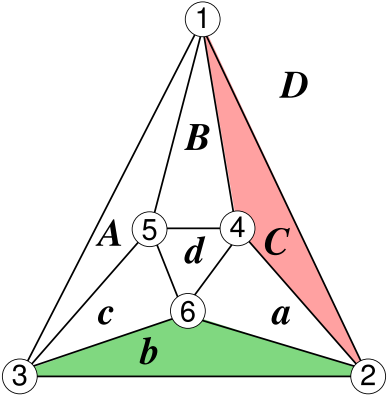
Room partition ***c*, *C***, sign ⊖: drop node **5**

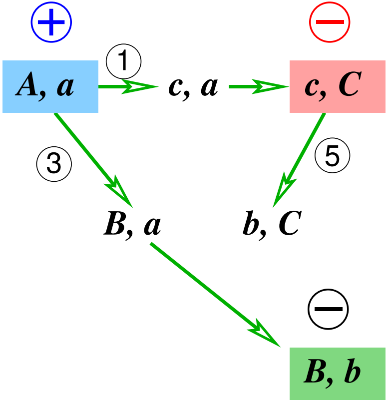




# How to orient room partitions?

Room partition ***c*, *C***, sign ⊖: drop node **5**





⊕ ⊖

***A*, *a*** ① ***c*, *a*** → ***c*, *C***

③ ***B*, *a***

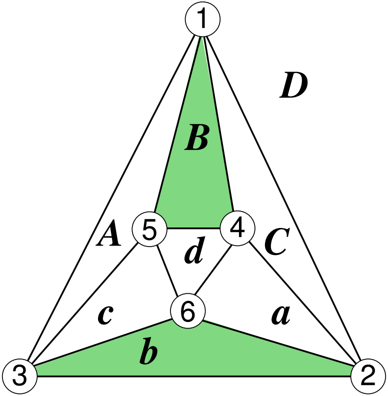
⑤ ***b*, *C***

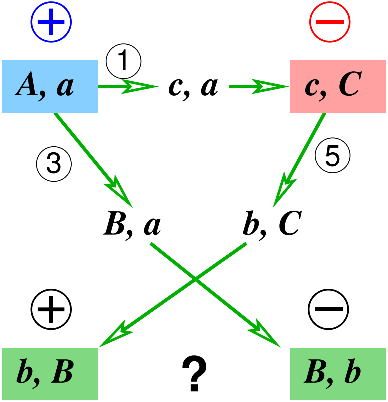
⊖

***B*, *b***

# How to orient room partitions?

Room partition ***c***, ***C***, sign ⊖: drop node **5** leads to ***b***, ***B***, sign ⊕





⊕ ⊖

***A, a*** ① ***c, a*** ***c, C***

③ ⑤

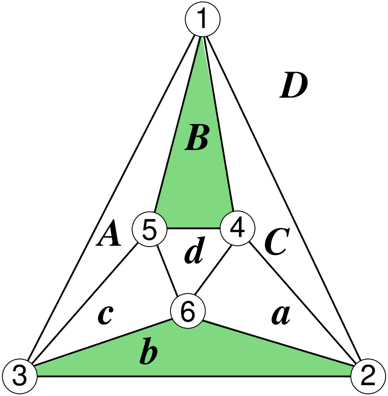
***B, a*** ***b, C***

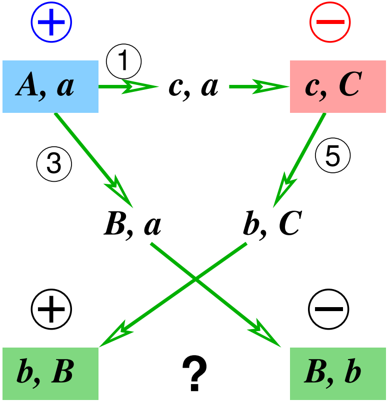
⊕ ⊖

***b, B*** **?** ***B, b***

# How to orient room partitions?

Which sign for $\{b, B\}$?





+

−

*A, a*

①

*c, a*

*c, C*

③

⑤

*B, a*

*b, C*

+

−

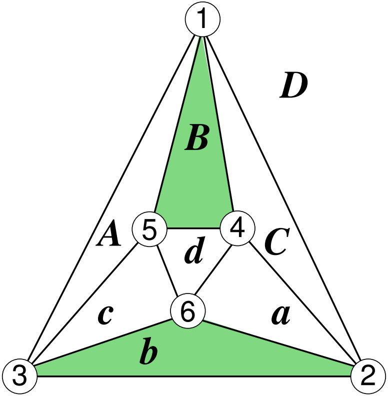
*b, B*

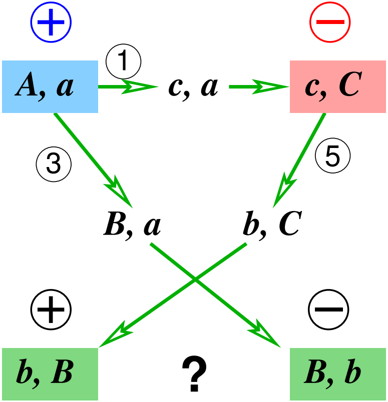
**?**

*B, b*

# How to orient room partitions?

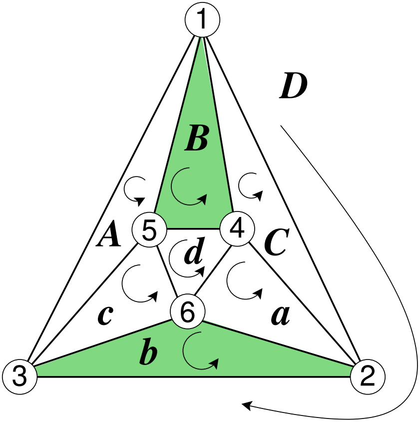
Which sign for {***b***, ***B***}? ⊕ for ***b***, ***B***, ⊖ for ***B***, ***b*** !





# How to orient room partitions?

⇒ for odd dimension (here $\boldsymbol{d} = \mathbf{3}$), order of rooms matters: permutations **263 154** (for ***b***, ***B***) and **154 263** (for ***B***, ***b***) have opposite parity.



⊖ ***b, B***

⊕ ***B, b***

# Ordered room partitions

**Theorem** [Végh / von Stengel 2014]

Let $\mathcal{R}$ be an oriented $d$-oik with node set $V$. Then the number of **ordered room partitions** $(R_1, \ldots, R_{|V|/d})$ is **even.**

Any two ordered room partitions connected by a pivoting path have opposite **sign**, and the respective unordered partitions are distinct.

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---

Proof uses “pivoting systems” with **labels** = nodes.

Pivoting systems generalize labeled polytopes, Lemke’s algorithm, Sperner’s lemma, room partitions in oiks, and more.

# Summary of results

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- if pivoting is sign-switching (orientability)
  $\Rightarrow$ endpoints of paths have opposite **signs** ⊕ ⊖
- opposite-signed **matching in Euler graph** found in linear time
- exponentially long paths for matchings in Euler graph emulate exponentially long Lemke–Howson paths in **games**
- we can orient oiks and room partitions (in odd dimension: need *ordered* partition).

Thank you!